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A Solution to the Navier–Stokes Inequality with an Internal Singularity

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Abstract. We consider weak solutions to the time dependent Navier–Stokes equations of incompressible fluid flow in three dimensional space with an external force that always acts against the direction of the flow. We show that there exists a solution with an internal singularity. The speed of the flow reaches infinity at this singular point. In addition, the solution has finite kinetic energy.

Section 1. Introduction

The purpose of this paper is to prove Theorem 1.1 below. The statement of this theorem is followed by an informal explanation of what it says.

Definition. If f is a function defined on an open subset of $R^3 \times R$, then the laplacian and the gradient of f will involve only the R^3 variables. Thus,

$$\Delta f(x,t) = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(x,t) \quad \text{and} \quad \nabla f(x,t) = \left(\frac{\partial f}{\partial x_1}(x,t), \frac{\partial f}{\partial x_2}(x,t), \frac{\partial f}{\partial x_3}(x,t) \right).$$

The norm $|f|$ will always be the euclidean norm. For example, in (1.7) we have $|\nabla u|^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2$.

Theorem 1.1. *There exist functions* $u: R^3 \times [0, \infty) \to R^3$ *and* $p: R^3 \times [0, \infty) \to R$ *with the following properties*:

$$\text{there is a compact set } K \subset R^3 \text{ such that } u(x,t) = 0 \quad \text{for all } x \notin K, \tag{1.1}$$

$$\text{for fixed } t, \text{ the function } u_t: R^3 \to R^3 \text{ defined by } u_t(x) = u(x,t) \text{ is a } C^\infty \text{ function,} \tag{1.2}$$

$$\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}(x,t) = 0, \tag{1.3}$$

$$p(x,t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}(y,t) \frac{\partial u_i}{\partial x_j}(y,t) (4\pi|x-y|)^{-1} dy, \tag{1.4}$$

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there exists $M<\infty$ *such that* $\|u_t\|_2<M$ *for all* t, (u_t is defined in (1.2)), (1.5)

$$|\nabla u|^2,\ |u|^3 \text{ and } |u|\,|p| \text{ are integrable,} \tag{1.6}$$

if $\phi: R^3\times(0,\infty)\to R$ *is a* C^∞ *function with compact support and* $\phi\geqq 0$, *then*

$$\int_0^\infty\int_{R^3}|\nabla u|^2\phi \leqq \int_0^\infty\int_{R^3}(2^{-1}|u|^2+p)u\cdot\nabla\phi+\int_0^\infty\int_{R^3}2^{-1}|u|^2\left(\frac{\partial\phi}{\partial t}+\Delta\phi\right), \tag{1.7}$$

u is not essentially bounded on any neighborhood of the point (0, 1) (*which is an interior point of the domain of u*). (1.8)

Before explaining this, we digress to place this theorem in the context of other papers. Let $S(u)$ be the set of points $(x,t)\in R^3\times(0,\infty)$ satisfying the following condition: If U is a neighborhood of (x,t) (in the natural topology of $R^3\times(0,\infty)$), then u is not essentially bounded on U. The first theorem in [4] implies the following: if $u: R^3\times[0,\infty)\to R^3$ and $p: R^3\times[0,\infty)\to R$ satisfy (1.1)–(1.7), then the Hausdorff dimension of S(u) is at most 2. The definition of Hausdorff dimension is given in [3], starting on page 171. Under slightly different conditions, *L.* Caffarelli, R. Kohn and L. Nirenberg have shown in [1] that the Hausdorff dimension of S(u) is at most 1. Theorem 1.1, which says $(0,1)\in$ S(u), shows that conditions (1.1)–(1.7) are not enough to imply that S(u) is empty.

I have found examples in which (1.1)–(1.7) hold and the Hausdorff dimension of S(u) is equal to $1-\varepsilon$ for any preassigned ε between 0 and 1. This shows that the Caffarelli, Kohn, Nirenberg estimate is the best possible. Since these examples are an order of magnitude more complex than what is presented in this paper, I am postponing publication until I can bring the exposition into a more readable form.

Now we give an informal discussion of the connection between (1.1)–(1.8) and the Navier–Stokes equations. If R^3 is space, $[0,\infty)$ is time, and u: $R^3\times[0,\infty)\to R^3$ is a solution to the Navier–Stokes equations of incompressible, time-dependent fluid flow with viscosity = 1, then we have the classical equations

$$\frac{\partial}{\partial t}u_i=-\sum_{j=1}^3 u_j\frac{\partial u_i}{\partial x_j}-\frac{\partial p}{\partial x_i}+\Delta u_i \quad\text{and}\quad \sum_{i=1}^3\frac{\partial u_i}{\partial x_i}=0, \tag{1.9}$$

where $p: R^3\times[0,\infty)\to R$ is the pressure. In this paper we consider solutions to the *Navier–Stokes inequality*, which can be written as (1.10) and (1.11):

$$\frac{\partial}{\partial t}u_i=-\sum_{j=1}^3 u_j\frac{\partial u_i}{\partial x_i}-\frac{\partial p}{\partial x_i}+\Delta u_i+f_i,\quad \sum_{i=1}^3\frac{\partial u_i}{\partial x_i}=0, \tag{1.10}$$

$$\sum_{i=1}^3 f_i u_i\leqq 0,\quad \sum_{i=1}^3\frac{\partial f_i}{\partial x_i}=0, \tag{1.11}$$

where $f: R^3\times[0,\infty)\to R^3$ is a function and u, p have the same domain and range as before. Condition (1.10) says that there is an external force f acting on the viscous, incompressible fluid. The first part of (1.11) says that this force does not increase the magnitude of the velocity vector at any point in space-time. We will give an informal proof of the following:

Assertion 1. Properties (1.3), (1.4), (1.7) imply that the Navier–Stokes inequality holds in a weak sense.

This says that Theorem 1.1 can be rewritten informally as follows:

Assertion 2. There exists a solution to the Navier–Stokes inequality with an internal singularity.

What we would really like to do is to prove the following conjecture:

Conjecture 1. There exists a solution to the Navier–Stokes *equations* with an internal singularity.

This paper is a step towards proving this conjecture. At first glance, it seems that the presence of f makes it more difficult to come up with an example of an interior singularity. After all, the inequality $\sum_{i=1}^{3} f_i u_i \leqq 0$ only makes it harder for u to become unbounded. The force f tends to push down on $|u|$. This argument seems to imply that Theorem 1.1 is at least as strong as Conjecture 1. Actually, the introduction of f gives us more freedom in constructing an example because we can decrease $|u|$ at some points in order to create a situation in which the pressure term causes an increase in $|u|$ at some other points at a later time. In summary, Conjecture 1 is more difficult to prove than Theorem 1.1, but this theorem suggests that Conjecture 1 should be true.

Now we present the informal proof of Assertion 1. The identity $u\cdot\Delta u = 2^{-1}\Delta|u|^2 - |\nabla u|^2$, (1.3), (1.7) and integration by parts give us

$$\int_0^\infty \int_{R^3} \left(u\cdot\frac{\partial u}{\partial t} + u\cdot\nabla(2^{-1}|u|^2 + p) - u\cdot\Delta u \right)\phi \leqq 0$$

if $\phi: R^3 \times (0,\infty) \to R$ is C^∞ with compact support and $\phi \geqq 0$. This is a weak form of

$$u\cdot\frac{\partial u}{\partial t} + u\cdot\nabla(2^{-1}|u|^2 + p) - u\cdot\Delta u \leqq 0,$$

which can be rewritten as

$$u\cdot\frac{\partial u}{\partial t} + u\cdot\left(\sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j}\right) + u\cdot\nabla p - u\cdot\Delta u \leqq 0.$$

Setting

$$f = \frac{\partial u}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p - \Delta u, \tag{1.12}$$

we conclude the first part of (1.11). If we take the divergence of (1.12) and use (1.3), then we obtain

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \Delta p.$$

Since (1.4) yields $\Delta p = -\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$, the above implies the second part of (1.11). Property (1.10) follows from (1.12) and (1.3).

It should be emphasized that the f in the example is very singular. This means that our solution is nonphysical. It is possible to smooth f a bit and make it C^∞ on $R^3 \times (0,1)$, but f would still be a bizarre function that would not come up in real life situations. Nevertheless, the example shows that energy-based methods are not enough to prove the regularity of weak solutions to the Navier–Stokes equations. After all, these methods are based on (1.7) and (1.3) rather than (1.9).

The key ideas in the construction are the following:

(A) *Changing the Direction of* u *at Discrete Times.* If u is a solution on a time interval (t_0, t_1) and u' is one on (t_1, t_2), then they combine to give a solution of the Navier–Stokes inequality on (t_0, t_2) iff

$$|u(x,t_1)|^2 \geqq |u'(x,t_1)|^2 \quad \text{for almost all } x \in R^3.$$

(B) *Changing the Viscosity.* We construct u in such a way that it is a solution to the Navier–Stokes inequality for all sufficiently small viscosities. Then we make the viscosity equal to 1 with a change of scale.

(C) *Self-Similarity.* If one could solve

$$2^{-1}w + 2^{-1}x\cdot\nabla w + w\cdot\nabla w - \nu\Delta w + \nabla q = g, \quad \nabla\cdot w = 0$$

for some $w(x)$, $g(x)$ with $w\cdot g \leqq 0$, then $u = (1-t)^{-1/2}\, w((1-t)^{-1/2}x)$ would be a singular solution of the Navier–Stokes inequality. The construction used in this paper is different, but similar in spirit.

(D) *Nonlocal Effects.* The example has compactly supported velocity $u = u' + u''$ with $\operatorname{spt}(u') \cap \operatorname{spt}(u'') = \emptyset$. The direction of u' is chosen to oscillate in a way that makes the magnitude of u'' grow, via the nonlocal effect of the pressure.

Section 2. Additional Definitions and Preliminaries

Definition. If U is an open subset of R^m, then $C_c^\infty(U, R^n)$ is the set of C^∞ functions from U into R^n with compact support. If $f \in C_c^\infty(U, R^n)$, then the support of f will be denoted $\operatorname{spt}(f)$. If $\{f_1, f_2\} \subset C_c^\infty(R^3, R)$, $f_i \geqq 0$, $a < b$ and $\nu > 0$ then the 5-tuple (f_1, f_2, a, b, ν) is called *admissible* when there exist functions $u: R^3 \times [a,b] \to R^3$ and $p: R^3 \times [a,b] \to R$ satisfying (1.1)–(1.6) and the condition

$$\int_{R^3} 2^{-1}(f_2(x))^2\phi(x,b)\,dx - \int_{R^3} 2^{-1}(f_1(x))^2\phi(x,a)\,dx + \int_a^b\int_{R^3} \nu|\nabla u|^2\phi$$
$$\leqq \int_a^b\int_{R^3} (2^{-1}|u|^2 + p)u\cdot\nabla\phi + \int_a^b\int_{R^3} 2^{-1}|u|^2\left(\frac{\partial\phi}{\partial t} + \nu\Delta\phi\right)$$

for every $\phi \in C_c^\infty(R^3 \times R, R)$ with the property $\phi \geqq 0$. We set

$$P = \{(x_1, x_2) \in R^2 : x_2 > 0\}.$$

If $c=(c_1,c_2)\in R^2$ and $c_1^2+c_2^2=1$, then $R_c\colon R^3\to R^3$ is the rotation

$$R_c(x_1,x_2,x_3)=(x_1,c_1x_2-c_2x_3,c_1x_3+c_2x_2)$$

about the x_1 axis. If $\{g_1,g_2\}\subset C_c^\infty(P,R)$, $g_i\geqq 0$, $a<b$ and $v>0$, then the 5-tuple (g_1,g_2,a,b,v) is called *P-admissible* if there exists $\{f_1,f_2\}\subset C_c^\infty(R^3,R)$ such that

If $c\in R^2$, $|c|=1$ and $(x_1,x_2)\in P$, then $f_i(R_c(x_1,x_2,0))=g_i(x_1,x_2)$, (f_1,f_2,a,b,v) is admissible. If $f\in C_c^\infty(P,R)$, $v=(v_1,v_2)\in C_c^\infty(P,R^2)$, $f\geqq 0$, and $f(x)>|v(x)|$ holds for all $x\in\operatorname{spt}(v)$, then $p^*[v,f]$ is the C^∞ function from R^3 into R defined by

$$p^*[v,f](x)=\int_{R^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[v,f]\right)(y)\left(\frac{\partial}{\partial x_j}u_i[v,f]\right)(y)(4\pi|x-y|)^{-1}dy,$$

where $u[v,f]=(u_1[v,f],u_2[v,f],u_3[v,f])\in C_c^\infty(R^3,R^3)$ is given by

$$u[v,f](x_1,x_2,0)=(v_1(x_1,x_2),v_2(x_1,x_2),((f(x_1,x_2))^2-|v(x_1,x_2)|^2)^{1/2})\quad\text{if}\,(x_1,x_2)\in P,$$

$$u[v,f](R_c(x_1,x_2,0))=R_c(u[v,f](x_1,x_2,0))\quad\text{if}\,c\in R^2,|c|=1,(x_1,x_2)\in P,$$

$$u[v,f](x_1,0,0)=0.$$

Note that u equals 0 in a neighborhood of the x_1 axis. We also set

$$p[v,f](x_1,x_2)=p^*[v,f](x_1,x_2,0)\quad\text{if }(x_1,x_2)\in P.$$

If A and B are sets we define $A\sim B=\{x\in A\colon x\notin B\}$. If $f\in C_c^\infty(P,R)$ we set

$$L(f)(x_1,x_2)=\Delta f(x_1,x_2)+x_2^{-1}\left(\frac{\partial f}{\partial x_2}(x_1,x_2)\right)-x_2^{-2}f(x_1,x_2)\quad\text{if }(x_1,x_2)\in P.$$

We will often encounter repeated indices (for example, in (2.6)). These indices are *not* summed unless there is a summation sign. In other words, the summation convention for repeated indices is never used. Furthermore, the function v_i in (2.3) and (3.2) is *not* a component of a vector valued function. Rather, we have two unrelated functions, v_1 and v_2, and these functions have components v_{11}, v_{12} and v_{21}, v_{22}, respectively.

Lemma 2.1. *Suppose that* a, b, J, C_1, C_2, C_1', C_2', S_1, S_2, η, v_1, v_2, q_1, q_2 *satisfy conditions* (2.1)–(2.9):

$$a<b,\quad J\text{ is an open set containing }[a,b],\tag{2.1}$$

$$C_i'\subset C_i\subset P,\quad C_i\text{ and }C_i'\text{ are compact}, C_1\text{ and }C_2\text{ are disjoint},\tag{2.2}$$

$$S_i\in\{1,-1\},\quad \eta>0,\quad v_i=(v_{i1},v_{i2})\in C_c^\infty(P,R^2),\quad \operatorname{spt}(v_i)\subset C_i',\tag{2.3}$$

$$q_i\colon P\times J\to R\text{ is a }C^\infty\text{ function},\quad q_i(x,t)=0\quad\text{if }x\notin C_i,\tag{2.4}$$

$$q_i\geqq 0,\quad q_i(x,t)>|v_i(x)|\quad\text{if }x\in\operatorname{spt}(v_i),\tag{2.5}$$

If $x\in C_i'$ then, using the notation $q_{i,t}(x)=q_i(x,t)$, we have $\dfrac{\partial}{\partial t}2^{-1}(q_i(x,t))^2$

$$\leqq -\eta-S_iv_i(x)\cdot\nabla(2^{-1}(q_{i,t})^2+p[v_1,q_{1,t}]+p[v_2,q_{2,t}])(x),\tag{2.6}$$

$$\frac{\partial}{\partial t}2^{-1}(q_i(x,t))^2 \leqq 0 \quad \textit{if } x \notin C_i', \tag{2.7}$$

$$L(q_{i,t})(x) \geqq 0 \quad \textit{if } x \notin C_i' \, (\textit{see } (2.6)), \tag{2.8}$$

$$x_2 \frac{\partial}{\partial x_1} v_{i1}(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} v_{i2}(x_1, x_2) + v_{i2}(x_1, x_2) = 0. \tag{2.9}$$

Then there exists $v_0 > 0$ *such that the 5-tuple*

$$(q_{1,a} + q_{2,a}, q_{1,b} + q_{2,b}, a, b, v)$$

is P-admissible for all v *satisfying* $0 < v < v_0$ *(where* $q_{i,t}(x) = q_i(x,t)$*).*

Proof. Hypotheses (2.1)–(2.5) and the notation in (2.6) allow us to define $u^i: R^3 \times J \to R^3$ and $p^i: R^3 \times J \to R$ for $i = 1, 2$ using the formulas

$$u^i(x,t) = S_i u[v_i, q_{i,t}](x), \quad p^i(x,t) = p^*[v_i, q_{i,t}](x). \tag{2.10}$$

If E is a subset of P we define $R(E) \subset R^3$ as follows:

$$R(E) = \{R_c(x_1, x_2, 0): c \in R^2, \quad |c| = 1, \quad (x_1, x_2) \in E\}. \tag{2.11}$$

Using (2.1)–(2.4) we obtain

$$u^i \text{ is a } C^\infty \text{ function on } R^3 \times J, \; u^i(x,t) = 0 \quad \text{if } x \notin R(C_i). \tag{2.12}$$

In addition, (2.11) and (2.2) imply

$$R(C_i') \subset R(C_i), R(C_i) \text{ and } R(C_i') \text{ are compact}, R(C_1) \text{ and } R(C_2) \text{ are disjoint.} \tag{2.13}$$

We let $p: R^3 \times J \to R$ be the C^∞ function given by

$$p(x,t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial}{\partial x_i}(u_j^1 + u_j^2)(y,t) \frac{\partial}{\partial x_j}(u_i^1 + u_i^2)(y,t)(4\pi|x-y|)^{-1} dy. \tag{2.14}$$

Using (2.12), (2.13) we find $\partial/\partial x_i u_j^m(y,t) \, \partial/\partial x_j u_i^n(y,t) = 0$ if $m \neq n$. This fact, the identity $(S_i)^2 = 1$ (see (2.3)), and (2.10), (2.14) imply

$$p(x,t) = p^*[v_1, q_{1,t}](x) + p^*[v_2, q_{2,t}](x) = (p^1 + p^2)(x,t), \text{ hence}$$

$$p(x_1, x_2, 0, t) = p[v_1, q_{1,t}](x_1, x_2) + p[v_2, q_{2,t}](x_1, x_2) \quad \text{if } (x_1, x_2) \in P. \tag{2.15}$$

From (2.12), (2.13), (2.10), (2.3) we conclude

$$|(u^1 + u^2)(x_1, x_2, 0, t)|^2 = |u^1(x_1, x_2, 0, t)|^2 + |u^2(x_1, x_2, 0, t)|^2$$
$$= (q_1(x_1, x_2, t))^2 + (q_2(x_1, x_2, t))^2 \quad \text{if } (x_1, x_2) \in P. \tag{2.16}$$

If $x \in C_1' \cup C_2'$ then (2.6), the disjointness of C_1 and C_2, and (2.2)–(2.4) yield

$$\frac{\partial}{\partial t} 2^{-1}((q_1(x,t))^2 + (q_2(x,t))^2)$$
$$\leqq -\eta - (S_1 v_1(x) + S_2 v_2(x)) \cdot \nabla(2^{-1}((q_{1,t})^2$$
$$+ (q_{2,t})^2))(x) - (S_1 v_1(x) + S_2 v_2(x)) \cdot \nabla(p[v_1, q_{1,t}] + p[v_2, q_{2,t}])(x).$$

Combining this with (2.16), the identity

$$u_k^i(x_1,x_2,0,t) = S_i v_{ik}(x_1,x_2) \quad \text{for} \quad k=1,2$$

(see (2.10)) and (2.15), we find that every $(x_1,x_2)\in C_1' \cup C_2'$ satisfies

$$\frac{\partial}{\partial t} 2^{-1}|u^1+u^2|^2(x_1,x_2,0,t)$$
$$\leqq -\eta - \sum_{k=1}^{2} (u_k^1+u_k^2)(x_1,x_2,0,t)\frac{\partial}{\partial x_k}(2^{-1}|u^1+u^2|^2+p)(x_1,x_2,0,t).$$

The rotational symmetry of $|u^1+u^2|^2$ and p about the x_1 axis implies

$$\frac{\partial}{\partial x_3}(2^{-1}|u^1+u^2|^2+p)(x_1,x_2,0,t)=0 \quad \text{if } (x_1,x_2)\in P. \tag{2.17}$$

Hence, using the 3-dimensional gradient ∇, we obtain

$$\frac{\partial}{\partial t} 2^{-1}|u^1+u^2|^2(x_1,x_2,0,t)$$
$$\leqq -\eta - (u^1+u^2)(x_1,x_2,0,t)\cdot\nabla(2^{-1}|u^1+u^2|^2+p)(x_1,x_2,0,t)$$

if $(x_1,x_2)\in C_1'\cup C_2'$. The rotational symmetry and (2.11) allow us to rewrite the above as

$$\frac{\partial}{\partial t} 2^{-1}|u^1+u^2|^2(x,t)$$
$$\leqq -\eta - (u^1+u^2)(x,t)\cdot\nabla(2^{-1}|u^1+u^2|^2+p)(x,t) \quad \text{if } x\in R(C_1')\cup R(C_2').$$

Since (2.3), (2.13) imply that η is positive and $R(C_1')\cup R(C_2')$ is compact, we can use (2.1), (2.12) to find $\nu_0>0$ with the following property:

$$\frac{\partial}{\partial t} 2^{-1}|u^1+u^2|^2(x,t) + \nu|\nabla(u^1+u^2)|^2(x,t)$$
$$\leqq -(u^1+u^2)(x,t)\cdot\nabla(2^{-1}|u^1+u^2|^2+p)(x,t) + \nu\Delta(2^{-1}|u^1+u^2|^2)(x,t)$$
$$\text{if } x\in R(C_1')\cup R(C_2'), t\in[a,b] \quad \text{and} \quad 0<\nu<\nu_0. \tag{2.18}$$

From (2.3), (2.5) we obtain

$$u[v_i, q_{i,t}](z_1,z_2,0) = (0,0,q_{i,t}(z_1,z_2)) \quad \text{if } (z_1,z_2)\in P\sim(C_1'\cup C_2'). \tag{2.19}$$

If $(x_1,x_2,x_3)=R_c(z_1,z_2,0)$ and $(z_1,z_2)\in P$, then we must have

$$c_1 = x_2(x_2^2+x_3^2)^{-1/2}, \quad c_2 = x_3(x_2^2+x_3^2)^{-1/2}, \quad z_1 = x_1, \quad z_2=(x_2^2+x_3^2)^{1/2}.$$

The above, (2.19) and (2.11) imply

$$u_3[v_i,q_{i,t}](x_1,x_2,x_3) = c_1 q_{i,t}(z_1,z_2) = x_2(x_2^2+x_3^2)^{-1/2}q_{i,t}(x_1,(x_2^2+x_3^2)^{1/2})$$

if $(x_1,x_2,x_3)\in R(P\sim(C_1'\cup C_2'))$. This fact and (2.10) yield

$$(u_3^1+u_3^2)(x_1,x_2,x_3,t) = x_2(x_2^2+x_3^2)^{-1/2}(S_1q_{1,t}+S_2q_{2,t})(x_1,(x_2^2+x_3^2)^{1/2})$$

for $(x_1, x_2, x_3) \in R(P \sim (C'_1 \cup C'_2))$. This gives us

$$\Delta(u_3^1 + u_3^2)(x_1, x_2, 0, t) = (S_1 L(q_{1,t}) + S_2 L(q_{2,t}))(x_1, x_2) \quad \text{if } (x_1, x_2) \in P \sim (C'_1 \cup C'_2). \tag{2.20}$$

Using (2.19) and (2.10) we get

$$(u^1 + u^2)(x_1, x_2, 0, t) = (0, 0, (S_1 q_{1,t} + S_2 q_{2,t})(x_1, x_2)) \quad \text{if } (x_1, x_2) \in P \sim (C'_1 \cup C'_2). \tag{2.21}$$

From (2.20), (2.21), (2.2), (2.4) we obtain

$$\begin{aligned} &((u^1 + u^2) \cdot \Delta(u^1 + u^2))(x_1, x_2, 0, t) \\ &= (S_1 q_{1,t} + S_2 q_{2,t})(x_1, x_2)(S_1 L(q_{1,t}) + S_2 L(q_{2,t}))(x_1, x_2) \\ &= \sum_{i=1}^{2} (S_i)^2 q_{i,t}(x_1, x_2) L(q_{i,t})(x_1, x_2) \end{aligned}$$

if $(x_1, x_2) \in P \sim (C'_1 \cup C'_2)$. Combining this with (2.3), (2.5), (2.8) we find

$$((u^1 + u^2) \cdot \Delta(u^1 + u^2))(x_1, x_2, 0, t) \geqq 0 \quad \text{if } (x_1, x_2) \in P \sim (C'_1 \cup C'_2)$$

The above, (2.11) and the rotational symmetry of $u^1 + u^2$ imply

$$((u^1 + u^2) \cdot \Delta(u^1 + u^2))(x, t) \geqq 0 \quad \text{if } x \in R(P \sim (C'_1 \cup C'_2)). \tag{2.22}$$

The general formula $\Delta(2^{-1}|f|^2) - |\nabla f|^2 = f \cdot \Delta f$ and (2.22) imply

$$\Delta(2^{-1}|u^1 + u^2|^2)(x, t) - |\nabla(u^1 + u^2)|^2(x, t) \geqq 0 \quad \text{if } x \in R(P \sim (C'_1 \cup C'_2)). \tag{2.23}$$

From (2.9) and (2.10) we conclude

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(u_i^1 + u_i^2) = 0. \tag{2.24}$$

Using (2.16) and (2.7) we find

$$\frac{\partial}{\partial t} 2^{-1}|u^1 + u^2|^2(x_1, x_2, 0, t) \leqq 0 \quad \text{if } (x_1, x_2) \in P \sim (C'_1 \cup C'_2).$$

The above, (2.11) and rotational symmetry yield

$$\frac{\partial}{\partial t} 2^{-1}|u^1 + u^2|^2(x, t) \leqq 0 \quad \text{if } x \in R(P \sim (C'_1 \cup C'_2)). \tag{2.25}$$

If $(x_1, x_2) \in P \sim (C'_1 \cup C'_2)$ then (2.3), (2.10) imply $(u_k^1 + u_k^2)(x_1, x_2, 0, t) = 0$ for $k = 1, 2$. This fact and (2.17) yield

$$((u^1 + u^2) \cdot \nabla(2^{-1}|u^1 + u^2|^2 + p))(x_1, x_2, 0, t) = 0 \quad \text{if } (x_1, x_2) \in P \sim (C'_1 \cup C'_2).$$

The above, (2.11) and rotational symmetry imply

$$((u^1 + u^2) \cdot \nabla(2^{-1}|u^1 + u^2|^2 + p))(x, t) = 0 \quad \text{if } x \in R(P \sim (C'_1 \cup C'_2)). \tag{2.26}$$

Combining (2.23), (2.25), (2.26) we obtain

$$\frac{\partial}{\partial t} 2^{-1}|u^1 + u^2|^2(x, t) + \nu|\nabla(u^1 + u^2)|^2(x, t)$$

$$\leqq -((u^1+u^2)\cdot\nabla(2^{-1}|u^1+u^2|^2+p))(x,t)+\nu\Delta(2^{-1}|u^1+u^2|^2)(x,t)$$
$$\text{if } x\in R(P\sim(C_1'\cup C_2')) \text{ and } \nu>0. \tag{2.27}$$

Suppose $\phi\in C_c^\infty(R^3\times R,R)$ satisfies $\phi\geqq 0$ and ν satisfies $0<\nu<\nu_0$. Multiplying (2.18) and (2.27) by $\phi(x,t)$, recalling (2.11) and (2.12), integrating over $R^3\times[a,b]$, and applying integration by parts and (2.24) we obtain

$$\int_{R^3} 2^{-1}|u^1+u^2|^2(x,b)\phi(x,b)dx-\int_{R^3} 2^{-1}|u^1+u^2|^2(x,a)\phi(x,a)$$
$$\cdot dx+\int_a^b\int_{R^3}\nu|\nabla(u^1+u^2)|^2\phi$$
$$\leqq\int_a^b\int_{R^3}(2^{-1}|u^1+u^2|^2+p)(u^1+u^2)\cdot\nabla\phi+\int_a^b\int_{R^3}2^{-1}|u^1+u^2|^2\left\{\frac{\partial\phi}{\partial t}+\nu\Delta\phi\right\}.$$

The conclusion follows from (2.2)–(2.4), (2.10)–(2.12), (2.14), (2.24), the definitions $f_1(x)=|u^1+u^2|(x,a)$ and $f_2(x)=|u^1+u^2|(x,b)$, the rotational symmetry of f_i and (2.16).

Lemma 2.2. *If $a<b<c$, (g_1,g_2,a,b,ν) is P-admissible and (g_2,g_3,b,c,ν) is P-admissible then (g_1,g_3,a,c,ν) is P-admissible.*

Proof. This is elementary.

Lemma 2.3. *If τ,T,ν are real numbers, $0<\tau<1,T>0,\nu>0,a\in R^3$, $\{f_1,f_2\}\subset C_c^\infty(R^3,R), f_i\geqq 0, f_1\neq 0, (f_1,f_2,0,T,\nu)$ is admissible and the inequality*

$$f_2(\tau x+a)\geqq\tau^{-1}f_1(x)\quad \text{if } x\in R^3 \tag{2.28}$$

is satisfied, then there exist functions $u:R^3\times[0,\infty)\to R^3$ and $p:R^3\times[0,\infty)\to R$ such that (1.1)–(1.6) *are satisfied and, in addition, we have* (2.29) *and* (2.30):

if $\phi:R^3\times(0,\infty)\to R$ is C^∞ with compact support and $\phi\geqq 0$ then $\int_0^\infty\int_{R^3}\nu|\nabla u|^2\phi$

$$\leqq\int_0^\infty\int_{R^3}(2^{-1}|u|^2+p)u\cdot\nabla\phi+\int_0^\infty\int_{R^3}2^{-1}|u|^2\left\{\frac{\partial\phi}{\partial t}+\nu\Delta\phi\right\}, \tag{2.29}$$

u is not essentially bounded on any neighborhood of the point

$$((1-\tau)^{-1}a,(1-\tau^2)^{-1}T). \tag{2.30}$$

Proof. Since $(f_1,f_2,0,T,\nu)$ is admissible, we can find functions $u^1:R^3\times[0,T]\to R^3$ and $p^1:R^3\times[0,T]\to R$ satisfying (1.1)–(1.6) (with u and p replaced by u^1 and p^1) and the condition

$$\int_{R^3}2^{-1}(f_2(x))^2\phi(x,T)dx-\int_{R^3}2^{-1}(f_1(x))^2\phi(x,0)dx+\int_0^T\int_{R^3}\nu|\nabla u^1|^2\phi$$
$$\leqq\int_0^T\int_{R^3}(2^{-1}|u^1|^2+p^1)u^1\cdot\nabla\phi+\int_0^T\int_{R^3}2^{-1}|u^1|^2\left\{\frac{\partial\phi}{\partial t}+\nu\Delta\phi\right\}$$
$$\text{if } \phi\in C_c^\infty(R^3\times R,R) \text{ and } \phi\geqq 0. \tag{2.31}$$

Recalling the hypothesis $0<\tau<1$, we set $T_j=\left(\sum_{k=0}^{j-1}\tau^{2k}\right)T$ for $j=0,1,2,\ldots$ (with the convention $T_0=0$) and $T_\infty=\lim_{j\to\infty}T_j=(1-\tau^2)^{-1}T$. The functions $f_i^j:R^3\to R, u^j:R^3\times[T_{j-1},T_j]\to R^3$ and $p^j:R^3\times[T_{j-1},T_j]\to R$ are defined inductively by the equations

$$f_i^1=f_i, f_i^{j+1}(x)=\tau^{-1}f_i^j(\tau^{-1}(x-a)) \quad \text{if } i=1,2,$$
$$u^{j+1}(x,t)=\tau^{-1}u^j(\tau^{-1}(x-a),\tau^{-2}(t-T)),$$
$$p^{j+1}(x,t)=\tau^{-2}p^j(\tau^{-1}(x-a),\tau^{-2}(t-T))$$

for $j\in\{1,2,3,\ldots\}$, where u^1 and p^1 are the functions that appear in (2.31). The required functions u and p are given by the following conditions:

$$\text{if } T_{j-1}\leqq t<T_j, \text{ then } u(x,t)=u^j(x,t) \quad\text{and}\quad p(x,t)=p^j(x,t),$$
$$\text{if } T_\infty\leqq t, \text{ then } u(x,t)=0 \quad\text{and}\quad p(x,t)=0.$$

From (2.31) we conclude

$$\int_{R^3}2^{-1}(f_2^j(x))^2\phi(x,T_j)dx-\int_{R^3}2^{-1}(f_1^j(x))^2\phi(x,T_{j-1})dx+\int_{T_{j-1}}^{T_j}\int_{R^3}\nu|\nabla u^j|^2\phi$$
$$\leqq\int_{T_{j-1}}^{T_j}\int_{R^3}(2^{-1}|u^j|^2+p^j)u^j\cdot\nabla\phi+\int_{T_{j-1}}^{T_j}\int_{R^3}2^{-1}|u^j|^2\left\{\frac{\partial\phi}{\partial t}+\nu\Delta\phi\right\}$$
$$\text{if } \phi\in C_c^\infty(R^3\times R,R) \quad\text{and}\quad \phi\geqq 0. \tag{2.32}$$

Since f_i is nonnegative and (2.28) implies $f_2^1(x)\geqq\tau^{-1}f_1^1(\tau^{-1}(x-a))=f_1^2(x)$ for all $x\in R^3$, we can use induction to show

$$(f_2^j(x))^2\geqq(f_1^{j+1}(x))^2 \quad \text{for all } x\in R^3 \quad\text{and}\quad j=1,2,3,\ldots. \tag{2.33}$$

The definition of u^1,p^1 and $0<\tau<1$ imply

$$\|\nabla u\|_2^2=\sum_{j=1}^\infty\|\nabla u^j\|_2^2=\sum_{j=1}^\infty\tau^{j-1}\|\nabla u^1\|_2^2=(1-\tau)^{-1}\|\nabla u^1\|_2^2<\infty,$$
$$\|u\|_3^3=\sum_{j=1}^\infty\|u^j\|_3^3=\sum_{j=1}^\infty\tau^{2(j-1)}\|u^1\|_3^3=(1-\tau^2)^{-1}\|u^1\|_3^3<\infty,$$
$$\|up\|_1=\sum_{j=1}^\infty\|u^jp^j\|_1=\sum_{j=1}^\infty\tau^{2(j-1)}\|u^1p^1\|_1=(1-\tau^2)^{-1}\|u^1p^1\|_1<\infty,$$
$$\int_{R^3}|u^j(x,t)|^2dx\leqq\tau^{j-1}\sup\left\{\int_{R^3}|u^1(x,t)|^2dx:0\leqq t\leqq T\right\}\leqq\tau^{j-1}M^2. \tag{2.34}$$

Now (2.32)–(2.34) and $\lim_{j\to\infty}\tau^{j-1}M^2=0$ yield

$$-\int_{R^3}2^{-1}(f_1^1(x))^2\phi(x,0)dx+\int_0^\infty\int_{R^3}\nu|\nabla u|^2\phi$$

$$\leqq \sum_{j=1}^{\infty}\left(\int_{R^3} 2^{-1}(f_2^j(x))^2 \phi(x,T_j)dx - \int_{R^3} 2^{-1}(f_1^j(x))^2\phi(x,T_{j-1})dx\right)$$

$$+\sum_{j=1}^{\infty}\int_{T_{j-1}}^{T_j}\int_{R^3} \nu|\nabla u^j|^2\phi \leqq \int_0^{\infty}\int_{R^3}(2^{-1}|u|^2+p)u\cdot\nabla\phi + \int_0^{\infty}\int_{R^3} 2^{-1}|u|^2\left\{\frac{\partial\phi}{\partial t}+\nu\Delta\phi\right\}$$

if $\phi\in C_c^\infty(R^3\times R,R)$ and $\phi\geqq 0$. The above, (2.34) and the properties of u^1,p^1 imply (2.29) and (1.1)–(1.6). It remains to prove (2.30). From (2.28), $f_i\geqq 0$ and $f_1\neq 0$ we get $f_2\neq 0$. Let $x'\in R^3$ satisfy $(f_2(x'))^2>0$. Choosing $\phi\geqq 0$ such that $\phi(x',T)>0$ and $\phi(x,0)=0$ for all $x\in R^3$ we see from (2.31) that u^1 cannot be zero almost everywhere on $R^3\times(0,T)$. Conclusion (2.30) follows from this fact, $0<\tau<1$ and the identity

$$u\left(\tau^j x+\left(\sum_{k=0}^{j-1}\tau^k\right)a,\tau^{2j}t+\left(\sum_{k=0}^{j-1}\tau^{2k}\right)T\right)=\tau^{-j}u^1(x,t)\quad \text{if } 0<t<T.$$

Lemma 2.4. *Suppose* τ,T,ν,a_1,a_2 *are real numbers,* $0<\tau<1, T>0, \nu>0$, $\{g_1,g_2\}\subset C_c^\infty(P,R), g_i\geqq 0, g_1\neq 0$, $(g_1,g_2,0,T,\nu)$ *is P-admissible and the following property holds*: If $(x_1,x_2,x_3)\in R^3$ *and* $(x_1,(x_2^2+x_3^2)^{1/2})\in \operatorname{spt}(g_1)$, *then* $(\tau x_2+a_2)^2+(\tau x_3)^2>0$ *and*

$$g_2(\tau x_1+a_1,((\tau x_2+a_2)^2+(\tau x_3)^2)^{1/2})\geqq\tau^{-1}g_1(x_1,(x_2^2+x_3^2)^{1/2}).$$

Then Theorem 1.1 *is true.*

Proof. Let f_1,f_2 be functions in $C_c^\infty(R^3,R)$ such that $f_i\geqq 0$, $f_i(R_c(x_1,x_2,0))=g_i(x_1,x_2)$ if $c\in R^2,|c|=1$ and $(x_1,x_2)\in P$, and $(f_1,f_2,0,T,\nu)$ is admissible. Then the property $f_i\geqq 0$ implies $f_2(\tau x+(a_1,a_2,0))\geqq\tau^{-1}f_1(x)$ for $x\in R^3$. Lemma 2.3 (with $a=(a_1,a_2,0)$) implies the existence of $u:R^3\times[0,\infty)\to R^3$ and $p:R^3\times[0,\infty)\to R$ such that (1.1)–(1.6), (2.29) and (2.30) are satisfied. We obtain Theorem 1.1 if we use u, p with a change of scale (i.e., $u(x,t)$ is replaced by $au(bx,ct)$ and $p(x,t)$ is replaced by $a^2p(bx,ct)$ for appropriate a,b,c) and a translation in the space coordinates.

Section 3. The Basic Construction

Throughout this section we fix $T,\theta,K_1,K_2,U_1,U_2,f_1,f_2,v_1,v_2$ such that $T>0$, $\theta>0$, and properties (3.1)–(3.7) are satisfied:

$$K_i\subset U_i\subset P,\ K_i \text{ is compact},\ U_i \text{ is open, closure}(U_i)\subset P, \tag{3.1}$$

$$f_i\in C_c^\infty(P,R),\quad v_i=(v_{i1},v_{i2})\in C_c^\infty(P,R^2), \tag{3.2}$$

$$\text{closure}(U_1) \text{ and closure}(U_2) \text{ are disjoint compact sets, } \operatorname{spt}(v_i)\subset K_i, \tag{3.3}$$

$$f_i\geqq 0,\quad f_i(x)=0\quad \text{if } x\notin U_i,\quad f_i(x)>|v_i(x)|\quad \text{if } x\in U_i, \tag{3.4}$$

$$x_2\frac{\partial}{\partial x_1}v_{i1}(x_1,x_2)+x_2\frac{\partial}{\partial x_2}v_{i2}(x_1,x_2)+v_{i2}(x_1,x_2)=0, \tag{3.5}$$

$$(f_2(x))^2-Tv_2(x)\cdot\nabla(p[v_1,f_1]-p[0,f_1])(x)>|v_2(x)|^2\quad \text{if } x\in U_2, \tag{3.6}$$

$$L(f_i)(x)\geqq 0\quad \text{if } x\notin K_i,\quad L(f_i)(x)>0\quad \text{if } x\in U_i\sim K_i. \tag{3.7}$$

Note that (3.1)–(3.4) imply that the functions $p[v_1, f_1]$ and $p[0, f_1]$ (which appear in (3.6)) are defined. We will see, in the course of the proof of Lemma 3.1, that the functions $p[v_1, h_{1,r}]$, $p[0, h_{1,r}]$ appearing in (3.13) are also defined. Assumptions (3.1), (3.3), (3.6) imply that the quantity under the square root sign in (3.14) is nonnegative.

Lemma 3.1. *There exist $\delta, g_1, g_2, h_1, h_2$ such that $\delta > 0$ and (3.8)–(3.15) are satisfied:*

$$g_i \in C_c^\infty(P, R), \quad h_i : P \times (-\delta, T+\delta) \to R \text{ is a } C^\infty \text{ function}, \tag{3.8}$$

$$\operatorname{spt}(g_i) \subset U_i, \quad \operatorname{spt}(v_i) \subset \{x : g_i(x) = 1\}, \quad 0 \leqq g_i(x) \leqq 1 \quad \text{if } x \in P, \tag{3.9}$$

$$h_i \geqq 0, \quad h_i(x,t) > |v_i(x)| \quad \text{if } x \in \operatorname{spt}(g_i), \tag{3.10}$$

$$h_i(x,t) = f_i(x) \quad \text{if } x \notin \operatorname{spt}(g_i), \tag{3.11}$$

$$(h_1(x,t))^2 = (f_1(x))^2 - 2t\delta g_1(x), \tag{3.12}$$

$$(h_2(x,t))^2 = (f_2(x))^2 - 2t\delta g_2(x) - \int_0^t v_2(x) \cdot \nabla(p[v_1, h_{1,r}] - p[0, h_{1,r}])(x)\,dr,$$

$$\text{where } h_{i,t}(x) = h_i(x,t), \tag{3.13}$$

$$h_2(x,T) + \theta > ((f_2(x))^2 - T v_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x))^{1/2} \quad \text{if } x \in P, \tag{3.14}$$

$$L(h_{i,t})(x) \geqq 0 \quad \text{if } g_i(x) < 1 \quad (\text{where } h_{i,t}(x) = h_i(x,t) \text{ as in (3.13)}). \tag{3.15}$$

Proof. Assumption (3.1) allows us to find an open set V_i with compact closure such that $K_i \subset V_i$ and $\operatorname{closure}(V_i) \subset U_i$. Let $g_i \in C_c^\infty(P, R)$ satisfy $\operatorname{spt}(g_i) \subset U_i, g_i(x) = 1$ if $x \in \operatorname{closure}(V_i)$, and

$$0 \leqq g_i(x) \leqq 1 \quad \text{for all } x \in P. \tag{3.16}$$

Let W_i be an open set with compact closure such that $\operatorname{spt}(g_i) \subset W_i$ and closure $(W_i) \subset U_i$. Using (3.3), (3.4), (3.2) we get

$$\begin{aligned} \operatorname{spt}(v_i) &\subset K_i \subset V_i \subset \operatorname{closure}(V_i) \subset \{x : g_i(x) = 1\} \subset \operatorname{spt}(g_i) \subset W_i \subset \operatorname{closure}(W_i) \\ &\subset U_i \subset \{x : f_i(x) > |v_i(x)|\} \subset \operatorname{spt}(f_i) \subset P. \end{aligned} \tag{3.17}$$

From (3.17) we conclude $f_i(x) > |v_i(x)|$ for all x in the compact set $\operatorname{closure}(W_i)$. From (3.7) and (3.17) we conclude $L(f_i)(x) > 0$ for all x in the compact set $\operatorname{closure}(W_i) \sim V_i$. Hence there exists $\varepsilon > 0$ such that

$$f_i(x) > (\varepsilon^2 + |v_i(x)|^2)^{1/2} \quad \text{if } x \in \operatorname{closure}(W_i), \tag{3.18}$$

$$L(f_i)(x) > \varepsilon \quad \text{if } x \in \operatorname{closure}(W_i) \sim V_i. \tag{3.19}$$

If $i \in \{1, 2\}$, $x \in P$ and $\delta \in R$ satisfy $2\delta g_i(x) \leqq (f_i(x))^2$ we define

$$k_i(x,\delta) = ((f_i(x))^2 - 2\delta g_i(x))^{1/2}, \quad k_{i,\delta}(x) = k_i(x,\delta). \tag{3.20}$$

If $x \in W_i$ and $\delta < \varepsilon^2/2$ then (3.16) and (3.18) imply $2\delta g_i(x) < \varepsilon^2 < (f_i(x))^2$. Hence k_i is C^∞ on $W_i \times (-\infty, \varepsilon^2/2)$. The function k_i is C^∞ on $(P \sim \operatorname{spt}(g_i)) \times R$ because (3.4) implies $k_i(x,\delta) = ((f_i(x))^2)^{1/2} = f_i(x)$ for (x,δ) in that set. All this and (3.17) (which implies $\operatorname{spt}(g_i) \subset W_i$ and $\operatorname{spt}(g_i) \subset \operatorname{spt}(f_i)$) yield

$$k_i \text{ is } C^\infty \text{ on } P \times (-\infty, \varepsilon^2/2), \quad k_i \geqq 0, \quad \operatorname{spt}(k_{i,\delta}) \subset \operatorname{spt}(f_i). \tag{3.21}$$

Setting $L(k_i)(x, \delta) = L(k_{i,\delta})(x)$ (see(3.20)), we conclude that $L(k_i)$ is a C^∞ function on $P \times (-\infty, \varepsilon^2/2)$. In particular, $L(k_i)$ is uniformly continuous on the compact set $(\operatorname{closure}(W_i) \sim V_i) \times [-\varepsilon^2/4, \varepsilon^2/4]$. Hence there exist $\delta_0 > 0$ such that $\delta_0 < \varepsilon^2/4$ and

$$|L(k_i)(x, \delta) - L(k_i)(x, 0)| < \varepsilon/2 \quad \text{if } x \in \operatorname{closure}(W_i) \sim V_i \quad \text{and} \quad |\delta| \leqq \delta_0.$$

Combining this with $L(k_i)(x, 0) = L(f_i)(x)$ (see (3.4)) and (3.19) we obtain

$$L(k_i)(x, \delta) > \varepsilon/2 \quad \text{if } x \in \operatorname{closure}(W_i) \sim V_i \quad \text{and} \quad |\delta| < \delta_0.$$

If $x \in P \sim \operatorname{closure}(W_i)$ then (3.17) yields $g_i(x) = 0$ and $x \notin K_i$. In this case, (3.4) and (3.7) imply $L(k_i)(x, \delta) = L(f_i)(x) \geqq 0$. All this implies

$$L(k_{i,\delta})(x) = L(k_i)(x, \delta) \geqq 0 \quad \text{if } x \in P \sim V_i \quad \text{and} \quad |\delta| < \delta_0. \tag{3.22}$$

If $x \in \operatorname{spt}(g_i)$ and $\delta < \varepsilon^2/2$ then (3.16)–(3.18) imply $(f_i(x))^2 - 2\delta g_i(x) > (f_i(x))^2 - \varepsilon^2 > |v_i(x)|^2$. We obtain (see (3.20))

$$k_{i,\delta}(x) > |v_i(x)| \geqq 0 \quad \text{if } x \in \operatorname{spt}(g_i) \quad \text{and} \quad \delta < \varepsilon^2/2. \tag{3.23}$$

Assertions (3.17), (3.21) and (3.23) imply that $p[v_1, k_{1,\delta}]$ and $p[0, k_{1,\delta}]$ make sense for $\delta < \varepsilon^2/2$. Hence, using the convention $\int_a^b = -\int_b^a$ and the notation

$$H(x, t, \delta) = (f_2(x))^2 - 2\delta t g_2(x) - \int_0^t v_2(x) \cdot \nabla(p[v_1, k_{1,r\delta}] - p[0, k_{1,r\delta}])(x)\, dr, \tag{3.24}$$

we can say

$$H \text{ is a } C^\infty \text{ function on } P \times (-2T, 2T) \times (-\varepsilon^2/(4T), \varepsilon^2/(4T)). \tag{3.25}$$

From (3.24), (3.20), (3.4) we obtain

$$H(x, t, 0) = (f_2(x))^2 - t v_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x). \tag{3.26}$$

If $x \in \operatorname{closure}(W_2)$ then (3.26), (3.17), (3.6) yield $H(x, 0, 0) > |v_2(x)|^2$ and $H(x, T, 0) > |v_2(x)|^2$. The linearity of (3.26) in the variable t implies $H(x, t, 0) > |v_2(x)|^2$ if $x \in \operatorname{closure}(W_2)$ and $t \in [0, T]$. Now the compactness of $\operatorname{closure}(W_2) \times [0, T]$, (3.2) and (3.25) imply that there exists $\alpha > 0$ such that $\alpha < T$, $\alpha < \varepsilon^2/(4T)$ and

$$H(x, t, \delta) > |v_2(x)|^2 \quad \text{if } x \in \operatorname{closure}(W_2), \quad t \in [-\alpha, T + \alpha], \quad |\delta| \leqq \alpha. \tag{3.27}$$

We will use the notation

$$h(x, t, \delta) = (H(x, t, \delta))^{1/2} \quad \text{if } H(x, t, \delta) \geqq 0. \tag{3.28}$$

If $x \notin \operatorname{spt}(g_2)$ then the properties $x \notin \operatorname{spt}(v_2)$ (see (3.17)), (3.28), (3.24), (3.4) imply $h(x, t, \delta) = ((f_2(x))^2)^{1/2} = f_2(x)$. Hence (3.25) implies that h is C^∞ on the set

$$(P \sim \operatorname{spt}(g_2)) \times (-2T, 2T) \times (-\varepsilon^2/(4T), \varepsilon^2/(4T)).$$

From (3.25), (3.27), (3.28) we obtain that h is C^∞ on the set $W_2 \times (-\alpha, T + \alpha) \times (-\alpha, \alpha)$. All this, (3.17), $\alpha < T$ and $\alpha < \varepsilon^2/(4T)$ imply

$$h \geqq 0, \quad h \text{ is } C^\infty \text{ on } P \times (-\alpha, T + \alpha) \times (-\alpha, \alpha). \tag{3.29}$$

Furthermore, (3.27), (3.28) and (3.17) yield

$$h(x,t,\delta) > |v_2(x)| \quad \textit{if } x \in \operatorname{spt}(g_2), \quad t \in (-\alpha, T+\alpha), \quad |\delta| < \alpha. \tag{3.30}$$

Using $\operatorname{spt}(v_2) \subset \operatorname{spt}(g_2) \subset \operatorname{spt}(f_2)$ (see (3.17)), (3.24), (3.28), (3.4) we find

$$h(x,t,\delta) = 0 \quad \text{if } x \notin \operatorname{spt}(f_2), \quad h(x,t,\delta) = f_2(x) \quad \text{if } x \notin \operatorname{spt}(g_2). \tag{3.31}$$

Recall that we fixed $\theta > 0$ at the beginning of this section. Now we use (3.29) and (3.2) to fix $\delta_1 > 0$ such that $(T+\delta_1)\delta_1 < \delta_0 < \varepsilon^2/4$, $\delta_1 < \alpha$ and the inequality $|h(x,T,\delta_1) - h(x,T,0)| < \theta$ holds for all $x \in \operatorname{spt}(f_2)$. From (3.31) we obtain that $|h(x,T,\delta_1) - h(x,T,0)| < \theta$ holds for all $x \in P$. This implies (see (3.26), (3.28), (3.29))

$$h(x,T,\delta_1) + \theta > ((f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x))^{1/2} \tag{3.32}$$

for all $x \in P$. We define $h_i\colon P \times (-\delta_1, T+\delta_1) \to R$ and $h_{i,t}$ by (see (3.21))

$$h_1(x,t) = k_1(x, t\delta_1), \quad h_2(x,t) = h(x,t,\delta_1), \quad h_{i,t}(x) = h_i(x,t).$$

If $(x,t) \in P \times (-\delta_1, T+\delta_1)$ satisfies $g_2(x) < 1$ then (3.17) implies $v_2(x) = 0$. Therefore, (3.28), (3.24), (3.20) yield

$$h_2(x,t) = h(x,t,\delta_1) = ((f_2(x))^2 - 2\delta_1 t g_2(x))^{1/2} = k_2(x, t\delta_1)$$

in this case. The definition of h_1 and the above imply

$$\text{if } g_i(x) < 1 \quad \text{and} \quad -\delta_1 < t < T+\delta_1 \quad \text{then} \quad h_i(x,t) = k_i(x, t\delta_1). \tag{3.33}$$

The number δ in the statement of the lemma will be δ_1. From the definition of g_i, $0 < \delta_1 < \alpha$, $(T+\delta_1)\delta_1 < \varepsilon^2/2$, (3.21), (3.29) we conclude (3.8). Assertion (3.9) is a consequence of (3.16), (3.17). Properties (3.23), (3.20), $(T+\delta_1)\delta_1 < \varepsilon^2/2$, (3.29), (3.30), $0 < \delta_1 < \alpha$ imply (3.10). Assertions (3.4), (3.20), (3.31) imply (3.11). We obtain (3.12)–(3.14) from (3.20), (3.24), (3.28) and (3.32). Finally, (3.15) follows from (3.22), (3.33), (3.17) and $(T+\delta_1)\delta_1 < \delta_0$.

Lemma 3.2. *Let δ, g_i, h_i be as in Lemma 3.1. Then there exist d, N, J^z for $z \in \{1,2,3\}$, and q_i^z for $i \in \{1,2\}$, $z \in \{1,2,3\}$ such that (3.34)–(3.41) are satisfied:*

$$N \text{ is a positive integer, } d = T/(4N), \tag{3.34}$$

$$J^z \text{ is an open subset of } (-\delta, T+\delta), q_i^z\colon P \times J^z \to R \text{ is } C^\infty, \tag{3.35}$$

$$[4nd, 4nd+d] \subset J^1, [4nd+d, 4nd+2d] \subset J^2,$$

$$[4nd+2d, 4nd+4d] \subset J^3 \quad \textit{if } n \in \{0,1,\dots,N-1\}, \tag{3.36}$$

$$q_i^1(x, 4nd+d) = q_i^2(x, 4nd+d), q_i^2(x, 4nd+2d) = q_i^3(x, 4nd+d),$$

$$q_i^3(x, 4nd+4d) = q_i^1(x, 4nd+4d) \quad \textit{if } n \in \{0,1,\dots,N-1\}, \tag{3.37}$$

$$q_i^z \geqq 0, \quad q_i^z(x,t) = h_i(x,t) \quad \textit{if } x \notin \operatorname{spt}(v_i), \tag{3.38}$$

$$q_i^z(x,t) > |v_i(x)| \quad \textit{if } x \in \operatorname{spt}(v_i), \tag{3.39}$$

$$q_i^1(x,0) = h_i(x,0), \quad q_i^3(x,T) = h_i(x,T), \tag{3.40}$$

$$\frac{\partial}{\partial t}2^{-1}(q_i^z(x,t))^2 \leqq \delta/2 - \delta g_i(x) - S_i^z v_i^z(x)\cdot\nabla(2^{-1}(q_{i,t}^z)^2 + p[v_1^z, q_{1,t}^z] + p[v_2^z, q_{2,t}^z])(x),$$

where $S_1^1 = S_2^1 = S_2^2 = S_1^3 = 1,\quad S_1^2 = S_2^3 = -1,\quad v_1^1 = v_1^2 = v_1,\quad v_1^3 = 0,\quad v_2^z = v_2,\quad$ *and* $\quad q_{i,t}^z(x) = q_i^z(x,t).$ (3.41)

Proof. If $i\in\{1,2\}$ and $j\in\{1,2\}$ we define $w_i^j\colon P\to R^2$ as follows:

$$w_i^j = v_1 \quad \textit{if } (i,j)\neq(2,2), \quad w_2^2 = 0. \tag{3.42}$$

From (3.9), (3.4) we obtain

$$\mathrm{spt}(v_i)\subset\mathrm{spt}(g_i)\subset\mathrm{spt}(f_i). \tag{3.43}$$

Recall the notation $h_{i,t}(x) = h_i(x,t)$ introduced in (3.13). Properties (3.8), (3.10), (3.42), (3.43), (3.2) imply that $p[w_i^j, h_{1,t}]$ and $p[v_2, h_{2,t}]$ make sense for $t\in(-\delta, T+\delta)$. This fact, (3.8), and the convention $\int_a^b = -\int_b^a$ allow us to use the notation

$$M_i^j(x,t,s) = (h_i(x,t))^2 - 2s\delta g_i(x) + (-1)^j\int_0^{2s} v_i(x)\cdot\nabla(2^{-1}(h_{i,t+r})^2 + p[w_i^j, h_{1,t+r}] + p[v_2, h_{2,t+r}])(x)\,dr \tag{3.44}$$

if $\{i,j\}\subset\{1,2\}$, $x\in P$, $-\delta/3 < t < T+\delta/3$ and $|s| < \delta/3$. We have

$$M_i^j \text{ is a } C^\infty \text{ function on } P\times(-\delta/3, T+\delta/3)\times(-\delta/3,\delta/3). \tag{3.45}$$

We will also use the notation

$$m_i^j(x,t,s) = (M_i^j(x,t,s))^{1/2} \quad \text{if } (x,t,s) \text{ satisfies } M_i^j(x,t,s)\geqq 0. \tag{3.46}$$

If (x,t) is in the compact set $\mathrm{spt}(g_i)\times[0,T]$ then (3.44), (3.45), (3.10) yield $M_i^j(x,t,0) = (h_i(x,t))^2 > |v_i(x)|^2$. Hence (3.45), (3.2) imply that there exist an open set $E_i\subset P$ and a number $\beta>0$ such that $\mathrm{spt}(g_i)\subset E_i$, $\beta<\delta/3$ and

$$M_i^j(x,t,s) > |v_i(x)|^2 \quad \text{if } (x,t,s)\in E_i\times(-\beta, T+\beta)\times(-\beta,\beta). \tag{3.47}$$

Using (3.45)–(3.47) we conclude

$$m_i^j \text{ is } C^\infty \text{ on } E_i\times(-\beta,T+\beta)\times(-\beta,\beta). \tag{3.48}$$

If $x\notin\mathrm{spt}(g_i)$ then (3.43) and (3.44) yield $M_i^j(x,t,s) = (h_i(x,t))^2$. Combining this with (3.45), (3.46), $h_i\geqq 0$ (see (3.10)) and $\beta<\delta/3$ we find

$$m_i^j(x,t,s) = h_i(x,t) \quad \text{if } (x,t,s)\in(P\sim\mathrm{spt}(g_i))\times(-\beta,T+\beta)\times(-\beta,\beta), \tag{3.49}$$

and hence (3.8) implies that m_i^j is C^∞ on the set mentioned in (3.49). This last assertion, (3.48), and $\mathrm{spt}(g_i)\subset E_i$ imply

$$m_i^j \text{ is } C^\infty \text{ on } P\times(-\beta,T+\beta)\times(-\beta,\beta). \tag{3.50}$$

Using (3.46), (3.47) and $\mathrm{spt}(g_i)\subset E_i$ we find

$$m_i^j(x,t,s) > |v_i(x)| \quad \text{if } (x,t,s)\in\mathrm{spt}(g_i)\times(-\beta,T+\beta)\times(-\beta,\beta). \tag{3.51}$$

For future reference, we use (3.44), (3.46), (3.50) to write

$$(m_i^j(x,t,s))^2 = (h_i(x,t))^2 - 2s\delta g_i(x) + (-1)^j \int_0^{2s} v_i(x)\cdot\nabla(2^{-1}(h_{i,t+r})^2 + p[w_i^j, h_{1,t+r}] + p[v_2, h_{2,t+r}])(x)dr$$
$$\text{if } (x,t,s)\in P\times(-\beta, T+\beta)\times(-\beta,\beta). \tag{3.52}$$

We will use the notation

$$m_{i,t,s}^j(x) = m_i^j(x,t,s). \tag{3.53}$$

Properties (3.49), (3.11), $\mathrm{spt}(g_i)\subset\mathrm{spt}(f_i)$ (which follows from (3.4) and (3.9)) and (3.46) imply

$$\mathrm{spt}(m_{i,t,s}^j)\subset\mathrm{spt}(f_i),\quad m_{i,t,s}^j\geqq 0\quad \text{if } (t,s)\in(-\beta,T+\beta)\times(-\beta,\beta). \tag{3.54}$$

Now (3.53), (3.51), (3.43), (3.54), (3.2) imply that $p[v_i, m_{i,t,s}^j]$ and $p[0, m_{i,t,s}^j]$ make sense if $(t,s)\in(-\beta, T+\beta)\times(-\beta,\beta)$. We define

$$F_i^{j,0}(x,t,s) = v_i(x)\cdot\nabla(2^{-1}(m_{i,t,s}^j)^2)(x),$$
$$F_i^{j,k}(x,t,s) = v_i(x)\cdot\nabla(p[v_k, m_{k,t,s}^j])(x)\ \textit{if}\ k=1,2,$$
$$F_2^{2,3}(x,t,s) = v_2(x)\cdot\nabla(p[0, m_{1,t,s}^2])(x) \tag{3.55}$$

for $(x,t,s)\in P\times(-\beta,T+\beta)\times(-\beta,\beta)$. We use (3.2), (3.50) to choose $d>0$ such that $8d<\beta/2$, $N=T/(4d)$ is an integer and

$$|F_i^{j,k}(x,t,s) - F_i^{j,k}(x,t',0)| < \delta/6$$
$$\text{if } \{t,t'\}\subset[-\beta/2, T+\beta/2],\quad |t-t'|<8d,\quad |s|<8d. \tag{3.56}$$

For $n\in\{0,1,\dots,N-1\}$ we set

$$J_n^1 = (4nd - d/3, 4nd + d + d/3),$$
$$J_n^2 = (4nd + d - d/3, 4nd + 2d + d/3),$$
$$J_n^3 = (4nd + 2d - d/3, 4nd + 4d + d/3),$$
$$J^z = \bigcup_{n=0}^{N-1} J_n^z \quad \text{if } z\in\{1,2,3\}. \tag{3.57}$$

If z is fixed then the intervals $J_0^z, J_1^z, \dots, J_{N-1}^z$ are disjoint. Hence the properties $N=T/(4d)$, $8d<\beta/2$, $\beta<\delta/3$, (3.8) and (3.50) allow us to define $q_i^z: P\times J^z\to R$ as follows:

$$\text{If } t\in J_n^1 \text{ then } q_1^1(x,t) = m_1^1(x, 4nd, t-4nd),$$
$$\text{If } t\in J_n^2 \text{ then } q_1^2(x,t) = m_1^2(x, 4nd+2d, t-(4nd+2d)),$$
$$\text{If } t\in J_n^3 \text{ then } q_1^3(x,t) = h_1(x,t),$$
$$\text{If } t\in J_n^1 \text{ then } q_2^1(x,t) = m_2^1(x, 4nd, t-4nd),$$

$$\text{If } t \in J_n^2 \text{ then } q_2^2(x,t) = m_2^1(x, 4nd, t - 4nd),$$
$$\text{If } t \in J_n^3 \text{ then } q_2^3(x,t) = m_2^2(x, 4nd + 4d, t - (4nd + 4d)). \tag{3.58}$$

Note that, in (3.58), m_i^j is evaluated only at points in $P \times (-\beta/2, T + \beta/2) \times (-\beta/2, \beta/2)$. Using (3.46), (3.10) we find

$$q_i^z \geqq 0. \tag{3.59}$$

If $x \notin \text{spt}(v_i)$ then (3.12), (3.13) imply that the identity in (3.52) reduces to

$$(m_i^j(x,t,s))^2 = (h_i(x,t))^2 - 2s\delta g_i(x) = (f_i(x))^2 - 2t\delta g_i(x) - 2s\delta g_i(x) = (h_i(x, t+s))^2.$$

The above and (3.58) yield $(q_i^z(x,t))^2 = (h_i(x,t))^2$ if $x \notin \text{spt}(v_i)$. Now (3.10) and (3.59) imply

$$q_i^z(x,t) = h_i(x,t) \quad \text{if } x \in P \sim \text{spt}(v_i) \quad \text{and} \quad t \in J^z. \tag{3.60}$$

From (3.58), (3.51) and (3.10) we conclude

$$q_i^z(x,t) > |v_i(x)| \quad \text{if } x \in \text{spt}(g_i) \quad \text{and} \quad t \in J^z. \tag{3.61}$$

Properties (3.58), (3.57), (3.52), (3.42), (3.12) and the argument

$$\int_0^{2d} F(r)dr = \int_{-2d}^{0} F(2d+r)dr = -\int_0^{-2d} F(2d+r)dr,$$

which is valid for an arbitrary F, yield

$$\begin{aligned}
(q_1^1(x, 4nd + d))^2 &= (m_1^1(x, 4nd, d))^2 = (h_1(x, 4nd))^2 - 2d\delta g_1(x) \\
&\quad - \int_0^{2d} v_1(x) \cdot \nabla (2^{-1}(h_{1,4nd+r})^2 + \sum_{k=1}^{2} p[v_k, h_{k,4nd+r}])(x)dr \\
&= (h_1(x, 4nd + 2d))^2 + 2d\delta g_1(x) \\
&\quad - \int_0^{2d} v_1(x) \cdot \nabla (2^{-1}(h_{1,4nd+r})^2 + \sum_{k=1}^{2} p[v_k, h_{k,4nd+r}])(x)dr \\
&= (h_1(x, 4nd + 2d))^2 + 2d\delta g_1(x) \\
&\quad + \int_0^{-2d} v_1(x) \cdot \nabla (2^{-1}(h_{1,4nd+2d+r})^2 + \sum_{k=1}^{2} p[v_k, h_{k,4nd+2d+r}])(x)dr \\
&= (m_1^2(x, 4nd + 2d, -d))^2 = (q_1^2(x, 4nd + d))^2.
\end{aligned} \tag{3.62}$$

From (3.13) we obtain

$$\begin{aligned}
&(h_2(x, 4nd + 4d))^2 - (h_2(x, 4nd))^2 \\
&= -8d\delta g_2(x) - \int_{4nd}^{4nd+4d} v_2(x) \cdot \nabla (p[v_1, h_{1,r}] - p[0, h_{1,r}])(x)dr \\
&= -8d\delta g_2(x) - \int_0^{4d} v_2(x) \cdot \nabla (p[v_1, h_{1,4nd+r}] - p[0, h_{1,4nd+r}])(x)dr.
\end{aligned} \tag{3.63}$$

Using (3.58), (3.57), (3.52), (3.42) and the argument

$$\int_0^{-4d} F(r)dr = -\int_{-4d}^{0} F(r)dr = -\int_0^{4d} F(r-4d)dr,$$

for arbitrary F, we find

$$\begin{aligned}
&(q_2^3(x,4nd+2d))^2-(q_2^2(x,4nd+2d))^2\\
&\quad=(m_2^2(x,4nd+4d,-2d))^2-(m_2^1(x,4nd,2d))^2\\
&\quad=(h_2(x,4nd+4d))^2+4d\delta g_2(x)+\int_0^{-4d} v_2(x)\cdot\nabla(2^{-1}(h_{2,4nd+4d+r})^2\\
&\qquad+p[0,h_{1,4nd+4d+r}]+p[v_2,h_{2,4nd+4d+r}])(x)dr-(h_2(x,4nd))^2+4d\delta g_2(x)\\
&\qquad+\int_0^{4d} v_2(x)\cdot\nabla(2^{-1}(h_{2,4nd+r})^2+p[v_1,h_{1,4nd+r}]+p[v_2,h_{2,4nd+r}])(x)dr\\
&\quad=(h_2(x,4nd+4d))^2-(h_2(x,4nd))^2+8d\delta g_2(x)\\
&\qquad+\int_0^{4d} v_2(x)\cdot\nabla(p[v_1,h_{1,4nd+r}]-p[0,h_{1,4nd+r}])(x)dr
\end{aligned}$$

The above and (3.63) yield

$$(q_2^2(x,4nd+2d))^2=(q_2^3(x,4nd+2d))^2. \tag{3.64}$$

Using (3.58), (3.57), (3.52) we obtain

$$\begin{aligned}
(q_1^2(x,4nd+2d))^2&=(m_1^2(x,4nd+2d,0))^2=(h_1(x,4nd+2d))^2\\
&=(q_1^3(x,4nd+2d))^2,
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
(q_1^3(x,4nd+4d))^2&=(h_1(x,4nd+4d))^2=(m_1^1(x,4nd+4d,0))^2\\
&=(m_1^1(x,4(n+1)d,0))^2=(q_1^1(x,4(n+1)d))^2,
\end{aligned} \tag{3.66}$$

$$q_2^1(x,4nd+d)=m_2^1(x,4nd,d)=q_2^2(x,4nd+d), \tag{3.67}$$

$$\begin{aligned}
(q_2^3(x,4nd+4d))^2&=(m_2^2(x,4nd+4d,0))^2=(h_2(x,4nd+4d))^2\\
&=(m_2^1(x,4nd+4d,0))^2=(m_2^1(x,4(n+1)d,0))^2\\
&=(q_2^1(x,4(n+1)d))^2.
\end{aligned} \tag{3.68}$$

Our construction yields (3.34)–(3.36). Assertion (3.37) is a consequence of (3.59), (3.62), (3.64)–(3.68). Properties (3.59), (3.60) yield (3.38). From (3.61) and (3.43) we obtain (3.39). Using (3.58), (3.57), $N=T/(4d)$, (3.52), (3.59), (3.10) we conclude (3.40). The proof of the lemma will be completed by showing (3.41).

From (3.52), (3.53)–(3.55), (3.10) we get

$$\begin{aligned}
&F_i^{j,0}(x,t,0)=v_i(x)\cdot\nabla(2^{-1}(h_{i,t})^2)(x),\\
&F_i^{j,k}(x,t,0)=v_i(x)\cdot\nabla(p[v_k,h_{k,t}])(x) \quad \text{if } k=1,2,\\
&F_2^{2,3}(x,t,0)=v_2(x)\cdot\nabla(p[0,h_{1,t}])(x).
\end{aligned} \tag{3.69}$$

The identity (3.52) gives us

$$\frac{\partial}{\partial s}2^{-1}(m_i^j(x,t,s))^2 = -\delta g_i(x) + (-1)^j v_i(x)\cdot\nabla(2^{-1}(h_{i,t+2s})^2 + p[w_i^j, h_{1,t+2s}] + p[v_2, h_{2,t+2s}])(x) \tag{3.70}$$

We will use the notation $q_{i,t}^z(x) = q_i^z(x,t)$. Assertions (3.59)–(3.61), (3.11), (3.43), (3.2) imply that $p[v_i, q_{i,t}^z]$ and $p[0, q_{i,t}^z]$ make sense. If $t\in J_n^1$ then (3.58), (3.53), (3.55) imply

$$v_i(x)\cdot\nabla(2^{-1}(q_{i,t}^1)^2)(x) = F_i^{1,0}(x, 4nd, t-4nd),$$
$$v_i(x)\cdot\nabla(p[v_k, q_{k,t}^1])(x) = F_i^{1,k}(x, 4nd, t-4nd) \quad \text{if } k=1,2.$$

The above, (3.69), (3.70), (3.58) and (3.42) yield the following for $t\in J_n^1$:

$$\begin{aligned}
&-\delta g_i(x) + \sum_{k=0}^{2}(F_i^{1,k}(x,4nd,t-4nd) - F_i^{1,k}(x,2t-4nd,0))\\
&\quad - v_i(x)\cdot\nabla(2^{-1}(q_{i,t}^1)^2 + p[v_1, q_{1,t}^1] + p[v_2, q_{2,t}^1])(x)\\
&= -\delta g_i(x) - \sum_{k=0}^{2} F_i^{1,k}(x, 2t-4nd, 0)\\
&= -\delta g_i(x) - v_i(x)\cdot\nabla(2^{-1}(h_{i,2t-4nd})^2 + \sum_{k=1}^{2} p[v_k, h_{k,2t-4nd}])(x)\\
&= \frac{\partial}{\partial t}2^{-1}(q_i^1(x,t))^2.
\end{aligned}$$

The above, (3.56) and the definition of J_n^1 imply

$$\frac{\partial}{\partial t}2^{-1}(q_i^1(x,t))^2 \leqq \delta/2 - \delta g_i(x) - v_i(x)\cdot\nabla(2^{-1}(q_{i,t}^1)^2 + p[v_1, q_{1,t}^1] + p[v_2, q_{2,t}^1])(x). \tag{3.71}$$

If $t\in J_n^2$ then (3.58), (3.53), (3.55) imply

$$v_1(x)\cdot\nabla(2^{-1}(q_{1,t}^2)^2)(x) = F_1^{2,0}(x, 4nd+2d, t-(4nd+2d)),$$
$$v_1(x)\cdot\nabla(p[v_1, q_{1,t}^2])(x) = F_1^{2,1}(x, 4nd+2d, t-(4nd+2d)),$$
$$v_1(x)\cdot\nabla(p[v_2, q_{2,t}^2])(x) = F_1^{1,2}(x, 4nd, t-4nd).$$

The above, (3.69), (3.70), (3.58) and (3.42) yield the following for $t\in J_n^2$:

$$\begin{aligned}
&-\delta g_1(x) + \sum_{k=0}^{1}(F_1^{2,k}(x,2t-4nd-2d,0) - F_1^{2,k}(x,4nd+2d,t-4nd-2d))\\
&\quad + (F_1^{1,2}(x,2t-4nd-2d,0) - F_1^{1,2}(x,4nd,t-4nd))\\
&\quad + v_1(x)\cdot\nabla(2^{-1}(q_{1,t}^2)^2 + p[v_1, q_{1,t}^2] + p[v_2, q_{2,t}^2])(x)\\
&= -\delta g_1(x) + \sum_{k=0}^{1} F_1^{2,k}(x,2t-4nd-2d,0) + F_1^{1,2}(x,2t-4nd-2d,0)
\end{aligned}$$

$$= -\delta g_1(x) + v_1(x)\cdot\nabla(2^{-1}(h_{1,2t-4nd-2d})^2 + \sum_{k=1}^{2} p[v_k, h_{k,2t-4nd-2d}])(x)$$

$$= \frac{\partial}{\partial t} 2^{-1}(q_1^2(x,t))^2.$$

The above, (3.56) and the definition of J_n^2 imply

$$\frac{\partial}{\partial t} 2^{-1}(q_1^2(x,t))^2 \leqq \delta/2 - \delta g_1(x) + v_1(x)\cdot\nabla(2^{-1}(q_{1,t}^2)^2 + p[v_1, q_{1,t}^2] + p[v_2, q_{2,t}^2])(x). \tag{3.72}$$

If $t \in J_n^2$ then (3.58), (3.53), (3.55) imply

$$v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^2)^2)(x) = F_2^{1,0}(x, 4nd, t-4nd),$$
$$v_2(x)\cdot\nabla(p[v_1, q_{1,t}^2])(x) = F_2^{2,1}(x, 4nd+2d, t-(4nd+2d)),$$
$$v_2(x)\cdot\nabla(p[v_2, q_{2,t}^2])(x) = F_2^{1,2}(x, 4nd, t-4nd).$$

The above, (3.69), (3.70), (3.58) and (3.42) yield the following for $t \in J_n^2$:

$$\begin{aligned}
&-\delta g_2(x) + (F_2^{1,0}(x, 4nd, t-4nd) - F_2^{1,0}(x, 2t-4nd, 0)) \\
&\qquad + (F_2^{2,1}(x, 4nd+2d, t-4nd-2d) - F_2^{2,1}(x, 2t-4nd, 0)) \\
&\qquad + (F_2^{1,2}(x, 4nd, t-4nd) - F_2^{1,2}(x, 2t-4nd, 0)) \\
&\qquad - v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^2)^2 + p[v_1, q_{1,t}^2] + p[v_2, q_{2,t}^2])(x) \\
&= -\delta g_2(x) - F_2^{1,0}(x, 2t-4nd, 0) - F_2^{2,1}(x, 2t-4nd, 0) \\
&\qquad - F_2^{1,2}(x, 2t-4nd, 0) \\
&= -\delta g_2(x) - v_2(x)\cdot\nabla(2^{-1}(h_{2,2t-4nd})^2 + \sum_{k=1}^{2} p[v_k, h_{k,2t-4nd}])(x) \\
&= \frac{\partial}{\partial t} 2^{-1}(q_2^2(x,t))^2.
\end{aligned}$$

The above, (3.56) and the definition of J_n^2 imply

$$\frac{\partial}{\partial t} 2^{-1}(q_2^2(x,t))^2 \leqq \delta/2 - \delta g_2(x) - v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^2)^2 + p[v_1, q_{1,t}^2] + p[v_2, q_{2,t}^2])(x). \tag{3.73}$$

Finally, if $t \in J_n^3$ then (3.58), (3.53), (3.55), (3.69) imply

$$v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^3)^2)(x) = F_2^{2,0}(x, 4nd+4d, t-(4nd+4d)),$$
$$v_2(x)\cdot\nabla(p[0, q_{1,t}^3])(x) = F_2^{2,3}(x, t, 0),$$
$$v_2(x)\cdot\nabla(p[v_2, q_{2,t}^3])(x) = F_2^{2,2}(x, 4nd+4d, t-(4nd+4d)).$$

The above, (3.69), (3.70), (3.58) and (3.42) yield the following for $t \in J_n^3$:

$$\begin{aligned}
&-\delta g_2(x) + (F_2^{2,0}(x, 2t-4nd-4d, 0) - F_2^{2,0}(x, 4nd+4d, t-(4nd+4d))) \\
&\qquad + (F_2^{2,3}(x, 2t-4nd-4d, 0) - F_2^{2,3}(x, t, 0)) \\
&\qquad + (F_2^{2,2}(x, 2t-4nd-4d, 0) - F_2^{2,2}(x, 4nd+4d, t-(4nd+4d))) \\
&\qquad + v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^3)^2 + p[0, q_{1,t}^3] + p[v_2, q_{2,t}^3])(x) \\
&= -\delta g_2(x) + F_2^{2,0}(x, 2t-4nd-4d, 0) + F_2^{2,3}(x, 2t-4nd-4d, 0) \\
&\quad + F_2^{2,2}(x, 2t-4nd-4d, 0) \\
&= -\delta g_2(x) + v_2(x)\cdot\nabla(2^{-1}(h_{2,2t-4nd-4d})^2 \\
&\quad + p[0, h_{1,2t-4nd-4d}] + p[v_2, h_{2,2t-4nd-4d}])(x) \\
&= \frac{\partial}{\partial t} 2^{-1}(q_2^3(x,t))^2.
\end{aligned}$$

The above, (3.56) and the definition of J_n^3 imply

$$\frac{\partial}{\partial t} 2^{-1}(q_2^3(x,t))^2 \leqq \delta/2 - \delta g_2(x) + v_2(x)\cdot\nabla(2^{-1}(q_{2,t}^3)^2 + p[0, q_{1,t}^3] + p[v_2, q_{2,t}^3])(x). \tag{3.74}$$

Using (3.58), (3.12) we obtain

$$\frac{\partial}{\partial t} 2^{-1}(q_1^3(x,t))^2 = \frac{\partial}{\partial t} 2^{-1}(h_1(x,t))^2 = -\delta g_1(x) \quad \text{if } t \in J_n^3. \tag{3.75}$$

Conclusion (3.41) follows from (3.71)–(3.75).

Lemma 3.3. *There exists $v_0 > 0$ such that the 5-tuple*

$$(h_{1,0} + h_{2,0}, h_{1,T} + h_{2,T}, 0, T, v)$$

is P-admissible when $0 < v < v_0$.

Proof. We recall Lemmas 3.1, 3.2. For $n \in \{0, 1, 2, \ldots, N-1\}$ we define $a_n^1 = 4nd$, $b_n^1 = a_n^2 = 4nd + d, b_n^2 = a_n^3 = 4nd + 2d, b_n^3 = 4nd + 4d$. Let $z \in \{1, 2, 3\}$ and n be fixed. From (3.9) and (3.4) we conclude $\operatorname{spt}(v_i) \subset \{x : g_i(x) = 1\} \subset \operatorname{spt}(f_i)$. Using (3.3), (3.4) we find that $\operatorname{spt}(f_1)$ and $\operatorname{spt}(f_2)$ are disjoint. Properties (3.11), (3.38), (3.9), (3.4) imply $q_i^z(x,t) = 0$ if $x \notin \operatorname{spt}(f_i)$. Using (3.38), (3.15), (3.9) we obtain $L(q_{i,t}^z)(x) \geqq 0$ if $x \notin \{x : g_i(x) = 1\}$. All this, the hypotheses of this section, and Lemmas 3.1, 3.2 imply that (2.1)–(2.9) are satisfied if a, b, J, C_i, C_i', S_i, η, v_i, q_i are a_n^z, b_n^z, J^z, $\operatorname{spt}(f_i)$, $\{x : g_i(x) = 1\}$, S_i^z, $\delta/2$, v_i^z, q_i^z, respectively. Now Lemma 2.1 implies that there exist positive numbers v_n^1, v_n^2, v_n^3 such that the 5-tuple

$$(q_{1,4nd}^1 + q_{2,4nd}^1, q_{1,4nd+d}^1 + q_{2,4nd+d}^1, 4nd, 4nd+d, v)$$

is P-admissible if $0 < v < v_n^1$, the 5-tuple

$$(q_{1,4nd+d}^2 + q_{2,4nd+d}^2, q_{1,4nd+2d}^2 + q_{2,4nd+2d}^2, 4nd+d, 4nd+2d, v)$$

is P-admissible if $0 < v < v_n^2$, and the 5-tuple

$$(q_{1,4nd+2d}^3 + q_{2,4nd+2d}^3, q_{1,4nd+4d}^3 + q_{2,4nd+4d}^3, 4nd + 2d, 4nd + 4d, v)$$

is P-admissible if $0 < v < v_n^3$. The conclusion follows from the above, Lemma 2.2, (3.34), (3.37), (3.40), and the choice of v_0 such that $v_0 > 0$ and $v_0 < v_n^z$ for all z and n.

Section 4. The Geometric Arrangement

Throughout this section, we fix F, A, B, C, D such that (4.1)–(4.4) are satisfied:

$$F = (F_1, F_2) \text{ is a } C^\infty \text{ function from } R^2 \text{ into } R^2, \tag{4.1}$$

$$A, B, C, D \text{ are real numbers, } B > 0, C > 0, D > 0, \tag{4.2}$$

$$F_1(A,0) = B, \quad \text{if } x \in R \quad \text{then } |F_1(x,0)| \leqq B \quad \text{and} \quad F_2(x,0) = 0, \tag{4.3}$$

$$\lim_{x \to \infty} x^4 F_1(x,0) = D, \quad \text{if } x \in R^2 \quad \text{then } |F(x)| \leqq C|x|^{-4}$$
$$\text{and} \quad |\nabla F(x)| \leqq C|x|^{-5}. \tag{4.4}$$

When $\alpha \in R, \rho > 0, \sigma > 0$ we let $F^{\alpha,\rho,\sigma} = (F_1^{\alpha,\rho,\sigma}, F_2^{\alpha,\rho,\sigma})$ be the C^∞ function from R^2 into R^2 which is defined by

$$F^{\alpha,\rho,\sigma}(x_1, x_2) = (\sigma^2/\rho) F((x_1 - \alpha)/\rho, x_2/\rho) \quad \text{if } (x_1, x_2) \in R^2. \tag{4.5}$$

From (4.1)–(4.3) we conclude the existence of G with the properties

$$0 < G < (6C^{1/4}B^{-1/4} + |A|)/8, \quad \text{if } |x - A| \leqq G \quad \text{then } F_1(x,0) > (.999)B. \tag{4.6}$$

Lemma 4.1. *If $\alpha \in R$, $\rho > 0$, $\sigma > 0$ and $\alpha = A - \rho A$ then the following two properties are satisfied*:

$$\text{if } |x - A| \geqq (6C^{1/4}B^{-1/4} + |A|)\rho, \quad \text{then } F_1^{\alpha,\rho,\sigma}(x,0) > -10^{-3}(\sigma^2/\rho)B;$$

$$\text{if } |x - A| \leqq G\rho, \quad \text{then } F_1^{\alpha,\rho,\sigma}(x,0) > (.999)(\sigma^2/\rho)B.$$

Proof. If $|x - A| \geqq (6C^{1/4}B^{-1/4} + |A|)\rho$, then the hypothesis $\alpha = A - \rho A$ implies

$$\begin{aligned}|x - \alpha| &= |(x - \alpha - \rho A) + \rho A| = |(x - A) + \rho A| \\ &\geqq |x - A| - \rho|A| \geqq (6C^{1/4}B^{-1/4} + |A|)\rho - \rho|A| = 6C^{1/4}B^{-1/4}\rho.\end{aligned}$$

The above and (4.5), (4.4), (4.2) yield

$$\begin{aligned}|F_1^{\alpha,\rho,\sigma}(x,0)| &= (\sigma^2/\rho)|F_1((x - \alpha)/\rho, 0)| \leqq (\sigma^2/\rho)C|x - \alpha|^{-4}\rho^4 \\ &\leqq (\sigma^2/\rho)C(6C^{1/4}B^{-1/4}\rho)^{-4}\rho^4 = (\sigma^2/\rho)6^{-4}B < 10^{-3}(\sigma^2/\rho)B.\end{aligned}$$

This proves the first property. If $|x - A| \leqq G\rho$, then $\alpha = A - \rho A$ implies $|(x - \alpha)/\rho - A| = |x - \alpha - \rho A|/\rho = |x - A|/\rho \leqq G$. This inequality, (4.5) and (4.6) imply

$$F_1^{\alpha,\rho,\sigma}(x,0) = (\sigma^2/\rho) F_1((x - \alpha)/\rho, 0) > (\sigma^2/\rho)(.999)B,$$

which yields the second property.

Let a', a'', r', r'', s', s'' be the real numbers determined by (4.7) and (4.8):

$$r' = G/(6C^{1/4}B^{-1/4} + |A|) > 0, \quad r'' = (r')^2 > 0, \tag{4.7}$$

$$a' = A - r'A, \quad a'' = A - r''A, \quad s' = (2r')^{1/2} > 0, \quad s'' = (4r'')^{1/2} > 0. \tag{4.8}$$

From (4.5), (4.3), (4.8) we conclude

$$(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(A,0) = 2B + 4B + B = 7B \tag{4.9}$$

Using (4.7) and (4.6) we find

$$0 < r' < 1/8, \quad 0 < r'' < r'/8. \tag{4.10}$$

Lemma 4.2. *If* $x \in R$, *then* $(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x,0) \geqq -(1.006)B$.

Proof. The fact $Gr' < G$ (see (4.6), (4.10)) allows us to separate the argument into the following three cases: $|x-A| \leqq Gr', Gr' \leqq |x-A| \leqq G, |x-A| \geqq G$. If $|x-A| \leqq Gr'$, then $|x-A| < G$ (see above). Hence (4.5), (4.7), (4.8), Lemma 4.1, (4.3), (4.6), (4.2) imply (in the first case)

$$\begin{aligned}(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x,0) &\geqq (.999)((s')^2/r')B - ((s'')^2/r'')B + (.999)B \\ &= (.999)(2B) - 4B + (.999)B > -(1.006)B.\end{aligned}$$

If $Gr' \leqq |x-A| \leqq G$, then (4.7) yields

$$|x-A| \geqq Gr' = (6C^{1/4}B^{-1/4} + |A|)(r')^2 = (6C^{1/4}B^{-1/4} + |A|)r''.$$

Hence (4.5), (4.3), (4.7), (4.8), Lemma 4.1, (4.6), (4.2) imply

$$\begin{aligned}(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x,0) &\geqq -((s')^2/r')B - 10^{-3}((s'')^2/r'')B + (.999)B \\ &= -2B - 10^{-3}(4B) + (.999)B > -(1.006)B.\end{aligned}$$

If $|x-A| \geqq G$, then (4.7), (4.10) yield

$$|x-A| \geqq G = (6C^{1/4}B^{-1/4} + |A|)r' > (6C^{1/4}B^{-1/4} + |A|)r''.$$

Hence (4.3), (4.7), (4.8), Lemma 4.1 imply

$$\begin{aligned}(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x,0) &\geqq -10^{-3}((s')^2/r')B - 10^{-3}((s'')^2/r'')B - B \\ &= -10^{-3}(2B) - 10^{-3}(4B) - B = -(1.006)B.\end{aligned}$$

Lemma 4.3. *There exists* E *such that* (4.11)–(4.13) *hold*:

$$E > 0, \quad E < r''/8 < r'/8 < 1/8, \tag{4.11}$$

$$\text{if } x_1 \in R \text{ and } |x_2| \leqq E, \text{ then } (F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1,x_2) > -(1.01)B, \tag{4.12}$$

$$\text{if } |x_1 - A| \leqq (10^4 C/D)E \text{ and } |x_2| \leqq E, \text{ then } (F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1,x_2) \geqq (6.99)B. \tag{4.13}$$

Proof. This follows from (4.10), Lemma 4.2, (4.5), (4.1), (4.4), (4.2), (4.9).

Lemma 4.4. *If* $x\in R^2$, $|x|>2|A|$, *then* $|(F^{a',r',s'}+F^{a'',r'',s''}+F)(x)|\leqq 2C|x|^{-4}$.

Proof. Using (4.8), (4.10), $|(x_1,x_2)|>2|A|$ we find

$$|(x_1-a',x_2)|\geqq|(x_1,x_2)|-|a'|=|(x_1,x_2)|-(1-r')|A|$$
$$\geqq|(x_1,x_2)|-|A|>|(x_1,x_2)|/2.$$

Then (4.5), (4.7), (4.8), (4.4), the above and (4.10) imply

$$|F^{a',r',s'}(x_1,x_2)|=((s')^2/r')|F((x_1-a')/r',x_2/r')|$$
$$=2|F((x_1-a')/r',\ x_2/r')|\leqq 2C|(x_1-a',x_2)|^{-4}(r')^4$$
$$\leqq 2C|(x_1,x_2)|^{-4}2^4(r')^4\leqq 2C|(x_1,x_2)|^{-4}4^{-4}$$

A similar argument yields $|F^{a'',r'',s''}(x_1,x_2)|\leqq 4C|(x_1,x_2)|^{-4}4^{-4}$ if $|(x_1,x_2)|>2|A|$. The conclusion follows from all this and (4.4).

Assumptions (4.2), (4.4) imply that we can choose M large enough to satisfy (4.14)–(4.16):

$$M\geqq 1+10^4C/D, \tag{4.14}$$

$$\text{if}\ m\geqq M\quad\text{and}\quad |x_1-m|\leqq 10^4C/D,\quad\text{then}\ |m^4F_1(x_1,0)-D|\leqq(.0005)D, \tag{4.15}$$

$$\text{if}\ m\geqq M,\quad\text{then}\ C(m-10^4C/D)^{-5}\leqq(.0005)m^{-4}D. \tag{4.16}$$

If $m\geqq M$, $|x_1-m|\leqq 10^4C/D$ and $|x_2|\leqq 1$ then the mean value theorem, (4.4), (4.14) and (4.16) yield

$$|F_1(x_1,x_2)-F_1(x_1,0)|\leqq|x_2||\nabla F_1(x_1,\xi)|\leqq|x_2|Cx_1^{-5}\leqq Cx_1^{-5}$$
$$\leqq C(m-10^4C/D)^{-5}\leqq(.0005)m^{-4}D.$$

The above and (4.15) give us

$$\text{if}\ m\geqq M,\quad |x_1-m|\leqq 10^4C/D,\quad |x_2|\leqq 1\ \ \text{then}\ |F_1(x_1,x_2)-m^{-4}D|$$
$$\leqq 10^{-3}m^{-4}D. \tag{4.17}$$

We choose $\varepsilon>0$ small enough to satisfy (4.18) and (4.19):

$$\varepsilon^{-1}(1+\varepsilon^2)^{1/2}E/10>2|A|+1,\quad 2C(\varepsilon^{-1}(1+\varepsilon^2)^{1/2}E/10)^{-4}<10^{-3}\varepsilon^2B, \tag{4.18}$$

$$0<\varepsilon<.01,\quad \varepsilon^{-1}10^3C/D>M,\quad \varepsilon^{-1}10^2C/D>10^4C/D,\quad (.99)(1+\varepsilon^2)^{1/2}<1. \tag{4.19}$$

We set

$$r=\varepsilon^{-1}(1+\varepsilon^2)^{1/2}E,\quad a=-r(\varepsilon^{-1}10^3C/D), \tag{4.20}$$

$$d=r(10^4C/D),\quad s=[((1.02)B/(.999))(\varepsilon^{-1}10^3C/D)^4(r/D)]^{1/2}. \tag{4.21}$$

Using (4.20), we can rewrite (4.18) and the last inequality in (4.19) in the form

$$r/10 > 2|A| + 1, \quad 2C(r/10)^{-4} < 10^{-3}\varepsilon^2 B, \quad (.99)r\varepsilon < E. \tag{4.22}$$

From (4.2), (4.4), (4.21) we get

$$C/D \geqq 1, \quad \text{hence } d \geqq 10^4 r. \tag{4.23}$$

If $|x_1 - A| \leqq (10^4 C/D)E$, then (4.22), (4.21), (4.23), (4.20), (4.19), (4.23) allow us to write

$$|x_1| \leqq |A| + (10^4 C/D)E < r/20 + d(E/r) < 10^{-4}d/20 + d\varepsilon < (1 - 10^{-4})d \leqq d - r.$$

The above, (4.19) and (4.20) imply

$$[A - (10^4 C/D)E, A + (10^4 C/D)E] \subset (r - d, d - r) \quad \text{and} \quad E < r/10. \tag{4.24}$$

Lemma 4.5. *If* $-d \leqq x_1 \leqq d$ *and* $0 \leqq x_2 \leqq r$, *then*

$$|F_2^{a,r,s}(x_1, x_2)| \leqq (.002)B\varepsilon \quad \text{and} \quad (1.02)B \leqq F_1^{a,r,s}(x_1, x_2) < (1.03)B.$$

Proof. Using (4.5), (4.20), (4.3), the mean value theorem, (4.4), our assumptions on x_i, (4.19) and (4.21) we obtain

$$\begin{aligned}
|F_2^{a,r,s}(x_1, x_2)| &= (s^2/r)|F_2(x_1/r + \varepsilon^{-1}10^3 C/D, x_2/r)| \\
&= (s^2/r)|F_2(x_1/r + \varepsilon^{-1}10^3 C/D, x_2/r) - F_2(x_1/r + \varepsilon^{-1}10^3 C/D, 0)| \\
&\leqq (s^2/r)(x_2/r)|\nabla F_2(x_1/r + \varepsilon^{-1}10^3 C/D, \xi)| \\
&\leqq (s^2/r)C(x_1/r + \varepsilon^{-1}10^3 C/D)^{-5} \\
&\leqq (s^2/r)C(\varepsilon^{-1}10^3 C/D - 10^4 C/D)^{-5} \leqq (s^2/r)C((.9)\varepsilon^{-1}10^3 C/D)^{-5} \\
&= ((1.02)B/(.999))(\varepsilon^{-1}10^3 C/D)^{-1}(C/D)(.9)^{-5} < (.002)B\varepsilon,
\end{aligned}$$

which gives us the first conclusion. Using (4.5), (4.20) we find

$$(s^2/r)^{-1}F_1^{a,r,s}(x_1, x_2) = F_1((x_1 - a)/r, x_2/r) = F_1(x_1/r + \varepsilon^{-1}10^3 C/D, x_2/r).$$

If we set $m = \varepsilon^{-1}10^3 C/D$, then the above, our assumptions on x_i, (4.17), (4.19) and (4.21) yield

$$|(s^2/r)^{-1}F_1^{a,r,s}(x_1, x_2) - (\varepsilon^{-1}10^3 C/D)^{-4}D| \leqq 10^{-3}(\varepsilon^{-1}10^3 C/D)^{-4}D.$$

Substituting (4.21) into this inequality, we obtain

$$|F_1^{a,r,s}(x_1, x_2) - (1.02)B/(.999)| \leqq 10^{-3}(1.02)B/(.999),$$

which implies the second conclusion.

Lemma 4.6. *If* $-d \leqq x_1 \leqq d$ *and* $0 \leqq x_2 \leqq E$, *then*

$$(F_1^{a,r,s} + F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1, x_2) \geqq (.01)B.$$

Proof. This follows from Lemma 4.5, (4.12) and (4.24).

Lemma 4.7. *If* $|x_1 - A| \leqq (10^4 C/D)E$ *and* $0 \leqq x_2 \leqq E$, *then*

$$(F_1^{a,r,s} + F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1, x_2) \geqq (8.01)B.$$

Proof. This is a consequence of Lemma 4.5, (4.13) and (4.24).

Lemma 4.8. *There exists* $w \in C_c^\infty(P, R^2)$ *such that we have*

$$\operatorname{spt}(w) \subset (-d, d) \times (10^{-3}\varepsilon r, r), \tag{4.25}$$

$$\operatorname{spt}(w) \quad \textit{and} \quad [r-d, d-r] \times [E, r/10] \quad \textit{are disjoint}, \tag{4.26}$$

$$x_2 \frac{\partial}{\partial x_1} w_1(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} w_2(x_1, x_2) + w_2(x_1, x_2) = 0, \tag{4.27}$$

and the function $h(x_1, x_2) = w(x_1, x_2) \cdot (F^{a,r,s} + F^{a',r',s'} + F^{a'',r'',s''} + F)(x_1, x_2)$ *satisfies* (4.28)–(4.30):

$$\textit{if} \quad |x_1 - A| \leqq (10^4 C/D)E \quad \textit{and} \quad (.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r, \quad \textit{then } h(x_1, x_2) \geqq (8.01)B, \tag{4.28}$$

$$\textit{if} \quad |x_1| \leqq d - r \quad \textit{and} \quad (.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r, \quad \textit{then} \quad h(x_1, x_2) \geqq (.01)B, \tag{4.29}$$

$$\textit{if} \quad x \in P, \quad \textit{then} \quad h(x) \geqq -\varepsilon^2 (1.032)B \tag{4.30}$$

Proof. We set (see Fig. 1)

$$H_1 = \{(x_1, x_2) : x_2 = \varepsilon(x_1 + d),\ (.01)\varepsilon r \leqq x_2 \leqq (.99)\varepsilon r\},$$
$$H_2 = \{(x_1, x_2) : (x_1 + d)x_2 = (.9801)\varepsilon r^2,\ (.99)\varepsilon r \leqq x_2 \leqq (.0199)^{1/2} r\},$$
$$H_3 = \{(x_1, x_2) : x_1 + d = \varepsilon(r^2 - x_2^2)/x_2,\ (.0199)^{1/2} r \leqq x_2 \leqq (.9999)^{1/2} r\},$$
$$H_4 = \{(x_1, x_2) : (x_1 + d)x_2 = (.0001)\varepsilon r^2,\ (.01)\varepsilon r \leqq x_2 \leqq (.9999)^{1/2} r\},$$
$$H_{i+4} = \{(x_1, x_2) : (-x_1, x_2) \in H_i\} \text{ for } i = 1, 2, 3, 4,$$

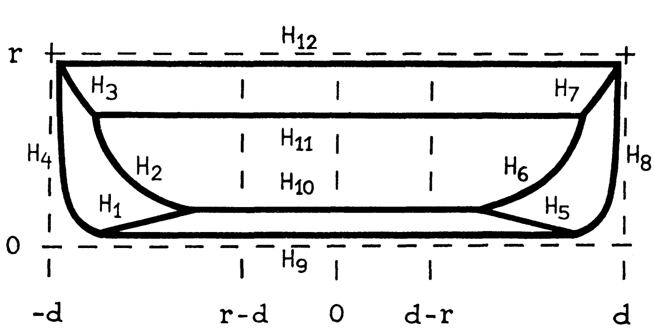

Fig. 1

$$H_9 = \{(x_1, x_2): |x_1| \leqq d - (.01)r, x_2 = (.01)\varepsilon r\},$$

$$H_{10} = \{(x_1, x_2): |x_1| \leqq d - (.99)r, x_2 = (.99)\varepsilon r\},$$

$$H_{11} = \{(x_1, x_2): |x_1| \leqq d - ((.9801)/(.0199)^{1/2})\varepsilon r, x_2 = (.0199)^{1/2} r\},$$

$$H_{12} = \{(x_1, x_2): |x_1| \leqq d - ((.0001)/(.9999)^{1/2})\varepsilon r, x_2 = (.9999)^{1/2} r\}.$$

The topological notions of this paragraph will always refer to the natural topology of P(not R^2). The sets $R_1, \ldots, R_6$ are open subsets of P determined by the following conditions: $R_1, \ldots, R_5$ are bounded, R_6 is unbounded,

$$H_1 \cup H_{10} \cup H_5 \cup H_9 = \text{boundary}(R_1), \quad H_5 \cup H_6 \cup H_7 \cup H_8 = \text{boundary}(R_2),$$

$$H_3 \cup H_{12} \cup H_7 \cup H_{11} = \text{boundary}(R_3), \quad H_1 \cup H_2 \cup H_3 \cup H_4 = \text{boundary}(R_4),$$

$$H_2 \cup H_{10} \cup H_6 \cup H_{11} = \text{boundary}(R_5), \quad H_9 \cup H_8 \cup H_{12} \cup H_4 = \text{boundary}(R_6).$$

The function $v = (v_1, v_2)$ from P into R^2 is defined almost everywhere by the equations

$$v(x_1, x_2) = (x_2, 0) \quad \text{if } (x_1, x_2) \in R_1,$$

$$v(x_1, x_2) = ((\varepsilon/2)(d - x_1), (\varepsilon/2)x_2) \quad \text{if } (x_1, x_2) \in R_2,$$

$$v(x_1, x_2) = (-\varepsilon^2 x_2, 0) \quad \text{if } (x_1, x_2) \in R_3,$$

$$v(x_1, x_2) = ((\varepsilon/2)(x_1 + d), -(\varepsilon/2)x_2) \quad \text{if } (x_1, x_2) \in R_4,$$

$$v(x_1, x_2) = (0, 0) \quad \text{if } (x_1, x_2) \in R_5 \cup R_6.$$

Let v^i be the continuous function on closure (R_i) that agrees with v on R_i. If $H_i = \text{closure}(R_j) \cap \text{closure}(R_k)$ then the restrictions of v^j and v^k to H_i have the same normal components with respect to H_i. If $f \in C_c^\infty(P, R)$ then $v \cdot \nabla f = \text{div}(fv)$ is valid on each R_i. Hence, applying the divergence theorem to each R_i and adding up the resulting six equations, we conclude

$$\int_P v(x_1, x_2) \cdot \nabla f(x_1, x_2) dx_1 dx_2 = 0 \quad \text{if } f \in C_c^\infty(P, R). \tag{4.31}$$

We have

$$0 < \varepsilon(d - x_1) < x_2 \quad \text{if } (x_1, x_2) \in R_2, \quad 0 < \varepsilon(x_1 + d) < x_2 \quad \text{if } (x_1, x_2) \in R_4.$$

This implies $0 < v_1(x_1, x_2) < x_2/2$ if $(x_1, x_2) \in R_2 \cup R_4$. Combining this with $0 < \varepsilon < 1$ (see (4.19)) we conclude

$$|v(x_1, x_2)| \leqq x_2, \quad v_1(x_1, x_2) \geqq -\varepsilon^2 x_2, \quad |v_2(x_1, x_2)| \leqq (\varepsilon/2)x_2 \tag{4.32}$$

for almost every $(x_1, x_2) \in P$. Using $(.99)\varepsilon r < E < r/10 < (.0199)^{1/2} r$ (see (4.22), (4.24)) we obtain

$$\text{spt}(v) \subset (-d, d) \times [10^{-2}\varepsilon r, r), \tag{4.33}$$

$$\text{spt}(v) \text{ and } [r - d, d - r] \times [E, r/10] \text{ are disjoint sets.} \tag{4.34}$$

Hence we can choose $g \in C_c^\infty(R^2, R)$ such that (extending v by 0 to all of R^2) we get

$$g(x) \geqq 0, \|g\|_1 = 1, g(x) = g(y) \quad \text{if } |x| = |y|, g(x) = 0 \quad \text{if } |x| \geqq 10^{-3}\varepsilon r, \tag{4.35}$$

$$\operatorname{spt}(v*g) \subset (-d,d) \times (10^{-3}\varepsilon r, r), \tag{4.36}$$

$$\operatorname{spt}(v*g) \quad \text{and} \quad [r-d, d-r] \times [E, r/10] \text{ are disjoint sets.} \tag{4.37}$$

From (4.32), (4.33) and (4.35) we obtain

$$|(v*g)(x_1,x_2)| \leqq x_2, \ (v_1*g)(x_1,x_2) \geqq -\varepsilon^2 x_2,$$
$$|(v_2*g)(x_1,x_2)| \leqq (\varepsilon/2)x_2 \quad \text{if } (x_1,x_2) \in P. \tag{4.38}$$

The definition of v on R_1, R_5, R_6, $0<\varepsilon<1$ (see (4.19)), $E<r/10$ (see (4.24)) and (4.35) give us

$$\text{if } |x_1| \leqq d-r \quad \text{and} \quad (.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r \quad \text{then } (v*g)(x_1,x_2)=(x_2,0), \tag{4.39}$$

$$\text{if } |x_1| \leqq d-r \quad \text{and} \quad 0<x_2<E \quad \text{then } (v_1*g)(x_1,x_2) \geqq 0, \quad (v_2*g)(x_1,x_2)=0. \tag{4.40}$$

Recalling (4.36), we define $w \in C_c^\infty(P, R^2)$ using the formula

$$w(x_1,x_2) = x_2^{-1}(v*g)(x_1,x_2), \tag{4.41}$$

and use (4.38), (4.39) to conclude

$$|w(x)| \leqq 1, \quad w_1(x) \geqq -\varepsilon^2, \quad |w_2(x)| \leqq \varepsilon/2 \quad \text{if } x \in P, \tag{4.42}$$

$$\text{if } |x_1| \leqq d-r \quad \text{and} \quad (.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r, \quad \text{then } w(x)=(1,0). \tag{4.43}$$

Conclusions (4.25), (4.26) follow from (4.41), (4.36), (4.37). Properties (4.31), (4.33) give us $\operatorname{div}(v*g)=0$, which implies (4.27). Using (4.43), (4.24), (4.22) and Lemma 4.7 we get (4.28). From (4.43), (4.22) and Lemma 4.6 we conclude (4.29). All we have to do now is prove (4.30).

We may assume $(x_1,x_2) \in \operatorname{spt}(w)$. If (x_1,x_2) satisfies $|(x_1,x_2)| > r/10$, then (4.42), (4.25), Lemma 4.5, $2|A| < r/10 < |(x_1,x_2)|$ (see (4.22)), Lemma 4.4, (4.22) imply

$$\begin{aligned}
&w(x_1,x_2)\cdot(F^{a,r,s} + F^{a',r',s'} + F^{a'',r'',s''} + F)(x_1,x_2) \\
&\quad = w_1(x_1,x_2)F_1^{a,r,s}(x_1,x_2) + w_2(x_1,x_2)F_2^{a,r,s}(x_1,x_2) \\
&\qquad + w(x_1,x_2)\cdot(F^{a',r',s'} + F^{a'',r'',s''} + F)(x_1,x_2) \\
&\quad > -\varepsilon^2(1.03)B - (\varepsilon/2)(.002)B\varepsilon - 2C|(x_1,x_2)|^{-4} \\
&\quad > -\varepsilon^2(1.031)B - 2C(r/10)^{-4} > -\varepsilon^2(1.031)B - 10^{-3}\varepsilon^2 B = -\varepsilon^2(1.032)B.
\end{aligned}$$

It remains to show that (4.30) holds if $|(x_1,x_2)| \leqq r/10$. In this case, properties (4.23), $(x_1,x_2) \in \operatorname{spt}(w)$, (4.25) imply

$$|x_1| \leqq r/10 \leqq 10^{-5}d < (1-10^{-4})d \leqq d-r, \quad 0<x_2 \leqq r/10.$$

The above and (4.26) imply $0<x_2<E$. Then (4.40), (4.41) yield $w_1(x_1,x_2) \geqq 0$, $w_2(x_1,x_2)=0$ and Lemma 4.6 gives us $h(x_1,x_2) \geqq 0$. The conclusion follows from this and (4.2).

Section 5. Edge Effects

Lemma 5.1. *If* $c \in R, s>0$ *and* $f:(c-s,c+s) \to R$ *is a* C^∞ *function such that*

$$f(x)=0 \quad \text{if } c-s<x \leqq c, \quad f^{(3)}(x)>0 \quad \text{if } c<x<c+s,$$

then every $x \in (c, c+s)$ *satisfies*

$$f''(x) > 0, \quad 0 < f(x) < (x-c)^2 f''(x), \quad 0 < f'(x) < (x-c) f''(x).$$

Proof. If $c < x < c+s$ then the theory of divided differences (see [2], pp. 41, 52) and $c - s < 2c - x < c$ imply that there exists $\xi \in (2c - x, x)$ such that

$$\begin{aligned} f(x) &= f(2c-x) - 2f(c) + f(x) = 2(x-c)^2 f[2c-x, c, x] \\ &= 2(x-c)^2 f''(\xi)/2 < (x-c)^2 f''(x). \end{aligned}$$

The remaining conclusions follow from the mean value theorem.

Lemma 5.2. *If* $c \in R, s > 0$ *and* $f:(c-s, c+s) \to R$ *is a* C^∞ *function such that*

$$f^{(3)}(x) < 0 \quad \text{if } c - s < x < c, \quad f(x) = 0 \quad \text{if } c \leqq x < c + s,$$

then every $x \in (c - s, c)$ *satisfies*

$$f''(x) > 0, \quad 0 < f(x) < (x-c)^2 f''(x), \quad (x-c) f''(x) < f'(x) < 0.$$

Proof. This is a consequence of Lemma 5.1 and a reflection of f.

Lemma 5.3. *Suppose* $a_1 < b_1$, $0 < a_2 < b_2$, $\varepsilon > 0$, $b_i - a_i > 2\varepsilon$ *for each* i, f_1 *and* f_2 *are* C^∞ *functions from* R *into* R, $h(x_1, x_2) = f_1(x_1) f_2(x_2)$ *and the following properties hold for each* $i \in \{1, 2\}$:

$$f_i(x) > 0 \quad \text{if } a_i < x < b_i, \quad f_i(x) = 0 \quad \text{if } x \notin (a_i, b_i),$$

$$f_i^{(3)}(x) > 0 \quad \text{if } a_i < x < a_i + \varepsilon, \quad f_i^{(3)}(x) < 0 \quad \text{if } b_i - \varepsilon < x < b_i.$$

Then there exists $\delta > 0$ *such that the inequality*

$$L(h)(x_1, x_2) \equiv \Delta h(x_1, x_2) + x_2^{-1} \frac{\partial}{\partial x_2} h(x_1, x_2) - x_2^{-2} h(x_1, x_2) > 0$$

is satisfied for every point (x_1, x_2) *that has the properties*

$$(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2), \quad (x_1, x_2) \notin [a_1 + \delta, b_1 - \delta] \times [a_2 + \delta, b_2 - \delta].$$

Proof. If $x_1 \in R$ and $x_2 > 0$, we set

$$g_1(x_1) = f_1''(x_1), \quad g_2(x_2) = f_2''(x_2) + f_2'(x_2)/x_2 - f_2(x_2)/x_2^2. \tag{5.1}$$

This notation implies

$$L(h)(x_1, x_2) = g_1(x_1) f_2(x_2) + f_1(x_1) g_2(x_2). \tag{5.2}$$

We may assume $\varepsilon/a_2 < 1/2, \varepsilon/(b_2 - \varepsilon) < 1/2$ without loss of generality. If $x_2 \in (a_2, a_2 + \varepsilon)$, then (5.1), Lemma 5.1 and $\varepsilon/a_2 < 1/2$ imply

$$\begin{aligned} g_2(x_2) &> f_2''(x_2) - f_2(x_2)/x_2^2 > f_2''(x_2) - (x_2 - a_2)^2 f_2''(x_2)/x_2^2 \\ &> (1 - (\varepsilon/a_2)^2) f_2''(x_2) > (3/4) f_2''(x_2). \end{aligned}$$

If $x_2 \in (b_2 - \varepsilon, b_2)$, then (5.1), Lemma 5.2 and $\varepsilon/(b_2 - \varepsilon) < 1/2$ imply

$$\begin{aligned} g_2(x_2) &> f_2''(x_2) + (x_2 - b_2) f_2''(x_2)/x_2 - (x_2 - b_2)^2 f_2''(x_2)/x_2^2 \\ &> (1 - \varepsilon/(b_2 - \varepsilon) - (\varepsilon/(b_2 - \varepsilon))^2) f_2''(x_2) > f_2''(x_2)/4. \end{aligned}$$

From the above and Lemmas 5.1, 5.2 we obtain

$$g_k(x_k) > f''_k(x_k)/4 > 0 \quad \text{if } x_k \in (a_k, a_k+\varepsilon) \cup (b_k - \varepsilon, b_k). \tag{5.3}$$

The positivity of f_k on (a_k, b_k), $b_k - a_k > 2\varepsilon$, (5.2) and (5.3) yield

$$L(h)(x_1, x_2) > 0 \quad \text{if } x_k \in (a_k, a_k+\varepsilon) \cup (b_k - \varepsilon, b_k). \tag{5.4}$$

There are numbers $m > 0$, $M < \infty$ such that

$$x_k \in [a_k + \varepsilon, b_k - \varepsilon] \text{ implies } f_k(x_k) > m \quad \text{and} \quad |g_k(x_k)| < M. \tag{5.5}$$

We can find $\delta > 0$ such that $\delta < \varepsilon$, $m/4 - \delta^2 M > 0$. Now suppose that (i,j) is a permutation of $(1,2)$ and the point (x_1, x_2) satisfies

$$x_i \in [a_i + \varepsilon, b_i - \varepsilon], \quad x_j \in (a_j, a_j + \delta) \cup (b_j - \delta, b_j). \tag{5.6}$$

The assumptions $b_i - a_i > 2\varepsilon$, $\delta < \varepsilon$ imply that $(a_j, a_j + \delta)$ and $(b_j - \delta, b_j)$ are disjoint. Hence we can define

$$c = a_j \quad \text{if } x_j \in (a_j, a_j + \delta), \quad c = b_j \quad \text{if } x_j \in (b_j - \delta, b_j). \tag{5.7}$$

The properties $\delta < \varepsilon$, (5.2), (5.3), (5.5), (5.6), (5.7) and Lemmas 5.1, 5.2 yield

$$\begin{aligned} L(h)(x_1, x_2) &= f_i(x_i) g_j(x_j) + f_j(x_j) g_i(x_i) > f_i(x_i) f''_j(x_j)/4 - f_j(x_j) M \\ &> m f''_j(x_j)/4 - (x_j - c)^2 f''_j(x_j) M > (m/4 - \delta^2 M) f''_j(x_j) > 0. \end{aligned}$$

This estimate and (5.4) imply that $L(h)$ is positive on the union of the sets

$$((a_1, a_1+\varepsilon) \cup (b_1 - \varepsilon, b_1)) \times ((a_2, a_2 + \varepsilon) \cup (b_2 - \varepsilon, b_2)),$$

$$[a_1 + \varepsilon, b_1 - \varepsilon] \times ((a_2, a_2 + \delta) \cup (b_2 - \delta, b_2)),$$

$$((a_1, a_1 + \delta) \cup (b_1 - \delta, b_1)) \times [a_2 + \varepsilon, b_2 - \varepsilon].$$

The conclusion of the lemma follows from this and $0 < \delta < \varepsilon$.

Lemma 5.4. *If* $a_1 < b_1$, $0 < a_2 < b_2$, $d > 0$ *and* $b_i - a_i > 2d$ *for each* i, *then there exist* $\delta > 0$ *and a* C^∞ *function* $h: R^2 \to [0,1]$ *such that* $\delta < d$,

$$\begin{aligned} h(x_1, x_2) &= 0 \quad \text{if } (x_1, x_2) \notin (a_1, b_1) \times (a_2, b_2), \\ h(x_1, x_2) &= 1 \quad \text{if } (x_1, x_2) \in [a_1 + d, b_1 - d] \times [a_2 + d, b_2 - d], \\ 0 < h(x_1, x_2) &\leqq 1 \quad \text{if } (x_1, x_2) \in (a_1, b_1) \times (a_2, b_2), \\ L(h)(x) &> 0 \quad \text{if } x \in (a_1, b_1) \times (a_2, b_2), x \notin [a_1 + \delta, b_1 - \delta] \times [a_2 + \delta, b_2 - \delta]. \end{aligned}$$

Proof. We can find $\varepsilon' > 0$ and C^∞ functions $f_i: R \to [0,1]$ for $i = 1, 2$ such that $0 < \varepsilon' < d$, $\mathrm{spt}(f_i) \subset [a_i, b_i]$, $f_i(x) > 0$ if $x \in (a_i, b_i)$, $f_i(x) = 1$ if $x \in [a_i + d, b_i - d]$,

$$f_i(x) = \exp(-(x - a_i)^{-2}) \quad \text{if } a_i < x < a_i + \varepsilon',$$

$$f_i(x) = \exp(-(b_i - x)^{-2}) \quad \text{if } b_i - \varepsilon' < x < b_i.$$

Setting $k(x) = \exp(-x^{-2})$ for $x > 0$, we can find ε such that $0 < \varepsilon < \varepsilon'$ and $k^{(3)}(x) > 0$ for all $x \in (0, \varepsilon)$. Then $f_i^{(3)}(x) > 0$ for $x \in (a_i, a_i + \varepsilon)$, $f_i^{(3)}(x) < 0$ for $x \in (b_i - \varepsilon, b_i)$. The conclusion follows from Lemma 5.3 and the definition $h(x_1, x_2) = f_1(x_1) f_2(x_2)$.

Lemma 5.5. *If $a_1 < b_1$ and $0 < d < a_2 < b_2$, then there exist $\delta > 0$ and a C^∞ function $g: R^2 \to [0,1]$ such that $\delta < d$,*

$$g(x_1,x_2)=0 \quad \text{if } (x_1,x_2)\in[a_1,b_1]\times[a_2,b_2],$$
$$g(x_1,x_2)=1 \quad \text{if } (x_1,x_2)\notin(a_1-d,b_1+d)\times(a_2-d,b_2+d),$$
$$0<g(x_1,x_2)\leqq 1 \quad \text{if } (x_1,x_2)\notin[a_1,b_1]\times[a_2,b_2],$$
$$L(g)(x)>0 \quad \text{if } x\in(a_1-\delta,b_1+\delta)\times(a_2-\delta,b_2+\delta), \quad x\notin[a_1,b_1]\times[a_2,b_2].$$

Proof. We can find $\varepsilon' > 0$ and C^∞ functions $f_i: R \to [0,1]$ for $i=1,2$ such that $\varepsilon' < d$, $f_i(x)=0$ if $x\in[a_i,b_i]$, $f_i(x)>0$ if $x\notin[a_i,b_i]$,

$$f_i(x)=\exp(-(a_i-x)^{-2}) \quad \text{if } a_i-\varepsilon'<x<a_i,$$
$$f_i(x)=\exp(-(x-b_i)^{-2}) \quad \text{if } b_i<x<b_i+\varepsilon'.$$

As in the proof of Lemma 5.4, there exists ε such that $0<\varepsilon<\varepsilon'$, $f_i^{(3)}(x)<0$ if $x\in(a_i-\varepsilon,a_i)$, $f_i^{(3)}(x)>0$ if $x\in(b_i,b_i+\varepsilon)$. Setting $h(x_1,x_2)=f_1(x_1)+f_2(x_2)$, we find

$$L(h)(x_1,x_2)=(f_1''(x_1)-f_1(x_1)/x_2^2)+(f_2''(x_2)+f_2'(x_2)/x_2-f_2(x_2)/x_2^2). \tag{5.8}$$

We may impose the conditions $\varepsilon/(a_2-\varepsilon)<1/2$, $\varepsilon/b_2<1/2$ on ε. If we use these conditions and interchange a_2, b_2 in the argument that led to (5.3), then we obtain

$$f_2''(x_2)+f_2'(x_2)/x_2-f_2(x_2)/x_2^2>0 \quad \text{if } x_2\in(a_2-\varepsilon,a_2)\cup(b_2,b_2+\varepsilon). \tag{5.9}$$

Our definitions (which imply $a_2-\varepsilon>0$) give us

$$f_2''(x_2)+f_2'(x_2)/x_2-f_2(x_2)/x_2^2=0 \quad \text{if } x_2\in[a_2,b_2], \tag{5.10}$$
$$f_1''(x_1)-f_1(x_1)/x_2^2=0 \quad \text{if } x_1\in[a_1,b_1] \quad \text{and} \quad x_2>a_2-\varepsilon. \tag{5.11}$$

Now suppose $x_1\in(a_1-\varepsilon,a_1)\cup(b_1,b_1+\varepsilon)$ and $x_2>a_2-\varepsilon$. If we set

$$c=a_1 \quad \text{if } x_1\in(a_1-\varepsilon,a_1), \quad c=b_1 \quad \text{if } x_1\in(b_1,b_1+\varepsilon),$$

then Lemmas 5.1, 5.2 and $\varepsilon/(a_2-\varepsilon)<1/2$ yield

$$f_1''(x_1)-f_1(x_1)/x_2^2>f_1''(x_1)-(x_1-c)^2 f_1''(x_1)/x_2^2>(1-(\varepsilon/(a_2-\varepsilon))^2)$$
$$\cdot f_1''(x_1)>(3/4)f_1''(x_1)>0.$$

Combining this with (5.8)–(5.11) and the definition of f_i, we find that $L(h)(x)$ and $h(x)$ are positive when x satisfies

$$x\in(a_1-\varepsilon,b_1+\varepsilon)\times(a_2-\varepsilon,b_2+\varepsilon), \quad x\notin[a_1,b_1]\times[a_2,b_2].$$

We can construct a C^∞ function $g: R^2\to[0,1]$ such that

$$g(x_1,x_2)=1 \quad \text{if } (x_1,x_2)\notin(a_1-d,b_1+d)\times(a_2-d,b_2+d),$$
$$0<g(x_1,x_2)\leqq 1 \quad \text{if } (x_1,x_2)\notin[a_1,b_1]\times[a_2,b_2],$$
$$g(x_1,x_2)=h(x_1,x_2) \quad \text{if } (x_1,x_2)\in(a_1-\delta,b_1+\delta)\times(a_2-\delta,b_2+\delta)$$

are satisfied for some δ with $0<\delta<\varepsilon$. This concludes the proof.

Section 6. Conclusion

Lemma 6.1. *Suppose* $u \in C_c^\infty(R^3, R^3)$ *satisfies* $\sum_{i=1}^{3} \partial u_i/\partial x_i = 0$ *and the functions* $p: R^3 \to R$, $G = (G_1, G_2, G_3): R^3 \to R^3$ *are defined by*

$$p(x) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}(y) \frac{\partial u_i}{\partial x_j}(y)(4\pi|x-y|)^{-1} dy \quad \text{and} \quad G = \nabla p. \tag{6.1}$$

Then we have

$$\lim_{x_1 \to \infty} x_1^4 G_1(x_1, 0, 0) = 3(4\pi)^{-1} \int_{R^3} (u_2(x))^2 + (u_3(x))^2 - 2(u_1(x))^2 dx, \tag{6.2}$$

and there exists $C \in R$ *such that* $C > 0$ *and*

$$|G(x)| \leqq C|x|^{-4}, \quad |\nabla G(x)| \leqq C|x|^{-5}. \tag{6.3}$$

Proof. We set $g(x) = (4\pi|x|)^{-1}$ for $x \in R^3$. Integration by parts and the assumption $\sum_{i=1}^{3} \partial u_i/\partial x_i = 0$ imply

$$G(x) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} u_j(y) u_i(y) \frac{\partial^2(\nabla g)}{\partial x_i \partial x_j}(x-y) dy \quad \text{if } x \notin \operatorname{spt}(u), \tag{6.4}$$

$$\frac{\partial G}{\partial x_k}(x) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} u_j(y) u_i(y) \frac{\partial^3(\nabla g)}{\partial x_i \partial x_j \partial x_k}(x-y) dy \quad \text{if } x \notin \operatorname{spt}(u). \tag{6.5}$$

Let r be a positive real number such that $u(x) = 0$ holds if $|x| \geqq r$. Then (6.4), (6.5) imply that $|x|^4|G(x)|$ and $|x|^5|\nabla G(x)|$ are bounded on the set $\{x: |x| \geqq 2r\}$. Since (6.1) implies that these functions are also bounded on the compact set $\{x: |x| \leqq 2r\}$, we conclude (6.3). Assertion (6.2) is a consequence of (6.4).

Lemma 6.2. *Let* U *be the open subset of* P *defined by*

$$U = \{(x_1, x_2) \in R^2 : |x_1| < 1, 1/8 < x_2 < 7/8\}. \tag{6.6}$$

There exist f, z, F, K, A, B, C, D *such that* (6.7)–(6.15) *are satisfied*:

$$f \in C_c^\infty(P, R), \quad z = (z_1, z_2) \in C_c^\infty(P, R^2), \quad F = (F_1, F_2) \in C^\infty(R^2, R^2), \tag{6.7}$$

$$K \text{ is compact}, \operatorname{spt}(z) \subset K \subset U, \tag{6.8}$$

$$f \geqq 0, \quad f(x) = 0 \quad \text{if } x \notin U, \quad f(x) > |z(x)| \quad \text{if } x \in U, \tag{6.9}$$

$$L(f)(x) \geqq 0 \quad \text{if } x \notin K, \quad L(f)(x) > 0 \quad \text{if } x \in U \sim K, \tag{6.10}$$

$$x_2 \frac{\partial z_1}{\partial x_1}(x_1, x_2) + x_2 \frac{\partial z_2}{\partial x_2}(x_1, x_2) + z_2(x_1, x_2) = 0, \tag{6.11}$$

$$\nabla(p[0, f] - p[z, f])(x) = F(x) \quad \text{if } x \in P, \tag{6.12}$$

$$A, B, C, D \text{ are real numbers}, B > 0, C > 0, D > 0, \tag{6.13}$$

$$F_1(A, 0) = B, \quad \text{if } x \in R, \quad \text{then } |F_1(x, 0)| \leqq B \quad \text{and} \quad F_2(x, 0) = 0, \tag{6.14}$$

$$\lim_{x\to\infty} x^4 F_1(x,0)=D, \quad \textit{if } x\in R^2, \quad \textit{then } |F(x)|\leqq C|x|^{-4} \quad \textit{and} \quad |\nabla F(x)|\leqq C|x|^{-5}. \tag{6.15}$$

Proof. Let $k:R^2\to R^2$ be the function defined by

$$k(x_1,x_2)=(-(x_2-1/2),x_1) \quad \text{if } 1/16<(x_1^2+(x_2-1/2)^2)^{1/2}<1/8$$

and $k(x_1,x_2)=0$ otherwise. Let $g\in C_c^\infty(R^2,R)$ satisfy

$$g(x)\geqq 0, \quad g(x)=0 \quad \text{if } |x|\geqq 1/16, \quad \|g\|_1=1, \quad g(x)=g(y) \quad \text{if } |x|=|y|.$$

Setting $z(x_1,x_2)=x_2^{-1}(k*g)(x_1,x_2)$ for $x\in P$, we obtain

$$z\in C_c^\infty(P,R^2), \quad \text{spt}(z)\subset[-3/16,3/16]\times[5/16,11/16], \quad \|z\|_\infty\leqq 2/5, \tag{6.16}$$

$$z_1(x_1,x_2)=z_1(-x_1,x_2), \quad z_2(x_1,x_2)=-z_2(-x_1,x_2), \tag{6.17}$$

$$x_2\frac{\partial z_1}{\partial x_1}(x_1,x_2)+x_2\frac{\partial z_2}{\partial x_2}(x_1,x_2)+z_2(x_1,x_2)=\frac{\partial}{\partial x_1}(k_1*g)(x_1,x_2) + \frac{\partial}{\partial x_2}(k_2*g)(x_1,x_2)=0. \tag{6.18}$$

Setting $a_1=-1$, $b_1=1$, $a_2=1/8$, $b_2=7/8$, we use Lemma 5.4 (with 1/16 in place of the d that appears there) to obtain the function h and $\delta<1/16$. We then define

$$f(x)=h(x) \quad \text{if } x\in P, \quad K=\{(x_1,x_2):|x_1|\leqq 1-\delta, 1/8+\delta\leqq x_2\leqq 7/8-\delta\}, \tag{6.19}$$

and observe that the properties of h, $\delta<1/16$, (6.6) and (6.16) imply

$$f\in C_c^\infty(P,R), \quad f(x)>0 \quad \text{if } x\in U, \quad f(x)=1 \quad \text{if } x\in\text{spt}(z), \tag{6.20}$$

$$L(f)(x)>0 \quad \text{if } x\in U\sim K, \quad f(x)=0 \quad \text{if } x\notin U, \quad \text{spt}(z)\subset K\subset U. \tag{6.21}$$

Using (6.16), (6.20) we find

$$f(x)>|z(x)| \quad \text{if } x\in U. \tag{6.22}$$

Assertions (6.16), (6.20)–(6.22) imply that the definition

$$F(x_1,x_2)=(F_1(x_1,x_2),F_2(x_1,x_2))=\left(\frac{\partial}{\partial x_1}(p^*[0,f]-p^*[z,f])(x_1,x_2,0), \frac{\partial}{\partial x_2}(p^*[0,f]-p^*[z,f])(x_1,x_2,0)\right) \tag{6.23}$$

makes sense for $(x_1,x_2)\in R^2$. Using (6.16), (6.18) we obtain $\sum_{i=1}^{3}(\partial/\partial x_i)u_i[z,f]=0$. This fact, the identity $\sum_{i=1}^{3}(\partial/\partial x_i)u_i[0,f]=0$, (6.16), (6.20) and Lemma 6.1 imply (6.24) and (6.25):

$$\lim_{x_1\to\infty} x_1^4F_1(x_1,0)=3(4\pi)^{-1}\int_{R^3}(u_2[0,f](x))^2+(u_3[0,f](x))^2-2(u_1[0,f](x))^2dx$$
$$-3(4\pi)^{-1}\int_{R^3}(u_2[z,f](x))^2+(u_3[z,f](x))^2$$
$$-2(u_1[z,f](x))^2dx$$

$$
\begin{aligned}
&= 3(4\pi)^{-1} \int_{R^3} (f(x_1,(x_2^2+x_3^2)^{1/2}))^2 dx_1 dx_2 dx_3 \\
&\quad - 3(4\pi)^{-1} \int_{R^3} (z_2(x_1,(x_2^2+x_3^2)^{1/2}))^2 dx_1 dx_2 dx_3 \\
&\quad - 3(4\pi)^{-1} \int_{R^3} (f(x_1,(x_2^2+x_3^2)^{1/2}))^2 \\
&\quad - |z(x_1,(x_2^2+x_3^2)^{1/2})|^2 dx_1 dx_2 dx_3 \\
&\quad - 3(4\pi)^{-1} \int_{R^3} -2(z_1(x_1,(x_2^2+x_3^2)^{1/2}))^2 dx_1 dx_2 dx_3 \\
&= 9(4\pi)^{-1} \int_{R^3} (z_1(x_1,(x_2^2+x_3^2)^{1/2}))^2 dx_1 dx_2 dx_3 > 0, \qquad (6.24)
\end{aligned}
$$

$$|F(x)| \leqq C|x|^{-4}, \quad |\nabla F(x)| \leqq C|x|^{-5} \quad \text{for some } C < \infty. \qquad (6.25)$$

Using (6.24), (6.25) we find that the function $F_1(x_1, 0)$ of the variable x_1 achieves a positive maximum at some number A. Let B be this maximum value. From (6.17) and (6.23) we obtain $F_1(-x_1, 0) = -F_1(x_1, 0)$ for all $x_1 \in R$. All this implies

$$F_1(A,0) = B, \quad |F_1(x,0)| \leqq B \quad \textit{if } x \in R. \qquad (6.26)$$

Finally, (6.20), (6.16), (6.23) imply (6.7); (6.19), (6.21) imply (6.8); (6.20)–(6.22) imply (6.9); (6.6), (6.21) imply (6.10); (6.18) implies (6.11); (6.23) implies (6.12); and (6.13)–(6.15) follow from (6.23)–(6.26) and the fact $p^*[v,f](R_c x) = p^*[v,f](x)$. The proof of the lemma is complete.

Now we tie together the results from Sects. 3, 4, 5. We will use the U, f, z, F, K, A, B, C, D from Lemma 6.2. Since (6.7), (6.13)–(6.15) say that the hypotheses of Sect. 4 are satisfied, we can use the machinery developed in that section. In particular, we will use the function w in Lemma 4.8 and the numbers a, r, s, a', r', s', a'', r'', s'', E, ε, d introduced in Sect. 4 (see (4.7), (4.8), (4.18)–(4.21), Lemma 4.3).

When $\alpha \in R$, $\rho > 0$, $\sigma > 0$ we let

$$f^{\alpha,\rho,\sigma} \in C_c^\infty(P,R) \quad \text{and} \quad z^{\alpha,\rho,\sigma} = (z_1^{\alpha,\rho,\sigma}, z_2^{\alpha,\rho,\sigma}) \in C_c^\infty(P,R^2) \qquad (6.27)$$

be given by

$$
\begin{aligned}
f^{\alpha,\rho,\sigma}(x_1,x_2) &= \sigma f((x_1-\alpha)/\rho, x_2/\rho), \quad z^{\alpha,\rho,\sigma}(x_1,x_2) \\
&= \sigma z((x_1-\alpha)/\rho, x_2/\rho). \qquad (6.28)
\end{aligned}
$$

If $\alpha \in R$ and $\rho > 0$, we set

$$
\begin{aligned}
U^{\alpha,\rho} &= \{x \in R^2 : ((x_1-\alpha)/\rho, x_2/\rho) \in U\}, \\
K^{\alpha,\rho} &= \{x \in R^2 : ((x_1-\alpha)/\rho, x_2/\rho) \in K\}. \qquad (6.29)
\end{aligned}
$$

We will use the notation

$$
\begin{aligned}
f_1 &= f^{a,r,s} + f^{a',r',s'} + f^{a'',r'',s''} + f, \\
v_1 &= z^{a,r,s} + z^{a',r',s'} + z^{a'',r'',s''} + z, \\
U_1 &= U^{a,r} \cup U^{a',r'} \cup U^{a'',r''} \cup U, \\
K_1 &= K^{a,r} \cup K^{a',r'} \cup K^{a'',r''} \cup K. \qquad (6.30)
\end{aligned}
$$

Properties (6.9), (6.28), (6.29) imply

$$\operatorname{spt}(f) = \operatorname{closure}(U), \quad \operatorname{spt}(f^{\alpha,\rho,\sigma}) = \operatorname{closure}(U^{\alpha,\rho}). \tag{6.31}$$

From (4.20), $0 < \varepsilon < .01$ (see (4.19)), (4.23) and (4.21) we conclude

$$a + r = -r\varepsilon^{-1}10^3 C/D + r < -r10^5 C/D + 10^{-4}d = -10d + 10^{-4}d < -d.$$

This inequality, (6.6), (6.29) and (6.31) imply

$$\operatorname{spt}(f^{a,r,s}) \quad \text{and} \quad [-d, d] \times [10^{-3}\varepsilon r, r] \text{ are disjoint.} \tag{6.32}$$

Using (4.10), (4.11), $r/10 > 1$ (see (4.22)), (6.6), (6.29), (6.31) we find

$$\operatorname{spt}(f^{a',r',s'}) \cup \operatorname{spt}(f^{a'',r'',s''}) \cup \operatorname{spt}(f) \subset R \times (E, r/10). \tag{6.33}$$

From (4.8), (4.10) we get $|a'| + r' < |A| + 1$, $|a''| + r'' < |A| + 1$. Properties (4.22), (4.23) imply $|A| + 1 < r/10 < d - r$. These facts, (6.6), (6.29), (6.31) and (6.33) yield

$$\operatorname{spt}(f^{a',r',s'}) \cup \operatorname{spt}(f^{a'',r'',s''}) \cup \operatorname{spt}(f) \subset (r - d, d - r) \times (E, r/10). \tag{6.34}$$

In addition, (4.10), (6.6), (6.29), (6.31) yield

$$\operatorname{spt}(f^{a',r',s'}), \quad \operatorname{spt}(f^{a'',r'',s''}) \quad \text{and} \quad \operatorname{spt}(f) \text{ are disjoint.} \tag{6.35}$$

From (4.22) and (4.24) we conclude

$$[r - d, d - r] \times [E, r/10] \subset (-d, d) \times (10^{-3}\varepsilon r, r). \tag{6.36}$$

Now (6.32), (6.34)–(6.36) yield

$$\operatorname{spt}(f^{a,r,s}), \quad \operatorname{spt}(f^{a',r',s'}), \quad \operatorname{spt}(f^{a'',r'',s''}) \quad \text{and} \quad \operatorname{spt}(f) \text{ are disjoint.} \tag{6.37}$$

Properties (6.6)–(6.9), (6.28), (6.30), (6.37) imply

$$p[v_1, f_1] = p[z^{a,r,s}, f^{a,r,s}] + p[z^{a',r',s'}, f^{a',r',s'}] + p[z^{a'',r'',s''}, f^{a'',r'',s''}] + p[z, f], \tag{6.38}$$

$$p[0, f_1] = p[0, f^{a,r,s}] + p[0, f^{a',r',s'}] + p[0, f^{a'',r'',s''}] + p[0, f]. \tag{6.39}$$

Using (6.12), (6.28), (4.5) we conclude

$$\nabla(p[0, f^{\alpha,\rho,\sigma}] - p[z^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}])(x) = F^{\alpha,\rho,\sigma}(x) \quad \text{if } x \in P. \tag{6.40}$$

From (6.12) and (6.38)–(6.40) we get

$$\nabla(p[0, f_1] - p[v_1, f_1])(x) = (F^{a,r,s} + F^{a',r',s'} + F^{a'',r'',s''} + F)(x) \quad \text{if } x \in P. \tag{6.41}$$

Recalling (6.36), we set

$$U_2 = ((-d, d) \times (10^{-3}\varepsilon r, r)) \sim ([r - d, d - r] \times [E, r/10]) \subset P. \tag{6.42}$$

The proof of (6.36) shows that we can find $d_1 > 0$ satisfying $2d_1 < d - (d - r)$, $2d_1 < E - 10^{-3}\varepsilon r$, $2d_1 < r - r/10$. Since Lemma 4.8 says $\operatorname{spt}(w) \subset U_2$ (see (6.42)), we can assume that d_1 is small enough to yield

$$\operatorname{spt}(w) \subset (-d + d_1, d - d_1) \times (10^{-3}\varepsilon r + d_1, r - d_1),$$

$$\operatorname{spt}(w) \text{ and } [r - d - d_1, d - r + d_1] \times [E - d_1, r/10 + d_1] \text{ are disjoint.}$$

We let h be the function of Lemma 5.4 which is obtained when $a_1 = -d$, $b_1 = d$, $a_2 = 10^{-3}\varepsilon r$, $b_2 = r$ and the number d in the statement of that lemma is equal to d_1. Similarly, we let g be the function of Lemma 5.5 which is obtained when $a_1 = r - d$, $b_1 = d - r$, $a_2 = E$, $b_2 = r/10$ and the number d in the statement of Lemma 5.5 is equal to d_1. Setting $k(x) = h(x)g(x)$, we find that there exists K_2 satisfying (6.43)–(6.45):

$$\operatorname{spt}(w) \subset K_2 \subset U_2, \quad K_2 \text{ is compact}, \quad k \in C_c^\infty(P, R), \quad 0 \leqq k \leqq 1, \tag{6.43}$$

$$k(x) = 0 \quad \text{if } x \notin U_2, \quad k(x) > 0 \quad \text{if } x \in U_2, \quad k(x) = 1 \quad \text{if } x \in \operatorname{spt}(w), \tag{6.44}$$

$$L(k)(x) \geqq 0 \quad \text{if } x \notin K_2, \quad L(k)(x) > 0 \quad \text{if } x \in U_2 \sim K_2. \tag{6.45}$$

We choose $\mu > 0$ large enough to satisfy

$$\mu^2 > 10(\|w\|_\infty^2 + 1), \tag{6.46}$$

$$\mu > 100 \| f^{a,r,s} \|_\infty \quad \text{and} \quad \mu > 10(\| f^{a',r',s'} \|_\infty + \| f^{a'',r'',s''} \|_\infty + \| f \|_\infty). \tag{6.47}$$

Noting that (6.46) implies $\mu^2 - \|w\|_\infty^2 - 1 > 0$, we set

$$f_2 = \mu k, \quad v_2 = w, \quad T = (\mu^2 - \|v_2\|_\infty^2 - 1)/(\varepsilon^2(1.032)B) > 0, \quad \tau = (.48)\varepsilon. \tag{6.48}$$

If $x \in \operatorname{spt}(v_2) = \operatorname{spt}(w)$, then (6.48), (6.44), (6.41), $v_2 = w$, $T > 0$ and Lemma 4.8 (in particular, (4.30)) yield

$$\begin{aligned}
&(f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x) \\
&= \mu^2 + Tw(x) \cdot (F^{a,r,s} + F^{a',r',s'} + F^{a'',r'',s''} + F)(x) \\
&\geqq \mu^2 - T\varepsilon^2(1.032)B = \| v_2 \|_\infty^2 + 1 > |v_2(x)|^2.
\end{aligned}$$

If $x \in U_2$ and $x \notin \operatorname{spt}(v_2) = \operatorname{spt}(w)$, then (6.48), (6.44) and $\mu > 0$ imply

$$(f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x) = (f_2(x))^2 = \mu^2(k(x))^2 > 0 = |v_2(x)|^2.$$

All this shows

$$(f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x) > |v_2(x)|^2 \quad \text{if } x \in U_2. \tag{6.49}$$

Conditions (6.46) and (6.48) imply

$$\mu^2 - \|v_2\|_\infty^2 - 1 > (.9)\mu^2, \quad \text{hence } T > (.9)\mu^2/(\varepsilon^2(1.032)B). \tag{6.50}$$

Now (6.50), (6.48), (6.47) yield

$$\begin{aligned}
&(\mu^2 + T(8.01)B)^{1/2} > (T(8.01)B)^{1/2} > [(.9)\mu^2(8.01)/(\varepsilon^2(1.032))]^{1/2} > ((.48)\varepsilon)^{-1}(1.1)\mu \\
&= \tau^{-1}(1.1)\mu > \tau^{-1}(\| f^{a',r',s'} \|_\infty + \| f^{a'',r'',s''} \|_\infty + \| f \|_\infty + \mu).
\end{aligned} \tag{6.51}$$

Similarly, (6.50), (6.48), (6.47) yield

$$\begin{aligned}
&(\mu^2 + T(.01)B)^{1/2} > (T(.01)B)^{1/2} > [(.9)\mu^2(.01)/(\varepsilon^2(1.032))]^{1/2} \\
&> ((.48)\varepsilon)^{-1}(\mu/100) = \tau^{-1}(\mu/100) > \tau^{-1} \| f^{a,r,s} \|_\infty.
\end{aligned} \tag{6.52}$$

Assertions (6.51), (6.52) allow us to choose $\theta > 0$ such that (6.53), (6.54) hold:

$$(\mu^2 + T(8.01)B)^{1/2} - \theta > \tau^{-1}(\| f^{a',r',s'} \|_\infty + \| f^{a'',r'',s''} \|_\infty + \| f \|_\infty + \mu). \tag{6.53}$$

$$(\mu^2 + T(.01)B)^{1/2} - \theta > \tau^{-1}\|f^{a,r,s}\|_\infty. \tag{6.54}$$

Lemma 6.3. *The parameters* $T, \theta, K_1, K_2, U_1, U_2, f_1, f_2, v_1, v_2$ *defined above satisfy* $T>0$, $\theta>0$ *and* (3.1)–(3.7).

Proof. This follows from (6.27)–(6.49) and the definition of θ.

The above lemma says that all the hypotheses of Sect. 3 are satisfied. We will now use the machinery developed in that section. In what follows, we refer to the functions h_i and $h_{i,t}$ that are featured in Lemmas 3.1, 3.3.

Lemma 6.4. *Let* h_1, h_2 *be the functions that were introduced in Lemma* 3.1. *Set* $h_3 = h_1 + h_2$ *and* $h_{3,t}(x) = h_3(x,t)$ *for* $t \in [0,T]$. *If* $(x_1, x_2, x_3) \in R^3$ *and* $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \mathrm{spt}(h_{3,0})$, *then* $(\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2 > 0$ *and*

$$h_{3,T}(\tau x_1 + A, ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2}) > \tau^{-1} h_{3,0}(x_1, (x_2^2 + x_3^2)^{1/2}).$$

Proof. Part (4.29) of Lemma 4.8, $B>0$, (6.48), (3.1) and (3.3) (see Lemma 6.3) imply

$$[r-d, d-r] \times [(.02)\varepsilon r, (.98)\varepsilon r] \subset \mathrm{spt}(w) = \mathrm{spt}(v_2) \subset U_2. \tag{6.55}$$

Properties (3.4), (3.9), (3.11) imply $h_1(y,t) = 0$ for $y \notin U_1$. This fact, (6.55), and the disjointness of U_1, U_2 (see (3.3)) imply

$$\text{if } |y_1| \leqq d - r \quad \text{and} \quad (.02)\varepsilon r \leqq y_2 \leqq (.98)\varepsilon r, \quad \text{then } h_1(y_1, y_2, T) = 0. \tag{6.56}$$

Using (3.12), (3.13), $f_i \geqq 0$, $h_i \geqq 0$ (see (3.4), (3.10)), (6.30) we find

$$h_{3,0} = f_1 + f_2 = f^{a,r,s} + f^{a',r',s'} + f^{a'',r'',s''} + f + f_2. \tag{6.57}$$

Suppose $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \mathrm{spt}(h_{3,0})$. Then (6.57), (6.31), (6.29), (6.6), (6.34), (6.36), (6.48), (6.44), (6.42) imply that either Case 1 or Case 2 is satisfied:

Case 1: $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \mathrm{spt}(f^{a,r,s}) = \mathrm{closure}(U^{a,r}) = [a-r, a+r] \times [r/8, 7r/8]$,

Case 2: $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \mathrm{spt}(f^{a',r',s'}) \cup \mathrm{spt}(f^{a'',r'',s''}) \cup \mathrm{spt}(f) \cup \mathrm{spt}(f_2) \subset [-d, d] \times [10^{-3}\varepsilon r, r]$.

In either case, we have $(x_2^2 + x_3^2)^{1/2} \leqq r$. This fact, (6.48), and the general inequality

$$-(x^2+z^2)^{1/2} + y \leqq ((x+y)^2 + z^2)^{1/2} \leqq (x^2+z^2)^{1/2} + y \quad \text{if } y > 0$$

imply

$$\begin{aligned} 0 < (.02)\varepsilon r &= -\tau r + \varepsilon r/2 \leqq -((\tau x_2)^2 + (\tau x_3)^2)^{1/2} + \varepsilon r/2 \\ &\leqq ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2} \leqq ((\tau x_2)^2 + (\tau x_3)^2)^{1/2} + \varepsilon r/2 \leqq \tau r + \varepsilon r/2 \\ &= (.98)\varepsilon r. \end{aligned} \tag{6.58}$$

In particular, (6.58) gives us $(\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2 > 0$. In both cases we also have $|x_1| < |a| + r + d$. Using this fact, (6.48), (4.20), (4.22), (4.21), (4.19) and (4.23) we find

$$\begin{aligned} |\tau x_1 + A| &\leqq \tau |x_1| + |A| \leqq \tau(|a| + r + d) + |A| \\ &\leqq (.48)\varepsilon r \varepsilon^{-1} 10^3 C/D + (.48)\varepsilon(r+d) + r/20 \\ &\leqq (.048)d + (.0048)(1.0001)d + 10^{-4}d/20 < d - r. \end{aligned} \tag{6.59}$$

From (6.58), (6.59), (6.55), (6.44), (6.48) we conclude

$$f_2(\tau x_1 + A, ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2}) = \mu. \tag{6.60}$$

Properties (6.48), (6.44), (6.42), (6.32), (6.34), (6.37) yield

$$\mathrm{spt}(f^{a,r,s}),\ \mathrm{spt}(f^{a',r',s'}),\ \mathrm{spt}(f^{a'',r'',s''}),\ \mathrm{spt}(f),\ \mathrm{spt}(f_2)\ \text{are disjoint.} \tag{6.61}$$

Recalling (6.58), we set

$$(y_1, y_2) = (\tau x_1 + A, ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2}) \in P. \tag{6.62}$$

From (6.62), (3.14), (6.60), (6.48) and (6.41) we conclude

$$\begin{aligned} h_2(y_1, y_2, T) + \theta &> ((f_2(y_1, y_2))^2 - T v_2(y_1, y_2)\cdot\nabla(p[v_1, f_1] - p[0, f_1])(y_1, y_2))^{1/2} \\ &= (\mu^2 + T w(y_1, y_2)\cdot(F^{a,r,s} + F^{a',r',s'} + F^{a'',r'',s''} + F)(y_1, y_2))^{1/2}. \end{aligned} \tag{6.63}$$

Properties (6.56), (6.58), (6.59), (6.62) imply

$$h_{3,T}(y_1, y_2) = h_1(y_1, y_2, T) + h_2(y_1, y_2, T) = h_2(y_1, y_2, T). \tag{6.64}$$

Now suppose that Case 1 holds. Using (6.63), part (4.29) of Lemma 4.8, (6.58), (6.59), (6.54), (6.61) and (6.57) we obtain

$$\begin{aligned} h_2(y_1, y_2, T) + \theta &> (\mu^2 + T(.01)B)^{1/2} > \theta + \tau^{-1}\|f^{a,r,s}\|_\infty \\ &\geqq \theta + \tau^{-1} f^{a,r,s}(x_1, (x_2^2 + x_3^2)^{1/2}) = \theta + \tau^{-1}(f_1 + f_2)(x_1, (x_2^2 + x_3^2)^{1/2}) \\ &= \theta + \tau^{-1} h_{3,0}(x_1, (x_2^2 + x_3^2)^{1/2}). \end{aligned}$$

The above and (6.64) yield

$$\text{Case 1 implies } h_{3,T}(y_1, y_2) > \tau^{-1} h_{3,0}(x_1, (x_2^2 + x_3^2)^{1/2}). \tag{6.65}$$

On the other hand, suppose that Case 2 holds. This implies $|x_1| \leqq d$. This fact, (6.48), (4.21) and (4.22) yield

$$|(\tau x_1 + A) - A| = \tau |x_1| \leqq \tau d = (.48)\varepsilon r 10^4 C/D < E(10^4 C/D).$$

Now (6.63), part (4.28) of Lemma 4.8, (6.58), the above inequality, (6.53), (6.43), (6.48), (6.61) and (6.57) yield

$$\begin{aligned} h_2(y_1, y_2, T) + \theta &> (\mu^2 + T(8.01)B)^{1/2} \\ &> \theta + \tau^{-1}(\|f^{a',r',s'}\|_\infty + \|f^{a'',r'',s''}\|_\infty + \|f\|_\infty + \|f_2\|_\infty) \\ &\geqq \theta + \tau^{-1}(f^{a',r',s'} + f^{a'',r'',s''} + f + f_2)(x_1, (x_2^2 + x_3^2)^{1/2}) \\ &= \theta + \tau^{-1}(f_1 + f_2)(x_1, (x_2^2 + x_3^2)^{1/2}) \\ &= \theta + \tau^{-1} h_{3,0}(x_1, (x_2^2 + x_3^2)^{1/2}). \end{aligned}$$

This inequality and (6.64) yield

$$\text{Case 2 implies } h_{3,T}(y_1, y_2) > \tau^{-1} h_{3,0}(x_1, (x_2^2 + x_3^2)^{1/2}). \tag{6.66}$$

The conclusion of the lemma follows from (6.58), (6.62), (6.65), (6.66).

Finally, Theorem 1.1 is a consequence of Lemma 6.4, Lemma 3.3 and Lemma 2.4.

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