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Hamiltonian Interpretation of Anomalies

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Abstract. A family of quantum systems parametrized by the points of a compact space can realize its classical symmetries via a new kind of nontrivial ray representation. We show that this phenomenon in fact occurs for the quantum mechanics of fermions in the presence of background gauge fields, and is responsible for both the nonabelian anomaly and Witten's SU(2) anomaly. This provides a hamiltonian interpretation of anomalies: in the affected theories Gauss' law cannot be implemented. The analysis clearly shows why there are no further obstructions corresponding to higher spheres in configuration space, in agreement with a recent result of Atiyah and Singer.

1. Introduction

We say we have an "anomaly" when a symmetry of a classical field theory is not reflected at all in those of the corresponding quantum theory, or more precisely when the full set of classical symmetries cannot be preserved in any of the many possible quantization schemes. When the symmetry in question is an ordinary one such as scale or chiral invariance, we have a straightforward interpretation for the effects of the anomaly in terms of states in Hilbert space: the symmetry in question is absent from the full theory. Coupling constants run; tunneling events do not conserve axial charge. These results are surprising, but not fatal to the theory.

The case of gauged symmetries is very different. Gauge symmetries are properly to be thought of as not being symmetries at all, but rather redundancies in our description of the system [1]. The true configuration space of a $(3+1)$-dimensional gauge theory is the quotient $\mathscr{C}^3=\mathscr{A}^3/\mathscr{G}^3$ of gauge potentials in $A_0=0$ gauge modulo three-dimensional gauge transformations[1]. When gauge degrees of freedom become anomalous, we find that they are not redundant after all.

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1 We will sometimes omit the superscript 3

Recently it has become clear that gauge theories with fermion display three different kinds of anomalies, all related to the global topology of the four-dimensional configuration space $\mathscr{C}^4$ by the family index of the Dirac operator $\not{D}^4$. These are the axial U(1) anomaly [the "$\pi_0(\mathscr{G}^3)$ anomaly"], Witten's SU(2) anomaly [2] [from $\pi_1(\mathscr{G}^3)$], and the nonabelian gauge anomaly [3] [from $\pi_2(\mathscr{G}^3)$]. The diversity of the manifestations of these anomalies seems to belie their common origin, however. In the first case we find particle production in the presence of instanton fields [4], breaking of a global symmetry, and no problem with gauge invariance. In the second we find no problem with chiral charge, but instead a nonperturbative failure of *gauge* symmetry, while in the latter the same thing occurs even perturbatively.

What is going on? In the following sections we will attempt to give a hamiltonian picture of the gauge anomalies as simple as the axial anomaly's particle-production interpretation. Essentially the answer will be that in anomalous theories we cannot formulate any Gauss law to constrain the physical states. Along the way we will try to make the above differences a bit less mystifying than they seem in the lagrangian picture. They will all turn out merely to reflect a simple fact about codimension: removing a point from a manifold can sever it into disconnected pieces only if its dimension equals one.

The aim of this paper is expository. We will not find any previously unknown anomalies, but instead will give an approach to understanding them which we have found illuminating. Our point of departure was a remark in [2] which we have generalized to embrace the anomaly of [3] as well[2]. In Sect. 2 we set up our framework and establish our criterion for a global anomaly to exist. In Sects. 3 and 4 we verify the criterion for the cases of [2, 3] respectively, making use of known results from the lagrangian approach. In Sect. 5 we conclude with remarks.

2. Setting Up

It may seem difficult to arrive at a physical interpretation of a problem which renders a gauge theory nonsensical. We know, however, that anomalies do not themselves originate in the gauge sector. We can therefore attempt to quantize a given theory in two steps, starting with the matter fields; at the intermediate point we will have a *family* of quantum systems parametrized by the space of classical background gauge field configurations $\mathscr{A}^3$. Furthermore, the whole collection should realize the classical gauge symmetry *via* unitary operators. The situation is not quite like the usual case of symmetry in quantum mechanics [6], however, since the transformations in question act both on Hilbert space $\mathscr{H}$ and on background configuration space $\mathscr{A}$. They are indeed *bundle maps* of a *family* of Hilbert spaces, $\mathscr{H} \xrightarrow{\pi} \mathscr{A}$. A simple example of such a situation is an ordinary quantum mechanics problem with a Schrödinger particle interacting with a classical rotor degree of freedom $\bar{\varphi}$: for fixed position of the rotor the system has no rotational symmetry, but the full family of theories does have an invariance expressed as a set of isometries, $U_\alpha : \mathscr{H}_{\bar{\varphi}} \to \mathscr{H}_{\bar{\varphi}+\alpha}$.

2 We were also influenced by the work of Rajeev [5]

The notion of families of quantum systems has recently appeared in several papers [7–9]. The phenomenon of "quantum holonomy" discussed in these papers will be crucial to our analysis.

When an ordinary quantum system realizes its classical symmetries, however, it need not do so in the obvious way, by a unitary action of the symmetry group G on $\mathscr{H}$. Instead, Wigner showed [6] that in general we can demand only that $\mathscr{H}$ furnish a projective, or ray, representation of G. When G is a multiply-connected topological group, $\mathscr{H}$ will thus in general have irreducible sectors transforming under $\tilde{G}$, the universal cover of G. This is, of course, the situation with the rotation group, where $\mathscr{H}$ has a sector of odd fermion number transforming as a "double-valued representation of $O(3)$," i.e., as a true representation of spin(3).

The same sort of thing can occur in parametrized families of quantum systems. As a simple example, let us return to the case of the Schrödinger particle and rotor. Constraining the particle to lie on a circle, we have the hamiltonian

$$H=-(\partial_\varphi)^2+b[\delta(\varphi-\tfrac{1}{2}\bar{\varphi})+\delta(\varphi-\tfrac{1}{2}\bar{\varphi}+\pi)]\,, \tag{1}$$

which is continuous for $\bar{\varphi}\in S^1$. For each $\bar{\varphi}$ half of the energy eigenstates of this system are odd under the translation $\varphi\to\varphi+\pi$. Now let $\bar{\varphi}$ vary, and for each value choose a real energy eigenfunction $\psi_{\bar{\varphi}}$ with fixed eigenvalue ε. For the odd states it will be impossible to choose $\psi_{\bar{\varphi}}$ smoothly; as $\bar{\varphi}$ completes a full circuit ψ goes over to its negative. In other words, the odd energy eigenspaces each form twisted line bundles over the parameter space S^1.

Let us attempt to find a unitary action of the symmetry group U(1) on a given odd energy eigenspace $\mathscr{H}_n$ of $\mathscr{H}$. Clearly U_g must map $\mathscr{H}_n$ to itself, but at each point a decision must be made: there is no canonical choice of sign. This raises the possibility that *no* smooth choice may exist. Indeed, any ordinary unitary action of the symmetry group U(1) must take any given ψ at $\bar{\varphi}=0$ and give a nonzero section of $\mathscr{H}$. Since no such section exists, this quantum system cannot realize its U(1) symmetry via an ordinary unitary action.[3] More formally, if the Hilbert bundle $\mathscr{H}$ admits an action of $G=\mathrm{U}(1)$ which projects to the usual action of U(1) on the parameter space S^1, we say it is a "G-bundle" [10]. In this case $\mathscr{H}$ reduces to a new bundle $\bar{\mathscr{H}}$ defined on the quotient $S^1/\mathrm{U}(1)=$point, and so is trivial. That is, any nontrivial bundle on the base (in our case an energy eigenspace) is not a G-bundle.

If the parameter space consists of many G-orbits it is sufficient to show that any one is nontrivial in order to rule out an ordinary G-action. In any case the key feature which makes possible the unremovable minus sign in the group action is the fact that the orbits are copies of G, which is not simply-connected.

Suppose now that we wish to quantize the rotor degree of freedom as well. The wavefunctions of the complete system can then be taken as complex functions of both φ, the particle position, and $\bar{\varphi}$, the rotor position. Alternatively, however, they can be taken as functions from S^1 into the *space of functions* of φ, that is, as *sections* of the Hilbert bundle $\mathscr{H}$. We will call the complete Hilbert space

3 The reader may well object that we have simply chosen a foolish normalization for the U(1) generator. Indeed the model has another classical $\widetilde{\mathrm{U(1)}}$ symmetry which *is* realized in the usual way. We will return to this point

$\dot{\mathscr{H}} \equiv \Gamma(\mathscr{H})$, the space of sections. It has a subspace spanned by the even eigenfunctions, and on this subspace we can define the unitary operator $\mathscr{U}_\alpha$ by $\mathscr{U}_\alpha \psi = \psi'$, where $\psi'_\varphi = U_\alpha \psi_{\varphi-\alpha}$. On the full $\dot{\mathscr{H}}$, however, we cannot in general define any $\mathscr{U}$.

This is the problem with gauge theory. When we quantize matter in the presence of background gauge fields, the resulting family of quantum theories in general realizes its classical gauge symmetry *via* a perfectly good ray representation. As far as the fermions are concerned there is *nothing wrong* with gauge symmetry. The phases in the ray representation are topologically unremovable; they prevent us from implementing the symmetry at all in the fully quantized theory, and in particular from imposing the constraint of gauge-invariance on the physical quantum states. Equivalently, in the temporal-gauge quantization of gauge theory [11, 12] we require that physical states obey

$$\left(\mathrm{Tr}\left\{T_\alpha \mathbf{D}\cdot\left(\frac{\delta}{\delta \mathbf{A}}\right)\right\} - i\psi^\dagger t_a \psi\right)\Psi = 0\,, \tag{2}$$

which is the infinitesimal version of

$$\Psi[\mathbf{A}^g] = U_g \Psi[\mathbf{A}]\,. \tag{3}$$

But this just says that physical elements of $\dot{\mathscr{H}}$ must be *equivariant* sections of $\mathscr{H}$, or in other words that they must define sections of the reduced bundle $\bar{\mathscr{H}}$ over the true configuration space $\mathscr{C}$. If U_g is only projectively defined, then $\bar{\mathscr{H}}$ is not defined and this requirement makes no sense. If, moreover, the phases which spoil U_g have global topological content and so cannot be removed, then there is no cure for the problem. The theory is then anomalous.

A few remarks are in order before closing this section. We have established the existence of a nontrivial ray representation in a toy model by solving it exactly and noting the behavior of various eigenspaces of the energy globally over the parameter space. This brute-force approach will of course have to be replaced by something more powerful in field theory. Having established that at least one subbundle of $\mathscr{H}$ twists on at least one orbit, we conclude that in the full theory the symmetry is "anomalous," *i.e.*, it cannot be implemented as a true representation. Since the energy eigenspaces were all one-dimensional the only possible twist was the Möbius twist over a noncontractible circle in the symmetry group G. More generally we have to look for twists of higher-dimensional subbundles of $\mathscr{H}$, which will appear over higher-dimensional subspaces of G. In gauge theory, however, it will turn out to be enough to obtain a G-action on the vacuum subbundle, which is one-dimensional, and so there will be no anomalies due to obstructions beyond the first.

One might object that quantum mechanics involves not the real numbers but the complex, and that there are no interesting complex bundles over S^1. We will answer this objection in two different ways in the sequel. For the $\pi_1(\mathscr{G}^3)$ anomaly, it is important for the G-action to preserve the real structure, while for the $\pi_2(\mathscr{G}^3)$ anomaly we indeed must consider two-spheres in $\mathscr{G}$ (as the name implies). The former case resembles the obstruction to placing a spin structure on a space [13], since nontrivial $\pi_1(\mathscr{G}^3)$ implies nontrivial two-cells in $\mathscr{C}^3$, while the latter resembles the obstruction to defining a spin^c structure, since it involves an *integer* (not Z_2) invariant and three-cells in $\mathscr{C}$.

3. Fermions

We begin for simplicity with the theory of [2], an SU(2) gauge theory with a single isodoublet of Weyl fermions. This theory has a Euclidean Dirac operator which is strictly real [2]. Thus the energy eigenstates of the first-quantized theory can be chosen real, and the full second-quantized Hilbert bundle $\mathscr{H}$ has a real structure.[4] Furthermore, the representation matrix appearing in Gauss' law is real, and so the required $\mathscr{G}$-action must respect this real structure. As in our example, it will now suffice to show that the vacuum subbundle, say, is a Möbius bundle over any gauge orbit in order to establish the anomaly.

At each point of gauge configuration space we must now quantize fermions in the given background. This is not, of course, the usual procedure, in which one quantizes *free* fermions and treats gauge interactions perturbatively. Since the SU(2) anomaly is nonperturbative, we must include the gauge fields from the start.

At the first-quantized level we encounter no difficulties. The Hilbert bundle is trivial, and the group action is $U_g v = v'$, where $v'(x) = g(x)v(x)$. Thus we expect any problems to come from second quantization, that is, from the definition of the Dirac sea. Accordingly let us focus our attention first on the vacuum subbundle $\mathscr{H}_0$; we will see that indeed once its $\mathscr{G}$-action has been defined there will be no further problems. Now the Dirac vacuum is defined as the state in Fock space in which all negative-energy states are filled. Since $\not{D}^3$ is gauge-covariant, all of its eigenvalues ε_i are gauge-invariant and $\mathscr{H}_0$ is mapped to itself by any gauge transformation (as indeed is any $\mathscr{H}_\varepsilon$ filled to another Fermi level ε). Actually, though, $\mathscr{H}_0$ is unambiguously defined only on the subset $\mathscr{A}'$ where none of the ε_i vanish. This turns out to be a small but crucial point, since unlike $\mathscr{A}$, which is contractible, $\mathscr{A}'$ has nontrivial topology and so admits the possibility that the vacuum $\mathscr{H}_0$ can be twisted.

To establish the twist we combine the result of Berry [7], which relates twist to degeneracies, with the result of Witten [2], which establishes those degeneracies. Our argument is summarized in Fig. 1. Following Witten, we begin with the generator g^4 of $\pi_4(\mathrm{SU}(2))$ and any point $A_{(0)}$ of $\mathscr{A}^4$, the space of four-dimensional gauge potentials. Take $A_{(0)\mu}(x,t) \equiv 0$. Since $\mathscr{A}^4$ is connected, we can join $A_{(0)}$ to $[A_{(0)}]^{(g^4)}$ by a smooth path $A_{(\tau)}$, $\tau = 0$ to 1. For each τ we now transform $A_{(\tau)}$ by a time-dependent gauge transformation $g_{(\tau)}$ to put it into temporal gauge; call the result $\mathbf{A}'_{(\tau)}$. In particular, $g_{(1)}$ is just $(g^4)^{-1}$, so instead of an open path of vector potentials each periodic in time we now have a closed loop of temporal-gauge histories, each of which ends at $\mathbf{A}'_{(\tau)}(t=\infty) = [0]^{g_{(\tau)}(t=\infty)}$, a three-dimensional gauge transform of $\mathbf{A}'_{(\tau)}(t=-\infty) \equiv 0$.

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4 In particular, the vacuum subbundle $\mathscr{H}_0$ gets a real structure

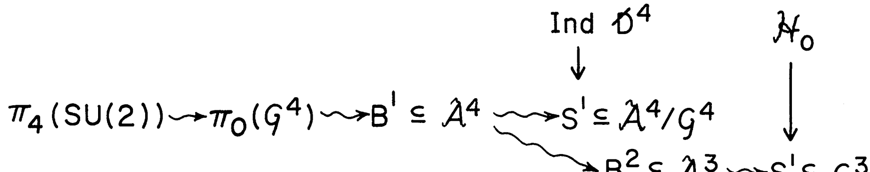

Fig. 1. Summary of the steps in Sect. 3

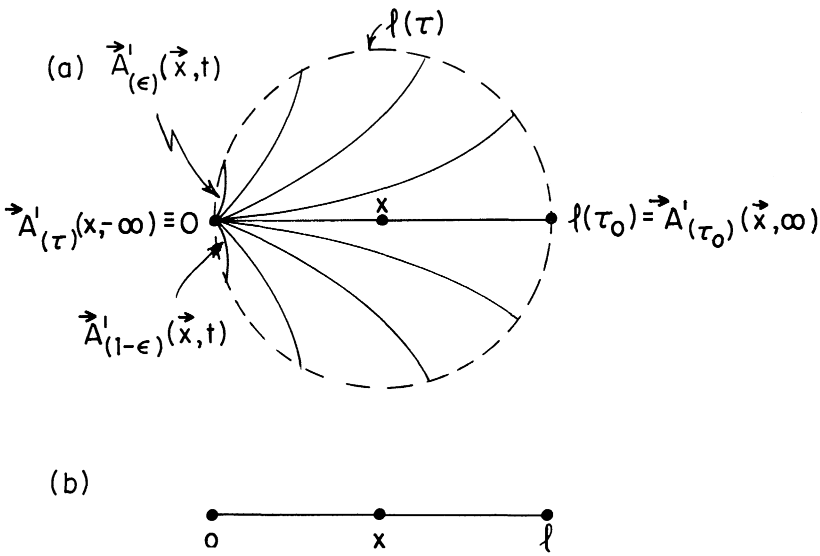

Fig. 2a and b. Disk in $\mathscr{A}^3$ associated to **(a)** SU(2) anomaly, **(b)** axial anomaly

The set of $\mathbf{A}'_{(\tau)}(t)$, $-\infty < t < \infty$, $0 < \tau < 1$ thus forms a disk in $\mathscr{A}^3$ whose rim is a loop $\ell(\tau)$ of gauge transforms of zero (see Fig. 2). Each $\ell(\tau)$ is in $\mathscr{A}'$, since $\not{D}_0$ has no zero modes on compactified space, and so we can restrict $\mathscr{H}_0$ to ℓ. We claim that $\mathscr{H}_0|_\ell$ is in fact twisted. For this to happen, there must be a point x on the disk excluded from $\mathscr{A}'$; that is, there must be a degeneracy at x.

The presence of such a degeneracy follows at once from Witten's argument [2]. From the mod2 index theorem, $\not{D}^4$ must have a pair of zero modes at some $A_{(\tau_0)}$, and hence for the corresponding $\mathbf{A}'_{(\tau_0)}$ as well. We can take these to be eigenstates $\phi_\pm$ of chirality. Taking Witten's argument one step further, if we choose each $\mathbf{A}'_{(\tau)}$ to vary slowly in t then $\phi_\pm$ must be slowly-varying functions of time times eigenfunctions $\eta^t_\pm$ of the Dirac hamiltonian $H_t \equiv \gamma_0 \not{D}^3_{(\tau_0,t)}$. The energy eigenvalues must pass through zero[5] at some t_0, since $\phi_\pm$ are normalizable zero modes of the Euclidean $\not{D}^4$. Then $x=(\tau_0, t_0)$. Moreover, η^t_- has a CT-conjugated partner of opposite energy and chirality ζ^t_+, leading to the conical arrangement of left-handed energy eigenvalues shown in Fig. 3a. The number of these crossings will be equal, modulo two, to the number of Weyl isodoublets present.

When we second-quantize, the Fermi vacuum ray at each point $\ell(\tau)$ is the ray in Fock space with all negative-energy states filled. Choose a state $|0\rangle_0$ in this ray at 0. We can now attempt to adduce a nonvanishing section of $\mathscr{H}_0|_\ell$ by evolving $|0\rangle_0$ in the slowly-varying backgrounds $\mathbf{A}'_{(\tau)}(t)$ for each τ. By the quantum adiabatic theorem [14], the final state will almost everywhere be almost pure vacuum, and we can project to $\mathscr{H}_0$. This trick fails, however, at y. Here the adiabatic evolution passes through the vertex of the cone in Fig. 3a, producing the particle associated to η_+ and the antiparticle associated to ζ_+. The resulting state has vanishing

5 Here is where the argument fails for the line bundle $\mathscr{H}_\varepsilon$ filled up to some level other than zero, since the index theorem tells us nothing about ε-crossings

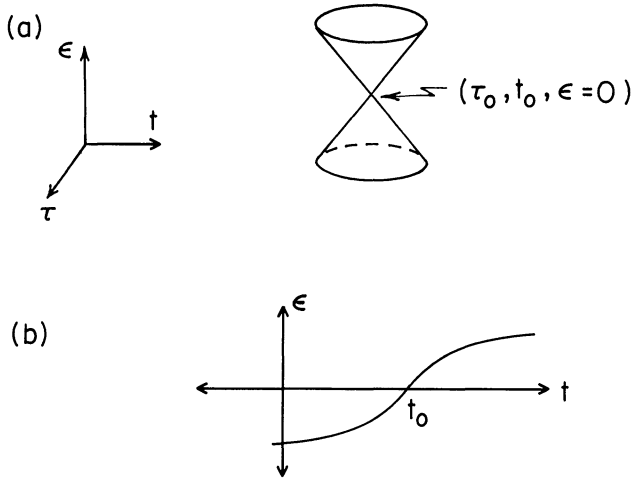

Fig. 3a and b. Eigenvalue behavior near $x=(r_0, t_0)$ for **(a)** SU(2) anomaly, **(b)** axial anomaly

projection to $\mathscr{H}_0$. That is, the putative section "rolls over" near y, out of the plane of $\mathscr{H}_0$ and into an orthogonal direction provided by $\mathscr{H}_{\text{pair}}$. When projected to $\mathscr{H}_0$, it vanishes at y. This reflects the twist of $\mathscr{H}_0$.

This can all be made more precise using the result of [7]: For a loop of real hamiltonians, adiabatic transport around the loop returns to a state which is $(-1)^n$ times the original, if the loop encircles n simple degeneracies. (Note that the adiabatically-continued wavesections of this paragraph and one preceding were chosen only for convenience. Once we know that one section twists, we know they all do.)

While Berry's result is elegant, we have given the pair-production picture as well in order to point up the physical similarities between the present case and the axial anomaly (see Figs. 2b and 3b). Usually the former is thought of in terms of phases, the latter in terms of particle production, but we can see that this really just a matter of emphasis. Particle production is crucial to *both*.[6] In the case of the SU(2) anomaly, however, it occurs only for a special value τ_0; since it is not the generic behavior we do not find an important effect on the vacuum structure. Nevertheless, production is important, as it gives the sign twist which characterizes the anomaly. In the axial anomaly, on the other hand, it is production which is important in suppressing vacuum tunneling [11] while the phases do not matter. After all (Fig. 2b), in this case the rim of the disk is two *points* and so admits no twisted bundles.

Another important qualitative defference between the anomalies also comes from the codimension of x. In the SU(2) case, the level crossing had to be absent for points τ not exactly on τ_0. For this to happen η^t_+ to have a partner ζ^t_+, leading to zero net chirality production for the SU(2) anomaly. No such considerations apply

6 J. Goldstone has pointed out to us that our argument for the SU(2) anomaly is similar to one of his, summarized in [12], in which particle production also plays a key role

in the axial anomaly, and indeed (Fig. 3b) only η^t_+ or ζ^t_+, not both, appears. Thus we get net production of chirality and a global symmetry is broken.

We can summarize the above discussion mathematically [15] by stating that the $\pi_0(\mathscr{G}^3)$ anomaly is given by the simplest invariant of the family index Ind $\not{D}^4$ (a real virtual bundle over $\mathscr{C}^4$), namely its net dimension. Histories $\mathbf{A}'(t)$ for which this is nonzero will have particle production with net change in chirality. The dimension is invariant to perturbations, so production is generic. The $\pi_1(\mathscr{G}^3)$ anomaly comes from the next invariant of Ind $\not{D}^4$, its twist over circles in $\mathscr{C}^4$. We have shown (Fig. 1) that this twist equals that of $\mathscr{H}_0$ over circles in $\mathscr{A}^3$ and so gives the obstruction to finding the $\mathscr{G}^3$-action needed to quantize the theory correctly.[7] For paths in $\mathscr{A}^3$ for which the lowest invariant vanishes, particle production gives no net chirality change and so comes from points where the null space of $\not{D}^4$ jumps; *i.e.*, production is not generic. No further invariants of $\not{D}^4$ are relevant to $\mathscr{H}_0$.

Unlike the $\pi_0(\mathscr{G}^3)$ anomaly, which is an integer, the $\pi_1(\mathscr{G}^3)$ anomaly can be cancelled by adding a second Weyl fermion of either chirality. Now a second pair state becomes degenerate with the vacuum and, by Berry's theorem, there is no sign change as we traverse ℓ.

We can also attempt to evade the anomaly by passing to the cover $\tilde{\mathscr{G}}^3$, as suggested in an earlier footnote. We now get a true $\tilde{\mathscr{G}}$-action on $\mathscr{H}_0$ provided we map the nontrivial element $\hat{g}$ covering the identity to the unitary operator -1, and hence a $\tilde{\mathscr{G}}$ representation on $\dot{\mathscr{H}}_0$ as in Sect. 2. If we take Gauss' law to mean that wave sections are equivariant under $\tilde{\mathscr{G}}$, however, we must in particular require that they be invariant under $\hat{g}$. Instead, all states have eigenvalue -1 under $\mathscr{U}_{\hat{g}}$! That is, we have succeeded only in defining on $\dot{\mathscr{H}}$ a ray realization of $\mathscr{G}$ of the type studied in [6]. All states of $\dot{\mathscr{H}}$ are "fermionic." This construction recovers the formulation of the anomaly given in [2].

We have suggested that the anomaly is a second-quantization phenomenon, preventing us from finding an appropriate family of vacuum states. To go further, let us suppose that we have cancelled the obstruction and so have a well-defined $\mathscr{G}$-action on $\mathscr{H}_0$. To get a $\mathscr{G}$-action on the rest of $\mathscr{H}$, we proceed as usual to define the Fock space creation operators $a^{\dagger i}_A$ on $\mathscr{H}_A$ associated to the eigenfunctions η^i_A with energy $\varepsilon^i_A > 0$. (Similarly, b^i_A creates the mode η^j_A with energy $\varepsilon^j_A < 0$, and we reinterpret b^i_A as a destruction operator.) We can choose η^i_A smoothly in an open set V in $\mathscr{A}^3$, and since there is an unambiguous $\mathscr{G}$-action on first-quantized states we can demand $\eta^i_{(A^g)}(x) = g(x)\eta^i_A(x)$.

Now define

$$U_g(a^{\dagger i_1}_A \dots a^{\dagger i_k}_A)|0\rangle_A \equiv a^{\dagger i_1}_{(A^g)} \dots a^{\dagger i_k}_{(A^g)} U_g|0\rangle_A\,, \tag{4}$$

for any vacuum state $|0\rangle_A$, and similarly with the $b^\dagger$. This definition is not arbitrary, but rather is dictated by the requirement that the quantum field built from a and $b^\dagger$ have the same unambiguous transformation law as its first-quantized counterpart. Since there are no phase choices to make, there is no possibility of any obstruction to making them smoothly. Equation (4) defines a $\mathscr{G}$-action on a dense subspace of $\mathscr{H}_A$, $A \in V$. Furthermore, if on some other patch V_1 we choose a different

7 Since these twists are pure torsion, we cannot establish this fact by the use of real characteristic classes

orthonormal expansion η^j_{A1} (still equivariant under $\mathcal{G}$), we end up defining the same $\mathcal{G}$-action for $\mathcal{H}_A$, $A \in V \cap V_1$. Even as we approach a degenerate point, where $\mathcal{H}_0$ is not defined, we can extend this definition. Thus a $\mathcal{G}$-action on $\mathcal{H}_0$ extends without further difficulty to $\mathcal{H}$, and thence as we have seen to the full $\dot{\mathcal{H}}$.

We have therefore found that the higher invariants of $\mathrm{Ind}\,\not{D}^4$, like the lowest one, are irrelevant to implementing Gauss' law. All that matters is the twist of the index over circles. For the case to be discussed in Sect. 4, this agrees with the result of Atiyah and Singer [16], who use the path-integral formulation. It disagrees, however, with [15].

4. The Nonabelian Anomaly

The nonabelian anomaly presents almost no new features. An example of an affected theory is massless QCD with a triplet of left-handed Weyl "quarks." Since $\pi_5(\mathrm{SU}(3)) = \pi_1(\mathcal{G}^4) = Z$, we can consider the loop [17] in $\mathcal{A}^4$ given by transforming zero with each one of the noncontractible loop of $4d$ gauge transformations given by the generator g^5. Again following the procedure outlined in Fig. 1, we then arrive at a three-ball in $\mathcal{A}^3$ whose boundary S^2 consists of three-dimensional gauge transformations of zero. Again by the family index theorem, D^4 generically has a pair of zero modes at one isolated value τ_0, again leading to a conical vanishing of a pair of energy eigenvalues at some x in the interior of the ball. As we follow the trajectory given by τ_0, we again find particle pair production obstructing the definition of a smooth nonvanishing vacuum section on the boundary of the ball. Berry's result for complex hamiltonians now says that indeed $\mathcal{H}_0$ is a twisted (monopole) line bundle over this S^2; its integer invariant is the nonabelian anomaly of the theory. Any action of $\mathcal{G}$ now must have a string singularity somewhere, and so no acceptable version of Gauss' law exists.

Let us now attempt to pass to $\mathcal{U}_g$ as before. Having established that $\mathcal{H}_0$ twists we can now forget about the interior of the orbit $\{\ell(\tau)\}$ and locally define our projective $\mathcal{G}$-action on $\mathcal{H}_0|_\ell$ as follows: Choose an S^2 metric on the orbit ℓ. If g is near the origin of $\mathcal{G}$ and takes P to Q, $P, Q \in \ell$, consider the geodesic from P to Q as a slowly-varying history and evolve any vacuum state $|0\rangle_P$ in this background. Call the result $U_g|0\rangle_P \in \mathcal{H}_0|_Q$. Now suppose that h takes Q to R, also on the orbit; then $(hg)^{-1}$ takes R back to P. By a redefinition of phases we can now arrange for the adiabatic transport on the geodesic triangle so defined to return $|0\rangle_P$ multiplied by $e^{i\Omega/2}$, where Ω is the solid angle subtended by PQR [7]. The $\frac{1}{2}$ is fixed by the requirement that the phase factor be smoothly defined even for large g, h since then Ω is ambiguous by 4π; this "Dirac quantization condition" on the normalization of the anomaly just reflects the fact that the anomaly is quantized due to its origin as a bundle twist.[8]

Thus $U_{hg}^{-1} U_h U_g |0\rangle_P = e^{i\Omega/2}|0\rangle_P$, and so $\mathcal{U}_{hg}^{-1}\mathcal{U}_h\mathcal{U}_g\Psi[P] = e^{i\Omega/2}\Psi[P]$. Choosing P to be any point where Ψ does not vanish we find once again that no state in $\dot{\mathcal{H}}$ is gauge-invariant.

Again we have seen that in the complex case the next-to-lowest invariant of the family index, in this case a two-form on $\mathcal{C}^4$, is the only thing obstructing the

8 See also [17]

definition of a $\mathcal{G}^3$-action on $\mathcal{H}$. Now, however, the obstruction is even more noticeable than in the previous case: since on a sphere we have nontrivial quantum holonomy even on infinitesimal loops,[9] we expect that the $\pi_2(\mathcal{G}^3)$ anomaly should be visible even in perturbation theory. This is of course the case.

5. Remarks

There is an even more direct way to relate the lagrangian derivations of the anomaly to the hamiltonian picture. While it is less physical than the one given above, it does give the quickest way to find the *sign* of the integer invariant in the previous section, something we cannot do by examining the behavior of the energy eigenvalues alone. This sign was irrelevant in the Z_2 case; now we need it in order to recover the anomaly cancellation condition.

The lagrangian derivations show that the fermion partition function $e^{-\Gamma[A]}$ is actually a twisted section on $\mathscr{C}^4$. In particular, it must vanish somewhere. But $e^{-\Gamma[A]}$ is just the vacuum expectation value of the time evolution operator $U(\infty, -\infty)$ in the presence of the time-dependent vector potential A_μ. Gauge transforming to temporal gauge as before, we get

$$\exp -\Gamma[A_{(\tau)}] = {}_{g_{(\tau)}}\langle 0|U_{\mathbf{A}'_{(\tau)}}(\infty, -\infty)|0\rangle_1 . \tag{5}$$

Here $\{|0\rangle_g\}$ are a set of vacuum states on the various $\mathcal{H}_{[0]^g}$. Now as τ makes a complete circuit in the $\pi_1(\mathcal{G}^3)$ case, $\mathbf{A}'_{(\tau)}$ returns to zero and so does its evolution operator. Since $e^{-\Gamma[A]}$ changes sign, it must be that the $\mathcal{G}$-action is twisted, as we found in Sect. 3. Furthermore, the single vanishing of $e^{-\Gamma[A]}$ which requires that it be twisted is just the signal of pair production again, since at τ_0 the evolved vacuum has no projection onto the transformed vacuum.

Repeating the argument in the case of the nonabelian $\pi_2(\mathcal{G}^3)$ anomaly, we find that not only must the $\mathcal{G}$-action be twisted, the twist in fact agrees in sign with that of the family index bundle. Hence the condition for the cancellation of the anomalous phases is that this bundle have no net twist, in agreement with [17]. In particular, ordinary QCD is safe.

From the hamiltonian point of view, the character of the gauge anomalies is determined by the structure of the possible real or complex line bundles over $\mathcal{G}^3$. Loosely speaking, if over a gauge orbit $\mathcal{H}_0$ contains a unit of "flux" then it cannot be "squeezed" to zero, *i.e.*, the theory does not factor through to one properly defined on $\mathscr{C}^3$. We have shown that the "flux" in a given theory's configuration space can be computed solely in terms of the second invariant of its Ind $\not{D}^4$. The fact that Witten's anomaly appears only for symplectic groups like SU(2), while the nonabelian anomaly appears for unitary groups like SU(3) also comes naturally

9 This is codimension once again: on S^1 there are no interesting paths in a neighborhood of 0 which do not intersect 0. Note however that we do not claim to have obviated the perturbative analysis of gauge anomalies. As is well known, there are anomalies which the global analysis fails to uncover, either in the hamiltonian or lagrangian form. All we are saying is that when the global obstruction *is* present, it is clear why it makes its presence felt in perturbation theory

from our construction, since in order to get interesting real (respectively complex) vacuum bundles over gauge orbits we needed nontrivial $\pi_1(\mathscr{G}^3)$ [respectively $\pi_2(\mathscr{G}^3)$]. This follows for the groups mentioned by the periodicity theorem.

While the higher invariants of the index are not related to gauge anomalies, they may still have interesting physical meaning, just as the lowest one does. The hamiltonian approach may yield further insight into this issue as well.

Note added. Some of the constructions in this paper have already been considered by I. M. Singer; see for example [19]. We thank the referee and Prof. Singer for bringing this work to our attention. After this paper was completed we also received the preprint by Faddeev [18], who discusses similar topics.

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