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The Aetiology of Sigma Model Anomalies

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Abstract. Certain nonlinear sigma models with fermions are ill-defined due to an anomaly which exhibits characteristics of both the nonabelian gauge theory anomaly and the SU(2) anomaly. The simplest way to diagnose the anomaly involves consideration of the global topology of the theory. We review the mathematical methods needed for this analysis and apply them to several supersymmetric sigma models. Some of these are found to be anomalous.

1. Introduction

Quantum field theories of fermions interacting with nonabelian gauge fields sometimes exhibit an anomaly in the gauge current [1, 2]. This anomaly has recently attracted much attention [3–20], since it has become clear that it is usually a manifestation of a global obstruction to defining the theory properly (i.e. gauge invariantly).

A slight rephrasing of this result clarifies the main issue. Instead of formulating gauge theories in terms of the space $\mathscr{A}^{(4)}$ of connections on a principal bundle over Euclidean spacetime X, we can instead formulate them in terms of the space $\mathscr{C}^{(4)} \equiv \mathscr{A}^{(4)}/\mathscr{G}^{(4)}$ of gauge orbits in $\mathscr{A}^{(4)}$ [21, 22].[1] Now there is no question of gauge-dependence of the effective action. Instead the anomaly shows up as a topological obstruction to defining the dynamics of the fermion fields throughout $\mathscr{C}$ in a smooth, consistent way.

Unlike perturbation theory, which simply gives the gauge variation of the fermion effective action $\Gamma_f[A]$, the topological approach gives a direct geometrical interpretation of this variation. The situation is analogous to what we would have in general relativity were we to treat a tensor quantity, like energy density, as a scalar. Things might look acceptable as long as we worked in one coordinate frame. But if our manipulations required us to integrate this density over spacetime, we would be disappointed to find that the resulting number had no

1 We will henceforth drop the superscript '4' when no confusion can arise

coordinate-invariant meaning. Similarly, in gauge theory $G_0[A] \equiv e^{-\Gamma_f[A]}$ does not reduce to any "scalar" function $G_0[\bar{A}]$ on $\mathscr{C}$, and so the functional integral makes no sense. Like energy density, however, G_0 does have a perfectly good geometrical meaning. It is a section of a *bundle over* $\mathscr{C}$. The anomaly is the statement that this bundle is twisted, so that $\int_{\mathscr{C}} G_0$ has no invariant meaning. If we stubbornly insist on viewing G_0 as an ordinary function on $\mathscr{C}$, for instance by choosing specific coordinates on $\mathscr{C}$, we find, as in general relativity, that this function generally becomes singular when the coordinate system does. This is not the sort of singularity familiar in quantum field theory, since it persists even when we regularize the theory.[2] Furthermore, its location is ambiguous, depending on the choice made. It is, in short, an unacceptable, unphysical pathology of the gauge theory.

The key fact allowing the obstruction we have described is the topological nontriviality of the configuration space $\mathscr{C}$. Since the above reformulation of the anomaly question does not involve gauge symmetry one can ask whether there are other theories with nontrivial configuration spaces (perhaps with no internal symmetries at all) which are anomalous in this generalized sense. We have already answered this question in the affirmative for certain nonlinear sigma models [23]. Models of this sort are of interest because they arise as low-energy approximations to strongly-interacting theories (such as preon models). In the present paper we will explain our results in detail, strengthen them slightly, and apply them to some sigma models which have been proposed as the low energy descriptions of supersymmetric preon physics. The anomaly is relevant to a nonrenormalizable theory such as a sigma model for the same reason that it is relevant in gravity: it can be interpreted as a low-energy phenomenon [18].

We begin with a geometric formulation of the action for nonlinear sigma models. A nonlinear sigma model is a field theory in which the (bosonic) dynamical variables φ take their values in a Riemannian manifold M. We call M the target space. The dynamics of φ are determined by the action functional

$$S_b = \int_X \langle d\varphi, d\varphi \rangle = \int_X g_{ab}(\varphi(x))\partial_\mu \varphi^a \partial_\mu \varphi^b d(\mathrm{vol}). \tag{1}$$

Here X is d-dimensional spacetime, $g_{ab}(\varphi)$ is the metric on M and the second integral gives the form of the Lagrangian in local coordinates (which must be specified patchwise).

How shall we couple matter fields, say left-handed fermions, to φ while maintaining an intrinsic geometrical meaning? One possible approach is motivated by supersymmetry [24]. If the fermion field ψ is to be a superpartner of φ, there must be a transformation law of the form $\delta\varphi = \bar{\varepsilon}\psi$, where ε is a spinor on spacetime. For this to make invariant sense, $\psi(x)$ must live both in the space of spinors at $x \in X$, $S^+|_x$, and the space of tangent vectors to M at $\varphi(x)$, $TM|_{\varphi(x)}$. As x varies the $S^+|_x$ fit together into a bundle over X, the (positive chirality) spinor bundle S^+, and the $TM|_{\varphi(x)}$ fit together into a bundle over X called the "pullback" $\varphi^*(TM)$ by φ of TM [25]. Thus, a complete classical field configuration is

2 See Sect. 2

specified by $\varphi \in \mathscr{C}$ and $\psi, \bar{\psi} \in \mathscr{H}^{\pm}$, where[3]

$$\mathscr{C} \equiv \{\mathrm{Maps}: X \to M\} \tag{2}$$

and

$$\mathscr{H}^{\pm} \equiv \{\text{sections of } S^{\pm} \otimes \varphi^*(TM)\}. \tag{3}$$

We will call $E_{\varphi}^{\pm} \equiv S^{\pm} \otimes \varphi^*(TM)$ and denote the vector space of sections either by $\Gamma(E_{\varphi}^{\pm})$ or by $\mathscr{H}^{\pm}$.

This suggests a generalization. If supersymmetry is not important we can replace TM and its Riemannian metric by an *arbitrary* vector bundle B over M with an arbitrary fixed fiber metric $\langle\, , \rangle$. Since the results of Sects. 2, 3, and 4 are not dependent on supersymmetry, we will state our anomaly criterion at this level of generality and only later specialize to $B = TM$. Similarly, we need not impose the requirement that M be Kähler, or even complex, until we apply our results to supersymmetry. For technical reasons we must, however, require that B be a *complex* vector bundle, as is the case in four-dimensional supersymmetry [26].

The above geometrical setting motivates a natural choice of an invariant Lagrangian for ψ. Given a fixed connection Θ on B (e.g. the canonical Hermitian connection [27]) define a connection $\omega \otimes 1 + 1 \otimes \varphi^* \Theta$ on $E_{\varphi}^{\pm}$, where ω is the usual spin connection on X. The Dirac operator $\not{D}_{\varphi} = \gamma^{\mu} D_{\mu\varphi}$, which maps $\Gamma(E_{\varphi}^{+})$ to $\Gamma(E_{\varphi}^{-})$, allows one to write down the invariant action

$$S_f \equiv \int_X h_{i^*j}(\varphi) \bar{\psi}^{i^*} (\delta^j_k \not{\partial} + \Theta^j_{ka}(\varphi) \partial \varphi^a) \psi^k \equiv \int_X \langle \bar{\psi}, \not{D}_{\varphi} \psi \rangle . \tag{4}$$

For brevity we have dropped the spin connection and will continue to do so. 'h' is the pulled back fiber metric of B and i,j are fiber coordinates.[4]

Thus there is no difficulty in defining a classical nonlinear sigma model with fermions in an invariant geometrical way. Quantization, however, requires a specific choice of frames for the Hilbert spaces $\mathscr{H}_{\varphi}^{\pm}$. In favorable circumstances all dependence on these choices drops out in the end and we are left with an invariant theory. Just as in gauge theory, though, the condition for this to happen is nontrivial and does eliminate some models.

The rest of this paper is organized as follows. Section 2 contains the heart of our argument. It is very short. In it we reexamine the well-known problem of defining the functional integral for chiral fermions in the context of the nonlinear sigma model. We give a heuristic treatment, characterizing the anomaly as an obstruction to a continuous definition of the functional integral on $\mathscr{C}$ [3, 7]. The condition we arrive at is that an integer ν [Eq. (17)] should vanish.[5]

In Sect. 3 we give a physical interpretation to this obstruction. We show that for $\nu \neq 0$ it is impossible to find well-behaved local counterterms which render

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3 Recall that in Euclidean space ψ and $\bar{\psi}$ are independent

4 In some cases, (e.g. in supersymmetry) one also adds quartic fermion interaction terms to S_f. These terms can be rewritten as quadratic terms by the introduction of scalar auxiliary fields. They do not change the index, and hence do not remove the anomaly. We will ignore such terms for simplicity

5 Atiyah and Singer [7] obtain this condition by more rigorous methods

different perturbative expansions of the same Green function physically equivalent.

In Sect. 4 we compute ν, arriving at our final anomaly criterion. (In Appendix A we review the corresponding derivation in gauge theory.) To state the result briefly we recall from Eq. (4) that given φ the connection Θ on B can be pulled back to $\varphi^*(\Theta)$ on $\varphi^*(B)$. As φ varies on $\mathscr{C}$, the various bundles $\{\varphi^*(B)\}$ can be regarded as constituting a single large bundle $\hat{\varphi}^*(B)$ over $\mathscr{C} \times X$.[6] The large bundle has a pulled-back connection $\hat{\varphi}^*(\Theta)$, which is like (4) but differentiates $\hat{\varphi}$ along $\mathscr{C}$ as well as along X. The curvature, or "field strength" $\mathscr{F}$ of $\hat{\varphi}^*(\Theta)$ is a 2-form on $\mathscr{C} \times X$. Our criterion Eq. (62) essentially says that $(\mathscr{F})^3$ should be an exact 6-form on $\mathscr{C} \times X$. The derivation of (62) requires mathematical tools which are perhaps unfamiliar to many physicists, and so we describe some of the foundations of the subject in some detail, since we know of no accessible discussion as yet in the physics literature.[7] Thus we briefly describe K-theory and the family index theorem. Further technical definitions appear in Appendix B.

We apply our results to models with Grassmannian target spaces in Sect. 5, showing that a large class of such models are anomalous. Grassmannian spaces, or spaces closely related to them, arise frequently in the literature as coset manifolds in theories of spontaneously broken symmetries. Fortunately such spaces also make our criterion especially easy to apply, since mathematically they are "universal" in a sense we will explain.

Sections 6 and 7 are perhaps the most accessible parts of the paper. In Sect. 6 we investigate further the anomaly for Grassmannian spaces and find that the analogy to the nonabelian gauge anomaly can be strengthened since there is an analog of the space $\mathscr{A}$ of gauge theory. In Sect. 7 we analyze some models which have arisen in the context of preon physics. One model, recently considered by Büchmuller et al. [28], involves the symmetry breakdown $\mathrm{U}(6) \to \mathrm{SU}(2) \times \mathrm{U}(4)$. Since a closely related model with $\mathrm{U}(6) \to \mathrm{U}(2) \times \mathrm{U}(4)$ *is* anomalous, one might suspect that the other is too. In fact it is not, as we demonstrate.

In Appendix C we give some explicit examples of families of configurations which exhibit the obstruction we will describe. Aside from being amusing, they are necessary to the arguments of Sect. 6. Finally, a technical lemma on the homotopy type of homogeneous spaces, which we need in Sect. 7, is relegated to Appendix D.

Note added. In this paper we consider only global obstructions to the existence of any sort of consistent fermion quantization. In general the vanishing of this obstruction is all one can require of a sigma model with arbitrary target space M. For the case of sigma models which actually arise as low-energy reductions of linear theories, we can demand more. In this case M is a homogeneous space G/H and we ask of a quantization scheme that it reproduce the (possibly anomalous) behavior of the underlying theory under the isometries of M. This leads to a *local* criterion for theories [29] which is simply the 't Hooft anomaly matching criterion: a linear model with fermions in the representation ϱ_G of a global symmetry group

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6 The notation is suggested by the "evaluation" function $\hat{\varphi} : \mathscr{C} \times X \to M$ which maps (φ, x) to $\varphi(x) \in M$

7 See, however, Alvarez, Singer, and Zumino, in preparation

G can reduce to a G/H sigma model with fermions in the representation ϱ_H of H if and only if the usual H-anomalies of ϱ_H match those of $\varrho_{G|H}$. In light of this result, the theories studied in Sect. 7 should be viewed only as illustrations of the global obstruction, since they can now be more conveniently treated by the local criterion. We thank L. Alvarez-Gaumé, P. Ginsparg, A. Manohar, and E. Witten for discussions on this point.

After this work was completed we also received some papers on related topics [30, 31].

2. The Chiral Functional Integral

Chiral anomalies for gauge symmetries are already well known. We can analyze them algebraically by considering all possible gauge-noninvariant terms in the theory's effective action, finding essentially one possibility up to local redefinitions of the bare action. This is the approach taken in [4, 5, 16], for example. This approach would be inconvenient for us, however, since *a priori* we have no gauge symmetries in sigma models[8]. Moreover, it does not tell us whether the anomalous terms do in fact arise in a given theory. To determine that we must have recourse either to perturbation theory or to the topology of the Dirac operator $\not{D}$ for the theory in question. We will take the latter approach in most of this paper. In the next section we will sketch the former as well.

The relation between chiral anomalies and the topological, or index, properties of $\not{D}$ was first discussed in [32–34]. These papers analyzed the axial U(1) anomaly and showed that it is given by the index density for $\not{D}$. The relation between anomalies in *gauged* nonabelian currents and index theory was given by Singer [3]. (See also [7–10, 20].) It is this derivation which we will generalize to sigma models. (In Appendix A we also reproduce the gauge derivation from this point of view.)

Let us try to quantize the theory given in Sect. 1 using the path integral formalism. We need only consider the fermionic path integrals for various *fixed* boson configurations φ:

$$G_p[\varphi;\bar{x}_1, x_1, \ldots, \bar{x}_p, x_p] \equiv \int d\bar{\psi} d\psi e^{-S_f[\bar{\psi},\psi,\varphi]} \bar{\psi}(\bar{x}_1)\psi(x_1) \ldots \bar{\psi}(\bar{x}_p)\psi(x_p) . \qquad (5)$$

It turns out that the main issue is the definition of the fermionic effective action

$$G_0[\varphi] \equiv \exp(-\Gamma_f[\varphi]) . \qquad (6)$$

We will focus on this Green function and return to the others later.

$G_0[\varphi]$ is the functional Grassmann integral of an action quadratic in the Fermi fields. Thus we expect

$$G_0[\varphi] \stackrel{?}{=} \det \not{D}_\varphi . \qquad (7)$$

Our goal is to find a reasonable interpretation of Eq. (7). Our answer is Eq. (9). We will relate it to more familiar expressions for the path integral in the next section.

8 See, however, Sect. 6

The expression $\det \not{D}_\varphi$ suffers from two problems. First, it must be regularized. Second, $\not{D}_\varphi$ maps spinors of positive chirality to spinors of negative chirality; that is, it is an operator between *different* Hilbert spaces. It follows that the eigenvalue problem, and hence the determinant, is not well-defined [18]. Failure to appreciate either of these difficulties would lead one to conclude that there is no anomaly. Furthermore, the lack of an intrinsic definition of the determinant indicates that choices must inevitably be made in giving meaning to $G_0[\varphi]$. This should alert us to the possibility of a global obstruction to a consistent set of choices.

One might try to rectify the chirality-flip of $\not{D}_\varphi$ by considering instead $\hat{D}_\varphi = \not{\partial}^{-1}\not{D}_\varphi$, where $\not{\partial}$ is the *free* Dirac operator, and so should contribute a factor independent of φ to $G_0[\varphi]$ [18]. But $\det\hat{D}$, or more generally $\det \not{D}_{\varphi_0}^{-1}\not{D}_\varphi$, makes no more sense than $\det\not{D}_\varphi$ itself since $\not{D}_\varphi : \mathscr{H}_\varphi^+ \to \mathscr{H}_\varphi^-$, while $\not{D}_{\varphi_0}^{-1} : \mathscr{H}_{\varphi_0}^- \to \mathscr{H}_{\varphi_0}^+$. We must therefore *choose* isomorphisms $T^{(\pm)}(\varphi, \varphi_0) : \mathscr{H}_{\varphi_0}^\pm \to \mathscr{H}_\varphi^\pm$ and take[9]

$$G_0[\varphi] \equiv \mathrm{Det}[T^{(+)}(\varphi, \varphi_0)\not{D}_{\varphi_0}^{-1} T^{(-)}(\varphi_0, \varphi)\not{D}_\varphi]\,. \tag{8}$$

We can now regularize this expression by choosing a smooth function f such that $f(0)=1$ and $f(\infty)=0$. We can finally define the regularized determinant by

$$\begin{aligned} G_0[\varphi; M] &\equiv \exp \mathrm{Tr}\left\{ f\left(\frac{\not{D}_\varphi^\dagger \not{D}_\varphi}{M^2}\right) \mathrm{Log}[T^{(+)}(\varphi, \varphi_0)\not{D}_{\varphi_0}^{-1} T^{(-)}(\varphi_0, \varphi)\not{D}_\varphi]\right\} \\ &\equiv \exp \mathit{Tr}_f \{\mathrm{Log}[T^{(+)}(\varphi, \varphi_0)\not{D}_{\varphi_0}^{-1} T^{(-)}(\varphi_0, \varphi)\not{D}_\varphi]\} \\ &\equiv \mathrm{Det}_f[T^{(+)}(\varphi, \varphi_0)\not{D}_{\varphi_0}^{-1} T^{(-)}(\varphi_0, \varphi)\not{D}_\varphi]\,. \end{aligned} \tag{9}$$

The regularization cuts off the contributions of the "high frequency" modes. The choice of the function f should not affect physical quantities[10] [34, 35].

Unfortunately the definition (9) ignores an important fact. The Hilbert spaces $\mathscr{H}_\varphi^\pm$ for different φ are not *naturally* isomorphic.[11] This means that appropriate $T^{(\pm)}(\varphi_0, \varphi)$ can only be defined in a neighborhood of φ_0. That is, we must cover $\mathscr{C}$ by patches $\{\mathscr{P}_\alpha\}$, choose a reference configuration $\varphi_\alpha \in \mathscr{P}_\alpha$ in each patch, and define the effective action patchwise: $G_0^\alpha[\varphi]$. If $\varphi \in \mathscr{P}_\alpha \cap \mathscr{P}_\beta$ we have

$$G_0^\alpha[\varphi] = g_{\alpha\beta}[\varphi] G_0^\beta[\varphi]\,, \tag{10}$$

where

$$g_{\alpha\beta}[\varphi] = \mathrm{Det}_f[\not{D}_{\varphi_\alpha}^{-1} T_\alpha^{(-)}(\varphi_\alpha, \varphi) T_\beta^{(-)}(\varphi, \varphi_\beta)\not{D}_{\varphi_\beta} T_\beta^{(+)}(\varphi_\beta, \varphi) T_\alpha^{(+)}(\varphi, \varphi_\alpha)]\,. \tag{11}$$

We are thus forced to conclude that (7) does make geometrical sense, but only if we give up thinking of it as a function. Instead we must think of it as a section of the complex *line bundle* L over configuration space $\mathscr{C}$ whose transition functions $g_{\alpha\beta}$ we have just written down.[12] Only if L is trivial can we regard G_0 as an ordinary function. We must therefore determine the twist of L.

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9 We assume, for the moment, that we can always choose φ_0 so that $\not{D}_{\varphi_0}$ is invertible

10 The nontrivial field dependence of the regulator modifies the Schwinger-Dyson equations. We do not know if any modifications survive the removal of the cutoff

11 While all Hilbert bundles are trivial under the structure group $GL(\mathscr{H}^\pm)$, a cutoff amounts to passing to a smaller group with the help of the operator $\not{D}$ [36, 37]. The next paragraphs explain this at our somewhat heuristic level

12 It follows from (10) that the $g_{\alpha\beta}$ satisfy the "cocycle conditions" $g_{\gamma\beta}g_{\beta\alpha} = g_{\gamma\alpha}$ on $\mathscr{P}_\alpha \cap \mathscr{P}_\beta \cap \mathscr{P}_\gamma$ necessary for a consistent definition of a bundle [22, 25, 38]

First we note that complex line bundles over $\mathscr{C}$ are characterized completely by their restrictions to nontrivial two-cells in $\mathscr{C}$ [25], so we lose no generality if we restrict attention to $L|_Y$, where $Y \subseteqq \mathscr{C}$ is a noncontractible two-sphere parametrized by $y \in S^2$.[13]

Next, let us recall some concepts of magnetic monopole theory. Bundles over S^2 can always be trivialized by choosing as patches the northern and southern hemispheres $\mathscr{P}_{N,S}$ and a transition function g_{NS} on the equator $\mathscr{E} = \mathscr{P}_N \cap \mathscr{P}_S$.

Thus a section σ of a line bundle L over S^2 is given by two complex functions $\sigma^{N,S}: \mathscr{P}_{N,S} \to C$ related by $\sigma^N(y) = g_{NS}(y)\sigma^S(y)$, where we can choose $|g_{NS}| = 1$. The twist, or "monopole number", of the line bundle is then given by the integer winding number of g_{NS}:

$$\nu = \frac{i}{2\pi} \int_{\mathscr{E}} g_{NS}^{-1} d(g_{NS}) = \frac{i}{2\pi} \int_{\mathscr{E}} d(\log g_{NS}). \tag{12}$$

Continuing the analogy to monopoles (although Y has nothing to do with ordinary space), we can introduce 1-forms $a^{N,S}$ on $\mathscr{P}_{N,S}$ which differ by the "gauge transformation" g_{NS} on $\mathscr{E}$ and reexpress ν in terms of the "field strength" $F = da^{N,S}$ by

$$\nu = \frac{i}{2\pi} \int_{\mathscr{E}} (a^N - a^S) = \frac{i}{2\pi} \int_Y F. \tag{13}$$

Thus ν depends only on the *cohomology class* of F in $H^2(Y)$. This class is called $ch_1(L)$, the first Chern character of L [25, 39].[14] That is, $ch_1(\cdot)$ sends *bundles* over Y to *classes* in $H^2(Y)$. From its definition as a winding number ch_1 is topologically invariant; from its definition in terms of F will come its important algebraic properties.

Note that the connection $a^{N,S}$ defines parallel transport on L and hence sets up families of isomorphisms $t_N(y; y_N)$ and $t_S(y; y_S)$ between the fibers at $y_{N,S}$ which are copies of C, and the one at $y \in \mathscr{P}_{N,S}$ (another copy of C). Conversely, choices of $t_{N,S}$ determine a connection by $a^{N,S} = d(t_{N,S})(t_{N,S})^{-1}$.

We can also extend the definition of $ch_1(\cdot)$ to bundles V of many dimensions, over complicated spaces. Since these can have more interesting structure than line bundles on S^2, we get a whole sequence of classes $ch_p(V) \in H^{2p}(Y)$. We will discuss $p > 1$ in Sect. 3. To generalize $ch_1(\cdot)$ in a useful way we will demand that $ch_1(V_1 + V_2) = ch_1(V_1) + ch_1(V_2)$, where on the left we have the direct sum of two vector bundles. Thus $ch_1(\cdot)$ is a homomorphism under direct sum. If V is a direct sum of many line (i.e. one-dimensional) bundles, its curvature $F^{N,S}$ can be taken diagonal in its internal indices. Then

$$ch_1 V = \sum_i ch_1(V^{(i)}) = \frac{i}{2\pi} \sum F^{(i)} = \frac{i}{2\pi} \mathrm{tr} F. \tag{14}$$

The Chern character is then defined so that this is true even for arbitrary V.

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13 For the rest of this paper we consider the bundles L and $\mathscr{H}^{\pm}$ restrict to Y

14 We will always speak of the Chern characters $ch_p(L)$ instead of the Chern classes $c_p(L)$. The two contain the same information [39], and indeed $c_1 = ch_1$, but ch will be more useful in Sect. 3 due to its ring property

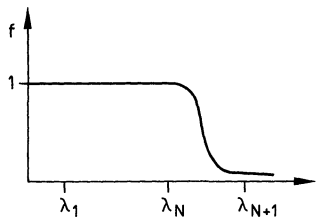

Fig. 2.1. A convenient choice of regulator. λ_i refer to the eigenvalues of $\not{D}_y^\dagger \not{D}_y$

Just as in the one-dimensional case we can trivialize V over patches with transition functions in the unitary group of the fiber and introduce connection forms which (like F) take values in the algebra of that group. On $Y=S^2$ the winding number is thus

$$\nu=\int_Y ch_1(V)=\frac{i}{2\pi}\int_Y \mathrm{tr}\,F=\frac{i}{2\pi}\int_{\mathscr{E}} \mathrm{tr}\,g_{NS}^{-1}dg_{NS}\,, \tag{15}$$

where $F=da^{N,S}+(a^{N,S})\wedge(a^{N,S})$. $t_{N,S}$ again satisfy $a^{N,S}=d(t_{N,S})(t_{N,S})^{-1}$.

Returning to G_0, we can now compute the twist of L using Eqs. (11) and (12):

$$\begin{aligned}\nu=&\frac{i}{2\pi}\int_{\mathscr{E}} d\{\mathrm{Tr}_f\,\mathrm{Log}[\not{D}_{y_N}^{-1}T_N^{(-)}(y_N,y)T_S^{(-)}(y,y_S)\not{D}_{y_S}T_S^{(+)}(y_S,y)T_N^{(+)}(y,y_N)]\}\\ =&\frac{i}{2\pi}\int_{\mathscr{E}}\mathrm{Tr}_f\{[dT_N^{(+)}(T_N^{(+)})^{-1}-dT_S^{(+)}(T_S^{(+)})^{-1}]\\ &-[dT_N^{(-)}(T_N^{(-)})^{-1}-dT_S^{(-)}(T_S^{(-)})^{-1}]\}\,.\end{aligned} \tag{16}$$

Note the similarity of Eqs. (16) and (15). In fact we can invoke the irrelevance of the choice of regulator f to give Eq. (16) an important interpretation as follows. Since Y is compact, we can find an integer N so large that for any $n \geqq N$, the n^{th} eigenmode of $\not{D}_y^\dagger \not{D}_y$ hardly feels the presence of the background boson field for any $y \in Y$, and in particular, never vanishes. We can then replace the eigenvalue cutoff $f(\not{D}_y^\dagger \not{D}_y/M^2)$ by a *mode* cutoff approximating $1-\theta(N-n)$ as in Fig. 2.1. Removing the cutoff means taking N to infinity.

With this choice of cutoff the trace in Eq. (16) becomes a finite-dimensional trace, and the forms $dT_{N,S}^{\pm}(T_{N,S}^{\pm})^{-1}$ become the connections for the *finite-dimensional* subbundles $\mathscr{H}_{\text{low}}^{\pm}$ of $\mathscr{H}^{\pm}$ spanned by the first N eigenfunctions of $\not{D}^\dagger \not{D}$ and $\not{D}\not{D}^\dagger$.[15]

Thus we can write

$$\nu=\int_Y [ch_1(\mathscr{H}_{\text{low}}^{+})-ch_1(\mathscr{H}_{\text{low}}^{-})] \tag{17}$$

or simply $ch_1(L)=ch_1(\mathscr{H}_{\text{low}}^{+})-ch_1(\mathscr{H}_{\text{low}}^{-})$. The homomorphism property of ch_1 now suggests that we express the anomaly in terms of the "defect" bundle

$$\mathscr{D}\equiv\mathscr{H}_{\text{low}}^{+}-\mathscr{H}_{\text{low}}^{-}\,.$$

15 Note that this choice of regulator justifies our not differentiating f in deriving Eq. (16)

But what is the difference of two bundles? We leave this question for Sect. 4. For now we simply remark that already in (17) we can see that $ch_1(L)$ is cutoff-independent. For, if we increase N then $\mathscr{H}^{\pm}_{\text{low}}$ acquire additional summands $\Delta\mathscr{H}^{\pm}_{\text{low}}$. But $\not{D}_y$ sets up an isomorphism between these, since given a normalized eigenmode $u_m \in \mathscr{H}^+$ we can let

$$v_m = \frac{1}{\sqrt{\lambda_m}} \not{D} u_m\,, \tag{18}$$

which is also normalized, has the same eigenvalue, and lives in $\mathscr{H}^-$. [See Eq. (19).] Hence raising N cannot change $ch_1(L)$. Moreover, given a homotopy between Y and some other Y' we can again choose N so large that our procedure is everywhere continuously well-defined on the compact parameter space $S^2 \times [0, 1]$. Since v is *a priori* an integer, it cannot change under such a deformation.

All this abstract nonsense must leave the reader feeling uneasy. How can (17) be nonzero? $\mathscr{H}^{\pm}$ involve $E^{\pm} = S^{\pm} \otimes \varphi^*(B)$; $S^{\pm}$ have no parameter dependence, while $\varphi^*(B)$ is common to both terms. How then can $\mathscr{H}^{\pm}$ have a relative twist? This objection is very similar to one we could raise concerning the axial anomaly, where the object in question is in a sense the difference k between the number of eigenmodes of $\not{D}$ and its adjoint [32–34]. How can these differ? The answer is that both have *infinitely* many eigenmodes, so that $k = \infty - \infty$ is not defined without some cutoff. When we regulate we find a mode imbalance at $\lambda = 0$. Roughly speaking, this happens because relative to the free $\not{\partial}$ the modes of one handedness have been shifted one step; this gives an imbalance at $\lambda = 0$ countered by one "at $\lambda = \infty$" which we throw away by pairing all modes with $\lambda \neq 0$. Thus we must define $k = \dim\ker\not{D} - \dim\ker\not{D}^{\dagger}$. This integer is called the *index* of the gauged Dirac operator in one given background field; it is a topological invariant of the field configuration. Since it depends only on the low-eigenvalue (long-distance) behavior of the theory, it is the same for any value of the cutoff M.

The same thing happens in our case. We argued that the obstruction is the difference of invariants of $\mathscr{H}^{\pm}$, but we still needed to regularize by passing to $\mathscr{H}^{\pm}_{\text{low}}$ defined by a cutoff. This again makes sense by the isomorphism argument for large eigenvalues, and shows that the obstruction involves the relative twists of only the low eigenspaces of $\not{D}_y^{\dagger}\not{D}_y$ and $\not{D}_y\not{D}_y^{\dagger}$. *These need not vanish.* They are measured by a generalization of the index, the so-called "index of the family of Dirac operators $\not{D}_{\varphi}$" [40]. Again this index and the obstruction it measures are present even for finite M, as mentioned in Sect. 1.

To get a feel for the family index, let us study the framing of $\mathscr{H}^{\pm}$ defined by diagonalizing the operators $\not{D}_y^{\dagger}\not{D}_y$ and $\not{D}_y\not{D}_y^{\dagger}$, $y \in Y$. Thus we choose the orthonormal eigenmodes

$$\begin{aligned} \not{D}_y^{\dagger}\not{D}_y u_n(y) &= \lambda_n(y) u_n(y)\,, \\ \not{D}_y\not{D}_y^{\dagger} v_n(y) &= \lambda_n(y) v_n(y)\,. \end{aligned} \tag{19}$$

Note that $\not{D}_y$ differentiates with respect to the (suppressed) spacetime coordinates x, while $y \in Y$ is a *parameter*. We have already mentioned that the nonzero λ_n are the same in each of the above equations.

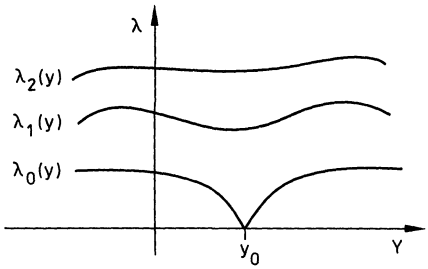

Fig. 2.2. Eigenvalue behavior which can lead to $\nu \neq 0$

If the ordinary index of $\not{D}_y$ vanishes for all y, then there is no imbalance in the *number* of zero-modes and generically $\not{D}_y$ will only have a zero-mode for isolated points on Y. Let us suppose that all eigenfunctions but the lowest, $\lambda_0(y)$, are nonvanishing on Y. Then we may take $\mathscr{H}^{\pm}_{\text{low}}$ as one-dimensional. In general $\mathscr{H}^{+}_{\text{low}}$ will be a nontrivial line bundle, and we must choose separate bases $u_0^{N,S}(y)$ on $\mathscr{P}_{N,S}$ related by a transition function g^{+}_{NS} on $\mathscr{E}$. If $\lambda_0(y)$ is always nonzero on Y, we can define bases for $\mathscr{H}^{-}_{\text{low}}$ using Eq. (18). Then $\mathscr{H}^{+}_{\text{low}}$ and $\mathscr{H}^{-}_{\text{low}}$ have the same transition function and there is no relative twist: ν of Eq. (17) is zero.

On the other hand, suppose the spectrum $\{\lambda_n(y)\}$ looks like Fig. 2.2. We can take y_0 as the north pole. Then Eq. (18) can be used to define $v_0^S(y)$ on $\mathscr{P}_S$, but no longer on $\mathscr{P}_N$. In fact, defining some smooth basis vector $v_0^N(y)$ on $\mathscr{P}_N$ we might find that the phase $\Psi = \left\langle v_0^N(y), \frac{1}{\sqrt{\lambda_0(y)}} \not{D}_y u_0^N(y) \right\rangle$, has a nontrivial winding number around $\mathscr{E}$. (This is possible since Ψ is only defined on the punctured hemisphere $\mathscr{P}_N - \{y_0\}$.) The winding number is the discrepancy between the winding numbers of the transition functions $g^{\pm}_{NS}$ of $\mathscr{H}^{\pm}_{\text{low}}$ and is thus the family index twist ν.

As another example we suppose that $\not{D}_y$ has nonzero ordinary index k. In this case, if $\not{D}_y^\dagger$ has no zero-modes for all y, then $\mathscr{H}^{+}_{\text{low}}$ will be the k-dimensional bundle of zero-modes of $\not{D}_y$ and $\mathscr{H}^{-}_{\text{low}}$ will be empty. Thus the family index measures the twist of $\mathscr{H}^{+}_{\text{low}}$ alone.[16]

To summarize, we have seen that while the fermion effective action $G_0[\varphi]$ makes no invariant sense as it stands for sigma models (or gauge theories) with chiral fermions, it can be interpreted as a section of a bundle L over $\mathscr{C}$. We have written an expression (16) for the winding number ν of L over a compact subspace Y of $\mathscr{C}$, and while we have as yet no idea how to compute it, we know it is a well-defined topological invariant of the theory. If ν vanishes, we can choose a representation of G_0 as a complex function on $\mathscr{C}$ and proceed to integrate it, obtaining a full quantum theory. This does not work if $\nu \neq 0$ since there is no invariant way to integrate a twisted section. Any attempt to interpret G_0 as a function will then require that we make choices, leading to the unphysical singularities mentioned in Sect. 1.

16 In general when $k \neq 0$ other eigenvalues $\lambda_n(y)$ will vanish at isolated points. In this case the "bundle of zero-modes" need not be well-defined, although $\mathscr{H}_{\text{low}}$ is

We should also mention the other Green functions. For this it is convenient to make the eigenmode expansions

$$\psi(x)=\sum a_n(\varphi)u_n(\varphi;x),\qquad \bar{\psi}(x)=\sum \bar{b}_n(\varphi)v_n^\dagger(\varphi;x). \tag{20}$$

These diagonalize $S_f[\bar{\psi},\psi;\varphi]$, allowing us to write (in the zero-instanton sector)

$$G_0[\varphi]=\mathscr{J}[\varphi]\int\prod_n(da_n d\bar{b}_n)\exp\Big(-\sum_n a_n\bar{b}_n\langle v_n,\not{D}_\varphi u_n\rangle\Big). \tag{21}$$

Here $\mathscr{J}$ is the Jacobian of the change of variables from $\psi,\bar{\psi}$ to $a_n,\bar{b}_n$. Symbolically we have

$$\mathscr{J}[\varphi]=[\det u_n(\varphi;x)\det v_n^\dagger(\varphi;x)]^{-1}, \tag{22}$$

where the "rows" of the determinants are labeled by n and the "columns" by x, spin, and internal indices. This expression is meaningless for the same reasons that (7) is. We can only define it patchwise, as

$$\mathscr{J}^\alpha[\varphi]=\{\mathrm{Det}_f\langle u_n(\varphi_\alpha),T_\alpha^{(+)}(\varphi_\alpha,\varphi)u_m(\varphi)\rangle\mathrm{Det}_f\langle v_n(\varphi),T_\alpha^{(-)}(\varphi,\varphi_\alpha)v_m(\varphi_\alpha)\rangle\}^{-1}, \tag{23}$$

where we have chosen fixed frames at one point φ_α in each $\mathscr{P}_\alpha$. The determinants are now over m,n and are regularized as before.

One perfectly good choice for $T_\alpha^\pm$, however, is simply[17]

$$T_\alpha^{(+)}(\varphi,\varphi_\alpha)=\sum_n|u_n(\varphi)\rangle\langle u_n(\varphi_\alpha)| \tag{24}$$

and similarly for $T_\alpha^{(-)}$. With this choice each $\mathscr{J}^\alpha=1$. Other choices will still give $\mathscr{J}$ as an ordinary untwisted function. Thus since

$$G_0^\alpha[\varphi]=\mathscr{J}^\alpha\prod_n\langle v_n^\alpha,\not{D}_\varphi u_n^\alpha\rangle\equiv\mathscr{J}^\alpha[\varphi]I_0^\alpha[\varphi], \tag{25}$$

we see that the twist of G_0 equals that of I_0. Similarly we define I_p by

$$G_p^\alpha[\varphi]\equiv\mathscr{J}^\alpha[\varphi]I_p^\alpha[\varphi], \tag{26}$$

where

$$I_p^\alpha[\varphi]=\int\prod_n(da_n d\bar{b}_n)e^{-\sum_n a_n\bar{b}_n\langle v_n^\alpha,\not{D}_\varphi u_n^\alpha\rangle}\Big(\sum_m a_m u_m^\alpha\Big)\wedge\ldots\wedge\Big(\sum_m\bar{b}_m v_m^\alpha\Big). \tag{27}$$

Note that by Fermi statistics, for each $\varphi, I_p[\varphi]$ is a *vector* in the antisymmetric subspace $\wedge^p(\mathscr{H}_\varphi^+)\otimes\wedge^p(\mathscr{H}_\varphi^-)$ of $(\mathscr{H}_\varphi^+)^{\otimes p}\otimes(\mathscr{H}_\varphi^-)^{\otimes p}$. To see whether it has any extra, anomalous twist we compare across patch boundaries.

If the transition functions for $\mathscr{H}^\pm$ are

$$u_n^\alpha(\varphi)=(g_{\alpha\beta}^+[\varphi])_{nm}u_m^\beta(\varphi),\quad\text{etc.}, \tag{28}$$

17 We made this choice in [23]

then the *integrands* in the expression (27) for I_p^α and I_p^β can be made identical by the change of variables

$$a'_n = a_m(g^+_{\alpha\beta}[\varphi])_{mn}, \quad \text{etc.}.$$

Taking into account the Jacobian for this transformation, together with Fermi statistics, we have

$$I^\alpha_p = (\mathrm{Det}\, g^-_{\alpha\beta})^{-1}(\mathrm{Det}\, g^+_{\alpha\beta}) I^\beta_p . \tag{29}$$

We regularize the determinant with a mode cutoff as usual. But the twist of the factor in (29) is just that of $\mathscr{H}^+_{\text{low}}$ minus that of $\mathscr{H}^-_{\text{low}}$, i.e. it is the twist of L. Hence *all* Green functions G_p, not just G_0, are twisted: they are sections of $\wedge^p(\mathscr{H}^+_\varphi) \otimes \wedge^p(\mathscr{H}^-_\varphi) \otimes L$.

We can even extend this result to the instanton sectors, $k>0$. If the ordinary index of $\not{D}_\varphi$ is k, there will always be k unpaired zero modes $u_{01}, \ldots, u_{0k}$, so that nonzero Green functions have more ψ's than $\bar{\psi}$'s. We call these $G^\alpha_{p+k,p}[\varphi; \ldots]$. The same reasoning that led to (29) applies here, so that the Green functions furnish sections of $\wedge^{p+k}(\mathscr{H}^+_\varphi) \otimes \wedge^p(\mathscr{H}^-_\varphi) \otimes L$, where again L has the family index twist.

In other words, *none* of the Green functions have the appropriate geometrical meaning in a chiral theory with a twisted family index. This is the sigma-model analog of the fact that in anomalous gauge theories the higher point functions are not gauge *co*varient, just as $G_0[A]$ is not gauge *in*variant. In particular we must search every connected component of $\mathscr{C}$ for anomalies, even though $G_0 \equiv 0$ whenever $k \neq 0$ and hence is no problem.

Since all Green functions have the same twist, one might ask whether the phase singularity on Y can be removed by a simple phase redefinition of the twisted Fermi measure. Such a redefinition must correspond to modifying the bosonic action by a counterterm. According to the philosophy adopted in this paper such a counterterm must have an intrinsic geometrical significance as a well defined function on $\mathscr{C}$, since counterterms simply redefine the bare action $S_b + S_f$, which is a function on $\mathscr{C}$. Removing the twist of L in some kind of singular way might define some kind of theory, but it will not be the original sigma model we set out to define. We will return to the counterterm issue at the end of the next section.

3. Symptoms of Sigma Model Anomalies

A natural question one might ask is whether the twist of the line bundle L introduced in the previous section has a conventional field-theoretic interpretation. Indeed there is such an interpretation, which we now describe.

We begin by relating the definition (9) of the Fermi effective action to the more familiar diagrammatic definition. Perturbation theory involves local expansions and ordinary functions (as opposed to sections), so to define it we must trivialize the N-dimensional bundle $B \to M$ by choosing a cover of M by contractible sets $\{\mathscr{U}_\varrho\}$ together with homeomorphisms

$$h_\varrho : \mathscr{U}_\varrho \times C^N \to B|_{\mathscr{U}_\varrho} .$$

The connection must be specified patchwise by Lie-algebra-valued differential forms Θ_ϱ. Transition functions $g_{\varrho\sigma}(p)$ for $p \in \mathscr{U}_\varrho \cup \mathscr{U}_\sigma$ are defined by

$$h_\varrho^{-1}(p, h_\sigma(p, v)) = (p, g_{\varrho\sigma}(p) \cdot v), \tag{30}$$

where v is a vector in C^N and $g_{\varrho\sigma}(p)$ is a matrix in $U(N)$. On the overlap $\mathscr{U}_\varrho \cap \mathscr{U}_\sigma$ the connections are related by a gauge transformation by the transition function:

$$\Theta_\varrho = (\Theta_\sigma)^{g_{\varrho\sigma}}. \tag{31}$$

Let $\varphi_0(x) \equiv p_0 \in M$ be a constant field configuration. We will refer to φ_0 as a vacuum field configuration and will set up a perturbation expansion about it. (If M is a homogeneous space each φ_0 corresponds to one of the equivalent vacua in a theory of spontaneous symmetry breaking.) We expect that when $\varphi: X \to M$ lies near φ_0 in $\mathscr{C}$ there will be a perturbative definition of Γ_f. We will say that φ is "near" φ_0 if there is some patch $\mathscr{U}_\varrho$ containing p_0 such that φ maps all of spacetime into $\mathscr{U}_\varrho$. Thus the patches $\{\mathscr{U}_\varrho\}$ determine corresponding patches in $\mathscr{C}$:

$$\mathscr{P}_\varrho = \mathscr{P}_\varrho[\varphi_0, \mathscr{U}_\varrho] \equiv \{\varphi : \varphi(X) \subset \mathscr{U}_\varrho\}. \tag{32}$$

(These need not cover all of $\mathscr{C}$.) Note that if $\mathscr{U}_\varrho$ is contractible then so is $\mathscr{P}_\varrho$.

To set up the perturbative expansion we will for simplicity take spacetime to be a d-dimensional torus, T^d, of finite volume V.[18] Then $\varphi_0^* B$ is trivial so the eigenmodes of (19) are simply the ordinary functions

$$u_n^{(0)}(x) \equiv u_n(\varphi_0; x) = \frac{1}{\sqrt{V}} \chi e^{-ik_n \cdot x},$$

$$v_n^{(0)}(x) \equiv v_n(\varphi_0; x) = \frac{1}{\sqrt{V}} \sigma \cdot \hat{k}_n \chi e^{-ik_n \cdot x},$$

where k_n is a 4-vector restricted by (anti-)periodic boundary conditions and χ carries spinor and internal indices, i.e., it is a vector in

$$C^{2^{(d/2-1)}} \otimes C^N.$$

We take $\sigma^\mu = (i, \sigma_k)$, while $\bar{\sigma}^\mu = (-i, \sigma_k)$. Furthermore, $\lambda_n(\varphi_0) = k_n^2$ and $\mathscr{H}_{\varphi_0}^{\pm} = \Gamma^{\pm}(S^{\pm} \otimes \varphi_0^* B)$ is an ordinary function space.

If $\varphi \in \mathscr{P}_\varrho$ then $B|_{\varphi(X)}$ is trivialized by h_ϱ. Since all the fibers of a trivial bundle are naturally isomorphic, h_ϱ induces a choice of the isomorphism $T^{(\pm)}(\varphi, \varphi_0)$ of the previous section which we can use in (8). More precisely, if we ignore spin indices, then for each $x \in X$, a section $u \in \mathscr{H}_{\varphi_0}^{\pm}$ defines a vector $u(x)$ in $B|_{\varphi_0(x)} = B|_{p_0}$. Then $h_\varrho^{-1}(u(x)) = (p_0, v)$ for some vector $v \in C^N$. We define T by

$$[T^{(\pm)}(\varphi, \varphi_0)u](x) = h_\varrho(\varphi(x), v) \in B|_{\varphi(x)}, \tag{33}$$

that is, $T^{(\pm)}(\varphi, \varphi_0)u \in \mathscr{H}_\varphi^{\pm}$. This choice of T is very different from the eigenfunction frame that was convenient in Sect. two.

18 This choice is convenient since the torus has trivial spinor bundles, while the finite volume eliminates irrelevant infrared divergences

Using these isomorphisms $T^{(\pm)}(\varphi, \varphi_0)$ we can now group the operators in (9) to obtain coordinate expressions for the relevant matrix elements. Thus, using[19]

$$\langle v_m^{(0)} | T^{(-)}(\varphi_0, \varphi) \not{D}_\varphi T^{(+)}(\varphi, \varphi_0) | u_n^{(0)} \rangle = \int \chi_m^\dagger \bar{\sigma} \cdot \not{k}_m e^{ik_m \cdot x} i\sigma \cdot (\partial_x + \Theta_{\varrho\alpha}(\varphi(x))\,(\partial_x \varphi^\alpha))\chi_n e^{-ik_n \cdot x},$$

one can show that the infinite volume limit of (9) is the infinite volume limit of

$$\exp(-\Gamma_f[\varphi]) = \exp \operatorname{Tr}_f \operatorname{Log}\left[\delta_{nm} - \left\langle u_n^{(0)} \left| \frac{\bar{\sigma} \cdot k_n}{k_n^2} \sigma \cdot A \right| u_m^{(0)} \right\rangle\right],$$

which is just the usual perturbative definition of the effective action:

$$\Gamma_f^\varrho[\varphi] = \sum_n \frac{1}{n} \int dx_1 \ldots dx_n \operatorname{tr}\{[f(\not{D}_{x_1}^\dagger \not{D}_{x_1}/M^2) S(x_1, x_2)]\sigma \cdot A(x_2) \ldots S(x_n, x_1)\sigma \cdot A(x_1)\} \tag{34}$$

(with a somewhat unconventional regulator). Here we have introduced the "gauge field"

$$A_{k\mu}^i(x) = (\Theta_\varrho)_{ka}^i(\varphi(x))\partial_\mu \varphi^a(x), \tag{35}$$

and, as usual the Euclidean propagator is

$$S(x_1, x_2) = \int e^{-ik\cdot(x_1 - x_2)} \frac{\bar{\sigma} \cdot k}{k^2} \frac{d^d k}{(2\pi)^d}.$$

The perturbation series (34) has an anomalous change under gauge transformations of the vector field A. However, the interpretation of A is different from that of gauge theory and we must re-investigate the consequence of the anomaly in the context of the nonlinear sigma model.

As we have emphasized, to arrive at the expansion (34) we had to make many choices: we chose the cover $\{\mathcal{U}_\varrho\}$, the trivializations h_ϱ and the vacuum φ_0. Let us now study the consequences of different choices for the trivialization and vacuum.

First, consider a unitary reparametrization of the fiber coordinates of B in one patch $\mathcal{U}_\varrho$. Thus we choose a set of local rotations $\lambda_\varrho : \mathcal{U}_\varrho \to U(N)$ which induce a change in trivialization by

$$h'_\varrho(p, v) = h_\varrho(p, \lambda_\varrho(p) \cdot v). \tag{36}$$

The collections $\{\mathcal{U}_\varrho, h_\varrho\}$ and $\{\mathcal{U}_\varrho, h'_\varrho\}$ are merely two different schemes for coordinatizing the *same* bundle B.[20] Therefore, such a change should have no physical effect.

It is useful to reformulate this condition, which is based on a passive transformation, to one based on an active transformation. The change (36) induces a change of bases for the Fermi fields which is equivalent to the replacement

19 We have locally set the fiber metric $h_{ij} = 1$

20 In the terminology of Steenrod [38] they define isomorphic "coordinate bundles"

$\Theta_\varrho \to \Theta_\varrho^{\lambda_\varrho}$, which in turn amounts to changing

$$A \to A^{(\varphi^* \lambda_\varrho)}. \tag{37}$$

We thus might naively demand that Γ_f be invariant under (37). Due to the nonabelian anomaly, however, the replacement (37) changes Γ_f^ϱ by the integrated anomaly [41, 6, 42, 43, 44, 45]

$$\mathscr{I}^d[\varphi^*\Theta_\varrho, \varphi^*\lambda_\varrho] = \int_0^1 ds \int_X \omega_d^1[\partial_s(\varphi^*\lambda_\varrho(s))(\varphi^*\lambda_\varrho(s))^{-1}, (\varphi^*\Theta_\varrho^{\lambda_\varrho(s)})]. \tag{38}$$

Here $\lambda_\varrho(s)$ is a one-parameter family of maps from $\mathscr{U}_\varrho$ to $U(N)$ such that $\lambda_\varrho(0)=1$ and $\lambda_\varrho(1)=\lambda_\varrho$, while ω_d^1 is the (appropriately normalized) differential form for the nonabelian anomaly [6, 4], and the expression $\mathscr{I}$ is independent of the choice of path. We will refer to (38) as a Wess-Zumino or WZ term for bundle reparametrizations. It measures the failure of naive bundle reparametrization invariance.

The WZ term contains only a finite number of derivatives of φ. In this sense it is a *local* functional of the scalar fields. Furthermore, adding the term (38) to a bosonic action has nontrivial physical consequences [41, 42]: it modifies the S-matrix of the theory, just as the WZ term in pion dynamics modifies the low-energy theorems for the reactions $\pi^0 \to 2\gamma$ and $K^+K^- \to \pi^+\pi^-\pi^0$. Thus we learn that different trivializations of B lead to inequivalent perturbative expansions, but that these expansions can be made equivalent by the addition of a compensating local counterterm in φ defined on $\mathscr{P}_\varrho$ [44].

Quantum theories are defined by their classical Lagrangians only up to the addition of such local counterterms. In this sense perturbation theory thus does have the invariance under local reparametrizations of B which we expect from the corresponding situation in classical sigma models.[21]

This is not the end of the story, however. While it might be that an anomalous theory makes sense and is coordinate-invariant *locally*, the fact remains that the full theory is defined not by one patch $(\mathscr{U}_\varrho, h_\varrho)$ but by many, all differing by recoordinatizations similar to the ones considered above, and in general we must perturb about many different vacua $\varphi_\alpha(x) \equiv p_\alpha$, where p_α lie in different patches $\mathscr{U}_\alpha$. Our experience with bundle reparametrizations might lead us to expect that with an appropriate choice of WZ terms in each $\mathscr{U}_\alpha$, all the Γ_f^α could be made physically equivalent on all the overlaps, but this is by no means assured. Instead the various discrepancies could fit together into an "obstruction cocycle" [38] which cannot be removed.

Consider the sets $\mathscr{P}_\alpha(\mathscr{U}_\alpha) \subseteqq \mathscr{C}$ defined as above for the various φ_α and define $\Gamma_f^\alpha[\varphi]$ on each according to (34). Focusing our attention on two patches $\mathscr{P}_\varrho, \mathscr{P}_\sigma$, note that if $\mathscr{P}_\varrho \cap \mathscr{P}_\sigma$ is not empty then there exist φ such that $\varphi(X) \subset \mathscr{U}_\varrho \cap \mathscr{U}_\sigma$. We can use (38) to find

$$\Gamma^\varrho[\varphi] = \Gamma^\sigma[\varphi] + \mathscr{I}^d[\varphi^*\Theta_\sigma, \varphi^* g_{\varrho\sigma}]. \tag{39}$$

21 In fact we *must* work at this level of generality, since otherwise even *free* (chiral) fermions are not reparametrization invariant

Now, in contrast to the local reparametrizations which are defined on a single patch, (39) holds only on $\mathscr{P}_\varrho \cap \mathscr{P}_\sigma$, and this overlap might be noncontractible.[22] This raises the possibility that the phase $\exp \mathscr{I}[\varphi^*\Theta, \varphi^* g_{\varrho\sigma}]$ might *wrap* as φ traverses a noncontractible loop in $\mathscr{P}_\varrho \cap \mathscr{P}_\sigma$.[23]

This is bad. It means that any WZ term we could add to Γ_f^ϱ, say, to fix the above discrepancy must be singular somewhere inside $\mathscr{P}_\varrho$, and hence not an acceptable WZ term at all. Thus when we go beyond one-patch perturbation theory and try to define our theory globally by fitting together perturbative expansions around several different vacua, we find that the various prescriptions give physically conflicting predictions which might not be reconcilable by the addition of bosonic counterterms.

We can make this scenario more concrete by considering a family of maps $X \to M$ parametrized by a two-sphere Y. Let $y_{N,S}$ be the north and south poles of Y. The family of maps defines a *single* map $\hat{\varphi} : Y \times X \to M$. For convenience let us take each $\hat{\varphi}(y, \cdot)$ to be homotopically trivial. (In particular this means that we consider a family of maps which lies in the zero instanton-number sector.) Then the restriction of B to the images $\hat{\varphi}[(Y - \{y_{N,S}\}) \times X]$ is trivial. Therefore we choose a cover on M which includes the patches

$$\mathscr{U}_\varrho = \hat{\varphi}[(Y - \{y_N\}) \times X],$$
$$\mathscr{U}_\sigma = \hat{\varphi}[(Y - \{y_S\}) \times X].$$

Also we choose a trivialization of B using these patches, with transition function $g_{\varrho\sigma}$. The corresponding $\mathscr{P}_{\varrho,\sigma}$ contain at least $Y - \{y_{N,S}\}$, and if $\hat{\varphi}$ is homotopically nontrivial then $\mathscr{P}_{\varrho,\sigma}$ cannot be extended to all of Y, so $\mathscr{P}_\varrho \cap \mathscr{P}_\sigma = S^2 - \{y_N\} - \{y_S\}$, which deforms to a circle.[24] The map

$$g_{\varrho\sigma} \circ \hat{\varphi} : (\mathscr{P}_\varrho \cap \mathscr{P}_\sigma) \times X \to U(N) \tag{40}$$

can then be homotopically nontrivial. If it is, then the map $\mathscr{P}_\varrho \cap \mathscr{P}_\sigma \to U(1)$ given by

$$\varphi \mapsto \exp \mathscr{I}[\varphi^*\Theta_\sigma, \varphi^* g_{\varrho\sigma}] \tag{41}$$

is homotopically nontrivial.[25]

All this is not idle speculation. If $M = S^6$ and $B = B_3$ (the Bott bundle on S^6), that is, the bundle with transition function the generator of $\pi_5[U(N)]$ for $N \geqq 3$, then the family given by a degree one map $\hat{\varphi} : S^2 \times S^4 \to S^6$ is of the type just discussed: perturbative expansions about the north and south poles on S^6 lead to inequivalent theories. In general, perturbative expansions around different points on a topologically interesting target space M can lead to inequivalent theories

22 Strictly speaking we should use patches small enough that all their intersections *are* contractible. The obstruction below would be unchanged, but its form would be more complicated

23 The anomaly $\mathscr{I}$ is always imaginary [18]

24 Actually, it is $Y \cap \mathscr{P}_\varrho \cap \mathscr{P}_\sigma$ which deforms to a circle. This distinction is not important to our argument

25 In Sect. 6 we will need the stronger assertion that the winding number of (41) is the same as the homotopy class of (40) for fermions in the fundamental representation. A proof of this statement can be found in many places, including [6, 42–45]

which *cannot* be made equivalent by the addition of a WZ term which is well-defined on the domain of validity of either expansion.

The obstruction we have described here is identical to the one found in Sect. 2 using an eigenmode framing of $\mathscr{H}^{\pm}$. The inequivalence of Γ^{ϱ} and Γ^{σ} means that $G_0[\varphi]$ is a section of a twisted line bundle L whose twist equals the winding number of the WZ term. Nevertheless, the characterization of the anomaly given in this section is awkward. In the following section we therefore return to the formulation (17) of the anomaly. We will be able to apply index theory to give a characterization of the obstruction v which involves only the *topology* of the spaces $\mathscr{C}$, B, X, and M. The condition (62) which we derive is then tractable in many cases of interest and facilitates the study of the epidemiology of sigma model anomalies.

We conclude this section with three remarks. First, we have seen that a choice of trivialization $\{h_\varrho\}$ corresponds to a choice of frames for $\mathscr{H}^{\pm}$. A bundle reparametrization corresponds to a particular change of such special frames. We have argued in this section that such changes cannot remove the anomaly. From this point of view the result of Sect. 2 is far more powerful than perturbation theory suggests, for the conclusions of Sect. 2 imply that there is *no* smooth set of local frame choices for $\mathscr{H}^{\pm}$ which can remove the anomaly.

Second, we can see that there is no smooth counterterm which can cancel the anomaly. Such a counterterm must have a perturbative formulation which is therefore uniquely determined by (39). The anomaly is precisely the obstruction to a smooth extension of this WZ term to $\mathscr{P}_{\varrho,\sigma}$.

Finally, note that the nonlinear sigma model anomaly has features similar to both the nonabelian gauge anomaly and Witten's SU(2) anomaly [46]. The necessity of cancelling the nonabelian gauge (and gravitational) anomalies can be seen purely within the framework of perturbation theory [2, 18]. On the other hand, the SU(2) anomaly can only be detected by considering the global topology of configuration space. In the case of the nonlinear sigma model, one can deduce the possibility of the anomaly within the framework of perturbation theory, but it is only the global topology of $\mathscr{C}$ which determines whether the anomaly is fatal to the theory in question.

4. The Family Index

We must now define precisely the index of a family of Dirac operators, and in particular its first Chern character. We can then evaluate the latter using the Atiyah-Singer index theorem [40]. The only result of this section which will be used in the sequel is the final answer (62). The reader willing to accept this result can skip the present section.

To get started we must sketch a framework in which the "defect" bundle

$$\mathscr{D} = \mathscr{H}^{+}_{\mathrm{low}} - \mathscr{H}^{-}_{\mathrm{low}} \tag{42}$$

mentioned in Sect. 2 makes sense. This framework is called K-theory.[26]

To describe topologically the possible complex bundles over a space Y, we can think in terms of the space $\mathrm{Vect}(Y)$ of isomorphism classes of bundles. This space

26 Readable introductions include [36, 39]

has naturally defined on it an addition operation, the direct sum: V_1+V_2 has for its fiber over y the vector space sum $V_1|_y \oplus V_2|_y$. Furthermore there is a multiplication operation, the pointwise tensor product, which is distributive with respect to addition. Finally, there is a map $\dim : \mathrm{Vect}(Y) \to Z$ with the homomorphism property $\dim(V_1+V_2)=\dim V_1+\dim V_2$. dim extracts from a given bundle its most obvious topological invariant, the (complex) dimension. In fact, if we regard the integers Z as the zeroth cohomology group $Z=H^0(Y)$, we see that both $ch_0 \equiv \dim$ and ch_1 are homomorphisms from $\mathrm{Vect}(Y)$ to the cohomology of Y. When Y is more complicated than S^2 there are indeed a series $ch_p(V)$ of $2p$-dimensional cohomology classes associated to a given $V \in \mathrm{Vect}(Y)$. All are topologically invariant and all can be written in terms of traces of various powers of a curvature of V [39], just like ch_1 and (trivially) ch_0. Explicitly,

$$ch(V) \equiv \sum_p ch_p(V) \equiv \mathrm{tr}\exp\left(\frac{i}{2\pi}F\right). \tag{43}$$

Moreover we have the multiplication property $ch(V_1 \otimes V_2)=ch(V_1) \wedge ch(V_2)$. Finally, if V is trivial the $ch_p(V)=0$ for all $p>0$.

While $\mathrm{Vect}(Y)$ has an addition, we cannot give it any subtraction operation. As a simple example, suppose for a moment that we repeat the above with real, not complex, bundles, and consider the tangent bundle $V=TS^2$. When we embed $S^2 \subseteqq R^3$ we can define the one-dimensional line bundle N normal to TS^2. N is trivial, that is, isomorphic to the trivial bundle $S^2 \times R^1$ over S^2. Now $TR^3|_{S^2} = TS^2+N$. But while TR^3 and N are trivial in $\mathrm{Vect}(Y)$, we cannot cancel them to conclude that TS^2 is trivial too. It isn't.

We would like to assign to Y an abelian group $K(Y)$ [much like the cohomology $H^*(Y)$] which is like $\mathrm{Vect}(Y)$ but *ignores* the difference between TS^2 and $S^2 \times R^2$. Such a group, it turns out, retains just the right amount of information to be of use in index theory. To construct it, we mimic the construction of the integers Z from the natural numbers $N=\{1,2,\ldots\}$ [47]. N, like Vect, has only a semigroup structure. But if we consider *pairs* $N \times N/\sim$, where we identify $(n,m) \sim (n+k, m+k)$ then we can construct the inverse operation $-(n,m) \equiv (m,n)$ and thus subtraction. For convenience we can then write $n-m$ for (n,m), n for $(n+k,k)$, and $-n$ for $(k,n+k)$.

In exactly the same way we can define $K(Y)=\mathrm{Vect}(Y) \times \mathrm{Vect}(Y)/\sim$ where $(V_1,V_2) \sim (V_1+V_3, V_2+V_3)$. We will refer to the elements of $K(Y)$ as "virtual bundles over Y," or more often simply as bundles. $K(Y)$ can be defined in terms of real or complex bundles. In the real case we do indeed have

$$\begin{aligned} TS^2 = (TS^2+N, N) &= (TR^3|_{S^2}, N) \\ &= (S^2 \times R^3, S^2 \times R) \\ &= S^2 \times R^2 . \end{aligned} \tag{44}$$

Here we split TR^3 into three trivial bundles. Therefore all that remains of TS^2 in real K-theory is its dimension, $\dim(TS^2)=2$. [Real K-theory, which classifies real vector bundles is quite different from complex K-theory, which classifies complex vector bundles. Indeed, TS^2 can be given a complex structure, and, in complex K-theory $(TS^2, 0)$ is *not* trivial. It is again true that in the complex case $\mathrm{Vect}(Y)$ is not a group; for that we must pass to $K(Y)$ with the above construction. Henceforth we consider only complex K-theory.]

In complex K-theory, the homomorphism properties of ch guarantee that the Chern characters make sense on $K(Y)$ if we define $ch((V_1, V_2)) = ch(V_1) - ch(V_2)$. Finally, if we define the product of differences in the obvious way then $K(Y)$ becomes a ring, and $ch: K(Y) \to H^*(Y)$ becomes a ring homomorphism. With these definitions, Eqs. (17) and (42) just say that the anomaly is measured by $ch_1(\mathscr{D})$. Far from being merely streamlined notation, however, K-theory will be crucial for the steps which follow.

We can now define the family index[27] $\mathrm{Ind}\,\not{D}$ and show that it equals $\mathscr{D}$ [40, 50, 51]. Following [8] we use capital "I" to distinguish the family index, which is a (virtual) bundle, from the ordinary index, which is an integer. Consider the ordinary index $\mathrm{ind}\,\not{D} = \dim\ker\not{D} - \dim\ker\not{D}^\dagger$. If $\not{D}_y$ actually belongs to a parametrized family, then as y moves we get a family of kernels moving around inside $\mathscr{H}^\pm$. Thus we are tempted to drop the dim's above, which discard all information about how the kernels move, and define instead

$$\mathrm{Ind}\,\not{D} \stackrel{?}{=} \ker\not{D} - \ker\not{D}^\dagger\,. \tag{45}$$

Then we recover $\mathrm{ind}\,\not{D}$ as $ch_0(\mathrm{Ind}\,\not{D})$.

This is not quite right. There will be, in general, points on Y where the dimension of $\ker\not{D}_y$ jumps. Thus $\ker\not{D}$ does not define a bundle on Y. Since the index $\mathrm{ind}\,\not{D}_y$ *is* defined (and constant) on Y it is plausible that there is a way to interpret the *difference* $\ker\not{D} - \ker\not{D}^\dagger$ as an element of $K(Y)$. This can be done as follows. Suppose we modify $\mathscr{H}^+$ by the addition of a trivial bundle $Y \times C^N$, and let $\bar{\not{D}} = (\not{D}, 0): \mathscr{H}^+ + (Y \times C^N) \to \mathscr{H}^-$. Then $\mathrm{ind}\,\bar{\not{D}} = (\mathrm{ind}\,\not{D}) + N$. Suppose we could now change $\bar{\not{D}}$ smoothly to a new $\tilde{\not{D}}$ with no cokernel, i.e. such that the image of $\tilde{\not{D}}_\varphi$ is $\mathscr{H}_\varphi^-$ for all φ. Then the kernel could not jump either, since $\mathrm{ind}\,\tilde{\not{D}}$ is constant, and hence $\ker\tilde{\not{D}}$ would be a bundle over Y as desired. Subtracting $Y \times C^N$ to correct the imbalance in dimension, we could thus define [40]

$$\mathrm{Ind}\,\not{D} \equiv \ker\tilde{\not{D}} - (Y \times C^N)\,; \tag{46}$$

again $\mathrm{ind}\,\not{D} = ch_0(\mathrm{Ind}\,\not{D})$. In fact, in the degenerate case $Y = \{point\}$ the two indices coincide, so Ind is a natural generalization of ind.

It turns out that we can indeed kill $\ker\not{D}^\dagger$ simply by choosing $\tilde{\not{D}}_y = (\not{D}_y, \boldsymbol{\sigma}(y))$, where σ_α, $\alpha = 1, \ldots, N$ are a fixed set of sections which always at least span $\ker\not{D}^\dagger$. That is,

$$\tilde{\not{D}}(u(y), \xi) = \not{D}_y u(y) + \sum \xi_\alpha \sigma_\alpha(y)\,, \tag{47}$$

and the second term fills out all of $\mathscr{H}_y^-$ missed by the first. $\mathrm{Ind}\,\not{D}$ does not depend on which sections we choose [40]. Furthermore, it equals the bundle $\mathscr{D} \equiv \mathscr{H}_{\mathrm{low}}^+ - \mathscr{H}_{\mathrm{low}}^-$, which makes it interesting to us. To see this, note that

$$\mathscr{H}_{\mathrm{low}}^+ - \mathscr{H}_{\mathrm{low}}^- = (\mathscr{H}_{\mathrm{low}}^+ + (Y \times C^N)) - (\mathscr{H}_{\mathrm{low}}^- + (Y \times C^N))\,,$$

27 For introductions to index theory see [39, 48, 49]. The constructions used in this section are actually applicable to a much wider class of differential operators than considered here

and consider

$$\hat{\not{D}} = \begin{pmatrix} \tilde{\not{D}} \\ 0 \end{pmatrix} = \begin{pmatrix} \not{D} & \sigma \\ 0 & 0 \end{pmatrix} \tag{48}$$

on $\mathscr{H}_{\text{low}} + (Y \times C^N)$. Since $\tilde{\not{D}}$ is onto, it furnishes, for each y, an isomorphism $(\ker \tilde{\not{D}}_y)^\perp \cong \mathscr{H}^-_{\text{low}}|_y$. So in K-theory

$$\begin{aligned}\mathscr{D} &= [\ker \tilde{\not{D}} + (\ker \tilde{\not{D}})^\perp] - [\mathscr{H}^-_{\text{low}} + (Y \times C^N)] \\ &= \ker \tilde{\not{D}} - (Y \times C^N) \\ &= \operatorname{Ind} \not{D} .\end{aligned} \tag{49}$$

Thus a sigma model is anomalous iff $\operatorname{Ind} \not{D}$ is twisted over some two-sphere Y in $\mathscr{C}$. We must now compute this twist.

At first sight the evaluation of $ch_1(\operatorname{Ind} \not{D})$ for arbitrary Y seems a hopeless task: We must solve an equation in arbitrary field configurations, search for zero modes, and establish their twists as we move around Y, a program which at best works only for very simple cases. The startling result of Atiyah and Singer is that none of this is necessary! Just as for the ordinary index, the family index of $\not{D}$ is completely determined by the *topology* of the spaces in question, and not at all by the particular metrics, connections, etc. we have chosen. More precisely, we say that the index of an elliptic operator depends only on its *symbol*, and all Dirac operators have essentially the same symbol.

To define the symbol of an elliptic operator we first expand it in coordinates and drop all but the leading derivative terms. For $\not{\partial}$ this yields $\gamma^\mu \partial_\mu$. Now replace the derivatives by symbolic "momentum" variables p_μ to get $\sigma(\not{\partial}) \equiv \gamma^\mu p_\mu$. For each point x in X and each value of momentum, $\sigma(\not{\partial})$ is a map from S^+_x to S^-_x [28]. We can state this concisely by defining the *pullback* $\pi^*(S^+)$, where π is the projection from the cotangent bundle T^*X to X. Just like $\varphi^*(\cdot)$ of Sect. 1, π^* fits together many vector spaces into a bundle over the total space of T^*X as follows: over the point (x, p) we place the vector space S^+_x. Then the previous statement becomes simply that $\sigma(\not{\partial}) : \pi^*(S^+) \to \pi^*(S^-)$.

The index theorem relates $\operatorname{Ind} \not{D}$, which loosely speaking is the difference of two bundles, $\mathscr{H}^+ - \mathscr{H}^-$, to a new bundle $\Sigma(\not{\partial})$ which loosely speaking is the difference $\pi^*S^+ - \pi^*S^-$. $\Sigma(\not{\partial})$ is a bundle over the total space of T^*X, and it depends only on the symbol $\sigma(\not{\partial})$. The details of the construction of $\Sigma(D)$ for an arbitrary elliptic operator D are given in Appendix B. For our purposes, though, all that matters is that $\Sigma(\not{\partial})$ is known, and a formula for the family index can be computed from it.

We begin with the ordinary index theorem for a single Dirac operator $\not{D}$ [not necessarily of the form (4)] [52, Theorem 2.12], which states that in even spacetime dimensions

$$\operatorname{ind} \not{D} = \int_{T^*X} ch \Sigma(\not{D}) \mathscr{T}(T^*X) . \tag{50}$$

28 In fact for every *nonzero* momentum $\sigma(\not{\partial})$ is an *isomorphism*. An operator with this property is called "elliptic." Index theory only works for elliptic operators, for reasons given in Appendix B

$$\begin{tikzcd} & Y\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y & & X \end{tikzcd}
\qquad
\begin{tikzcd} Y\times T^{*}X \arrow[d, "\pi"'] \arrow[dr, "p_2"] & \\ Y\times X \arrow[d, "p_1"'] & T^{*}X \\ Y & \end{tikzcd}$$

Fig. 4.1. Projections used in the text

Here $\mathscr{T}(T^*X)$ is a cohomology class of T^*X depending only on the topology of X. Its definition will not be important for us. $ch\Sigma(\not{D})$ is also in $H^*(T^*X)$, since $\Sigma(\not{D})$ is a bundle on T^*X. We can rewrite (50) in a way which is susceptible to generalization as follows. If P is any bundle projection, it sets up a local product structure which lets us define P_*, the operation of integrating forms along the fibers of P [25]. For example, if $P: R^3 \to R^1$ with $P(\mathbf{x}) = x^1$ then

$$P_*(f(\mathbf{x})dx^2 \wedge dx^3) = g(x^1) \quad \text{and} \quad P_*(f(\mathbf{x})dx^1 \wedge dx^2 \wedge dx^3) = g(x^1)dx^1\,,$$

where $g(x^1) = \int dx^2 dx^3 f(\mathbf{x})$. By convention we also define $P_*(dx^2) = P_*(dx^1 \wedge dx^2) = 0$, etc. Defining projection maps as in Fig. 3.1 and taking, for the moment, the case $Y = \{point\}$, the above integral can be written $(p_1 \circ \pi)_*$, since $p_1 \circ \pi$ projects T^*X all the way down to a point. Moreover we can perform integrals in succession, to get

$$(p_1 \circ \pi)_* = (p_1)_* \circ (\pi)_*\,. \tag{51}$$

Here $\pi_*: H^*(T^*X) \to H^*(X)$ and $(p_1)_*: H^*(X) \to H^*(point)$. Also we have that

$$\pi_*((\pi^*\omega) \wedge \eta) = \omega \wedge \pi_*\eta \tag{52}$$

for any class ω on the base and η on the total space, where π^* is the pullback of forms [25]. (We apologize for using the same symbol to denote both this and bundle pullback – this is standard usage.) This just means that ω can be pulled outside an integral if it doesn't depend on the integration variable. These simple "covariance" (51) and "module" (52) properties of $(\cdot)_*$ are the key to our computation. They enable us to get the desired index formula from (50).

Thus for the free Dirac operator (50) becomes

$$\operatorname{ind}\not\partial = (p_1)_*[\pi_*(ch\Sigma(\not\partial) \wedge \mathscr{T}(T^*X))]\,. \tag{53}$$

The class in square brackets is called $\hat{A}(X)$. For $X = S^4$ it is known to be $1 \in H^0(S^4)$ [39], and so the index vanishes.[29]

In the nonlinear sigma model we are interested in the Dirac operator $\not{D}_\varphi$ coupled to $\varphi^*(B)$. Then its symbol $\sigma(\not{D}_\varphi)$ is again $\gamma^\mu p_\mu$, or rather $\gamma^\mu p_\mu \otimes 1$, where the unit matrix acts on internal indices. The symbol still "knows" that it is coupled to

29 In fact, the free Dirac operator on S^4 has no zero modes of *either* chirality, by Lichnerowicz's theorem

φ, but only via its domain and range $\pi^*E^{\pm}$, which contain $\varphi^*(B)$. Since $\Sigma(\not{D}_\varphi)$ is in a sense the difference between the domain and range, we can factor out the common $\varphi^*(B)$ to get (see Appendix B)

$$\Sigma(\not{D}_\varphi)=\Sigma(\not{\partial})\otimes\varphi^*(B)\,, \tag{54}$$

or more explicitly

$$\Sigma(\not{D}_\varphi)=\Sigma(\not{\partial})\otimes\pi^*\varphi^*(B)\,. \tag{55}$$

[We need the pullback π^* since $\varphi^*(B)$ is a bundle on X and must be trivially extended along the fibers of T^*X before we can take the indicated tensor product.] This factorization simplifies our problem immensely, since at the level of K-theory the index is essentially known from the properties of the free Dirac operator. For, we now have that

$$\operatorname{ind}\not{D}_\varphi=(p_1\circ\pi)_*[ch\Sigma(\not{D})\wedge\mathscr{T}(T^*X)]\,. \tag{56}$$

Using (55) and the remark following (43) we get

$$\operatorname{ind}\not{D}_\varphi=(p_1\circ\pi)_*[\pi^*(ch\varphi^*(B))\wedge(ch\Sigma(\not{\partial}))\wedge\mathscr{T}(T^*X)]\,. \tag{57}$$

But by (51), (52) this is

$$\begin{aligned}&=(p_1)_*[(ch\varphi^*(B))\wedge\pi_*(ch\Sigma(\not{\partial})\wedge\mathscr{T}(T^*X))]\\&=\int_X(ch\varphi^*(B))\wedge\hat{A}(X)\,,\end{aligned} \tag{58}$$

and we have recovered the usual index theorem. When spacetime is S^4, $\hat{A}=1$ and the expression (58) is the familiar formula for the instanton number of the "gauge field" $\varphi^*(\Theta)$.

The point of the above approach is that the family case is quite similar. A family of Dirac operators $\not{D}_y$ gives a symbol $\sigma(\not{D}_y)$ for each y. These combine to define a single virtual bundle $\Sigma(\not{D})\in K(Y\times T^*X)$. But again $\sigma(\not{D})$ is completely independent of y, so all information about the family twist of $\not{D}$ is again encoded in the domain and range bundles $\pi^*(E_y^{\pm})$. The left-hand side of Eq. (56) should now be thought of as $ch_0\operatorname{Ind}\not{D}$, and for arbitrary Y we finally have the family index theorem [40],

$$ch\operatorname{Ind}\not{D}=(p_1)_*\pi_*[ch(\Sigma(\not{D}))\wedge\mathscr{T}(p_2^*(T^*X))]\,. \tag{59}$$

Now $(p_1\circ\pi)$ projects not to a point but to Y, so that both sides are differential forms on Y.

In the case of the nonlinear sigma model we consider a family of maps $\varphi_y: X\to M$ which combine into a single map $\hat{\varphi}: Y\times X\to M$. We can use Eq. (59) to find the index of the family $\not{D}_{\varphi_y}$, which we will just call $\not{D}$. Since $\mathscr{T}$ knows nothing of the twisting bundle $\hat{\varphi}^*(B)$, it turns out that $\mathscr{T}(p_2^*(T^*X))$ is trivial along Y, i.e. it is just $p_2^*\mathscr{T}(T^*X)$. Since we can integrate along the fibers of T^*X either before or after applying p_2^*, we again use Eq. (55) to get

$$\begin{aligned}ch\operatorname{Ind}\not{D}&=(p_1)_*[ch(\hat{\varphi}^*(B))\wedge p_2^*\hat{A}(X)]\\&=\int_X ch\hat{\varphi}^*(B)\hat{A}(X)\,.\end{aligned} \tag{60}$$

Finally, the operation of taking the Chern character can be done either before or after taking a pullback, since the curvature of $\hat{\varphi}^*(B)$ is just the pullback of the curvature form of B itself. Taking $X=S^4$ the above expression becomes

$$\int_X \hat{\varphi}^*(chB), \tag{61}$$

which is a differential form on Y.

We can at last evaluate the anomaly (17) of an arbitrary sigma model. Given a two-sphere $Y\in\mathscr{C}$, we can extract ch_1 from all of $ch\,\mathrm{Ind}\,\not{D}$ by simply integrating over Y. The result is then the anomaly $ch_1(\mathscr{D})$, since we have already shown that $\mathrm{Ind}\,\not{D}=\mathscr{D}$. Thus

$$anomaly=\nu=\int_{Y\times X}\hat{\varphi}^*(ch_3 B). \tag{62}$$

Note that only ch_3 appears in (62) because $Y\times X$ has six real dimensions. If we consider two-dimensional spacetime, then the anomaly involves ch_2. If ν is nonzero for any Y then the theory is inconsistent. This completes the derivation promised in [23].

We will refer to a family $\hat{\varphi}$ such that $\nu\neq 0$ as an anomalous family, with family index equal to ν. For example, in the model introduced in the previous section with $M=S^6$, $B=B_3$ the family $\hat{\varphi}$ is anomalous since $ch_3(B_3)$ can be taken to be the solid angle $\omega^{(6)}$ on S^6 and

$$anomaly=\int_{Y\times X}\hat{\varphi}^*(\omega^{(6)})=(\deg\hat{\varphi})\int_{S^6}\omega^{(6)}=1. \tag{63}$$

We will discuss more interesting models in the next few sections.

We cannot resist closing this section with a remark on the meaning of Eq. (62) [51, 23]. The reader has probably noticed a similarity between Eq. (62) and Eq. (58): the twist of the family index equals the *ordinary* index of a *six*-dimensional Dirac operator $\not{D}^{Y\times X}$ on $Y\times X$. This is no accident. We can measure the twist of a bundle L on S^2 by writing down a Dirac operator $\not{D}^Y_L$ on S^2 coupled to L, a fact well known from magnetic monopole theory. The notation means that this operator differentiates y and is coupled to L by some connection. Our above observation then amounts to saying,

$$\mathrm{ind}\,\not{D}^Y_{[\mathrm{Ind}\,\not{D}^X]}=\mathrm{ind}\,\not{D}^{Y\times X}_{\hat{\varphi}^*(B)}. \tag{64}$$

This formula is essentially the one proved in [9] using an adiabatic argument. In fact it expresses a deep algebraic property of the family index.[30]

Consider the operation which takes a bundle β on $Z\times Y\times X$ to the family index of $\not{D}^X$ coupled to β. Call this map $(p_1)_!$, where p_1 is the projection from $Z\times Y\times X$ to $Z\times Y$. (In our case $Z=\{point\}$.) So

$$(p_1)_!(\beta)\equiv\mathrm{Ind}\,\not{D}^X_\beta \tag{65}$$

with

$$(p_1)_!: K(Z\times Y\times X)\to K(Z\times Y). \tag{66}$$

30 We thank R. Bott and D. Quillen for enlightening us on this point

Consider also $\varrho: Z\times Y\to Z$ and $\varrho_!$ which takes the index of $\not{D}^Y$. Then Eq. (64) simply says

$$\varrho_!\circ(p_1)_!=(\varrho\circ p_1)_! \tag{67}$$

The analogy to Eq. (51) is evident. In fact, K-theory can itself be regarded as a form of cohomology [36, 53], in which $(\cdot)_!$ plays the same algebraic role as $(\cdot)_*$. [We also have the analogy of Eq. (52).] Thus the mysterious formal connection between gauge (or sigma model) anomalies in d dimensions and chiral U(1) anomalies in $d+2$ dimensions simply reflects the composition properties of $(\cdot)_!$, i.e. that K, like H is "covariant." The special role of 2 comes about since we are interested only in ch_1, a two-form on Y.

5. Applications to Supersymmetry

In this section[31] we will show that the four-dimensional supersymmetric G_{pq} model exhibits the topological anomaly for p, q both $\geqq 3$. We recall that this model is of the form discussed in the introduction, with the specific choice of $M=G_{pq}$, $B=T_c(G_{pq})$, where G_{pq} is the Grassmannian manifold defined below, and $T_c(G_{pq})$ is its holomorphic tangent space.[32] We can also consider such chiral G_{pq} models in two spacetime dimensions. These models are not supersymmetric, since in $2d$, superpartner fermions are not Weyl but Dirac. Nevertheless, we shall include them for completeness.

By our criterion (62) we need only find a map $\hat{\varphi}: S^2\times S^d\to G_{pq}$ such that

$$\int_{S^2\times S^d}[ch(\hat{\varphi}^*T_cG_{pq})]\neq 0 \tag{68}$$

The goal of this section is merely to establish the *existence* of such $\hat{\varphi}$ by using the theory of classifying spaces [25, 55]. Here is a brief exposition of the relevant facts.

The manifold

$$G_{pq}=U(p+q)/U(p)\times U(q) \tag{69}$$

can be defined as the space of all p-dimensional subspaces W of C^{p+q}. We will alternate between viewing W as a *point* in G_{pq} and as a *vector space*, or "p-plane," in C^{p+q}. For example, over G_{pq} there is a canonical p-plane bundle γ_p whose fiber over $W\in G_{pq}$ consists of the vectors contained in the space W. That is,

$$\gamma_p=\{(W,v): v\in W\}\subset G_{pq}\times C^{p+q}. \tag{70}$$

The vector bundle γ_p is associated via the fundamental representation to the principal $U(p)$-bundle $V_{pq}\to G_{pq}$. Here $V_{pq}\equiv U(p+q)/U(q)$ is called a Stiefel manifold. It can be shown [38] that

$$\pi_i(V_{pq})=0, \tag{71}$$

for $i\leqq 2q$ and thus V_{pq} is $(2q+1)$-universal, in the sense of Steenrod. The bundle classification theorem [38] then states that any p-plane bundle over a compact

31 We thank V. Della Pietra and T. Parker for helpful discussions on classifying spaces

32 A good introduction to G_{pq} is [54]

manifold Q (more technically, a finite CW-complex) of real dimension $\leqq 2q$ is isomorphic to the pullback of γ_p under some map $f: Q \to G_{pq}$. Furthermore, it can be shown that the map f is determined up to homotopy.

We will use this theorem by expressing $T_c(G_{pq})$ in terms of the canonical bundle γ_p and then expressing bundles over $S^2 \times S^d$ with nonvanishing Chern character as pullbacks of γ_p. This will establish the existence of families $\hat{\varphi}$ satisfying (68).

Consider the tangent space to G_{pq} at a point given by a p-plane W. A neighborhood of W is given by the set of p-dimensional subspaces of C^{p+q}, V such that

$$V \cap W^{\perp} = \{0\}. \tag{72}$$

This neighborhood can be coordinatized as follows [56]. Choose an orthonormal basis $w_1, \ldots, w_{p+q}$ for C^{p+q} such that $w_1, \ldots, w_p$ span W. The decomposition $C^{p+q} = W \oplus W^{\perp}$ defines a projection $p: V \to W$ which, by condition (72), is an isomorphism and defines a basis v_i for V by the equation $p(v_i) = w_i$. Then

$$v_i = w_i + \sum_{j=1}^{q} a_{ij}(V) w_{p+j} \tag{73}$$

defines a $p \times q$ matrix a_{ij} which is the desired coordinate system. A path of vector spaces V_t such that $V_0 = W$ therefore determines a tangent vector

$$\frac{d}{dt} a_{ij}(V_t)|_0 \in \mathrm{Hom}_c(W, W^{\perp}). \tag{74}$$

Here $\mathrm{Hom}_c(V_1, V_2)$ denotes the complex vector space of linear transformations (homomorphisms) from V_1 to V_2. But $\mathrm{Hom}_c(W, W^{\perp})$ is the fiber over W of the *bundle*

$$\mathrm{Hom}_c(\gamma_p, \gamma_p^{\perp}), \tag{75}$$

and so the tangent to G_{pq} is just

$$T_c G_{pq} = \mathrm{Hom}_c(\gamma_p, \gamma_p^{\perp}). \tag{76}$$

We are actually interested in relating the Chern classes of $T_c G_{pq}$ to those of γ_p. This can be done using the following trick. Note that

$$\begin{aligned} T_c G_{pq} \oplus \mathrm{Hom}(\gamma_p, \gamma_p) &\cong \mathrm{Hom}(\gamma_p, G_{pq} \times C^{p+q}) \\ &\cong \bigoplus_1^{p+q} \mathrm{Hom}(\gamma_p, G_{pq} \times C). \end{aligned}$$

Each summand is the dual to γ_p. Using the metric we then get

$$T_c G_{pq} \oplus \mathrm{Hom}(\gamma_p, \gamma_p) \cong \bigoplus_1^{p+q} \bar{\gamma}_p, \tag{77}$$

where $\bar{\gamma}_p$ denotes the conjugate bundle. Now we apply ch to (77), use the homomorphism properties discussed in Sect. 3, and apply the identity

$$\mathrm{Hom}(\gamma_p, \gamma_p) \cong \gamma_p \otimes \bar{\gamma}_p \tag{78}$$

to obtain the desired relation:[33]

$$ch T_c G_{pq} = (p+q) ch\bar{\gamma}_p - ch\gamma_p ch\bar{\gamma}_p . \tag{79}$$

For example, we can expand out (79) to obtain

$$\begin{aligned} ch_3 T_c G_{pq} &= (p+q) ch_3 \bar{\gamma}_p - (ch_0 \gamma_p ch_3 \overline{\gamma_p} + ch_1 \gamma_p ch_2 \overline{\gamma_p} + ch_2 \gamma_p ch_1 \overline{\gamma_p} + ch_3 \gamma_p ch_0 \overline{\gamma_p}) \\ &= -(p+q) ch_3 \gamma_p , \end{aligned}$$

where we have used the fact that $ch_p(\bar{V}) = (-1)^p ch_p(V)$. Therefore, the condition (68) becomes

$$\int_{S^2 \times S^4} ch(\hat{\varphi}^* T_c G_{pq}) = -(p+q) \int_{S^2 \times S^4} ch_3 \hat{\varphi}^* \gamma_p \neq 0 \tag{80}$$

in four spacetime dimensions. Similarly expanding (79) we find an anomaly in two dimensions if there exists a family $\hat{\varphi}$ with

$$\int_{S^2 \times S^2} ch(\hat{\varphi}^* T_c G_{pq}) = \int_{S^2 \times S^2} (q-p) ch_2 \hat{\varphi}^* \gamma_p + (ch_1 \hat{\varphi}^* \gamma_p)^2 \neq 0 . \tag{81}$$

The next step is to construct bundles over $S^2 \times S^d$ with nontrivial Chern characters using the "external product" construction which is described as follows. Given two vector bundles $E_i \to X_i, i=1,2$, define

$$E_1 \boxtimes E_2 = \pi_1^* E_1 \otimes \pi_2^* E_2 , \tag{82}$$

where $\pi_i : X_1 \times X_2 \to X_i$ is the projection. Thus, $E_1 \boxtimes E_2$ is a ($\dim E_1 \dim E_2$)-plane bundle over $X_1 \times X_2$ with Chern character

$$ch(E_1 \boxtimes E_2) = \pi_1^* ch E_1 \wedge \pi_2^* ch E_2 . \tag{83}$$

In particular, one can choose $B_1 \to S^2$ to be the line bundle associated to the Hopf bundle, and $B_2 \to S^4$ to be the 2-plane bundle associated to the instanton bundle. One can show that these bundles have Chern characters[34]

$$\begin{aligned} ch B_1 &= 1 + \omega^{(2)} , \\ ch B_2 &= 2 + \omega^{(4)} , \end{aligned} \tag{84}$$

where $\omega^{(2)}, \omega^{(4)}$ denote the volume forms on the spheres S^2, S^4.

By the classification theorem quoted above with $Q = S^2 \times S^4$ or $S^2 \times S^2$, we know that there exist maps

$$\begin{aligned} f &: S^2 \times S^4 \to G_{2q} \quad q \geqq 3 , \\ g &: S^2 \times S^2 \to G_{1q} \quad q \geqq 2 , \end{aligned} \tag{85}$$

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33 We will henceforth drop the wedge product symbol. The expression (74) chooses one of two possible complex structures for $T_c G_{pq}$. We would have obtained the other had we used the canonical q-plane bundle γ_q. Taking this into account one can show that Eq. (79) is actually symmetric under the interchange $p \leftrightarrow q$, as expected from Eq. (69)

34 In general $B_n \to S^{2n}$ is the "Bott bundle" with transition function the generator of $\pi_{2n-1}(U(N))$, where N is large, and $ch B_n = n + \omega^{(2n)}$

such that

$$\begin{aligned} f^*\gamma_2 &= B_1 \boxtimes B_2 \,, \\ g^*\gamma_1 &= B_1 \boxtimes B_1 \,. \end{aligned} \tag{86}$$

We may then take $\hat{\varphi} = f$ for $d=4$, $p=2$, and $\hat{\varphi} = g$ for $d=2$, $p=1$ to obtain nonzero integrals in (80) and (81). For larger values of p, one can add trivial bundles to the above external products to obtain p-plane bundles. The classifying maps for these bundles then furnish anomalous families. By the arguments of Sects. 2 and 4 we can conclude that the only four-dimensional supersymmetric Grassmannian sigma models which do *not* have a topological obstruction are those with target space CP^n or G_{22}. Similarly, the only two-dimensional chiral Grassmannian sigma model free of obstructions has target space $CP^1 = S^2$. These results are slightly stronger than those of [23].

Using Eqs. (79) and (83)–(86) one can show that each member of the anomalous families we have constructed is an instanton, i.e. the families lies in nontrivial elements of $\pi_0(\mathscr{C})$. A simple modification of the above procedure allows us to construct anomalous families of maps which are not instantons for a slightly restricted class of models.[35]

For example, consider once more the Bott bundle $B_3 \to S^6$ with $chB_3 = 3 + \omega^{(6)}$. If $p, q \geqq 3$, the classification theorem guarantees the existence of a map $f: S^6 \to G_{pq}$ such that $f^*\gamma_p = B_3 \oplus I_{p-3}$, where I_{p-3} is the trivial $(p-3)$-plane bundle. Composing f with a degree one map $r: S^2 \times S^4 \to S^6$ gives an anomalous family of the required type. Similarly, in two dimensions $G_{pq}, p, q \geqq 2$ admit anomalous families of maps in the zero instanton sector.

In the following two sections we will continue to explore the nature of the anomaly for Grassmannian target spaces.

6. An Analogy to Gauge Theory

In Sects. 2 and 4 we gave a global characterization of the anomaly which is mathematically similar to the global formulation of the anomaly of gauge theory. (See Appendix A.) A peculiarity of the topological interpretation of non-abelian anomalies is that it is not entirely equivalent to the perturbative characterization of the anomaly. Indeed, in the case of a chiral U(1) gauge theory the global obstruction vanishes, although the theory is anomalous, and hence nonsensical. We will show in this section that supersymmetric Grassmannian sigma models have a formulation which displays an anomaly similar to the perturbative gauge anomaly. Just as for gauge theory, we will find that in some models there is a perturbative anomaly although the global obstruction we have discussed thus far vanishes.

The perturbative gauge anomaly shows up when the theory is formulated in terms of the affine space $\mathscr{A}^{(4)}$ of connections on a principal bundle. The anomaly is then the nontrivial variation of $\Gamma_f[A]$ along the gauge group fibers of the bundle

35 These will be useful in Sect. 7 and Appendix C

$\mathscr{A}^{(4)} \to \mathscr{C}^{(4)}$. We will see how a very similar situation occurs in the case of Grassmannian sigma models.

We begin with the four-dimensional supersymmetric Grassmannian sigma models considered in the previous section. Following Ong [57] we formulate the theory in terms of the linear space of scalar and spinor $p \times (p+q)$ matrix fields A and χ, and the nondynamical scalar, vector, and spinor $p \times p$ matrix fields D, V_μ, and λ. Using the covariant derivative

$$D_\mu = \partial_\mu + \frac{i}{2} V_\mu , \tag{87}$$

we form the Lagrangian

$$\mathscr{L}^I = \mathrm{Tr}\left\{ -\frac{1}{2} D + \frac{1}{2} A^\dagger D A + (D_\mu A)^\dagger (D_\mu A) + \frac{i}{2} \bar{\chi}^T \bar{\sigma}^\mu \overleftrightarrow{D}_\mu \chi + \frac{i}{\sqrt{2}} (A^\dagger \lambda \chi - \bar{\chi}^T \bar{\lambda} A) \right\} . \tag{88}$$

Classically, the equations of motion serve to eliminate D, V_μ, and λ, thereby inducing constraints on the fields A and χ appropriate to the G_{pq} supersymmetric sigma model. For example, the equation of motion for D imposes $AA^\dagger = 1_p$, which forces the scalar fields to lie on the Stiefel manifold

$$V_{pq} = U(p+q)/U(q) . \tag{89}$$

Next, the equations of motion for $\lambda, \bar{\lambda}$ imply that the fermions take values such that

$$\chi A^\dagger = A \bar{\chi} = 0 . \tag{90}$$

Finally, elimination of V_μ yields the Lagrangian

$$\mathscr{L}^{II} = \mathscr{L}_b + \mathrm{tr}\,\{ i \bar{\chi}^T \bar{\sigma}^\mu [\partial_\mu + (A \overleftrightarrow{\partial}_\mu A^\dagger)] \chi + \tfrac{1}{4} (\chi \bar{\sigma}^\mu \bar{\chi}^T)^2 \} , \tag{91}$$

where

$$\mathscr{L}_b = \mathrm{tr}\,[\partial_\mu A^\dagger \partial_\mu A + \tfrac{1}{4} (A \overleftrightarrow{\partial}_\mu A^\dagger)^2] . \tag{92}$$

The Lagrangian of Eq. (91) has a gauge invariance: left multiplication of A and χ by a unitary matrix U leaves the Lagrangian and the constraints unchanged. Thus, certain degrees of freedom of the maps of spacetime into the Stiefel manifold are spurious, and the true configuration space consists of gauge-equivalence classes of maps. Since the equivalence class of matrices $A, A \sim UA$, which satisfy $AA^\dagger = 1$ defines a point in G_{pq}, the Lagrangian of Eq. (91) describes the dynamics of a nonlinear sigma model with target space G_{pq}.

The analogy with gauge theories can now be clarified. The constraint $AA^\dagger = 1_p$ is not a gauge constraint, so that the proper analog of the space $\mathscr{A}^{(4)}$ of gauge theory is not the linear space of $p \times (p+q)$ matrix fields A, but the space

$$\mathscr{A}^{(d)}_{pq} = \{\mathrm{Maps} : S^d \to V_{pq}\} .$$

The principal $U(p)$ fibration $r : V_{pq} \to G_{pq}$ induces a $\mathscr{G}_p$-fibration $R : \mathscr{A}_{pq} \to \mathscr{C}_{pq}$, where

$$\mathscr{G}^{(d)}_p = \{\mathrm{Maps} : S^d \to U(p)\} .$$

The introduction of fermions is slightly different from the gauge analog. It can be shown that Eq. (90) implies that the fermions take their values in the tangent bundle $T_c G_{pq}$, and that the covariant derivative in Eq. (91) corresponds to the pullback of a connection on $T_c G_{pq}$, as required. The anomaly we will find results from quantum effects which prevent the "gauge modes" $\mathscr{G}_p$ from decoupling from the fermions.

In quantum mechanics the necessary constraints on A and χ are obtained by functional integration over D, V_μ, and λ in the partition function. That is, one proceeds from

$$Z = \int [dA dA^\dagger d\chi d\bar{\chi} d\lambda d\bar{\lambda} dV_\mu dD] e^{-\int \mathscr{L}^I} \tag{93}$$

to

$$\begin{aligned} Z &= \int [dA dA^\dagger][d\chi d\bar{\chi}] \delta(AA^\dagger - 1_p) \delta(\chi A^\dagger) \delta(A\bar{\chi}) e^{-\int \mathscr{L}^{II}} \\ &= \int [dA dA^\dagger] \delta(AA^\dagger - 1_p) e^{-\int \mathscr{L}_b} e^{-\Gamma_f[A]} . \end{aligned} \tag{94}$$

Note that the delta function constraint in Eq. (94) does not eliminate the gauge degrees of freedom from the measure. We must therefore study the (possible) dependence of the integrand on these degrees of freedom.

In perturbation theory, one can parametrize A by $A = UC$ where $U \in U(p)$ and C is $p \times (p+q)$ with the first p columns forming a diagonal positive-definite matrix. The field U does not enter into $\mathscr{L}_b$, but does couple to the fermions through the vector field

$$\tfrac{1}{2} A \overleftrightarrow{\partial}_\mu A^\dagger = U(\tfrac{1}{2} C \overleftrightarrow{\partial}_\mu C^\dagger) U^\dagger + U \partial_\mu U^\dagger = (\tfrac{1}{2} C \overleftrightarrow{\partial}_\mu C^\dagger)^U , \tag{95}$$

which is a gauge transform of the vector potential $\frac{1}{2} C \overleftrightarrow{\partial}_\mu C^\dagger$ by U. We can consider the matrix χ to be $(p+q)$ Fermi fields in the fundamental representation of the gauge group $U(p)$. Since the fermions are chiral, the anomaly implies that $\Gamma_f[A]$ depends on U:

$$\exp(-\Gamma_f[A]) = \exp(i(p+q)\mathscr{I}^4[C, U])) \exp(-\Gamma_f[C]) , \tag{96}$$

where $\mathscr{I}^4$ again denotes the integrated four-dimensional anomaly for a fermion in the fundamental representation of $U(p)$. Having isolated the dependence of the integrand in Eq. (94) on the gauge modes U we can now perform the functional integral over these degrees of freedom. This integration imposes constraints inappropriate to the G_{pq} model.

Some of these constraints can be exhibited more explicitly by noting that in perturbation theory one can factor the measure $[dU]$ into $[d\theta][d\mu]$, where $\theta(x)$ denotes the U(1) degree of freedom in $\mathscr{G}_p$ and $[d\mu]$ is everything else. The result of integrating out the $\theta(x)$ degree of freedom can be shown to be

$$\prod_x \delta[\varepsilon^{\mu\nu\alpha\beta} \operatorname{tr}(\partial_\mu C \partial_\nu C^\dagger \partial_\alpha C \partial_\beta C^\dagger + \tfrac{1}{2} \partial_\mu C \partial_\nu C^\dagger C \partial_\alpha C^\dagger C \partial_\beta C^\dagger)] . \tag{97}$$

This is an extra, unwanted constraint if $p \geqq 2$. Clearly the theory defined by Eq. (94) is not the G_{pq} sigma model. Note, in particular, that there is an anomaly for the CP^n models, $n \geqq 2$, although the global obstruction vanishes in that case.[36]

36 In the special case of $CP^1 = S^2$ the above anomaly also vanishes. We thank H. Schnitzer for pointing this out

One can also consider analogous models in two dimensions. These can be defined by the Lagrangian of Eq. (88) where $\bar{\sigma}^{\mu}$ are the 1×1 matrices 1 and i. (Again these models are not supersymmetric.) The elimination of the nondynamical fields proceeds as before, except that the quartic fermion interactions vanish. Again, the chiral anomaly implies

$$e^{-\Gamma_f[A]} = \exp(i(p+q)\mathcal{I}^2[C, U]) \exp(-\Gamma_f[C]) \tag{98}$$

with

$$\mathcal{I}^2 = \frac{1}{2\pi} \int_0^1 ds \int_{S^2} \omega_2^1 \left[\partial_s g g^{-1}(s), \left(\frac{1}{2} C \overleftrightarrow{\partial}_\mu C^\dagger \right)^{g(s)} \right] \tag{99}$$

for fermions in the fundamental representation of $\mathrm{U}(p)$ and the $\mathrm{U}(1)$ degree of freedom imposes the constraint

$$\prod_x \delta(\varepsilon^{\mu\nu} \operatorname{tr} \partial_\mu C \partial_\nu C^\dagger), \tag{100}$$

wich again is inappropriate for *all* Grassmannian manifolds.

We can continue the analogy with gauge theory by relating the global obstruction to defining $e^{-\Gamma_f}$ as a smooth function on $\mathscr{C}_{pq}$ to the variation of $e^{-\Gamma_f}$ along the fibers of $\mathscr{A}_{pq}$. First, recall an example from magnetic monopole theory which is mathematically similar to our case. On S^2 we can consider an abelian gauge theory defined with respect to some principal $\mathrm{U}(1)$ bundle $R: P \to S^2$. If P has one twist we say that there is a monopole inside the sphere; then $P = S^3$ and R is the Hopf map. Sections of a line bundle associated to P (e.g. the monopole harmonics) correspond exactly to ordinary *functions* on P itself which satisfy the "equivariance condition"

$$f(\alpha+\delta, x) = e^{it\delta} f(\alpha, x), \tag{101}$$

where α, x are coordinates for the fiber and base and t is an integer called the equivariance of f.

In gauge theory one can form a Hopf bundle $\tilde{P} \subset \mathscr{A}^{(4)}$ which projects to a nontrivial two-sphere Y in $\mathscr{C}^{(4)}$ if there is a nontrivial loop g_θ in $\mathscr{G}_p^4$. One way to construct $\tilde{P}$ [9] is by forming the disk in $\mathscr{A}^{(4)}$ given by

$$A^{t,\theta} = t A^{g_\theta} + (1-t) A, \quad 0 \leqq \theta \leqq 2\pi; 0 \leqq t \leqq 1.$$

This disk projects to a two-sphere Y in $\mathscr{C}$, and can be viewed as a (singular) section of a Hopf bundle $R: \tilde{P} \to Y$ with $\tilde{P} \subset \mathscr{A}^{(4)}$. The "group-loop" g_θ is actually a $\mathrm{U}(1)$ subgroup of $\mathscr{G}_p^{(4)}$ (see Appendix C) and the fibration R is thus a principal $\mathrm{U}(1)$ fibration. It can be shown that the twist ν of the family index $\operatorname{Ind} \not{D}|_Y$ is the same as the equivariance t of $\exp[-\Gamma_f(A)]$ along the fibers of $\tilde{P}$ [7, 9]. Thus the intrinsically defined fermion determinant has a singularity on Y which can only be smoothed out by viewing it as an equivariant function on $\tilde{P}$ (and, more generally, on $\mathscr{A}^{(4)}$) with equivariance $\nu = t$.

In the case of Grassmannian sigma models we have constructed the analog of $\tilde{P}$ for most of the cases which have a global anomaly. The details of the constructions are given in Appendix C. There we give explicit examples of maps $\hat{\varphi}: S^2 \times S^d \to G_{pq}$

and $\hat{\Phi}: S^3 \times S^d \to V_{pq}$ and $\pi: S^3 \to S^2$ such that the diagram

$$\begin{array}{ccc} S^3 \times S^d & \xrightarrow{\hat{\Phi}} & V_{pq} \\ \Big\downarrow{\scriptstyle (\pi,1)} & & \Big\downarrow{\scriptstyle r} \\ S^2 \times S^d & \xrightarrow{\hat{\varphi}} & G_{pq} \end{array} \tag{102}$$

commutes. Thus we have a family in $\mathscr{C}_{pq}^{(d)}$ parametrized by $Y = S^2$ and a family in $\mathscr{A}_{pq}^{(d)}$ parametrized by S^3. The map π is the restriction of the projection R to $S^3 \subset \mathscr{A}_{pq}^{(d)}$. From the construction of Appendix C one can see that this three-sphere can be regarded as the total space $\tilde{P}$ of a Hopf bundle $R: \tilde{P} \to Y$ with the principal U(1)-fibration given by a homotopically nontrivial U(1) subgroup of $\mathscr{G}_p^{(d)}$, which is, in fact, the generator of $\pi_1(\mathscr{G}_p^{(d)})$.

From (96) and (98) we can now find the equivariance of $\exp(-\Gamma_f[A])$ along the fibers of $\tilde{P}$. For fermions in the fundamental representation of U(p) the restriction of $\exp \mathscr{I}^d$ to a circle in a gauge orbit is (homotopic to) an equivariant function with equivariance t equal to the homotopy class of that circle (see footnote 23). Therefore the equivariance of $\exp(-\Gamma_f[A])$ along the fibers of $\tilde{P}$ is $t = (p+q)$. For the explicit families given in Appendix C we show that the twist of $\mathrm{Ind}\, \not{D}|_Y$ is $\nu = (p+q)$. Therefore, as in gauge theory, the intrinsically defined fermion determinant has a singularity on Y which can only be smoothed out by viewing it as an equivariant function on $\tilde{P}$ (and, more generally, on $\mathscr{A}_{pq}^{(d)}$) with equivariance $t = \nu$.

Thus far we have emphasized the similarities of the sigma model anomaly to the gauge theory anomaly. Indeed, as far as index theory is concerned they are almost identical. What we have just shown is that the physical interpretation is different. We have illustrated the failure of the attempt to define the path integral for the Grassmannian sigma models by imposing constraints on the linear fields. In contrast to gauge theory, for which the phase variation along $\mathscr{A}$ (not necessarily homotopically nontrivial) renders the theory ill-defined [2], the logical possibility remains that there exists some other way to define the quantum sigma model. In some cases the global analysis of the previous sections precludes this possibility. In other cases, e.g. the four-dimensional CP^n models, the global obstruction vanishes, and our results are not powerful enough to exclude the existence of an intrinsically defined theory. A more refined version of this obstruction might eliminate that possibility as well.

7. Applications to Preon Physics

While four-dimensional nonlinear sigma models are of interest in their own right [24, 58, 26], they also arise as the low-energy approximations to strongly interacting gauge theories. In a vectorlike nonsupersymmetric theory such as QCD, the relevant sigma model contains only bosonic degrees of freedom (the Goldstone modes corresponding to dynamically broken symmetries) and the considerations of this paper are irrelevant. In a theory with unbroken supersymmetry, however, some fermions must remain massless.[37] Some authors have attempted to identify these massless fermions with quarks and leptons in the context of supersymmetric preon models [59, 60, 61, 28].

A preon model consists of chiral and gauge superfields together with the dynamical assumption that a gauge singlet order parameter superfield Φ takes on a symmetry-breaking vacuum expectation value. Unbroken supersymmetry then requires that only the scalar component φ of Φ develops a vev. One further assumes that at energies lower than the confinement scale Λ_{HC} the full theory is well approximated by a linear sigma model with superfields Φ and an effective superpotential respecting those symmetries (and only those symmetries) of the underlying theory [62–65]. Finally, at low energies one eliminates all degrees of freedom other than those which describe M, the space of absolute minima of the potential.

There is an important qualitative difference between the nature of the space M in supersymmetric and in ordinary sigma models. In the latter the potential V is required to be invariant under the group G of all symmetry transformations of the full theory, so that M contains (at least) the homogeneous space G/H, where $H=\mathrm{stab}_G\langle\varphi\rangle$ is the little group. In supersymmetry G is still the symmetry group of V, but M possesses a larger symmetry. The reason is that a supersymmetric potential has the special form

$$V(\varphi,\varphi^*)=\left(\frac{\partial F}{\partial\varphi}\right)^{\dagger} J\left(\frac{\partial F}{\partial\varphi}\right), \tag{103}$$

where

$$(J^{-1})_{ij}=\frac{\partial^2 D(\varphi,\varphi^*)}{\partial\varphi^i\partial\varphi^{j*}}, \tag{104}$$

and F, D are the functions appearing in the supersymmetric linear sigma-model. In principle, they are computable from the dynamics of the original preon theory. Thus, if J is nonsingular,

$$M=\{\varphi:\partial_\varphi F=0\} \tag{105}$$

for some analytic function F. This implies that M is invariant under the larger Lie group $\bar{G}$ whose Lie algebra is the complexification of the Lie algebra of G,

$$\pounds(\bar{G})=\pounds(G)\otimes C$$

[66–68]. Roughly speaking, if G consists locally of elements $\exp(i\pi\cdot\mathbf{T})$, then we get $\bar{G}$ by letting π become complex. For example, $\overline{\mathrm{U}(1)}=C^*$, the nonzero complex numbers, while $\overline{\mathrm{U}(n)}=\mathrm{GL}(n,C)$. These examples illustrate the important fact that the complexifications of compact groups are noncompact.

Thus in supersymmetry M always contains at least $\bar{G}/H'$, where $H'=\mathrm{stab}_{\bar{G}}\langle\varphi\rangle$. Clearly H' contains $\bar{H}$, although as we will see it can be much larger, depending on the nature of Φ. In any case, just as in QCD we must add the assumption that the

37 There are other ways to get massless fermions coupled to nonlinear fields, but we will not consider them here

effective potential $V(\varphi, \varphi^*)$ has no flat directions other than those required by symmetry, since presumably nonperturbative quantum effects will give masses to every unprotected mode. Thus M in fact equals $\bar{G}/H'$ and the Hessian $\partial^2 F/\partial\varphi^i \partial\varphi^j$ on M is nondegenerate in all directions other than those generated by $\bar{G}$ (i.e. F is a "holomorphic equivariant Morse function").

There is an additional subtlety here. To conclude that $M = \bar{G}/H'$ we must assume (as do most authors [60, 61, 28, 64, 67–69]) that G acts transitively on M. This can happen if the strong dynamics chooses either F or D so as to eliminate fixed points of $\bar{G}$ in φ-space. (The origin is such a point if G acts linearly.) It should be borne in mind that this is a dynamical question which can radically affect the topology we will discuss.

Thus, two important features about effective supersymmetric models stand out. First, it is not enough to specify the unbroken symmetry group $H \subsetneqq G$ of the theory since H' need not be the complexification of H. One must instead assume a particular vev $\langle\varphi\rangle$ and find the stability group explicitly. Second, some homogeneous spaces $\bar{G}/H'$ cannot be realized for *any* choice of $\langle\varphi\rangle$. In particular, if G/H is a symmetric space then Lerche [67] has shown that $H' = \bar{H}$, so that $\bar{G}/H'$ has real dimension twice that of G/H, a situation he refers to as "full doubling."

In the remainder of this section we will examine in detail three sigma models which have appeared in the literature and a fourth of our own. Two of these will prove to be anomalous, and hence untenable (as they stand) as the low-energy limit of any well-defined theory. The other two turn out to be anomaly-free. We conjecture that all nonlinear models which arise by setting to zero the nondegenerate potential of a well-defined, renormalizable supersymmetric model are anomaly-free. We have not proved this statement. Indeed our fourth example is designed as a counterexample to the stronger assertion that whenever M is analytically imbedded in a linear space (*not* necessarily as a critical surface) then M is anomaly-free.

As a first example [57], consider the symmetry breakdown

$$\mathrm{U}(p+q) \to \mathrm{U}(p) \times \mathrm{U}(q). \tag{106}$$

Since G_{pq} is itself a Kähler manifold it is possible that the low-energy theory exhibits no doubling at all. Then the considerations of Sect. 4 show that this leads to an ill-defined theory. Thus the no-doubling theory cannot be realized as the low-energy effective theory of some preon model. Actually, since G_{pq} is a symmetric space [70], the result of Lerche gives an independent reason for believing that the no-doubling scenario is impossible.

The second example is the fully doubled G_{24} model, which has been proposed in the literature as being phenomenologically interesting [61]. We will now show that this model has no anomaly. More generally, we will show that whenever the sigma-model is fully doubled there is no anomaly. Heuristically, the fermions tangent to $\bar{G}/\bar{H}$ are nonchiral when restricted to G/H.

In Appendix D we show that if G is compact then the inclusion of G/H into $\bar{G}/\bar{H}$ has a homotopy inverse. That is, there is a map $R: \bar{G}/\bar{H} \to G/H$ such that $i \circ R$ and $R \circ i$ are homotopic to the identity, where $i: gH \mapsto g\bar{H}$ is the inclusion. This result is reasonable, since we can think of deforming $\bar{G}/\bar{H}$ along its noncompact directions until it fits onto G/H, just as we can shrink C^* onto S^1. Now define the pullbacks of bundles R^* and i^*. In particular i^* just takes bundles on the larger space and *restricts* them to the subspace. Hence $i^*R^*=1$ and $R^*i^*=1$, and the topology of $T_c\bar{G}/\bar{H}$ is determined by its restriction to G/H:

$$R^*(T_c\bar{G}/\bar{H}|_{G/H})=R^*i^*(T_c\bar{G}/\bar{H})\cong T_c\bar{G}/\bar{H}\,. \tag{107}$$

So, if $\hat{\varphi}:S^2\times S^4\to\bar{G}/\bar{H}$ is a family of configurations, then

$$\hat{\varphi}^*(chT_c\bar{G}/\bar{H})=(R\circ\hat{\varphi})^*ch(T_c\bar{G}/\bar{H})|_{G/H}\,. \tag{108}$$

Along G/H the fermions are nonchiral,

$$T_c\bar{G}/\bar{H}|_{G/H}=T_RG/H\otimes C\,, \tag{109}$$

where T_RG/H denotes the real tangent bundle to G/H considered as a real manifold. If E is a real vector bundle then $E\otimes C$ is a complex vector bundle with real transition functions, so [25, 39]

$$ch_i(E\otimes C)=ch_i(\overline{E\otimes C})=(-1)^i ch_i(E\otimes C)\,. \tag{110}$$

[See the remark following Eq. (79).] Therefore $ch_3(T_RG/H\otimes C)=0$ and four-dimensional supersymmetric sigma models with full doubling have no topological anomaly.

For our third example we consider a model which is neither fully doubled nor fully undoubled. This model has been proposed by Büchmuller et al. as a preon theory reproducing the weak interactions of quarks and leptons [28].

Büchmuller et al. consider a supersymmetric SU(2) hypercolor model with six doublet chiral superfields χ_p^α $\alpha=1,\ldots,6$; $p=1,2$. The global symmetry is U(6).[38] These authors further assume that the gauge-invariant superfield operator

$$\Phi^{\alpha\beta}=\varepsilon_{pq}\chi_p^\alpha\chi_q^\beta \tag{111}$$

develops a vacuum expectation value, e.g. $\langle\varphi^{56}\rangle\neq0$ while the other $\langle\varphi^{\alpha\beta}\rangle=0$, so the pattern of symmetry breaking is $\mathrm{U}(6)\to\mathrm{SU}(2)\times\mathrm{U}(4)$.

An effective theory for $\Phi^{\alpha\beta}$ will have a superpotential which is $\overline{\mathrm{U}(6)}=\mathrm{GL}(6,C)$ invariant. The stability group H' of $\langle\varphi\rangle$ has a Lie algebra which can be represented by complex 6×6 matrices of the form

$$\begin{pmatrix} A & 0\\ B & C\end{pmatrix}, \tag{112}$$

where A is 4×4 while C is 2×2 and traceless. The low-energy theory has target space

$$\mathscr{D}=\mathrm{GL}(6,C)/H'\,. \tag{113}$$

To decide if this model has an anomaly we need to consider the geometry of $\mathscr{D}$.

38 The U(1) subgroup of U(6) does not generate a symmetry in the quantum theory since it is anomalous. However, the theory also has an unbroken $\mathrm{U}(1)_R$ symmetry and a linear combination of U(1) and $\mathrm{U}(1)_R$ is nonanomalous. Taking proper account of these U(1) factors gives a low energy manifold which is the same as the one we consider

First, note that one can enlarge the above Lie algebra by dropping the condition that C be traceless. This new Lie algebra generates a group K and $\mathrm{GL}(6,C)/K = G_{24}$ follows from considering the transitive action of $\mathrm{GL}(6,C)$ on G_{24}. Since the Lie algebras of H' and K differ by a single generator we learn that $\mathscr{D}$ can also be regarded as the total space of a C^*-bundle over G_{24}. We will use this interpretation of $\mathscr{D}$ below. Furthermore, given a principal C^*-bundle like $\Pi : \mathscr{D} \to G_{24}$, we can introduce yet another space $\mathscr{D}'$ by contracting each C^* fiber of $\mathscr{D}$ to a circle. Then $\mathscr{D}' \to G_{24}$ is a principal U(1)-bundle.

The geometry of $\mathscr{D}$ is most easily understood by considering the exterior algebra on C^6 [55]. In particular, in $\Lambda^2(C^6) \cong C^{15}$ consider the space of nonzero "decomposable" two-forms, i.e. those which can be written as products of single vectors

$$\mathbf{v} \wedge \mathbf{w}, \qquad \mathbf{v}, \mathbf{w} \in C^6 . \tag{114}$$

$\mathrm{GL}(6,C)$ acts transitively on this space and the stability group of a point is H'. Hence the space is precisely $\mathscr{D}$. In fact the map

$$\Pi : C^{15} - \{0\} \to CP^{14}, \tag{115}$$

which projects a vector to its equivalence class under identification by a complex factor, projects $\mathscr{D}$ to G_{24}. Since $\mathscr{D}$ is holomorphically imbedded in the Kähler manifold $C^{15} - \{0\}$, it is Kähler [55, 27].

Considering $\Pi : \mathscr{D} \to G_{24}$ as a C^*-bundle, tangent vectors to the total space $\mathscr{D}$ can lie along the fiber direction or along the base direction, that is

$$T\mathscr{D} = \Pi^* S \oplus \Pi^* TG_{24} . \tag{116}$$

Here S refers to the restriction to G_{24} of the canonical line bundle $\gamma_1 \to CP^{14}$. The bundles S and $\mathscr{D}$ are both associated to the circle bundle $\mathscr{D}'$ mentioned above; explicitly

$$\mathscr{D}' = \mathrm{U}(6)/\mathrm{SU}(2) \times \mathrm{U}(4) . \tag{117}$$

One can show [55] that $ch_1(\mathscr{D}')$ is the same as $ch_1(\gamma_2)$.

We are now in a position to demonstrate that the $\mathscr{D}$-sigma-model has no anomaly. Suppose these exists an anomalous family

$$\hat{\varphi} : S^2 \times S^4 \to \mathscr{D} . \tag{118}$$

Consider the projected map $\tilde{\varphi} = \Pi \circ \hat{\varphi} : S^2 \times S^4 \to G_{24}$. Then $\hat{\varphi}$ is a lift of $\tilde{\varphi}$. See Fig. 7.1. Since

$$\hat{\varphi}^* ch_3 T\mathscr{D} = \hat{\varphi}^* ch_3 (\Pi^* S \oplus \Pi^* TG_{24}) = \tilde{\varphi}^* ch_3 TG_{24} , \tag{119}$$

$$\begin{tikzcd} & \mathscr{D} \arrow[d] \\ S^2 \times S^4 \arrow[ur, "\hat{\phi}"] \arrow[r, "\tilde{\phi}"'] & G_{24} \end{tikzcd}$$

Fig. 7.1. Some maps used in the text

we have that $\hat{\varphi}$ is anomalous iff $\tilde{\varphi}$ is. On the other hand, the cohomology of G_{24} is generated by the Chern classes $ch_1(\gamma_2)$ and $ch_2(\gamma_2)$ [25]. That is, if $\hat{\varphi}$ is anomalous then

$$\tilde{\varphi}^* ch_1(\mathscr{D}') = ch_1(\tilde{\varphi}^* \mathscr{D}') \neq 0, \tag{120}$$

so the circle bundle $\tilde{\varphi}^*(\mathscr{D}')$ is twisted and does not admit a section. This is incompatible with Fig. 7.1, for if $\tilde{\varphi}$ has a lift $\hat{\varphi}$, then $\tilde{\varphi}^*(\mathscr{D}')$ must have a section. (Recall that $\mathscr{D}$ retracts to $\mathscr{D}'$.) Thus, there is no anomalous map $\hat{\varphi}$.

This example suggests a generalization which leads to an interesting family of anomalous target spaces. Consider $\mathscr{D}_{pq}$, the space of nonvanishing decomposable p-forms of vectors in C^{p+q}. Then $\mathscr{D}_{pq}$ lies in $C^{\zeta}-\{0\}$, $\zeta = \binom{p+q}{p}$ and the map

$$\Pi : C^{\zeta}-\{0\} \to CP^{\zeta-1} \tag{121}$$

projects $\mathscr{D}_{pq}$ to G_{pq}. Thus $\mathscr{D}_{pq}$ is a C^*-bundle over G_{pq}.

We will construct anomalous maps to $\mathscr{D}_{pq}$ using a classifying map $\tilde{\varphi} : S^6 \to G_{pq}$, $p, q \geqq 3$, for the bundle $B_3 \oplus I_{p-3}$ considered at the end of Sect. 4. Now $\tilde{\varphi}^*(\mathscr{D}'_{pq})$ is a U(1)-bundle over S^6, but all such bundles are trivial, so $\tilde{\varphi}$ has a lift $\hat{\varphi} : S^6 \to \mathscr{D}_{pq}$. Composing $\hat{\varphi}$ with a degree-one map from $S^2 \times S^4$ to S^6 gives an anomalous family.[39]

This last example raises the question of the existence of theories which predict a $\mathscr{D}_{pq}$-sigma-model at low energies. If such models exist then either naive decoupling, or the assumed pattern of chiral symmetry breakdown, or the assumption of unbroken supersymmetry must fail. Indeed, there exist explicit superpotentials for which the manifold of supersymmetric vacua is an anomalous target space (a vector bundle over $\mathscr{D}_{pq}$). These potentials are degenerate: they have quadratically (but not quartically) flat directions not associated with the vacuum manifold. Thus naive decoupling breaks down. We conjecture that this is a general rule.

8. Conclusion

The topological interpretation of the anomaly is simple, even though the analysis needed to back it up has been difficult. An anomalous theory is one in which we cannot regard the fermionic effective action as an ordinary complex function on boson configuration space because the Green functions have a true geometrical meaning different from their naive one. We think this is the most illuminating way to think about the *global* sigma model anomalies; the alternate approach of Sect. 6, when available, is somewhat artificial. On the other hand, the latter approach may be needed to resolve the issue of *local* obstructions (see below).

While the interpretation we have emphasized is similar to the topological interpretation of the gauge anomaly, the physical meaning is somewhat different. When the fermionic bundle B over the target manifold is twisted, perturbative

39 This trick did not work for G_{24} because the latter is too simple: every 6-cycle in G_{24} is cohomologous to a product of lower-dimensional ones

expansions around different points of $\mathscr{C}$ (in particular, around different choices of vacua) lead to discrepancies which have nontrivial physical consequences for the low-energy behavior of the theory. The addition of compensating WZ counterterms which are well-defined in the domain of validity of perturbation expansions will alter these discrepancies but cannot eliminate them if the obstruction $\nu \neq 0$. In other words, the local bosonic counterterm needed to untwist $\exp - \Gamma_f$ cannot be smoothly extended even over the regions on which the fermion effective action *is* smoothly defined. It is in this respect that our situation differs from analogous cases involving the parity anomaly in odd-dimensional spacetimes [71, 11] and the SU(2) anomaly in four dimensions for a theory with an odd number of both quarks and leptons [42, 72]. In both of these cases the bosonic counterterm is ill-defined *only* in those regions where $\exp(-\Gamma_f[A])$ is ill-defined.

We have shown that the topological approach leads naturally to index theory, where powerful results already exist. Part of the reason for their power lies in the "universal" property of the Dirac operator: since its symbol always looks the same in any coordinate system and for any connection and metric, the index depends solely on the topology of the spaces involved. This is evident in Eq. (61), which makes no reference to connections or metrics. Thus it is possible and desirable to compute the anomaly without writing down specific field configurations. We did this in Sects. 5 and 7, and for gauge theory in Appendix A.

The index is also easy to work with due to its simple algebraic properties. These arise because Ind is a natural construction in K-theory, as we described.

We have seen that the anomaly for Grassmannian sigma models is almost identical to that of nonabelian gauge theory. This analysis raised a problem: Is there a consistent theory for the four-dimensional supersymmetric CP^n model? We have noted that the global obstruction measured by ν vanished for this model, but we have suggested, based on the analogy to chiral U(1) gauge theories, that a more refined obstruction might show that the CP^n model is inconsistent.

Sigma models are of interest primarily as the low-energy approximations to strongly-interacting gauge theories with certain assumed patterns of symmetry breakdown. By ruling out some sigma models, then, we can rule out some symmetry-breaking patterns. We did this for some preon models in Sect. 7.

The physical considerations of Sect. 7 suggested the mathematical conjecture that all nonlinear models which arise by setting to zero the nondegenerate potential of a well-defined, renormalizable, supersymmetric model are anomaly-free. The validity of this conjecture is still an open question. Finally we note that we have by no means analyzed all interesting supersymmetric sigma models. For example, some recently considered models involve $M = E_6/\mathrm{Spin}(10) \times \mathrm{SO}(2)$, $E_7/\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{U}(1)$, and $E_8/\mathrm{SO}(10) \times \mathrm{SU}(3) \times \mathrm{U}(1)$ [73]. We do not know whether these models are anomalous.

Appendix A: Anomalies in Gauge Theory

With the machinery developed in the text we can easily describe gauge anomalies [7]. The gravitational case is only slightly more subtle [8]. What we must do is to find the analogs of $E^{\pm}$. A general reference for this section is [22].

Consider the principal $\mathscr{G}$-bundle $\mathscr{P}: \mathscr{A} \to \mathscr{C} = \mathscr{A}/\mathscr{G}$, where $\mathscr{A} = \{$connections on $P\}$, and P is a principal bundle with gauge group G. Suppose for concreteness that $G = \mathrm{SU}(n)$ $n \geqq 3$, $X = S^4$, and that P is untwisted; that is, we work in the zero-instanton sector. Then $\mathscr{G} = \{\mathrm{Maps}: X \to G\}$ and we can obtain a generator of $\pi_1(\mathscr{G})$ as follows (see also Appendix C): take a generator $\hat{g}$ of $\pi_5(G)$. Compose this with the "pinch" map $r(\theta, x)$ from $S^1 \times X$ to S^5, which has degree one. Let $g_\theta(x) = (\hat{g} \circ r)(\theta, x)$.

Now $\not{D}_A$ in gauge theory is defined on $\mathscr{A}$, not $\mathscr{C}$. It takes sections of E_A^+ to those of E_A^-, where $E_A^\pm = S^\pm \otimes \mathscr{B}$ are completely independent of A. Here $\mathscr{B}$ is a bundle associated to P by the matter representation ϱ. Since we are assuming P is trivial, $\mathscr{B}$ is also trivial and the action of $\mathscr{G}$ on $\mathscr{B}$ is $g \cdot (x, v) = (x, \varrho(x))v)$.

Before anything interesting can happen, then, we must pass to $\mathscr{C}$. This is the mathematical way of enforcing gauge-invariance.[40] Define a bundle $\bar{B}$ over $\mathscr{C} \times X$ by $\bar{B} = (\mathscr{A} \times B)/\sim$, where

$$(A, x, v) \sim (A^g, x, \varrho(g(x))v). \tag{122}$$

Here $v \in \mathscr{B}|_x$, $g \in \mathscr{G}$. In other words, $\bar{B} = \mathscr{A} \times_{\mathscr{G}} \mathscr{B}$. Just like $\hat{\varphi}^*(B)$ in the sigma model case, we can define $\mathscr{H}_{\bar{A}}^\pm = \Gamma(S^\pm \otimes \bar{B}_{\bar{A}})$ to obtain bundles $\mathscr{H}^\pm$ over $\mathscr{C}$. Unlike $\mathscr{B}$, $\bar{B}$ need not be trivial over $\mathscr{C}$, since (122) mixes up the vector-bundle structure of $\mathscr{B}$ with the parameter space.

The form of (122) is fixed by the requirement that $\not{D}$ descend to an operator $\not{D}_{\bar{A}}: \mathscr{H}_{\bar{A}}^+ \to \mathscr{H}_{\bar{A}}^-$. Since $\not{D}$ is gauge covariant, (122) correctly eliminates the gauge redundancy of $\mathscr{B}$, and so we get an elliptic family on $\mathscr{H}^\pm$. We can now repeat the arguments of Sects. 2 and 4 to conclude that the gauge theory will be anomalous if this family has an index which twists over any two-sphere $Y \subset \mathscr{C}$.

We proceed as usual to construct a noncontractible Y as in Sect. 6 [9]. Consider the loop $A_0^{(g_\theta)}$ of gauge transforms of some initial A_0. Extend this smoothly to a disk $A^{r,\theta}$, with $A^{r=1,\theta} = A_0^{(g_\theta)}$ and project to get $Y \subset \mathscr{C}$. The principal bundle $\mathscr{P}: \mathscr{P}^{-1}(Y) \to Y$ then has transition function (homotopic to) g_θ on the equator of S^2, and $\mathscr{H}^\pm$ have transition functions $\varrho(g_\theta)$. The anomaly (17) is then just $\int_{Y \times X} ch_3(\bar{B})$.

For example, let ϱ contain an $\underline{n}$ of left-handed fermions, i.e. $\varrho(g)$ is the $\mathrm{SU}(n)$ matrix g. Then $\bar{B}$ has transition functions $g_\theta(x)$, so it is the pullback r^*B_3 of the Bott bundle B_3 over S^6 with transition function $\hat{g}$, and

$$\textit{anomaly} = \int_{Y \times X} ch_3(r^*B_3) = \int_{Y \times X} r^* ch_3(B_3) = \int_{S^6} ch_3(B_3). \tag{123}$$

In fact, B_3 generates $K(S^6)$, so this expression equals one and the theory is anomalous.

Appendix B: The Symbol Bundle

In this appendix we define precisely the symbol bundle $\Sigma(\not{D})$ used in Sect. 3. This will give the factorization (55).

40 The appropriate construction has already been given in a somewhat more general form in [50]

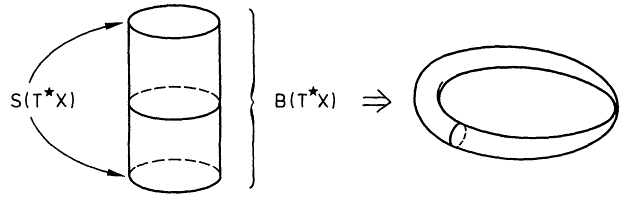

Fig. B.1. If X is a circle T^*X is an infinite cylinder, $B(T^*X)$ is a finite cylinder, and $S(T^*X)$ is the rim of this cylinder. Thus $B(T^*X)/S(T^*X)$ is the pinched torus shown

We consider an elliptic differential operator $D: E_1 \to E_2$ between two vector bundles E_1, E_2 over a compact spacetime X. The symbol bundle is to be defined as an element of $K(T^*X)$. Unfortunately, the definition of K-theory given in the text is not quite the one appropriate for noncompact spaces like T^*X. Consider for example the space R^N, on which all bundles are trivial. In order to get any interesting K-theory on this space (and similarly to get a K-theory on T^*X containing more information than that on X itself) we must modify our definitions slightly to get "K-theory with compact supports." This modified definition turns out to be the one relevant to index theory.

If N is any locally compact space one might try to define

$$K(N) \stackrel{?}{\equiv} K(N^+), \tag{124}$$

where N^+ denotes the one-point compactification of N, obtained by identifying all points at infinity. This is almost right, but if N is in fact compact then N^+ is the disjoint union of N with the point at infinity and $K(N^+) = K(N) \oplus Z$, so (124) does not agree with the definition of $K(N)$ given in the text. This difficulty can be overcome by eliminating the trivial information contained in K concerning the dimensions of the bundles involved. That is, for compact Q we can define $\tilde{K}(Q)$ as the kernel of $\dim: K(Q) \to Z$. Then

$$K(Q) = \tilde{K}(Q) \oplus Z, \tag{125}$$

and for arbitrary N we can consistently define $K(N) \equiv \tilde{K}(N^+)$.[41]

We are interested in the case $N = T^*X$ for compact spacetime X. In this case there is a particularly convenient description of the one-point compactification. If X has a Riemannian metric then cotangent vectors, i.e. elements of T^*X, can be assigned a length. Let $B(T^*X)$, the "unit ball bundle," be the set of elements of length ≤ 1, and let $S(T^*X)$, the unit sphere bundle, consist of elements of length exactly 1. If we identify the subset $S(T^*X)$ of $B(T^*X)$ to a point we obtain $(T^*X)^+$, that is,

$$(T^*X)^+ = B(T^*X)/S(T^*X). \tag{126}$$

For example, if $X = S^1$, then $T^*X = S^1 \times R$ and $(T^*X)^+$ is the pinched torus illustrated in Fig. B.1. Note that we do not compactify each fiber *separately*: $(T^*S^1)^+ = S^1 \mp T^2$, the torus.

41 Note that now $K(R^{2n}) = \tilde{K}(S^{2n}) = Z$, by the Bott periodicity theorem

The characterization of $(T^*X)^+$ given in (126) suggests an extension of K-theory we will need called "relative K-theory." If A is a closed subset of N such that N/A is compact define the relative K group by $K(N,A) \equiv \tilde{K}(N/A)$, the equivalence classes of bundle pairs which have zero net dimension and are trivial on A. In particular,

$$K(T^*X) = K(B(T^*X), S(T^*X)) . \tag{127}$$

We want to define $\Sigma(D)$ as an element of $K(T^*X)$. The definition is slightly awkward if we use (127) directly, and so we will use an equivalent description of $K(N,A)$ which takes as its basic objects the triples $J^{\cdot} = [\alpha : J_1 \to J_2]$, where J_1, J_2 are vector bundles on N and α is a homomorphism between them. The "support" of a triple is the set of points x where $\alpha_x : J_1|_x \to J_2|_x$ *fails* to be an isomorphism. We will consider triples which have support in a compact subset of $N - A$. Two triples $J^{\cdot}$ and $(J')^{\cdot}$ are considered the same if there exist isomorphisms ξ_1, ξ_2 such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\alpha} & E_2 \\ \Big\downarrow{\scriptstyle \xi_1} & & \Big\downarrow{\scriptstyle \xi_2} \\ E_1' & \xrightarrow{\alpha'} & E_2' \end{array}$$

commutes on A.

The set of isomorphism classes of such triples, $L(N,A)$, is a semigroup under the addition

$$J^{\cdot} + (J')^{\cdot} = (\alpha : E_1 \to E_2) \oplus (\alpha' : E_1' \to E_2') \equiv (\alpha \oplus \alpha' : E_1 \oplus E_1' \to E_2 \oplus E_2') ,$$

but there is no obvious subtraction. Thus $L(N,A)$ is much like Vect $\times$ Vect in Sect. 3, and as in that case we can pass to a group by dividing out an equivalence relation. In this case the relation can be defined in terms of "elementary triples." These are triples $R^{\cdot}$ with *empty* support, i.e. whose map is everywhere an isomorphism. Two triples $J^{\cdot}$ and $(J')^{\cdot}$ are then considered equivalent if they become isomorphic upon the addition of elementary triples. That is, $J^{\cdot} \sim (J')^{\cdot}$ if there exist elementary triples $R^{\cdot}$ and $S^{\cdot}$ such that $J^{\cdot} \oplus R^{\cdot} \cong (J')^{\cdot} \oplus S^{\cdot}$. It turns out that $L(N,A)/\sim$ has a subtraction and is in fact isomorphic to $K(N,A)$ [36, 74–76]. The correspondence between triples and elements of K involves the "clutching construction," which we must now describe.

Given a triple $J^{\cdot}$ we will define an element of $\tilde{K}(N/A)$. We begin by glueing together two copies of N (call them N_1, N_2) along A to produce a space $N_1 \cup_A N_2$. Note that

$$(N_1 \cup_A N_2)/N_2 = N/A . \tag{128}$$

Next we construct a *bundle* $\tilde{J}$ on $N_1 \cup_A N_2$ from the triple $J^{\cdot}$ by taking J_1 over N_1, J_2 over N_2 and identifying fibers over A using α, which is an isomorphism there since the triple has compact support in $N - A$. See Fig. B.2. Now in light of (128) we would like $\tilde{J}$ to be trivial and of dimension zero on N_2. Since this is not necessarily the case, we finally consider not $\tilde{J}$ but $\tilde{J} - J_2 \in \tilde{K}(N_1 \cup_A N_2)$. Since this *is* trivial on

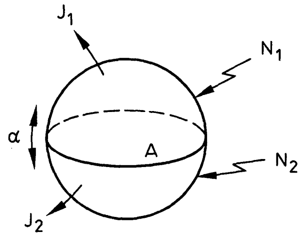

Fig. B.2. The clutching construction

N_2 it defines an element of $\tilde{K}(N/A) = K(N, A)$. This element corresponds to the original triple $J^{\cdot}$.

We can now return to index theory by letting $N = B(T^*X)$, $A = S(T^*X)$. A general elliptic operator $D: \Gamma(E_1) \to \Gamma(E_2)$ has a symbol $\sigma(D)$ as described in Sect. 3. This symbol together with its domain and range constitutes a triple $\Sigma^{\cdot} \equiv [\sigma(D): \pi^*(E_1) \to \pi^*(E_2)]$ over T^*X as described above, since by definition an elliptic operator is one whose symbol has as support the zero section of T^*X, a compact set not touching the sphere of unit radius. The element of $L(B(T^*X), S(T^*X))/\sim \;\cong K(B(T^*X), S(T^*X)) = K(T^*X)$ is the symbol bundle $\Sigma(D)$.[42]

We are finally ready to demonstrate (55). The free Dirac operator has symbol represented by the triple $\sigma(\partial\!\!\!/): \pi^*S^+ \to \pi^*S^-$. The coupled Dirac operator $D\!\!\!\!/: S^+ \otimes \varphi^*(B) \to S^- \otimes \varphi^*(B)$ has symbol $\sigma(D\!\!\!\!/) = \sigma(\partial\!\!\!/) \otimes 1$ and so defines the triple

$$\sigma(\partial\!\!\!/) \otimes 1: \pi^*(S^+ \otimes \varphi^*(B)) \to \pi^*(S^- \otimes \varphi^*(B)).$$

Working through the clutching construction given above now shows that $\pi^*\varphi^*(B)$ factors out of $\Sigma(D\!\!\!\!/)$, as stated in the text.

Appendix C: Anomalous Grassmannian Families

In this appendix we give the construction of the maps $\hat{\varphi}$ and $\hat{\Phi}$ which we used in Sect. 6 for various target spaces G_{pq}. We give two basic examples. The first example is a family for the two-dimensional chiral CP^n model with $B = T(CP^n)$, and $n \geqq 2$. Each member of the family is an instanton, that is, the family lies in a nontrivial component of $\mathscr{C}$. The second example is an anomalous family for the four-dimensional G_{pq} models with $p, q \geqq 4$. The family lies in the zero instanton sector. The method used in the second example can be used to construct two-dimensional G_{pq} families when $p, q \geqq 2$. We have not constructed $\hat{\varphi}$ and $\hat{\Phi}$ for the four-dimensional G_{2n} and G_{3n} models with $n \geqq 3, 2$, although these have a global obstruction.

(a) The Double Instanton. By representing CP^n as equivalence classes of nonzero $(n+1)$-tuples of complex numbers: $[(z_1, \ldots, z_{n+1})]$ one can define the "double instanton" family $\Delta: S^2 \times S^2 \to CP^n$ by

$$\Delta(s, t) = [(1, s+t, st, 0, \ldots, 0)]. \tag{129}$$

42 Using the above clutching construction this definition of $\Sigma(D)$ is equivalent to that given in [39,48]

Here s, t are complex numbers obtained from y, x by stereographic projection $S^2 \to C$ on each of the two S^2 factors. Thus we have only given Δ on the product of northern hemisphere patches $\mathscr{P}_N \times \mathscr{P}_N$, but Eq. (129) can be extended consistently to the other patches.

One can compute the twist of the family index

$$\nu = \int_{S^2 \times S^2} \Delta^* ch_2 TCP^n = n+1 \tag{130}$$

and the instanton number

$$k = \int_{S^2} \Delta_s^* ch_1 TCP^n = n+1 \tag{131}$$

for fixed s, where $\Delta_s \equiv \Delta(s, \cdot)$, using the following observations. The canonical line bundle $\gamma_1 \to CP^n$ is associated to the principal U(1) fibration $r: S^{2n+1} \to CP^n$ by the fundamental representation. This allows one to compute $\Delta^* c_1(\gamma_1)$ since the natural connection on S^{2n+1},

$$\Theta = \frac{1}{2\pi}\frac{i}{2}(\mathbf{z} \cdot d\mathbf{z}^* - \mathbf{z}^* \cdot d\mathbf{z}) \tag{132}$$

gives $r^* c_1 = d\Theta$. Here S^{2n+1} is considered as the set of complex $(n+1)$-tuples constrained to satisfy $\mathbf{z}^* \cdot \mathbf{z} = 1$. It is sufficient to compute $c_1(\gamma_1)$ because [25]

$$ch_1(TCP^n) = -(n+1)c_1 ,$$

$$ch_2(TCP^n) = \frac{n+1}{2} c_1^2 .$$

After some computation one then finds $\nu = (n+1)$, while the generalization

$$\Delta^{l_1 l_2}(s, t) = [(1, s^{l_1} + t^{l_2}, s^{l_1} t^{l_2}, 0, \ldots, 0)] \tag{133}$$

has $\nu = (n+1) l_1 l_2$ and instanton number

$$k = \int_{S^2} (\Delta_s^{l_1 l_2})^* ch_1 TCP^n = (n+1) l_2 . \tag{134}$$

(These "ring homomorphism" properties are a consequence of the external product construction of Sect. 4.)

If we consider the first factor of S^2 in the domain of Δ as the parameter two-sphere Y, then one can show that fixing Landau gauge

$$\partial_\mu \left(\frac{i}{2} \mathbf{z}^* \cdot \overleftrightarrow{\partial}_\mu \mathbf{z} \right) = 0 ,$$

which eliminates all but constant gauge transformations, defines a Hopf bundle $\tilde{P} \subset R^{-1}(Y) \subset \mathscr{A}_{1n}^2$. This is the bundle described in Sect. 6 since the U(1) subgroup of $\mathscr{G}_1^2$ which generates $\pi_1(\mathscr{G}_1^2)$ is the group of constant gauge transformations.

If one replaces $B = TCP^n$ by $B = \gamma_1$, the zero-modes of $\not{D}_y : E_y^+ \to E_y^-$ can be readily found for each $y \in Y = S^2$. One finds that $\not{D}_y$ always has exactly one zero-mode, and that the line in the Hilbert space $\mathscr{H}_y^+$ spanned by this zero-mode fits into a one-dimensional subbundle of the Hilbert bundle $\mathscr{H}^+$. In this case the family index just measures the twist of this line bundle. For $B = \gamma_1$ the twist is one and the "zero-mode-bundle" $\mathscr{H}_0^+$ is isomorphic to the associated Hopf bundle $B_1 \to S^2$.

(b) The Group-loop Family. We will construct a family $\mathscr{L}: S^2 \times S^4 \to G_{pq}$, $p, q \geqq 4$, by first constructing a family $\hat{\Phi}: S^3 \times S^4 \to V_{pq}$. Recall that the principal $\mathrm{U}(p)$ fibration $r: V_{pq} \to G_{pq}$ induces a $\mathscr{G}_p$-fibration $R: \mathscr{A}_{pq} \to \mathscr{C}_{pq}$ where $\mathscr{A}_{pq} = \{\mathrm{Maps}: S^4 \to V_{pq}\}$ and $\mathscr{G}_p = \{\mathrm{Maps}: S^4 \to \mathrm{U}(p)\}$. We will use this fibration to construct $\mathscr{L}$ from $\hat{\Phi}$ realizing the diagram (102) with $\hat{\varphi} = \mathscr{L}$.

We begin with a representative of the generator of $\pi_1(\mathscr{G}_p)$ for $p \geqq 4$. The nontrivial generator of $\pi_5((\mathrm{U}(4)))$ can be represented in terms of five anti-hermitian γ-matrices γ_i, $i = 1, \ldots, 5$ as

$$\boldsymbol{\xi} \mapsto \xi_0 + \sum_{i=1}^{5} \xi_i \gamma_i, \tag{135}$$

where $\boldsymbol{\xi}$ is a unit vector in R^6 [74]. We may compose Eq. (135) with a degree *two* map $S^1 \times S^4 \to S^5$ given by $(e^{i\psi}, \boldsymbol{\varrho}) \mapsto (\cos\psi, \sin\psi \boldsymbol{\varrho})$ (here $\boldsymbol{\varrho}$ is a unit vector in R^5) to get

$$(e^{i\psi}, \boldsymbol{\varrho}) \mapsto e^{-i\psi} P_+ + e^{i\psi} P_-, \tag{136}$$

where $P_\pm = \frac{1}{2}(1 \pm i\varrho \cdot \gamma)$ are projection operators: $P_\pm^2 = P_\pm$, $P_\pm P_\mp = 0$. However, we can write

$$(e^{-i\psi} P_+ + e^{i\psi} P_-) = (e^{-i\psi} P_+ + P_-)(P_+ + e^{i\psi} P_-). \tag{137}$$

Each of the two factors on the right-hand side of (137) must wind the same way since they are mapped into one another by the orientation-preserving involution $(e^{i\psi}, \boldsymbol{\varrho}) \to (e^{-i\psi}, -\boldsymbol{\varrho})$. We conclude that the generator of $\pi_1(\mathscr{G}_p)$ is represented by $(e^{i\psi}, \boldsymbol{\varrho}) \to P_+ + e^{i\psi} P_-$.[43] In fact, the group-loop is a $\mathrm{U}(1)$ subgroup of $\mathscr{G}_p$ since

$$(P_+ + e^{i\psi} P_-)(P_+ + e^{i\psi'} P_-) = P_+ + e^{i(\psi + \psi')} P_-.$$

This will be useful below.

We are now in a position to write an interesting family $\hat{\Phi}: S^3 \times S^4 \to V_{pq}$ for $p = q = 4$. One can trivially embed this into spaces with larger p and q. Let $A_0 = (1_{4\times 4}, 0_{4\times 4})$ and $B_0 = (0_{4\times 4}, 1_{4\times 4})$ be 4×8 matrices representing two standard elements in V_{44}. [Recall that V_{pq} can be regarded as the set of complex $p \times (p+q)$ matrices A satisfying $AA^\dagger = 1_p$.] Represent S^3 as 2-dimensional $\mathrm{SU}(2)$ matrices q. These act on the "2-vector"

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{138}$$

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43 More precisely, a representative α of the fundamental generator of $[S^1 \times S^4, \mathrm{SU}(N)]$ can be deformed into a map from S^5 to $\mathrm{SU}(N)$. Therefore [74, 77] the integral $w(\alpha) = \frac{1}{240\pi^2} \int_{S^1 \times S^4} \mathrm{tr}(\alpha^{-1} d\alpha)^5$, which is a homotopy invariant, is an integer. If we compose the map (135) with a degree one map $r: S^1 \times S^4 \to S^5$ we obtain a map $\beta: S^1 \times S^4 \to \mathrm{SU}(N)$ with $w(\beta) = 1$. For maps into Lie groups the composition of maps is homotopic to the pointwise product of maps. For the pointwise product of two maps $\gamma_{1,2}: S^1 \times S^4 \to \mathrm{SU}(N)$, w is a homomorphism: $w(\gamma_1 \cdot \gamma_2) = w(\gamma_1) + w(\gamma_2)$. Therefore $w(\alpha) = 1$ and if $w(\zeta) = 1$ for some ζ then ζ represents the fundamental generator. The map (136) has $w = 2$ so each of the factors in (137) must have $w = 1$. The fact that each factor maps into $\mathrm{U}(4)$ rather than $\mathrm{SU}(4)$ is irrelevant, again by the homomorphism property

Then each component of

$$(P_+ \otimes 1_{2\times 2} + P_- \otimes q)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \tag{139}$$

defines an element in V_{44} for every ϱ, q. That is, either component describes an interesting map (with the two components wrapping oppositely). Let $\hat{\Phi}$ be the upper component obtained by taking the "inner product" of the above 2-vector with the vector (1 0). If we project $\hat{\Phi}$ with R defined above the parametrization of the resulting family by S^3 is partially redundant.

Indeed, we can consider S^3 as the total space of a principal Hopf bundle $\tilde{P} \subseteq \mathscr{A}_{pq}$. The U(1) action on SU(2), $e^{i\psi} \cdot q \equiv e^{i\sigma_3 \psi} q$,[44] is represented by the principal U(1) action of the group-loop on $\hat{\Phi}$. That is, if $q \to e^{i\sigma_3\psi} q$, then $\hat{\Phi}$ is rotated to

$$\begin{aligned}(1\ \ 0)(P_+ \otimes 1_{2\times 2} + P_- \otimes e^{i\sigma_3\psi} q)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} &= (P_+ + e^{i\psi} P_-)(1\ \ 0)(P_+ \otimes 1 + P_- \otimes q)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \\ &= (P_+ + e^{i\psi} P_-)\hat{\Phi}\,. \end{aligned} \tag{140}$$

Thus $\mathscr{L} = R \circ \hat{\Phi}$ is unambiguously parametrized not only by $\tilde{P}$ in $\mathscr{A}_{pq}$ but also by its projection $Y \subset \mathscr{C}_{pq}$, where Y is a copy of S^2. The fibration $R : \tilde{P} \to Y$ is the Hopf fibration with a *principal* U(1) action given be the nontrivial loop in $\mathscr{G}_p$.

We claim that $\mathscr{L} : S^2 \times S^4 \to G_{44}$ and its embeddings into higher G_{pq} are anomalous families in the zero instanton sector. Thus we need to compute,

$$\nu = \int_{S^2\times S^4} ch_3 \mathscr{L}^* TG_{pq} = -(p+q)\int_{S^2\times S^4} ch_3 \mathscr{L}^* \gamma_p \tag{141}$$

and[45]

$$k = \int_{S^4} ch_2 \mathscr{L}^* TG_{pq} = (q-p)\int_{S^4} ch_2 \mathscr{L}^* \gamma_p\,. \tag{142}$$

As in the case of the double instanton we note that γ_p is associated to the principal U(p) bundle $r : V_{pq} \to G_{pq}$ by the defining representation and V_{pq} has a natural connection

$$\Theta = \tfrac{1}{2}(dAA^\dagger - AdA^\dagger) \tag{143}$$

which allows one to compute $\Omega = d\Theta - \Theta^2$ [46] and hence $r^* ch_k \gamma_p = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \operatorname{tr} \Omega^k$.

A little computation shows that $\hat{\Phi}^* \operatorname{tr} \Omega^2 = 0$ so that $\mathscr{L}$ is in the zero instanton sector and,

$$\hat{\Phi}^* \operatorname{tr} \Omega^3 = -\frac{9}{64} \sin^4 \frac{\theta}{2} (\sin\theta d\theta d\phi) \operatorname{tr}(\gamma \cdot d\varrho)^4 \gamma \cdot \varrho \tag{144}$$

so that for $\mathscr{L}$, and $B = T(CP^n)$, the family index is $\nu = (p+q)$.

---

44 Note this is a *left* action. We do this since for our representation of the Stiefel manifold $r : V_{pq} \to G_{pq}$ projects by a *left* U(p) action

45 For G_{44} (142) is trivially zero. However this is not true for arbitrary G_{pq}, or for bundles other than $T_c G_{pq}$

46 The minus sign is a consequence of the left U(p) action

An explicit solution of the zero modes of the Dirac operator would show that for at least one point y on $Y=S^2$ the lowest eigenvalue $\lambda_1(y)$ of $\not{D}_y^\dagger \not{D}_y$ drops to zero (Fig. 2.2), and that around this point the eigenmodes u_1 and v_1 have a relative twist, as described in Sect. 2.

Appendix D: The Homotopy Type of $\bar{G}/\bar{H}$

In this appendix we prove that $\bar{G}/\bar{H}$ has the same homotopy type as G/H, where G is a compact Lie group and H is a subgroup.

We need the Cartan decomposition [70, 78] which in this case is

$$\mathcal{L}(\bar{G})=\mathcal{L}(G)\oplus i\mathcal{L}(G)\,. \tag{145}$$

If G is a compact Lie group there is a diffeomorphism

$$\mathcal{L}(G)\times G\to\bar{G} \tag{146}$$

with $\mathrm{Exp}(i\mathcal{L}(G))\cdot G=\bar{G}$.

Lemma 1. *G/H deformation retracts to G/H.*

Proof. Define maps

$$\bar{G}/H \underset{i_1}{\overset{r}{\rightleftarrows}} G/H \tag{147}$$

by

$$\begin{aligned} i_1(gH)&=gH\,,\\ r(\bar{g}H)&=gH\,, \end{aligned} \tag{148}$$

where in (148) $\bar{g}$ has the unique decomposition $\bar{g}=\mathrm{Exp}(p)g$. We also see that $r\circ i_1=1$, and $i_1\circ r\cong 1$ because we can define the homotopy

$$F(\bar{g}H,t)=\mathrm{Exp}(tp)\cdot gH \tag{149}$$

as was to be shown. □

Now we have a fiber bundle

$$\pi:\bar{G}/H\to\bar{G}/\bar{H}\,, \tag{150}$$

where $\pi(\bar{g}H)=\bar{g}\bar{H}$ with fiber $\bar{H}/H$ and structure group $\bar{H}$. Note that the fiber is contractible by the Cartan decomposition

$$\bar{H}/H\overset{\text{diff}}{\approx}\mathrm{Exp}(i\mathcal{L}(H))\,. \tag{151}$$

Since $\bar{G}/\bar{H}$ is triangulable, and hence a CW complex, and $\bar{H}/H$ is contractible, there is a section

$$\bar{G}/H \underset{s}{\overset{\pi}{\rightleftarrows}} \bar{G}/\bar{H}\,. \tag{152}$$

Lemma 2. $s\circ\pi\cong 1$.

Proof. We use induction on the dimension of the cell complex.

On the zero cells $\{v\}$ we have a diffeomorphism

$$\bar{G}/H|_{\{v\}} \xrightarrow{\varphi} \{v\} \times \bar{H}/H . \tag{153}$$

Since $\bar{H}/H$ is contractible we can choose $F: \bar{H}/H \times I \to \bar{H}/H$ with

$$\begin{aligned} &F(\bar{h}H, 0) = \bar{h}H , \\ &F(\bar{h}H, 1) = \pi_2(\varphi(s(v))) , \end{aligned} \tag{154}$$

where π_2 is the projection $\pi_2: \{v\} \times \bar{H}/H \to \bar{H}/H$. Define

$$\bar{F}(\bar{g}H, t) = \varphi^{-1}(v, F(\pi_2 \circ \varphi(\bar{g}H), t)) . \tag{155}$$

This defines the homotopy on the 0-cells.

Assume we have a continuous map

$$\bar{F}: \bar{G}/H|_{K^{n-1}} \times I \to \bar{G}/H \tag{156}$$

with

$$\begin{aligned} &\bar{F}(\bar{g}H, 0) = \bar{g}H , \\ &\bar{F}(\bar{g}H, 1) = s(\bar{g}\bar{H}) = s\pi(\bar{g}H) , \end{aligned} \tag{157}$$

where K^{n-1} is the $(n-1)$-skeleton, which satisfies the further property that if $\pi(\bar{g}H)$ lies in some cell E, then the curve $\pi(\bar{F}(\bar{g}H, t))$ remains in $\bar{E}$. (We will use this condition.)

Let E be a closed n-cell, then

$$E \cap K^{n-1} = \partial E = S \approx S^{n-1} \tag{158}$$

is a union of closed $(n-1)$-cells. Choose a diffeomorphism

$$\bar{G}/H|_E \xrightarrow{\varphi} E \times \bar{H}/H \tag{159}$$

and define

$$\mathscr{F}: (S \times \bar{H}/H \times I) \cup (E \times \bar{H}/H \times \{0\}) \cup (E \times \bar{H}/H \times \{1\}) \to E \times \bar{H}/H \tag{160}$$

as follows. For $0 < t < 1$, $x \in S$,

$$\mathscr{F}(x, \bar{h}H, t) = \varphi \bar{F}(\varphi^{-1}(x, \bar{h}H), t) . \tag{161}$$

Since $\pi(\varphi^{-1}(x, \bar{h}H))$ is in some $(n-1)$-cell, $\bar{F}(\varphi^{-1}(x, \bar{h}H), t)$ remains in that cell so the composition with φ is defined. Also define for $e \in E$

$$\begin{aligned} &\mathscr{F}(e, \bar{h}H, 0) = (e, \bar{h}H) , \\ &\mathscr{F}(e, \bar{h}H, 1) = (e, \pi_2 \circ \varphi \circ s \circ \pi \circ \varphi^{-1}(e, \bar{h}H)) . \end{aligned} \tag{162}$$

This gives a map

$$\mathscr{F}: \partial D^{n+1} \times \bar{H}/H \to E \times \bar{H}/H \tag{163}$$

for some $(n+1)$-cell D. Since $E \times \bar{H}/H$ is contractible there is an extension

$$\hat{\mathscr{F}}: D^{n+1} \times \bar{H}/H \to E \times \bar{H}/H \tag{164}$$

$$
\begin{tikzcd}
& \bar{G}/H \arrow[dl, shift right, "r"'] \arrow[d, shift right, "\pi"'] \\
G/H \arrow[ur, shift right, "i_1"'] \arrow[dr, shift left, "i"] & \\
& \bar{G}/\bar{H} \arrow[ul, shift left, "R"] \arrow[u, shift right, "s"']
\end{tikzcd}
$$

and defining (take $D^{n+1}=E\times I$)

$$\hat{\bar{F}}=\varphi^{-1}\hat{\bar{\mathscr{F}}}\varphi \tag{165}$$

on $\bar{G}/H|_E\times I$ we can extend $\bar{F}$ in this way on all the n-cells satisfying the appropriate conditions. Thus there is a homotopy of $s\circ\pi$ with 1. □

Consider the maps in Fig. D.1. We have $i(gH)=g\bar{H}$ and $R=r\circ s$. Note that $i=\pi\circ i_i$. Thus $R\circ i: G/H\to G/H$ satisfies $R\circ i=r\circ s\circ\pi\circ i_1\cong r\circ i_1=1$ and $i\circ R:\bar{G}/\bar{H}\to\bar{G}/\bar{H}$ satisfies $i\circ R=\pi\circ i_1\circ r\circ s\cong\pi\circ s=1$. Note that R is *not* a deformation retract, but it is a homotopy equivalence, which is sufficient for our purposes.

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