Spaces of Regular Gauge Field Configurations on a Lattice and Gauge Fixing Conditions

T. Bałaban★

Department of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. We consider spaces of lattice gauge field configurations satisfying gauge invariant regularity conditions, and intersections of these spaces with a surface given by gauge fixing conditions. We prove that if these conditions are chosen properly then configurations belonging to the intersection are small and regular.

In this paper we continue our investigation of the renormalization group method for lattice approximations of gauge field theories. A notion of regularity and spaces of regular gauge field configurations have appeared already in a natural way in [3]. There the compositions of averaging operations were considered and we have proved that they are regular (analytic) functions of the configurations U if the following regularity conditions are satisfied

$$|U(\partial p) - 1| < \alpha_0 \eta^2, \qquad \eta = L^{-k} \tag{0.1}$$

for α_0 sufficiently small and for plaquettes p contained in a subdomain $\Omega \subset \eta Z^d$. We refer the reader to [1, 3] for an explanation of notations and notions used here. We will use also almost all the results of those papers.

The regularity conditions (0.1) are the most fundamental conditions we impose on gauge field configurations. They are invariant with respect to gauge transformations

$$U(x, x') \rightarrow U^u(x, x') = u(x) U(x, x') u^{-1}(x'), \tag{0.2}$$

so the space of configurations satisfying (0.1) is decomposed into a union of orbits determined by the group of all gauge transformations.

In this paper we will consider more complicated regularity conditions, which are gauge invariant also; but let us explain our problem and results on the example (0.1).

Our renormalization group procedure will be based on solutions of some variational problems. To consider these problems we will have to fix gauge

conditions, which means that we have to choose one element from each orbit, or choose a surface intersecting each orbit at exactly one element. Of course there are many such surfaces and we have to specify further our problem. We choose a configuration U_0 satisfying (0.1) and we ask if there exists a surface passing through the element U_0 and intersecting each orbit at an element sufficiently close to U_0. More precisely, this means that if U is an intersection of an orbit satisfying (0.1) with the surface, then it satisfies the condition

$$UU_0^{-1} = e^{i\eta A}, \qquad |A| < 0(\alpha_0), \tag{0.3}$$

A is a Lie algebra valued configuration.

   This problem has a solution, i.e. there exists a surface having the above described property, but we have to introduce some further restrictions. An example of such restrictions, appearing naturally in the variational problem, is given by averaging operations

$$\bar{U}^k = V \quad \text{on} \quad \Omega^{(k)} = \Omega \cap Z^d, \tag{0.4}$$

where V is a fixed configuration on $\Omega^{(k)}$. This changes somewhat the notion of an orbit, because the conditions (0.4) are invariant with respect to gauge transformations u satisfying the restrictions $u = 1$ on $\Omega^{(k)}$. Taking into account this change, the remaining formulations are the same.

   An example of the gauge fixing conditions, having the above property, is given by the generalized Landau conditions investigated in [1, 2, 4]. In this paper we will consider these conditions only.

   Actually the regularity conditions and the results are much more complicated and we refrain from the formulations in the introduction.

   This paper is based on the definitions and the results of [1, 3, 4].

A. Regularity Conditions

Let us recall the definition of a covariant derivative on a lattice. For a function F defined on a subset of the lattice T_η, with values in complex $N \times N$ matrices, we define

$$\begin{aligned}(D^\eta_{U,\mu}F)(x) &= \eta^{-1}(R(U(x, x+\eta e_\mu))F(x+\eta e_\mu) - F(x)),\\(D^{\eta*}_{U,\mu}F)(x) &= \eta^{-1}(R(U(x, x-\eta e_\mu))F(x-\eta e_\mu) - F(x)),\end{aligned} \tag{1.1}$$

where U is an arbitrary gauge field configuration, and $R(U)X = UXU^{-1}$ for a unitary matrix U and an arbitrary X. Let F be a function defined at plaquettes of a subset of T_η. Let us denote by $p_{\mu\nu}(x)$ a plaquette determined by the point x and vectors e_μ, e_ν, $\mu < \nu$, i.e., $p_{\mu\nu}(x) = \langle x, x+\eta e_\mu, x+\eta e_\mu+\eta e_\nu, x+\eta e_\nu \rangle$, and let $F_{\mu\nu}(x) = F(p_{\mu\nu}(x))$. We define

$$(D^{\eta*}_U F)(x, x+\eta e_\mu) = (D^{\eta*}_U F)_\mu(x) = \sum_{\nu < \mu}(D^{\eta*}_{U,\nu}F_{\nu\mu})(x) - \sum_{\nu > \mu}(D^{\eta*}_{U,\nu}F_{\mu\nu})(x). \tag{1.2}$$

Of course these definitions hold for an arbitrary lattice with an arbitrary scale instead of η.

To formulate the notion of regularity we have to describe at first a geometric setting. We consider a sequence of domains (see also [2.II, 4])

$$\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \ldots \supset \Omega_k, \qquad \Omega_j \subset T_\eta, \qquad j = 0, 1, \ldots, k, \tag{1.3}$$

which satisfy the following conditions:

$$\Omega_j = B^j(\Omega_j^{(j)}), \ \Omega_j \text{ is a sum of cubes of a size } M_1 L^j \eta,$$
$$(L^j \eta)^{-1} \operatorname{dist}(\Omega_j^c, \Omega_{j+1}) > R M_1. \tag{1.4}$$

The number M_1 is a size of big blocks and was fixed in [4]. It is much bigger than δ_0^{-1}, where δ_0 is a decay rate of propagators, and it depends on d and L only. We assume that R is a sufficiently large positive integer (a power of L), so that all the theorems on propagators in [2, 4] hold for RM_1. The sets Ω_j are identified with sets of bonds, or sets of plaquettes, in the following way. If $\Omega \subset T_\eta$, then we denote by Ω also the set of bonds $\bigcup_{x \in \Omega} st(x) = \{\text{bonds } b \subset T_\eta: \text{ at least one end-point of } b$ belongs to $\Omega\}$. Similarly for the corresponding set of plaquettes. This convention applies to an arbitrary lattice. Let us denote

$$\Lambda_j = \Omega_j^{(j)} \backslash \Omega_{j+1}^{(j)}, \qquad j = 0, 1, \ldots, k-1, \qquad \Lambda_k = \Omega_k^{(k)}, \tag{1.5}$$

thus we have

$$\Omega = \bigcup_{j=0}^{k} B^j(\Lambda_j), \quad \text{where} \quad B^0(\Lambda_0) = \Lambda_0. \tag{1.6}$$

Let us notice that we admit the case where some domains Ω_j are equal to T_η, for example $\Omega_j = T_\eta$ for $j = 0, 1, \ldots, l, \ l \leq k$.

Now we can introduce the following spaces of regular field configurations: for a sequence (1.3) and a positive number α_0 we define $\mathfrak{U}_k(\{\Omega_j\}, \alpha_0)$ as a set of all gauge field configurations U on T_η satisfying the conditions

$$|U(\partial p) - 1| < \alpha_0 L^{-2j} \quad \text{for} \quad p \in \Omega_j, \qquad j = 0, 1, \ldots, k, \tag{1.7}$$

$$\text{or} \quad |U(\partial p) - 1| < \alpha_0 \eta^2 (L^j \eta)^{-2} \quad \text{for} \quad p \subset \Omega_j, \tag{1.8}$$

$$|(D_U^{\eta *} \partial U)(b)| < \alpha_0 L^{-2j}(L^j \eta)^{-1} \quad \text{for} \quad b \in \Omega_j, \qquad j = 0, 1, \ldots, k. \tag{1.9}$$

We have denoted $U(\partial p) = (\partial U)(p)$. These conditions may be written in many equivalent ways, for example (1.7) may be replaced by

$$\eta^{-4}[1 - \operatorname{Re} \operatorname{tr} U(\partial p)] < \alpha_0^2 (L^j \eta)^{-4}, \qquad p \in \Omega_j. \tag{1.10}$$

Of course it is enough to assume that (1.7), (1.9) hold for $p, b \in B^j(\Lambda_j)$, because if these conditions hold for some $j = l$, then they hold for all $j < l$. These conditions are invariant with respect to gauge transformations. It is obvious for (1.7), (1.8). To see that (1.9) is invariant also let us notice that $(\partial U^u)(p_{\mu\nu}(x)) = R(u(x))(\partial U)(p_{\mu\nu}(x))$, and if a function F transforms as $F^u(x) = R(u(x))F(x)$, then from (1.1) we get

$$(D_{U^u, \nu}^{\eta *} F^u)(x) = R(u(x))(D_{U, \nu}^{\eta *} F)(x). \tag{1.11}$$

This and (1.2) imply the invariance. Thus the space $\mathfrak{U}_k(\{\Omega_j\}, \alpha_0)$ is invariant with respect to gauge transformations.

Let us introduce further notations and definitions. We denote

$$\mathfrak{B}_k = \bigcup_{j=0}^{k} \Lambda_j \tag{1.12}$$

(the same for the sets of bonds), and for an arbitrary configuration V on $\mathfrak{B}_k$ we define

$\mathfrak{B}_k(\mathfrak{B}_k, V)$ is a set of all gauge field configurations U satisfying the conditions

$$\bar{U}^j = V \quad \text{on} \quad \Lambda_j, \quad j = 0, \ldots, k. \tag{1.13}$$

Because $(\overline{U^u})_b = (\bar{U}^j)_b^u = u(b_-)(\bar{U}^j)_b u^{-1}(b_+)$ for $b \subset T^{(j)}$, hence the set $\mathfrak{B}_k(\mathfrak{B}_k, V)$ is invariant with respect to gauge transformations u satisfying the conditions

$$u(y) = 1 \quad \text{for} \quad y \in \mathfrak{B}_k. \tag{1.14}$$

They form a subgroup of the group of all gauge transformations and we are interested in spaces of orbits of this subgroup.

B. Gauge (Fixing) Conditions

Generally speaking a gauge condition is a surface in a space of gauge field configurations, intersecting each orbit at exactly one point. Usually such a surface is given by equations $F(U) = 0$.

There are many gauge conditions. An axial gauge is defined by the equations

$$\text{for } x_j \in \Lambda_j, j = 1, \ldots, k, \quad \text{we put} \quad \bar{U}^{j-1}(\Gamma_{x_j, x_{j-1}}) = 1 \quad \text{for} \quad x_{j-1} \in B(x_j),$$

$$\bar{U}^{j-2}(\Gamma_{x_{j-1}, x_{j-2}}) = 1 \quad \text{for} \quad x_{j-2} \in B(x_{j-1}), \ldots, \tag{1.15}$$

$$\bar{U}(\Gamma_{x_2, x_1}) = 1 \quad \text{for} \quad x_1 \in B(x_2), U(\Gamma_{x_1, x}) = 1 \quad \text{for} \quad x \in B(x_1).$$

It is easy to see that these equations together with (1.14) for gauge transformations determine uniquely an element in each orbit.

We will choose gauge conditions relative to some fixed configuration U_0. This means that if we consider new variables U' defined by

$$U = U' U_0, \quad \text{or} \quad U' = U U_0^{-1}, \tag{1.16}$$

then the conditions are imposed on the variables U', and the surfaces pass through the element U_0. If we apply a gauge transformation u to U and if we demand that in the above representation for U^u the configuration U_0 is unchanged, then we get the following transformation law for the configurations U':

$$U'^u(x, x') = u(x) U'(x, x') R(U_0(x, x')) u^{-1}(x'). \tag{1.17}$$

If we are interested in a dependence on U_0, then we may consider another point of view. We demand that U and U_0 are transformed in the usual way, and then we get for $^uU' = U^u(U_0^u)^{-1}$:

$$^uU'(x, x') = R(u(x)) U'(x, x'). \tag{1.18}$$

Defining gauge conditions we will use the first point of view expressed in the transformation law (1.17).

Now let us define the axial gauge conditions relative to U_0. They were introduced in [3], (64–72) by the conditions:

for $x_0 \in B^j(x_j)$, $x_j \in \Lambda_j$, $1 \leq j \leq k$, we define a sequence of points $x_0, x_1, \ldots, x_{j-1}$, x_j by the conditions $x_n \in B(x_{n+1})$, $n = 0, 1, \ldots, j-1$, and we put

$$(\bar{R}^n_{0, x_{n+1}} \tilde{U}'^m)(\Gamma_{x_{n+1}, x_n}) = \prod_{b \subset \Gamma_{x_{n+1}, x_n}} R(\bar{U}^n_0(\Gamma_{x_{n+1}, b-})) \tilde{U}'^m_b = 1 . \tag{1.19}$$

Let us recall that the average $\tilde{U}'^m$ was defined in [3] as

$$\tilde{U}'^m = (\overline{U'U_0})^n (\bar{U}^n_0)^{-1} . \tag{1.20}$$

The conditions (1.19) determine uniquely an element in each orbit given by the subgroup (1.14). We denote this gauge condition by $Ax_k(\mathfrak{B}_k, U_0)$.

Let us analyze it for $k = 1$. More exactly we want to understand implications of (1.19) and (1.7). We have for a plaquette $p = \langle x, y, z, w \rangle$

$$U(\partial p) = (U'U_0)(\partial p)$$
$$= U'(x, y) R(U_0(x, y)) U'(y, z) U_0(\partial p) R(U_0(x, w)) U'(z, w) U'(w, x), \tag{1.21}$$

and let us denote

$$(\partial_{U_0} U')(p) = U'(x, y) R(U_0(x, y)) U'(z, y) R(U_0(x, w)) U'(z, w) U'(w, x). \tag{1.22}$$

Let us introduce a set of bonds. For a bond $c \subset T^{(j)}$ we take a set of bonds connecting the blocks $B^j(c_-)$, $B^j(c_+)$ and denote it by $B^j(c)$, thus

$$B^j(c) = \{b \subset T : b_- \in B^j(c_-), b_+ \in B^j(c_+)\} . \tag{1.23}$$

Later we will need to consider a general situation which is described in the lemma

Lemma 1. *Let* $V_0, V'V_0$ *satisfy the condition* (1.7) *for* $k = 1$ *and* L *arbitrary, and let*

$$(R(V_0)V')(\Gamma_{y, x}) = 1 \quad \text{for} \quad x \in B(y), |\overline{V'V_0} - \bar{V}_0| < \alpha_1 \quad \text{on} \quad \Omega^{(1)}_1 . \tag{1.24}$$

Then for α_0, α_1 *small the configuration* V' *is also small, more precisely we have the bound*

$$|V' - 1| < 4d^2\alpha_0 + \alpha_1 \quad \text{on} \quad \Omega_1 . \tag{1.25}$$

We will prove the lemma. From (1.22) and the assumptions we have

$$|(\partial_{V_0} V')(p) - 1| \leq |V_0(\partial p) - 1| + |(V'V_0)(\partial p) - 1| < 2\alpha_0 L^{-2} . \tag{1.26}$$

The conditions $(R_0 V')(\Gamma_{y, x}) = 1$, $x \in B(y)$, imply $V'_b = 1$ for $b \subset \Gamma_{y, x}$. This and the above estimate imply $|V'_b - 1| < (d-1)(L-1)2\alpha_0 L^{-2}$ for $b \subset B(y)$ by the same reasoning as in [3] (between (44) and (46)). For a plaquette p connecting two neighboring blocks $B(c_-)$, $B(c_+)$ we take two bonds $b', b'' \subset \partial p$, $b', b'' \in B(c)$ and from (1.26) and the bounds on $|V'_b - 1|$ for b', b'' belonging either to $B(c_-)$, or to $B(c_+)$, we get $|R(V_0(b'_-, b''_-)) V'_{b''} - V'_{b'}| < 2\alpha_0 L^{-2} + 4(d-1)\alpha_0 L^{-1} < 4d\alpha_0 L^{-1}$. Thus $|R(V_0(\Gamma_{b_0, -, b-})) V'_b - V'_{b_0}| < (d-1)(L-1)4d\alpha_0 L^{-1} < 4(d-1)d\alpha_0$ for an arbitrary bond $b \in B(c)$ and b_0 being the unique bond of c belonging to $B(c)$. The condition $|\overline{V'V_0} - \bar{V}_0| = |\tilde{V}' - 1| < \alpha_1$ implies $|R(V_0([c_-, b_0, -])) V'_{b_0} - 1| \leq 4(d-1)$

$(L-1)L\alpha_0 L^{-2}+\alpha_1<4(d-1)\alpha_0+\alpha_1$, hence finally we have the bound (1.25) for an arbitrary bond b.

From this proof it follows that the result is local in the sense that the bound (1.25) for a bond b depends on bounds (1.7), (1.24) on $B(c_-)\cup B(c_+)$, if b belongs to this set.

The bound (1.25) holds for arbitrary L, unfortunately for L large it is too weak. We expect that choosing properly a gauge condition we can get $V'=e^{iL^{-1}A}$ with a bound $|A|\leqq O(\alpha_0+\alpha_1)$. The good gauge condition for which this is true is the Landau gauge condition considered in [2, 4].

Let us recall the definition of this gauge, as it was given in [2, 4]. We consider the subspace $N(Q'(U_0))=\{\lambda:Q'(U_0)\lambda=0\}$ of the space $L^2(\Omega_0,\mathfrak{g})$ of functions on Ω_0 with values in the Lie algebra $\mathfrak{g}$. We assume always that U_0 is a sufficiently regular configuration, so that operators depending on U_0 are well defined and all necessary properties hold for them. Later we will state precisely the assumptions. An operator $R(U_0)$ is defined as an orthogonal projection in this real Hilbert space, onto the subspace $\Delta_{U_0}^\eta N(Q'(U_0))$. The Landau gauge condition can be defined formally on the whole space of gauge field configurations in terms of the variables U' by the equation

$$R(U_0)D_{U_0}^{\eta*}\frac{1}{i}\log U'=0. \tag{1.27}$$

This definition has a sense really only for configurations in a small neighborhood of U_0, thus for $A'=\frac{1}{i}\log U'$ sufficiently small.

We may formulate our problem in the form of the following statement: each orbit in the space $\mathfrak{U}_k(\{\Omega_j\},\alpha_0)\cap\mathfrak{B}_k(\mathfrak{B}_k,V)$ intersects the surface given by (1.27) at exactly one point A' satisfying the bounds $|A'|<O(\alpha_0 L^{-j})$ on Ω_j. Unfortunately, such a statement may not be true. Having in view also the future applications to the variational problem, we are interested in a slightly changed problem. The allowed gauge transformations u are restricted by the conditions (1.14): $u(y)=1$, $y\in\mathfrak{B}_k$. These conditions are very hard to work with analytically and we have to replace them by conditions imposed on some averages of u rather than on values of u at the points of $\mathfrak{B}_k$. We choose the averages which were described in detail in [3], formulas (78)–(80), and denoted by $\overline{R_0 u^j}$. Let us notice that for $u(x)=e^{i\lambda(x)}$ and λ small a linear part of the function $\frac{1}{i}\log(\overline{R_0 u^j})(y)$ is equal to $(Q_j'(U_0)\lambda)(y)$, and the definition of the Landau gauge depends on these averaging operations. The restrictions on gauge transformations are now given by the conditions $(\overline{R_0 u^j})(y)=1$, $y\in\Lambda_j$, $j=0,1,\ldots,k$. The problem with these conditions is that they do not agree with the group structure of the space of gauge transformations, that is if u_1,u_2 are two transformations satisfying these conditions, then generally the product $u_1 u_2$ does not satisfy them. This implies that the notion of orbit is not well defined and depends on an element U chosen to generate the set $\{U^{u^{-1}}\}$. Let us remark that we have changed the convention how to define an orbit. From now on we will use inverse elements u^{-1}, in accordance with the definitions and formulas of the paper [3]. A way out of this difficulty is to choose in some way an element U

from each orbit defined by (1.14), that is to choose some preliminary gauge condition, next to generate the set $\{U^{u^{-1}}\}$ for all u satisfying the conditions $\overline{R_0 u^j} = 1$, and finally to find an intersection of this set with the surface determined by the Eq. (1.27). We would like to prove that this intersection is unique and that the corresponding configuration A satisfies the bounds $|A| < O(\alpha_0 L^{-j})$ on Ω_j.

As the preliminary gauge condition we choose the axial gauge given by the Eq. (1.19). Now we can formulate the problem precisely. Let us take an arbitrary U' such that

$$U' U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0) \cap \mathfrak{B}_k(\mathfrak{B}_k, V) \cap A x_k(\mathfrak{B}_k, U_0) \tag{1.28}$$

and let us consider the set $\{U'^{u^{-1}}\}$ for gauge transformations u satisfying the conditions

$$(\overline{R_0 u^j})(y) = 1 \quad \text{for} \quad y \in \Lambda_j, \quad j = 0, 1, \dots, k. \tag{1.29}$$

Our problem is to prove that there exists a gauge transformation u such that $U'^{u^{-1}}$, or rather $A = \dfrac{1}{i} \log U'^{u^{-1}}$, satisfies the Landau gauge condition (1.27) and the bounds $|A|, |V_{U_0}^{L^{-j}} A| < O(\alpha_0 L^{-j})$ on Ω_j. We would like to prove also that such a gauge transformation is unique, i.e. the set $\{U'^{u^{-1}}\}$ intersects the part of the surface (1.27) satisfying the above bounds at exactly one point.

The rest of this paper is devoted to a proof of this statement.

Let us make few remarks assuming that this statement is true. The unique gauge transformations described in it define a mapping of the set (1.28) into the regular part of the surface (1.27). Let us denote the image set of this mapping by Σ_k, and $U'^{u^{-1}} = U_1$. The set Σ_k is not contained in $\mathfrak{B}_k(\mathfrak{B}_k, V)$ because the gauge transformations u do not satisfy the necessary conditions (1.14). The configurations U' satisfy the equations $\tilde{U}'^j = V(\bar{U}_0^j)^{-1}$ on Λ_j, hence U_1 satisfies

$$(\tilde{U}_1^{u^j})_b = u(b_-)(\tilde{U}_1^j)_b \bar{R}_{0,b}^j u^{-1}(b_+) = V_b(\bar{U}_0^j)_b^{-1}, \quad b \in \Lambda_j. \tag{1.30}$$

Let us notice that we do not assume that U_0 satisfies (1.13), i.e. that $U_0 \in \mathfrak{B}_k(\mathfrak{B}_k, V)$, hence the right-hand side in (1.30) generally is not equal to 1. We will assume only that it is close to 1.

In [3] we have determined the gauge transformation u in terms of the configuration U_1. We refer the reader especially to the formulas (87), (99), (97), and (105) of that paper. If both end-points b_-, b_+ of a bond b belong to Λ_j, then the expression on the left-hand side of (1.30) is equal to $(\bar{U}_1^j)_b$ by (92) of [3]. If a bond b crosses the boundary of Λ_j, then one of the end-points belongs to Λ_j, e.g. $b_+ \in \Lambda_j$, and another to Λ_{j-1}, $b_- \in \Lambda_{j-1}$. In this case the formulas (97), (99), and (87) of [3] imply

$$(\tilde{U}_1^{u^j})_b = \overline{\bar{R}_{0,b_-}^{j-1}} \overline{\bar{U}_1^{j-1}}^{-1}(\bar{U}_1^j)_b = V_b(\bar{U}_0^j)_b^{-1},$$

where by the formula (105) of [3]

$$\overline{\bar{R}_{0,b_-}^{j-1}} \overline{\bar{U}_1^j}^{-1} = \exp\left[i \sum_{x \in B(b_-)} L^{-d} \frac{1}{i} \log(\bar{R}_{0,b_-}^{j-1} \overline{\bar{U}_1^{j-1}}^{-1})(\Gamma_{b_-, x}) \right].$$

All sites of the contours $\Gamma_{b_-,x}$ belong to Λ_{j-1}, hence $\bar{U}_1^{j-1}$ in the above formula is equal to $V(\bar{U}_0^{j-1})^{-1}$. Let us denote $V(\bar{U}_0^j)^{-1} = V'$. We can write Eq. (1.30) as

$$(\bar{U}_1^j)_b = V_b' = \exp iB_b, \quad \text{if} \quad b \subset \Lambda_j (\text{i.e.}, b_-, b_+ \in \Lambda_j),$$

$$(\bar{U}_1^j)_b = \exp\left[-i \sum_{x \in B(b_-)} L^{-d} \frac{1}{i} \log(\bar{R}_{0,b_-}^{j-1} V')(\Gamma_{b_-,x})\right] V_b'$$

$$= \exp iB_b, \quad \text{if} \quad b_- \in \Lambda_{j-1}, b_+ \in \Lambda_j, \tag{1.31}$$

or

$$\frac{1}{i} \log \bar{U}_1^j = B \quad \text{on} \quad \Lambda_j, \quad j = 0, 1, ..., k,$$

where the configuration B is defined by the above equations. Thus we have

$$\Sigma_k \subset \{\text{a set of configurations } U = U_1 U_0, U \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0), U_1 = e^{i\eta A},$$

$$|A| < O(\alpha_0 (L^j \eta)^{-1}), |\nabla_{U_0}^{\eta} A| < O(\alpha_0 (L^j \eta)^{-2}) \quad \text{on} \quad \Omega_j, Q(U_0, \eta A) = B,$$

$$R(U_0) D_{U_0}^{\eta *} A = 0\}. \tag{1.32}$$

Let us remark also that we have chosen the preliminary gauge condition Ax_k not only for the reason of simplicity and convenience, but for deeper reasons connected with properties of the averaging operations $\bar{U}_1^j$.

C. Formulation of Theorems on Transformation from the Axial Gauge to the Landau Gauge

At first let us formulate precisely the assumptions we have to make on the configurations U_0, $U = U' U_0$. We assume that:

$$U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0), \quad U_0 \text{ satisfies the regularity condition (3.35) in [4]}. \tag{1.33}$$

$$U' U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0) \cap Ax_k(\mathfrak{B}_k, U_0), \tag{1.34}$$

$$|(\overline{U' U_0})^j - \bar{U}_0^j| < \alpha_1 \quad \text{on} \quad \Lambda_j, \quad j = 0, 1, ..., k. \tag{1.35}$$

We need the second condition in (1.33) to apply all the results of [4] on propagators with the background gauge field configuration U_0. We will see later that this condition is a consequence of the first one in (1.33), so eventually we will drop it out of the assumptions. The condition (1.35) replaces the equalities for the averages in the conditions (1.28). It is satisfied if V is close to $\bar{U}_0^j$, more precisely if $|V - \bar{U}_0^j| < \alpha_1$.

Now our problem is to construct a gauge transformation u satisfying the equalities (1.29) and such that $U_1 = U'^{u^{-1}}$ satisfies the conditions

$$U_1 = e^{i\eta A}, |A| < B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-1}, |\nabla_{U_0}^{\eta} A| < B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-2},$$

$$\|A\|_{1,\beta} < B_2(\beta_0)(\alpha_0 + \alpha_1)(L^j \eta)^{-2-\beta}, \beta \leq \beta_0 < 1, \quad \text{on} \quad \Omega_j, \quad j = 0, 1, ..., k, \tag{1.36}$$

$Q_j(U_0, \eta A) = B$ on Λ_j, $j = 0, 1, ..., k$, B is given by formula (1.31) with

$$V' = \tilde{U}'^j,$$

$$|B| < 2dL\alpha_1 \text{ by the assumption (1.35),} \tag{1.37}$$

$$R(U_0) D_{U_0}^{\eta *} A = 0. \tag{1.38}$$

Of course the last condition is the most important constraint on A which assures that the others are satisfied. It holds by the construction. Also the condition (1.37) is basically of an algebraic character and it follows from the construction, as in (1.30), (1.31). The only fact we need to write it this way is the representation of U_1 and the first condition on A in (1.36). Thus (1.36) contains all the results we have to prove about A, besides its existence. This condition describes fully regularity properties of the first order derivatives. Such information is unavailable for the second order derivatives, but we have the following bounds for the second order operators acting on A:

$$|D^{\eta*}_{U_0} D^{\eta}_{U_0} A|, |\Delta^{\eta}_{U_0} A| < B_1(\alpha_0 + \alpha_1)(L^j \eta)^{-3} \quad \text{on} \quad \Omega_j, \quad j = 0, 1, \ldots, k. \quad (1.39)$$

Of course we want to prove that the constants B_1, $B_2(\beta_0)$ in (1.36), (1.39) are absolute constants depending on d and L only, $B_2(\beta_0)$ on β_0 also. We expect that we have to assume that $\alpha_0 + \alpha_1$ is sufficiently small.

We can formulate now the main result of the paper.

Theorem 2. *There exist constants B_1, $B_2(\beta_0)$, c_1 such that for arbitrary U_0, $U'U_0$ satisfying (1.33)–(1.35) with $\alpha_0 + \alpha_1 \leq c_1$ there exists exactly one gauge transformation u satisfying (1.29) and such that the conditions (1.36)–(1.39) hold for the configuration $U_1 = U'^{u^{-1}}$.*

We will be proving this theorem, and its reformulations, in the rest of this paper.

Let us separate some aspects of the proof. It will be by induction with respect to k and we will construct the configuration U_1 perturbatively, the perturbation around a configuration constructed in a previous step. This method will not give us good constants in (1.36), or (1.39), and we have to prove the bounds in (1.36), (1.39) by some other method, assuming that we have constructed U_1 satisfying (1.36)–(1.39) with worse constants. This raises the question of what are the best constants in these bounds. This question may be answered independently of a proof of Theorem 2.

Let us consider a somewhat more general situation. We assume that we have configurations U_0, $U_1 U_0$ satisfying the following conditions

$$U_0, U_1 U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0), U_0$$

satisfies the additional regularity condition (3.35) in [4], $\qquad$ (1.40)

$$U_1 = e^{i\eta A}, |A| < \alpha_2 (L^j \eta)^{-1} \quad \text{on} \quad \Omega_j, \qquad (1.41)$$

$$R(U_0) D^{\eta*}_{U_0} A = 0, \quad Q_j(U_0, \eta A) = B \quad \text{on} \quad \Lambda_j, \quad |B| < 2dL\alpha_1, \qquad (1.42)$$

($j = 0, 1, \ldots, k$ above). Thus we have apparently much weaker regularity now. We will prove that in fact these conditions imply all the other conditions in (1.36), (1.39), and we will find the constants B_1, B_2, and restrictions on $\alpha_0, \alpha_1, \alpha_2$. We start investigating implications of (1.40).

We derive formulas and estimates which will have other applications in the future, so we formulate them now in the most general way needed. We have from (1.21), (1.22),

$$(U_1 U_0)(\partial p) - 1 = [U_1(x, y) R(U_0(x, y)) U_1(y, z) - 1][U_0(\partial p) - 1]$$
$$\cdot R(U_0(x, w)) U_1(z, w) U_1(w, x)$$
$$+ [U_0(\partial p) - 1][R(U_0(x, w)) U_1(z, w) U_1(w, x) - 1]$$
$$+ [U_0(\partial p) - 1] + [(\partial_{U_0} U_1)(p) - 1]. \qquad (1.43)$$

The first two terms on the right-hand side can be bounded by $4\alpha_2\alpha_0(L^j\eta)^{-3}\eta^3$ for $p \subset \Omega_j$. If we apply the derivative $D^{\eta*}_{U_1U_0}$ to $\partial U_1 U_0$, then the derivative of these two terms can be bounded by $8d\alpha_2\alpha_0(L^j\eta)^{-3}\eta^2$. Further we have

$$|D^{\eta*}_{U_1U_0}\partial U_0 - D^{\eta*}_{U_0}\partial U_0| < 2d\alpha_0\alpha_2(L^j\eta)^{-3}\eta^2 \quad \text{on} \quad \Omega_j.$$

Let us denote by $O_1(\alpha)$ an expression which can be estimated by α, i.e. $|O_1(\alpha)| \leq \alpha$. Thus we have

$$D^{\eta*}_{U_1U_0}\partial U_1 U_0 = D^{\eta*}_{U_0}\partial U_0 + D^{\eta*}_{U_1U_0}\partial_{U_0}U_1 + O_1(10d\alpha_0\alpha_2(L^j\eta)^{-3}\eta^2). \quad (1.44)$$

Let us consider the second term on the right-hand side above. From the definition (1.1), (1.2) we obtain that a value of this expression at a bond $\langle x, x+\eta e_\mu\rangle \in \Omega_j$ is equal to

$$(D^{\eta*}_{U_1U_0}\partial_{U_0}U_1)(x, x+\eta e_\mu) = (D^{\eta*}_{U_1U_0}(\partial_{U_0}U_1 - 1))(x, x+\eta e_\mu)$$

$$= (D^{\eta*}_{U_0}(\partial_{U_0}U_1 - 1))(x, x+\eta e_\mu)$$

$$+ \sum_{\nu < \mu} \eta^{-1}R(U_0(x, x-\eta e_\nu))$$

$$\cdot (R(U_1(x, x-\eta e_\nu)) - 1)((\partial_{U_0}U_1)(p_{\nu\mu}(x-\eta e_\nu)) - 1)$$

$$- \sum_{\nu > \mu} \eta^{-1}R(U_0(x, x-\eta e_\nu))(R(U_1(x, x-\eta e_\nu)) - 1)((\partial_{U_0}U_1)(p_{\mu\nu}(x-\eta e_\nu)) - 1).$$

$$(1.45)$$

We expand $R(U_1(x, x-\eta e_\nu)) - 1$ in A. Because $\partial_{U_0}U_1 - 1 = O_1(4\alpha_2(L^j\eta)^{-1}\eta)$, so it is enough to consider a term of the first order, a remainder is $O_1(3|\eta A|^2) = O_1(3\alpha_2^2(L^j\eta)^{-2}\eta^2)$. The term of the first order is $i\eta \, \mathrm{ad}_{A(x, x-\eta e_\nu)} = O_1(2\alpha_2(L^j\eta)^{-1}\eta)$. Similarly, if we take this term and expand $\partial_{U_0}U_1 - 1$, then it is enough to consider a term of the first order only, a remainder is $O_1\left(\frac{1}{2!}\eta^2(\partial|A|(p))^2\right) = O_1(8\alpha_2^2(L^j\eta)^{-2}\eta^2)$. The term of the first order is $i\eta^2(D^\eta_{U_0}A)(p)$. This gives

$$(D^{\eta*}_{U_1U_0}\partial_{U_0}U_1)(x, x+\eta e_\mu) = (D^{\eta*}_{U_0}(\partial_{U_0}U_1 - 1))(x, x+\eta e_\mu)$$

$$+ \eta^2 \sum_{\nu \neq \mu} R(U_0(x, x-\eta e_\nu)) \, \mathrm{ad}_{A(x, x-\eta e_\nu)}(D^\eta_{U_0}A)(p_{\mu\nu}(x-\eta e_\nu))$$

$$+ O_1(28d\alpha_2^3(L^j\eta)^{-3}\eta^2). \quad (1.46)$$

Let us consider the first term on the right-hand side of the above equality. We expand it in A up to the third order and we get

$$(\partial_{U_0}U_1)(p) - 1 = i\eta^2(D^\eta_{U_0}A)(p) - \frac{1}{2}\eta^2 V_2(U_0, A, \partial p)$$

$$+ O_1\left(\frac{1}{3!}\eta^3(\partial|A|(p))^3\right), \quad (1.47)$$

where V_2 is a polynomial of the second order in A. Thus

$$D^{\eta*}_{U_0}(\partial_{U_0}U_1 - 1) = i\eta^2 D^{\eta*}_{U_0}D^\eta_{U_0}A - \frac{1}{2}\eta^2 D^{\eta*}_{U_0}V_2(U_0, A) + O_1\left(\frac{4^3}{3}d\alpha_2^3(L^j\eta)^{-3}\eta^2\right).$$

$$(1.48)$$

We find easily the polynomial V_2:

$$
\begin{aligned}
V_2(U_0, A, \partial p) = &(A(x,y))^2 + 2A(x,y)R(U_0(x,y))A(y,z) \\
&+ 2A(x,y)R(U_0(x,w))A(z,w) \\
&+ 2A(x,y)A(w,x) + (R(U_0(x,y))A(y,z))^2 \\
&+ 2R(U_0(x,y))A(y,z)R(U_0(x,w))A(z,w) \\
&+ 2R(U_0(x,y))A(y,z)A(w,x) \\
&+ (R(U_0(x,w))A(z,w))^2 \\
&+ 2R(U_0(x,w))A(z,w)A(w,x) + (A(w,x))^2 \\
= &\eta^2((D^\eta_{U_0}A)(p))^2 + \{[A(x,y), R(U_0(x,y))A(y,z)] \\
&+ [A(x,y), R(U_0(x,w))A(z,w)] + [A(x,y), A(w,x)] \\
&+ [R(U_0(x,y))A(y,z), R(U_0(x,w))A(z,w)] \\
&+ [R(U_0(x,y))A(y,z), A(w,x)] \\
&+ [R(U_0(x,w))A(z,w), A(w,x)]\} .
\end{aligned}
\tag{1.49}
$$

This equality holds for arbitrary A in the complexified Lie algebra, but if A is Hermitian, then the first expression above is a real part of V_2, and the second is an imaginary part of V_2, multiplied by i. Let $p = p_{\mu\nu}(x)$, then the expression in $\{\dots\}$ can be written as

$$
\begin{aligned}
\{\dots\} = &2[A_\mu(x), A_\nu(x)] + 2\eta[A_\mu(x), (D_\nu A_\mu)(x)] \\
&+ 2\eta[(D_\nu A_\mu)(x), A_\nu(x)] - \eta[A_\mu(x), (D_\nu A_\mu)(x)] \\
&- \eta[(D_\mu A_\nu)(x), A_\nu(x)] - \eta^2[(D_\mu A_\nu)(x), (D_\nu A_\mu)(x)],
\end{aligned}
\tag{1.50}
$$

where D_μ denotes the derivative $D^\eta_{U_0,\mu}$. From these equalities we see that all the terms in V_2, except the first term on the right-hand side above, have some powers of η. If we apply $D^{\eta*}_{U_0}$ to V_2, we use these factors to cancel η^{-1} and to replace $D^{\eta*}_{U_0}$ by $D^{1*}_{U_0}$, which is a very simple bounded operator. Thus

$$
D^{\eta*}_{U_0} V_2(U_0, A) = -2 \sum_{\nu=1}^d D^*_\nu[A_\mu, A_\nu] + O_1(48d|A| |\nabla^\eta_{U_0}A|).
\tag{1.51}
$$

Further

$$
(D^*_\nu[A_\mu, A_\nu])(x) = \eta[(D^*_\nu A_\mu)(x), (D^*_\nu A_\nu)(x)] + [(D^*_\nu A_\mu)(x), A_\nu(x)] \\
+ [A_\mu(x), (D^*_\nu A_\nu)(x)],
\tag{1.52}
$$

hence, taking into account the identity $(D^*_\mu A)(x) = -R(U_0(x-\eta e_\mu, x))(D_\mu A)$ $(x-\eta e_\mu)$, we get

$$
\tfrac{1}{2}\eta^2 D^{\eta*}_{U_0} V_2(U_0, A) = O_1(32d|A| |\nabla^\eta_{U_0}A|\eta^2) = O_1(32d\alpha_2(L^j\eta)^{-1}|\nabla^\eta_{U_0}A|\eta^2).
\tag{1.53}
$$

Gathering together the equalities and the estimates obtained until now we get

$$
\begin{aligned}
D^{\eta*}_{U_1 U_0} \partial U_1 U_0 = &D^{\eta*}_{U_0} \partial U_0 + i\eta^2 D^{\eta*}_{U_0} D^\eta_{U_0} A + O_1(36d\alpha_2(L^j\eta)^{-1}|\nabla^\eta_{U_0}A|\eta^2) \\
&+ O_1(50d\alpha_2^3(L^j\eta)^{-3}\eta^2) + O_1(10d\alpha_0\alpha_2(L^j\eta)^{-3}\eta^2) \quad \text{on } \Omega_j.
\end{aligned}
\tag{1.54}
$$

This is the basic estimate. It will be applied in several different situations in the future. Let us recall that it was derived under the assumption (0.1) for U_0 and (1.41) for U_1.

We will apply it to get regularity results for A, assuming (1.40)–(1.42). From (1.40) the term on the left-hand side and the first term on the right-hand side can be estimated by $\alpha_0 (L^j \eta)^{-3} \eta^2$. This implies that

$$D^{\eta *}_{U_0} D^\eta_{U_0} A = J,$$

$$|J| \leqq (2\alpha_0 + 36 d\alpha_2 (L^j \eta)^2 |V^\eta_{U_0} A| + 50 d\alpha_2^3 + 10 d\alpha_0 \alpha_2) (L^j \eta)^{-3}, \qquad (1.55)$$

or

$$|J|_{(-3)} \leqq 2\alpha_0 + 36 d\alpha_2 |V^\eta_{U_0} A|_{(-2)} + 50 d\alpha_2^3 + 10 d\alpha_0 \alpha_2,$$

where the norms $|\cdot|_{(\alpha)}$ were introduced in [4]. For the reader's convenience let us recall the definition: $|A|_{(\alpha)} = \sup_j \sup_{\Omega_j} (L^j \eta)^{-\alpha} |A|$.

The configuration A satisfies (1.42) also. Proposition 4 from [3] implies that

$$Q_j(U_0, \eta A) = L^j \eta Q_j A + C_j(L^j \eta A), \ |C_j(L^j \eta A)| \leqq C_2 |L^j \eta A|^2 < C_2 \alpha_2^2, \qquad (1.56)$$

hence $L^j \eta Q_j A = B_1$, $B_1 = B - C_j(L^j \eta A)$ on Λ_j, $|B_1| < 2dL\alpha_1 + C_2 \alpha_2^2$. We make the translation $A = A_1 + H(U_0) B_1$, where the operator $H(U_0)$ was defined in [4], and we get the equations

$$D^{\eta *}_{U_0} D^\eta_{U_0} A_1 = J - D^{\eta *}_{U_0} D^\eta_{U_0} H(U_0) B_1,$$
$$Q_j A_1 = 0 \quad \text{on} \quad \Lambda_j, \quad R(U_0) D^{\eta *}_{U_0} A_1 = 0. \qquad (1.57)$$

They imply finally

$$A = G(U_0) J - G(U_0) D^{\eta *}_{U_0} D_{U_0} H(U_0) B_1 + H(U_0) B_1$$
$$= G(U_0) J + G(U_0) \sum_j Q^*_j \Lambda_j (L^j \eta)^{-3} B_1, \qquad (1.58)$$

where the operator $G(U_0)$ was introduced and investigated in [4]. Let us recall only the definition:

$$G(U_0) = \left(D^{\eta *}_{U_0} D^\eta_{U_0} + D^\eta_{U_0} R(U_0) D^{\eta *}_{U_0} + \sum_j Q^*_j \Lambda_j (L^j \eta)^{-2} Q_j \right)^{-1}.$$

Theorem 3.3 of [4] implies the bounds:

$$|A|_{(-1)}, |V^\eta_{U_0} A|_{(-2)}, |D^{\eta *}_{U_0} D^\eta_{U_0} A|_{(-3)}, |\Delta^\eta_{U_0} A|_{(-3)} \leqq B_0(|J|_{(-3)} + |B_1|)$$
$$\leqq B_0(2\alpha_0 + 36 d\alpha_2 |V^\eta_{U_0} A|_{(-2)} + 50 d\alpha_2^3 + 10 d\alpha_0 \alpha_2 + 2dL\alpha_1 + C_2 \alpha_2^2). \qquad (1.59)$$

Let us take this bound for $|V^\eta_{U_0} A|_{(-2)}$ on the left-hand side, and let us assume that $B_0 36 d\alpha_2 \leqq 1/2$. This gives us a bound for $|V^\eta_{U_0} A|_{(-2)}$, equal to the right-hand side above without this term and multiplied by 2. Using this bound we get

$$|A|_{(-1)}, |V^\eta_{U_0} A|_{(-2)}, |D^{\eta *}_{U_0} D^\eta_{U_0} A|_{(-3)}, |\Delta^\eta_{U_0} A|_{(-3)}$$
$$< B_0(4\alpha_0 + 4dL\alpha_1 + 2\alpha_2^2 + 20 d\alpha_0 \alpha_2 + 2C_2 \alpha_2^2). \qquad (1.60)$$

Now we assume further that

$$2\alpha_2^2 + 20 d\alpha_0 \alpha_2 + 2C_2 \alpha_2^2 \leqq \alpha_0 + \alpha_1. \qquad (1.61)$$

This and the previous inequality give finally

$$|A| < 5dLB_0(\alpha_0 + \alpha_1)(L^j\eta)^{-1}, \quad |V_{U_0}^\eta A| < 5dLB_0(\alpha_0 + \alpha_1)(L^j\eta)^{-2},$$

$$\|A\|_{1,\beta} < 5dLB_0(\beta)(\alpha_0 + \alpha_1)(L^j\eta)^{-2-\beta}, \quad |D_{U_0}^{\eta*}D_{U_0}^\eta A|, |\Delta_{U_0}^\eta A|$$

$$< 5dLB_0(\alpha_0 + \alpha_1)(L^j\eta)^{-3} \quad \text{on} \quad \Omega_j. \tag{1.62}$$

Let us formulate the results in

Proposition 3. *If U_0, U_1U_0 satisfy (1.40)–(1.42) with α_0, α_1, α_2 bounded by a constant depending on d and L only, and α_2 satisfies the additional restriction (1.61), then U_1 satisfies (1.36)–(1.39) with $B_1 = 5dLB_0$, $B_2(\beta_0) = 5dLB_0(\beta_0)$, where B_0, $B_0(\beta_0)$ are the corresponding norms of the operators $G(U_0)$, $H(U_0)$, and depend on d and L only, $B_0(\beta_0)$ on β_0 also.*

This proposition explains how the constants B_1, B_2 may be chosen. Let us make some comments on the restriction (1.61). In the future we will get a configuration A satisfying (1.41) with $\alpha_2 = B_1'(\alpha_0 + \alpha_1) = C_1'B_1(\alpha_0 + \alpha_1)$. The restriction follows from the bounds

$$20d\alpha_2\alpha_0 + 2\alpha_2^2 + 2C_2\alpha_2^2 < (20dB_1C_1' + 2B_1^2C_1'^2 + 2C_2B_1^2C_1'^2)(\alpha_0 + \alpha_1)^2 \leqq \alpha_0 + \alpha_1, \tag{1.63}$$

hence from

$$(20dB_1C_1' + 2(1 + C_2)B_1^2C_1'^2)(\alpha_0 + \alpha_1) \leqq 1. \tag{1.64}$$

This is a restriction on $\alpha_0 + \alpha_1$ depending on d, L, and C_1', and in the future considerations we will have to make sure that C_1' depends on d and L only.

We will prove Theorem 2 by induction. To use the inductive assumptions easily we have to reformulate the conditions (1.35). The first two conditions (1.33), (1.34) have the property that they hold for $k-1$, if they hold for k. We will replace (1.35) by an equivalent condition which has this property also. By Proposition 2 of [3] the bounds $|\partial U - 1| < \alpha_0\eta^2$ imply $|\partial \bar{U}^j - 1| < \alpha_0 L^{2j}\eta^2 + 2C_1(\alpha_0 L^{2j}\eta^2)^2 < 2\alpha_0 L^{2j}\eta^2$, $j = 0, 1, \ldots, k$ for α_0 small. Let us take $j = k-1$ and let us apply Lemma 1 to $V_0 = \bar{U}_0^{k-1}$, $V'V_0 = (\overline{U'U_0})^{k-1} = \tilde{U}'^{k-1}\bar{U}_0^{k-1}$. Of course all the assumptions are satisfied and we obtain $|V' - 1| = |\tilde{U}'^{k-1} - 1| < 8d^2\alpha_0 + \alpha_1$. We apply again the Lemma 1 to $V_0 = \bar{U}_0^{k-2}$, $V'V_0 = \tilde{U}'^{k-2}\bar{U}_0^{k-2}$ and the constant α_1 replaced by $8d^2\alpha_0 + \alpha_1$, and we obtain $|\tilde{U}'^{k-2} - 1| < 8d^2\alpha_0 L^{-2} + 8d^2\alpha_0 + \alpha_1$. Continuing these arguments we get $|\tilde{U}'^j - 1| < 8d^2\alpha_0 L^{-2(k-j-1)} + \ldots + 8d^2\alpha_0 L^{-2} + 8d^2\alpha_0 + \alpha_1$ $< 8d^2\alpha_0 \dfrac{1}{1 - L^{-2}} + \alpha_1 < 11d^2\alpha_0 + \alpha_1$. Thus the conditions (1.33)–(1.35) imply

$$|(\overline{U'U_0})^j - \bar{U}_0^j| = |\tilde{U}'^j - 1| < 11d^2\alpha_0 + \alpha_1 \quad \text{on} \quad \Omega_j^{(j)}. \tag{1.65}$$

We will assume that the above bounds hold instead of (1.35). For simplicity let us write α_1 instead of the right-hand side above, so we assume that

$$|(\overline{U'U_0})^j - \bar{U}_0^j| = |\tilde{U}'^j - 1| < \alpha_1 \quad \text{on} \quad \Omega_j^{(j)}, \quad j = 0, 1, \ldots, k. \tag{1.66}$$

Now we are ready to give an inductive proof of the following

Theorem 4. *There exists a constant c_1 such that for arbitrary U_0, $U'U_0$ satisfying (1.33), (1.34), (1.66) with $\alpha_0 + \alpha_1 \leqq c_1$ there exists exactly one gauge transformation u satisfying (1.29) and such that the conditions (1.37), (1.38), (1.62) hold for the configuration $U_1 = U'^{u^{-1}}$.*

Of course this theorem implies Theorem 2. Proposition 3 implies that it is enough to prove (1.37), (1.38) and

$$|A| < B_1'(\alpha_0 + \alpha_1)(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, \, B_1' = C_1' 5 B_0, \qquad (1.67)$$

with an absolute constant C_1', i.e. a constant depending on d and L only. In the rest of this paper we will be occupied with a proof of these conditions.

A proof of the first step, for $k = 1$, will be included in a proof of the general step. Thus let us assume that Theorem 4 holds for some $k-1$ and we will prove it for k. Let us take configurations U_0, $U'U_0$ satisfying the conditions (1.33), (1.34), (1.66). It was already noticed that they imply the same conditions for $k-1$ (we do not consider the set Ω_k). Applying the inductive hypothesis we get a gauge transformation u_1 such that

$$(\overline{R_0 u_1})^j(y) = 1 \quad \text{for} \quad y \in \Lambda_j, \quad j = 0, 1, \ldots, k-2, \text{ and}$$
$$\text{for} \quad y \in \Lambda_{k-1} \cup B(\Lambda_k), \quad j = k-1, \qquad (1.68)$$

$$U_1 = U'^{u_1^{-1}} = e^{iL\eta A'} = e^{i\eta A}, \quad |A| < B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1},$$

$$|V_{U_0}^\eta A| < B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-2} \quad \text{on} \quad \Omega_j, \quad j = 0, 1, \ldots, k-1. \qquad (1.69)$$

The other conditions in (1.37), (1.38), and (1.62) hold also, but we will not need them. It is clear from (1.69) that we have to improve the bounds on the domain Ω_k.

Let us consider the gauge transformation u_1. From (64)–(87) of [3] it follows that u_1 is determined uniquely in terms of U_1 and is given by (106). From (105) and (106) we have for $x \in B^j(\Lambda_j)$,

$$|u_1(x) - 1| \leqq \sum_{n=j-1}^{0} (|(\overline{R_{0, x_{n+1}}^n} \overline{\overline{U}}_1^n) - 1| + |(\overline{R_{0, x_{n+1}}^n} \overline{\overline{U}}_1^n)(\Gamma_{x_{n+1}, x_n}) - 1|)$$

$$\leqq 2 \sum_{n=0}^{j-1} \max_{x \in B(x_{n+1})} \left| \frac{1}{i} \log(\overline{R_{0, x_{n+1}}^n} \overline{\overline{U}}_1^n)(\Gamma_{x_{n+1}, x}) \right|. \qquad (1.70)$$

Using (130) and (1.69) we obtain

$$\left| \frac{1}{i} \log(\overline{R_{0, x_{n+1}}^n} \overline{\overline{U}}_1^n)(\Gamma_{x_{n+1}, x}) \right| \leqq |Q_n(U_0, \eta A)|(\Gamma_{x_{n+1}, x})$$

$$+ O((|Q_n(U_0, \eta A)|(\Gamma_{x_{n+1}, x}))^2) < 2B_1(\alpha_0 + \alpha_1)L^{n-j}dL$$

$$+ O((2B_1(\alpha_0 + \alpha_1)L^{n-j}dL)^2) \leqq 4dB_1(\alpha_0 + \alpha_1)L^{n+l-j} \qquad (1.71)$$

for $\alpha_0 + \alpha_1$ sufficiently small, hence

$$|u_1(x) - 1| < 8dB_1(\alpha_0 + \alpha_1)\frac{1}{1 - L^{-1}} \leqq 16dB_1(\alpha_0 + \alpha_1), \quad x \in \Omega_0. \qquad (1.72)$$

The same argument can be applied to the averages $\overline{R_0 u_1^n}$, and we get

$$|(\overline{R_0 u_1^n})(x_n) - 1| < 16dB_1(\alpha_0 + \alpha_1), \quad x_n \in \Omega_n^{(n)}. \qquad (1.73)$$

From (108) and (1.71) we obtain also

$$|(\overline{R_0 u_1^n})^{-1}(x_{n+1})(\overline{R_{0,x_{n+1}}^n}\overline{R_0 u_0^n})(x_n) - 1| < 4dB_1(\alpha_0 + \alpha_1)L^{n+1-j} \qquad (1.74)$$

for $x_n \in B(x_{n+1})$, $x_{n+1} \in B^{j-n-1}(\Lambda_j)$, $n = 0, ..., j-1$, $j = 1, ..., k-1$ (we take $\Lambda_{k-1} \cup B(\Lambda_k)$ as Λ_{k-1}).

We will use only the properties (1.69), (1.73), (1.74) of the configurations u_1, U_1 in the future. They are satisfied in the case $k = 1$ also, if we take $u_1 = 1$, $U' = U_1$, because then by Lemma 1 we have $|U_1 - 1| < 4d^2\alpha_0 + \alpha_1$ and $U_1 = e^{iA'} = e^{iL^{-1}A}$ with A satisfying the bounds (1.69), at least for B_1 not too small, e.g. for B_0 satisfying $5dLB_0 \geq 8d^2$. The conditions (1.73), (1.74) are important because we have analyzed the averaging operations for gauge transformations in [3] assuming that they satisfy these conditions. We will use the results of that paper.

The gauge transformation u_1 transforms the configuration U' into the configuration U_1 which is sufficiently regular and close to 1. We want to find another gauge transformation u' which transforms U_1 into a configuration $U_1^{u'^{-1}}$ satisfying the conditions (1.37), (1.38), (1.67) and which satisfies the conditions $\overline{R_0 u' u_1}^j = 1$ on Λ_j. The composition $u = u' u_1$ of these two gauge transformations is the transformation we are looking for. The conditions which have to be satisfied for u' lead to restrictions on u', e.g. u' has to be a small configuration also, and they lead to equations on u'. We will analyze these equations and we will prove that under proper conditions they have a unique solution.

D. Equations for the Gauge Transformation u'

We fix a configuration U_0 and we omit it in formulas below, for example we write D instead of $D_{U_0}^\eta$. We are looking for a configuration u' such that $U_1^{u'^{-1}}$ satisfies the conditions (1.37), (1.38), (1.62). In particular (1.62) and (1.69) imply

$$|u'^{-1}(b_-)R_{0,b}u'(b_+) - 1| \leq |u'^{-1}(b_-)U_{1,b}R_{0,b}u'(b_+) - 1|$$
$$+ |u'^{-1}(b_-)(U_{1,b} - 1)R_{0,b}u'(b_+)| < 2B_1(\alpha_0 + \alpha_1)\eta(L^j\eta)^{-1}$$
$$\text{on} \quad \Omega_j, \quad j = 1, ..., k-1. \qquad (1.75)$$

Reasoning as in (1.70)–(1.72) we have $|u'u_1 - 1| < 16dB_1(\alpha_0 + \alpha_1)$. From the inequality (1.72) for u_1 we get

$$|u' - 1| < 32dB_1(\alpha_0 + \alpha_1). \qquad (1.76)$$

The estimates (1.75), (1.76) imply that u' satisfies the regularity conditions (176), (177) [3] on $\Omega_j \subset T_{L-j}$ with $\alpha_4 = 32dB_1(\alpha_0 + \alpha_1)$. The estimates (1.73), (1.74) together with the condition $\overline{R_0 u_1}^j = 1$ on Λ_j imply that u_1 satisfies (166), (167) [3] on $B^j(\Lambda_j)$ $\subset T_{L-j}$ with $\alpha_3 = 16dB_1(\alpha_0 + \alpha_1)$, thus for $\alpha_0 + \alpha_1$ sufficiently small the assumptions of Proposition 10 are satisfied and we have the representation (213) and the bounds (214) [3].

We have to construct the configuration u', so we have to change somewhat our point of view. The above estimates show that it is natural to assume the regularity conditions (176), (177) on $\Omega_j \subset T_{L-j}$, with some constant α_4 to be determined later. Here it is more convenient to work with regularity conditions of the type (207) [3].

Thus we consider the set of configurations $u' = e^{i\lambda}$ with λ satisfying

$$|\lambda| < \alpha_4, \, |(D\lambda)(b)| < \alpha_4 (L^j \eta)^{-1} \quad \text{for} \quad b \in \Omega_j, \quad j = 0, 1, \ldots, k-1, \quad (1.77)$$

where α_4 is sufficiently small, but otherwise arbitrary (i.e., not necessarily given by $\alpha_4 = 32 d B_1(\alpha_0 + \alpha_1)$). These conditions imply the regularity conditions (176), (177) on $\Omega_j \subset T_{L^{-j}}$, with $4\alpha_4$ instead of α_4.

The configuration u_1 is fixed and determined by U_1, hence we keep the same meaning of α_3 as before (i.e. $\alpha_3 = 16 d B_1(\alpha_0 + \alpha_1)$). The transformation $u'u_1$ has to satisfy the conditions (1.29), and u_1 satisfies (1.68). Of course $(\overline{R_0 u_1}^{k-1})(x) = 1$ for $x \in B(\Lambda_k)$ implies $(\overline{R_0 u_1}^k)(y) = 1$ for $y \in \Lambda_k$, hence u_1 satisfies the conditions (1.29) and we may write these conditions for $u'u_1$ in the following way

$$\tilde{u}'^j = \overline{R_0 u' u_1}^j (\overline{R_0 u_1}^j)^{-1} = 1 \quad \text{on} \quad \Lambda_j, \quad j = 0, 1, \ldots, k, \quad (1.78)$$

or equivalently as

$$Q'(u_1, \lambda) = 0 \quad \text{on} \quad \mathfrak{B}_k, \quad \lambda = \frac{1}{i} \log u', \quad (1.79)$$

where

$$Q'(u_1, \lambda, y) = Q'_j(u_1, \lambda, y) \quad \text{for} \quad y \in \Lambda_j \subset \mathfrak{B}_k.$$

We get a second equation for λ using (1.38):

$$RD^* \frac{1}{i\eta} \log U_1^{u'^{-1}} = 0. \quad (1.80)$$

We have

$$(U_1^{u'^{-1}})_b = u'^{-1}(b_-) U_{1,b} R_{0,b} u'(b_+) = e^{i\eta R(u'^{-1}(b_-))A_b} e^{-i\lambda(b_-)} e^{iR_{0,b}\lambda(b_+)}$$

$$= e^{i\eta R(u'^{-1}(b_-))A_b} e^{-i\lambda(b_-)} e^{i\lambda(b_-)+i\eta(D\lambda)(b)}, \quad (1.81)$$

and using the formulas (32), (36)–(41) [3] we obtain

$$(U_1^{u'^{-1}})_b = e^{i\eta R(u'^{-1}(b_-))A_b} e^{i\eta g(i \, \mathrm{ad}_{\lambda(b_-)})(D\lambda)(b) + i\eta^2 \mathfrak{F}_1(\lambda(b_-),(D\lambda)(b))}, \quad (1.82)$$

where the Lie algebra valued function $\mathfrak{F}_1$ satisfies the bound

$$|\mathfrak{F}_1(\lambda(b_-),(D\lambda)(b))| \leqq O(1)|(D\lambda)(b)|^2. \quad (1.83)$$

We assume of course that α_4 is sufficiently small. Using (28) and (31) [3], we get

$$\frac{1}{i\eta} \log(U_1^{u'^{-1}})_b = R(u'^{-1}(b_-))A_b + g(i \, \mathrm{ad}_{\lambda(b_-)})(D\lambda)(b) + \eta \mathfrak{F}_2(\lambda(b_-),(D\lambda)(b),A_b),$$

$$(1.84)$$

where

$$|\mathfrak{F}_2(\lambda(b_-),(D\lambda)(b),A_b)| \leqq O(1)|A_b||(D\lambda)(b)|. \quad (1.85)$$

Now we apply the derivative D^* to the expression on the right-hand side of (1.84). ηD^* is a bounded operator, hence $D^* \eta \mathfrak{F}_1$ satisfies the bound (1.85). The operators $R(u'^{-1}(b_-)) = e^{-i \, \mathrm{ad}_{\lambda(b_-)}}$ and $g(i \, \mathrm{ad}_{\lambda(b_-)})$ are given by power series in $\mathrm{ad}_{\lambda(b_-)}$. Let us

consider for example a term $(\mathrm{ad}_{\lambda(b_-)})^n A_b$,

$$\eta D_\mu^* (\mathrm{ad}_{\lambda(x)})^n A_\mu(x) = R(U_0(x, x - \eta e_\mu)) (\mathrm{ad}_{\lambda(x - \eta e_\mu)})^n A_\mu(x - \eta e_\mu) - (\mathrm{ad}_{\lambda(x)})^n A_\mu(x)$$

$$= (\mathrm{ad}_{R(U_0(x, x - \eta e_\mu)) \lambda(x - \eta e_\mu)})^n R(U_0(x, x - \eta e_\mu)) A_\mu(x - \eta e_\mu)$$

$$- (\mathrm{ad}_{\lambda(x)})^n A_\mu(x)$$

$$= (\eta \, \mathrm{ad}_{(D_\mu^* \lambda)(x)} + \mathrm{ad}_{\lambda(x)})^n (\eta (D_\mu^* A_\mu)(x) + A_\mu(x))$$

$$- (\mathrm{ad}_{\lambda(x)})^n A_\mu(x), \tag{1.86}$$

hence

$$D_\mu^* (\mathrm{ad}_{\lambda(x)})^n A_\mu(x) = (\mathrm{ad}_{\lambda(x)})^n (D_\mu^* A_\mu)(x) + P_n(\lambda(x), D_\mu^* \lambda(x), A_\mu(x), A_\mu(x - \eta e_\mu)), \tag{1.87}$$

and

$$\left(D^* \frac{1}{i\eta} \log U_1^{u'-1} \right)(x) = e^{-i \, \mathrm{ad}_{\lambda(x)}} (D^* A)(x) + g(i \, \mathrm{ad}_{\lambda(x)})(D^* D\lambda)(x) + \mathfrak{F}_3(\lambda(x), D\lambda, A), \tag{1.88}$$

where $\mathfrak{F}_3$ depends on $(D\lambda)(b)$, A_b for $b \in st(x)$ and satisfies

$$|\mathfrak{F}_3(\lambda(x), D\lambda, A)| \leq O(1)|D\lambda| |A|. \tag{1.89}$$

Equations (1.79), (1.80) can be written as

$$Rg(i \, \mathrm{ad}_\lambda) D^* D\lambda + \mathrm{Re}^{-i \, \mathrm{ad}_\lambda} D^* A + R\mathfrak{F}_3(\lambda, D\lambda, A) = 0,$$

$$Q'(u_1, \lambda) = 0. \tag{1.90}$$

The function $Q'(u_1, \lambda)$ is almost equal to $Q'\lambda$, the linear averaging operator used in the definition of the operator R, the error is of second order in α_3, α_4. If it was equal to $Q'\lambda$, then $D^* D\lambda = \Delta\lambda$ would belong to the subspace $R = \Delta N(Q')$ and we would have

$$Rg(i \, \mathrm{ad}_\lambda) \Delta\lambda = R\Delta\lambda + O(\alpha_4) \Delta\lambda = (\Delta + Q'^* a Q') \lambda + O(\alpha_4) \Delta\lambda.$$

Thus the main linear part in λ of the left-hand side of (1.90) would be given by the invertible operator, the remaining terms being small. We will reduce the problem (1.90) to such a situation by a change of variables. We want to find a transformation of variables λ which linearizes the function $Q'(u_1, \lambda)$, more exactly transforms it into the function $Q'\lambda$. This transformation has to be applied to Eqs. (1.90) also, hence it has to be sufficiently regular. Of course such a problem is highly indeterminate because a number of variables involved is much bigger than a number of conditions. In the next section we will find one solution of this problem. Let us introduce the operators

$$H' = G'^2 Q'^* (Q' G'^2 Q'^*)^{-1}, \qquad G' = (\Delta + Q'^* a Q')^{-1}. \tag{1.91}$$

They were investigated in [4], and the following inequality can be obtained from the results of this paper:

$$|(H'X)(x)|, |(\nabla H'X)(x)| \leq B_0'[1, (L^j \eta)^{-1}]|X| \quad \text{for} \quad x \in \Omega_j, \tag{1.92}$$

where

$$(H'X)(x) = \sum_{y' \in \mathfrak{B}_k} (L^{j'}\eta)^d H'(x, y') X(y'),$$

and B_0' is an absolute constant (depending on d and L only).

In the next section we will prove that for α_3, α_4 sufficiently small there exists a function $D'(u_1, \lambda)$ such that the change of variables $\lambda \to \lambda - H'D'(u_1, \lambda)$ transforms the function $Q'(u_1, \lambda)$ into $Q'\lambda$. The function $D'(u_1, \lambda)$ is analytic in λ, defined on a set of λ satisfying (1.77) with $1/2\,\alpha_4$ instead of α_4, and is bounded by $C_2'(\alpha_3 + \alpha_4)\alpha_4$, and also by $\dfrac{1}{2B_0'}\alpha_4$. This change of variables applied to Eqs. (1.90) gives the following equations

$$Rg(i\,\mathrm{ad}_{\lambda - H'D'(u_1, \lambda)})\Delta(\lambda - H'D'(u_1, \lambda)) - R\exp(-i\,\mathrm{ad}_{\lambda - H'D'(u_1, \lambda)})D^*A$$
$$- R\mathfrak{F}_3(\lambda - H'D'(u_1, \lambda), D\lambda - DH'D'(u_1, \lambda), A) = 0,$$
$$Q'\lambda = 0. \tag{1.93}$$

The first term in the first equation can be written as

$$[I + R[g(i\,\mathrm{ad}_{\lambda - H'D'(u_1, \lambda)}) - 1]R]\Delta\lambda, \tag{1.94}$$

because $R\Delta\lambda = \Delta\lambda$ for λ satisfying $Q'\lambda = 0$. For α_4 small the second operator above has a small norm. Let us recall that from Theorems 3.1, 3.2 of [4] it follows that $|Rf| \le B_0'|f|$, hence the operator $R[\ldots]R$ has a L^∞ norm bounded by $O(\alpha_4)B_0'^2$. We assume that α_4 is so small that $O(\alpha_4)B_0'^2 \le \frac{1}{2}$. Then the operator in the square bracket in (1.94) is invertible. Multiplying Eq. (1.93) by an inverse operator we obtain the equations

$$\Delta\lambda - RD^*A - R\mathfrak{F}_4(\lambda, D\lambda, A, D^*A) = 0, \qquad Q'\lambda = 0. \tag{1.95}$$

Now we will investigate properties of the function $R\mathfrak{F}_4$. Let us denote by V the operator $g(\ldots) - 1$ in the internal bracket $[\ldots]$ in (1.94). It is a very simple local operator of the form $(Vf)(x) = V(x)f(x)$, where $V(x)$ is a linear operator on the Lie algebra $\mathfrak{g}$, satisfying the bound $|V| \le O(\alpha_4)$. The operator in (1.94) acting on $\Delta\lambda$ is equal to $I + RVR$ and its inverse is given by the Neumann series

$$(I + RVR)^{-1} = I + \sum_{n=1}^{\infty} (-1)^n (RVR)^n = I + \sum_{n=1}^{\infty} (-1)^n R(VR)^n. \tag{1.96}$$

Let us consider the term with the function $\mathfrak{F}_3$ in Eq. (1.93). Using the inequality (1.89) we can bound a value of this function at a point $x \in \Omega_j$ by

$$O(1)\max_{b \in st(x)} |(D\lambda)(b) - (DH'D'(u_1, \lambda))(b)|\,|A_b| < O(1)\alpha_4(L^j\eta)^{-1} B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1}$$

$$= O(1)B_1(\alpha_0 + \alpha_1)\alpha_4(L^j\eta)^{-2}. \tag{1.97}$$

Thus it is a function with a bounded norm $|\cdot|_{(-2)}$.

The operators R and V are bounded in this norm, and we have

$$|Rf|_{(-2)} \le B_0'|f|_{(-2)}, \qquad |Vf|_{(-2)} \le O(\alpha_4)|f|_{(-2)}. \tag{1.98}$$

This and (1.96) implies that the operator $(I + RVR)^{-1}$ applied to the last term in (1.93) gives a term bounded by $O(1)B_1(\alpha_0 + \alpha_1)\alpha_4(L^j\eta)^{-2}$ on Ω_j. We get the same bound for the term with D^*A (but with RD^*A subtracted), and with $\Delta H'D'(u_1, \lambda)$. Thus we have the bound

$$|R\mathfrak{F}_4(\lambda, D\lambda, A, D^*A)| \leqq C_4'B_1(\alpha_0 + \alpha_1)\alpha_4(L^j\eta)^{-2} \quad \text{on} \quad \Omega_j \qquad (1.99)$$

for $j < k$.

Equations (1.95) can be changed into the equivalent equation

$$\lambda = G'RD^*A + G'R\mathfrak{F}_4(\lambda, D\lambda, A, D^*A). \qquad (1.100)$$

If λ is a solution of this equation, then λ satisfies both Eqs. (1.95) because $Q'G'R = Q'G'(I - G'Q'^*(Q'G'^2Q'^*)^{-1}Q'G') = 0$. Of course (1.95) imply (1.100). A solution of this equation may be interpreted as a fixed point of the transformation given by the right-hand side of the equation. Let us denote it by $\mathfrak{F}(\lambda)$. We will prove that for $\alpha_0 + \alpha_1$ sufficiently small and for α_4 chosen properly there exists exactly one fixed point of $\mathfrak{F}$.

At first we will find domains in λ which are invariant under the transformation $\mathfrak{F}$. One of the results of [4], Theorem 3.1, tells us that G' is a bounded operator from a space with the norm $|\cdot|_{(-2)}$ into a space with the norm $|\cdot|$ for functions, and the norm $|\cdot|_{(-1)}$ for their first derivatives. Thus we have

$$|\mathfrak{F}(\lambda)| \leqq B_0'B_1(\alpha_0 + \alpha_1) + B_0'C_4'B_1(\alpha_0 + \alpha_1)\alpha_4,$$
$$|D\mathfrak{F}(\lambda)| \leqq (B_0'B_1(\alpha_0 + \alpha_1) + B_0'C_4'B_1(\alpha_0 + \alpha_1)\alpha_4)(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, \qquad (1.101)$$

if λ belongs to the space

$$\{\lambda : |\lambda| < \beta\alpha_4, |D\lambda| < \beta\alpha_4(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, \quad j = 0, 1, ..., k-1\}, \quad \beta \leqq \tfrac{1}{2},$$

or simply
$$\{\lambda : |\lambda|, |D\lambda|_{(-1)} < \beta\alpha_4\}. \qquad (1.102)$$

The function $\mathfrak{F}$ is obviously an analytic function in λ and we consider configurations λ with values in the complexified algebra. The function $\mathfrak{F}$ maps the domain (1.102) into itself if $C_4'\alpha_4 \leqq 1$ and

$$B_0'(1 + C_4'\alpha_4)B_1(\alpha_0 + \alpha_1) \leqq 2B_0'B_1(\alpha_0 + \alpha_1) \leqq \beta\alpha_4,$$

hence if

$$\alpha_4 \leqq \frac{1}{C_4'}, \, \alpha_0 + \alpha_1 \leqq \frac{1}{2B_0'B_1}\beta\alpha_4. \qquad (1.103)$$

Now we will prove that for α_4 sufficiently small the transformation is contractive. We have

$$\mathfrak{F}(\lambda_1) - \mathfrak{F}(\lambda_2) = G'(R\mathfrak{F}_4(\lambda_1, D\lambda_1, A, D^*A) - R\mathfrak{F}_4(\lambda_2, D\lambda_2, A, D^*A))$$

$$= G'\int_0^1 dt \frac{d}{dt}R\mathfrak{F}_4(t\lambda_1 + (1-t)\lambda_2, tD\lambda_1 + (1-t)D\lambda_2, A, D^*A)$$

$$= G'\int_0^1 dt \frac{d}{d\tau}R\mathfrak{F}_4(t\lambda_1 + (1-t)\lambda_2 + \tau(\lambda_1 - \lambda_2), tD\lambda_1$$

$$+ (1-t)D\lambda_2 + \tau(D\lambda_1 - D\lambda_2), A, D^*A)|_{\tau=0}. \qquad (1.104)$$

Let us take λ_1, λ_2 from the domain (1.102) with $\beta = \frac{1}{4}$ and let us apply the Cauchy formula for the derivative of the analytic function in τ above. We have

$$\mathfrak{F}(\lambda_1) - \mathfrak{F}(\lambda_2) = G' \int\limits_0^1 dt \frac{1}{2\pi i} \int\limits_{|\tau| = r} \frac{d\tau}{\tau^2} R \mathfrak{F}_4(t\lambda_1 + (1-t)\lambda_2 + \tau(\lambda_1 - \lambda_2), t D\lambda_1$$
$$+ (1-t)D\lambda_2 + \tau(D\lambda_1 - D\lambda_2), A, D^*A) \tag{1.105}$$

with $r = (4 \max\{|\lambda_1 - \lambda_2|, |D\lambda_1 - D\lambda_2|_{(-1)}\})^{-1}\alpha_4$. The argument $t\lambda_1 + (1-t)\lambda_2 + \tau(\lambda_1 - \lambda_2)$ belongs to the domain (1.102) with $\beta = \frac{1}{2}$, and we use the inequality (1.99). We get

$$|\mathfrak{F}(\lambda_1) - \mathfrak{F}(\lambda_2)|, |D(\mathfrak{F}(\lambda_1) - \mathfrak{F}(\lambda_2))|_{(-1)}$$
$$\leq B_0' C_4' B_1(\alpha_0 + \alpha_1) 4 \max\{|\lambda_1 - \lambda_2|, |D\lambda_1 - D\lambda_2|_{(-1)}\}, \tag{1.106}$$

and by (1.103)

$$4 C_4' B_0' B_1(\alpha_0 + \alpha_1) \leq 2 C_4' \beta \alpha_4 = \tfrac{1}{2} C_4' \alpha_4 \leq \tfrac{1}{2} \quad \text{for} \quad \beta = \tfrac{1}{4}.$$

Thus the transformation $\mathfrak{F}$ maps the domain (1.102) with $\beta = \frac{1}{4}$ into itself and is contractive if the inequality (1.103) holds with $\beta = \frac{1}{4}$. The contraction mapping theorem implies that there exists a unique fixed point of this transformation in the considered domain. To get a best uniqueness result we have to take a largest possible α_4. It is independent of $\alpha_0 + \alpha_1$. To get best bounds on the solution we have to take a smallest possible α_4, hence $\alpha_4 = 8 B_0' B_1(\alpha_0 + \alpha_1)$, and then we have to assume that $\alpha_0 + \alpha_1$ is so small that all the conditions on $\alpha_0 + \alpha_1$, α_4 are satisfied.

The solution λ of Eq. (1.100) determines the configuration $u' = \exp[i(\lambda - H'D'(u_1, \lambda))]$ we are looking for. Let us formulate the obtained results in the following

Proposition 5. *There exist positive constants c_2, c_3, depending on d and L only, such that for an arbitrary configuration U_1 satisfying (1.69), and for the configuration u_1 determined by U_1 and satisfying (1.68), (1.73), (1.74), if $\alpha_0 + \alpha_1 \leq c_2$, then there exists a configuration $u' = e^{i\lambda}$ satisfying the equations*

$$RD^* \frac{1}{i\eta} \log U_1^{u'^{-1}} = 0, \quad \overline{R_0 u' u_1}^j = 1 \quad \text{on} \quad \Lambda_j, j = 0, 1, \ldots, k \tag{1.107}$$

and the bounds

$$|\lambda|, |D\lambda|_{(-1)} < 8 B_0' B_1(\alpha_0 + \alpha_1). \tag{1.108}$$

Such a configuration u' is unique in the domain

$$|\lambda|, |D\lambda|_{(-1)} < c_3. \tag{1.109}$$

The part of this proposition concerning the existence of u' was proved completely. The uniqueness follows from the fact that the image of a set $\{\lambda : |\lambda|, |D\lambda|_{(-1)} < \frac{1}{2}\alpha_4\}$ by the transformation $\lambda \to \lambda - H'D'(u_1, \lambda)$ contains the set $\{\lambda' : |\lambda'|, |D\lambda'|_{(-1)} < \frac{1}{4}\alpha_4$ for α_4 sufficiently small. This follows from results of the next section.

Now let us consider implications of the proposition concerning the inductive proof of the Theorem 4. It gives us a gauge transformation $u = u' u_1$ such that the conditions (1.29), (1.37), (1.38) are satisfied. This follows from the inductive

construction of U_1 and from (1.107). The bounds (1.108) and the representation (1.84) imply

$$\left|\frac{1}{i\eta}\log(U_1^{u'^{-1}})_b\right| \leqq |A_b| + |g(i\,\mathrm{ad}_{\lambda(b_-)})(D\lambda)(b)| + O(1)\eta|(D\lambda)(b)|\,|A_b|$$

$$\leqq 2|A_b| + 2|(D\lambda)(b)| < 2B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1}$$
$$+ 16B_0' B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1}$$
$$= (2 + 16B_0')B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, j = 0, 1, ..., k-1,$$

$$(1.110)$$

hence

$$\left|\frac{1}{i\eta}\log(U_1^{u'^{-1}})_b\right| < (2 + 16B_0')LB_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1}$$

$$= C_1' B_1(\alpha_0 + \alpha_1)(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, j = 0, 1, ..., k, \qquad (1.111)$$

where C_1' is an absolute constant depending on d and L only. Thus all the assumptions of Proposition 3 are satisfied and for $\alpha_0 + \alpha_1$ sufficiently small it implies Theorem 4, except the uniqueness statement.

To prove the uniqueness let us assume that there are two transformations u_1, u_2 satisfying the conditions of the Theorem 4. Then we have two configurations $U_1 = U^{u_1^{-1}}, U_2 = U^{u_2^{-1}}, U_1 = e^{i\eta A_1}, U_2 = e^{i\eta A_2}$. They satisfy all the conditions of the theorem, hence A_2 satisfies (1.37), (1.38), and both configurations A_1, A_2 satisfy the bounds (1.62). This implies that $u' = u_2 u_1^{-1}$ satisfies the regularity conditions (1.73), (1.74) with k instead of $k-1$ and with a worse constant. This follows from Proposition 8 of [3]. We have

$$u'u_1 = u_2, \quad \text{hence} \quad \overline{R_0 u'u_1}^j = 1 \quad \text{on} \quad \Lambda_j, \quad j = 0, 1, ..., k, \quad U_1^{u'^{-1}} = U_2,$$

$$(1.112)$$

thus u' is a solution of the problem described in Proposition 5. For $\alpha_0 + \alpha_1$ sufficiently small the assumptions of this proposition are satisfied and we have the uniqueness property. In the considered case a configuration identically equal to 1 is a solution also, because U_1 satisfies the same conditions as U_2. The uniqueness implies $u' = 1$, or $u_1 = u_2$. Thus we have completed the proof of Theorem 4, and Theorem 2.

E. A Construction of the Linearizing Transformation

Let us consider the function $Q'(u_1, \lambda')$. The configuration u_1 is fixed and we omit u_1 in the notations below. We want to construct a function $D'(\lambda)$ for α_3, α_4 sufficiently small, whose values are configurations $X : \mathfrak{B}_k \to \mathfrak{g}$, such that the transformation

$$\lambda' = \lambda - H'D(\lambda) \tag{1.113}$$

changes the function $Q'(\lambda')$ into the linear function $Q'\lambda$:

$$Q'(\lambda') = Q'(\lambda - H'D'(\lambda)) = Q'\lambda. \tag{1.114}$$

Using the equality (213) [3] the above equation can be written in the following form:

$$Q'\lambda - Q'H'D'(\lambda) + C'(\lambda - H'D'(\lambda)) = Q'\lambda, \qquad (1.115)$$

hence

$$-D'(\lambda) + C'(\lambda - H'D'(\lambda)) = 0. \qquad (1.116)$$

The function $D'(\lambda)$ is a solution of the equation

$$C'(\lambda - H'X) = X, \qquad (1.117)$$

or a fixed point of the transformation

$$X \to C'(\lambda - H'X). \qquad (1.118)$$

The configurations u_1, λ are treated as parameters and a solution of Eq. (1.117) depends on them.

At first we will find a set of functions X which is transformed into itself by the transformation (1.118). Let us assume that

$$|\lambda| < \tfrac{1}{2}\alpha_4, |D\lambda| < \tfrac{1}{2}\alpha_4(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, \quad |X| < \frac{1}{2B_0'}\alpha_4. \qquad (1.119)$$

This implies

$$|\lambda - H'X| < \alpha_4, |D(\lambda - H'X)| < \alpha_4(L^j\eta)^{-1} \quad \text{on} \quad \Omega_j, \qquad (1.120)$$

and by the inequality (214) we have

$$|C'(\lambda - H'X)| < C_2'(\alpha_3 + \alpha_4)\alpha_4, \qquad (1.121)$$

where the constant $O(1)$ in (214) [3] was denoted by C_2'. The transformation (1.118) maps the set (1.119) of X's into itself if

$$C_2'(\alpha_3 + \alpha_4)\alpha_4 \leqq \frac{1}{2B_0'}\alpha_4, \quad \text{or} \quad \alpha_3 + \alpha_4 \leqq \frac{1}{2B_0'C_2'}.$$

We may admit configurations λ, X with values in the complexified algebra $\mathfrak{g}^c$ and all the above equations and inequalities are valid also.

We will prove that the mapping (1.118) is a contraction on the set (1.119) for α_3, α_4 sufficiently small. We have

$$C'(\lambda - H'X_1) - C'(\lambda - H'X_2) = \int_0^1 dt\, \frac{d}{dt} C'(\lambda - H'(tX_1 + (1-t)X_2))$$

$$= -\int_0^1 dt\, \left\langle \frac{\delta}{\delta\lambda} C'(\lambda - H'(tX_1 + (1-t)X_2)), H'(X_1 - X_2) \right\rangle, \qquad (1.122)$$

where the functional derivative $\dfrac{\delta}{\delta\lambda} C'(\lambda)$ is defined as the linear mapping

$$\left\langle \frac{\delta}{\delta\lambda} C'(\lambda), \lambda_0 \right\rangle = \frac{d}{d\tau} C'(\lambda + \tau\lambda_0)|_{\tau=0}. \qquad (1.123)$$

Using the analyticity properties of $C'(\lambda)$, the derivative above can be written as

$$\left\langle \frac{\delta}{\delta\lambda} C'(\lambda), \lambda_0 \right\rangle = \frac{1}{2\pi i} \int_{|\tau|=r} d\tau \frac{1}{\tau^2} C'(\lambda + \tau\lambda_0). \tag{1.124}$$

Taking $r = (2\max\{|\lambda_0|, |D\lambda_0|\})^{-1}\alpha_4$, we get the estimate

$$\left| \left\langle \frac{\delta}{\delta\lambda} C'(\lambda), \lambda_0 \right\rangle \right| \leq C'_2 \, 2\max\{|\lambda_0|, |D\lambda_0|\} (\alpha_3 + \alpha_4). \tag{1.125}$$

Applying it to the expression (1.122) we have

$$|C'(\lambda - H'X_1) - C'(\lambda - H'X_2)| \leq C'_2 \, 2B'_0(\alpha_3 + \alpha_4)|X_1 - X_2|,$$

hence the mapping (1.118) is contractive if e.g. $\alpha_3 + \alpha_4 \leq \dfrac{1}{4B'_0 C'_2}$. If the last condition is satisfied, then the mapping transforms the set $\left\{|X| < \dfrac{1}{2B'_0}\alpha_4\right\}$ into itself and is contractive on this set. Thus by the contraction mapping theorem there exists exactly one solution of Eq. (1.117). This solution is an analytic function of λ defined on the set of λ satisfying (1.119). We take $D'(\lambda)$ equal to this solution. From Eq. (1.116) we can get much better bounds on it:

$$|D'(\lambda)| = |C'(\lambda - H'D'(\lambda))| < C'_2(\alpha_3 + \alpha_4)\alpha_4.$$

They imply in particular that the mapping (1.113) transforms the set $\{\lambda : |\lambda| < \frac{1}{2}\alpha_4,$ $|D\lambda| < \frac{1}{2}\alpha_4(L^j\eta)^{-1}$ on $\Omega_j\}$ onto a set containing $\{\lambda' : |\lambda'| < \frac{1}{4}\alpha_4, |D\lambda'| < \frac{1}{4}\alpha_4(L^j\eta)^{-1}$ on $\Omega_j\}$ for α_3, α_4 sufficiently small.

F. Removal of the Additional Regularity Assumption

We will prove that the additional regularity assumption (3.35) in (1.33) is a consequence of the fundamental assumption $U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0)$, thus of the regularity conditions (1.7)–(1.9). In fact we will prove a more general, and much more precise, theorem which will have other important applications in subsequent papers.

At first let us notice that if $\alpha_0 + \alpha_1 \leq c_1$ and

$$U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0), |\bar{U}_0^j - 1| < \alpha_1 \quad \text{on} \quad \Lambda_j, \ j = 0, 1, \ldots, k, \tag{1.126}$$

then Theorem 2 implies that there exists a gauge transformation u such that

$$U_0^{u^{-1}} = e^{i\eta A}, |A|, |\nabla^\eta A| < B_1(\alpha_0 + \alpha_1)[(L^j\eta)^{-1}, (L^j\eta)^{-2}] \quad \text{on} \quad \Omega_j, \ j = 0, 1, \ldots, k. \tag{1.127}$$

We have to transform U_0 into a configuration satisfying the axial gauge conditions (1.15), i.e. we get a configuration belonging to $Ax_k(\mathfrak{B}_k, 1)$. It is achieved by applying a gauge transformation u_0 satisfying the conditions $u_0(y) = 1$ for $y \in \mathfrak{B}_k$, hence the configuration $U_0^{u_0^{-1}}$ satisfies (1.126). Then we apply Theorem 2 to the configurations $U_0^{u_0^{-1}}, 1$ in the place of $U'U_0, U_0$, and we get (1.127). The configuration

identically equal to 1 satisfies, of course, all possible regularity conditions. Let us notice that the gradient $V^\eta A$ in (1.127) is the ordinary finite difference gradient. Of course (1.127) implies the regularity condition (3.35). We will apply this remark to specially chosen $\{\Omega_j\}$.

Now let us take a cube $\square \subset \Omega_0$ for which we want to prove the condition (3.35). A size of this cube depends on an index j indicating a scale we are interested in. We assume that the cube $\square$ is contained in Ω_j, and not in Ω_{j+1}. Further, we assume that it is a union of cubes of the size $R_1 M_1 L^j \eta$, where R_1, M_1 are smallest integers for which all the theorems of the papers [2, 4] are valid. To prove condition (3.35) it is enough to take one such cube. Having in view future applications we take a size of $\square$ equal to $ML^j \eta$, where M is a multiple of $R_1 M_1$. Without a loss of generality we can assume also that $j = k$, because we can drop out the domains $\Omega_{j'}, j' > j$, from our assumptions, and change a scale. Then we have to admit the value $k = 0$. In this case the reasoning is much simpler and will be covered by our analysis of a general case, so we can assume that $k \geq 1$.

Let us take a sequence of cubes $\square_0, \square_1, ..., \square_{k-1}, \square_k, \square$, such that $\square_j \supset \square_{j+1}$ and a distance between boundaries of these cubes is equal to $R_1 M_1 L^j \eta$. Thus a distance of the boundary of $\square_0$ to $\square$ is equal to

$$\sum_{j=0}^{k} R_1 M_1 L^j \eta < \frac{1}{1 - L^{-1}} R_1 M_1 \leq 2 R_1 M_1 .$$

We cover $\square_0$ by a smallest family of cubes of the size $R_1 M_1$. A sum of these cubes is a cube which we denote by $\tilde{\square}$. A distance of its boundary to $\square$ is equal to $2 R_1 M_1$. We assume that RM is bigger than $R_1 M_1$, for example $L^{-1} RM > 2 d R_1 M_1$, thus we have $\tilde{\square} \subset \Omega_{k-1}$. Of course we assume also that these constructions are compatible with the block structure of the lattice T, i.e. for every j the cube $\square_j$ is a sum of the big blocks of the lattice T_{L-j}.

Let us consider the configuration U_0 on the cube $\tilde{\square}$. By the assumptions (1.7)–(1.9) and Proposition 2 from [3] we have

$$|\bar{U}_0^j(\partial p) - 1| < 2 \alpha_0 L^2 (L^j \eta)^2, \quad p \subset \tilde{\square}^{(j)}, \quad j = 0, 1, ..., k. \tag{1.128}$$

We apply a gauge transformation to U_0, such that the gauge transformed configuration U_0' satisfies the axial gauge conditions (1.15), and $\bar{U}_0'^k$ satisfies the global axial gauge conditions on $\tilde{\square}^{(k)}$ of the type introduced in the proof of Proposition 1 in [3], i.e. $\bar{U}_0'^k(\Gamma_{y,x}) = 1$ for $x \in \tilde{\square}^{(k)}$, where y is a center of $\tilde{\square}^{(k)}$. These last conditions imply

$$|\bar{U}_0'^k(x, x') - 1| < |x - y| 2 L^2 \alpha_0 \leq (M + 4 R_1 M_1) d L^2 \alpha_0, \quad \langle x, x' \rangle \subset \tilde{\square}^{(k)}. \tag{1.129}$$

We assume that α_0 is so small that the number on the right-hand side above is still small and all the theorems on averaging operations are valid. In particular this means $\alpha_0 \leq O(1) M^{-1}$, where $O(1)$ has to be a small number. If we are proving the condition (3.35), then $M = R_1 M_1$ and the restriction is $\alpha_0 \leq O(1)(R_1 M_1)^{-1}$. R_1, M_1 are absolute constants, hence the last restriction is of the same kind as the other restrictions on α_0. Applying Lemma 1 many times, as in the proof of (1.65), (1.66),

we get

$$|\bar{U}_0'^j(x, x') - 1| < 8d^2 L^2 (L^{-2(k-j-1)} + \ldots + L^{-2} + 1)\alpha_0$$
$$+ (M + 4R_1 M_1)dL^2\alpha_0 < 11d^2 L^2\alpha_0 + 5dL^2 M\alpha_0 < 6dL^2 M\alpha_0, \langle x, x' \rangle \subset \tilde{\square}^{(j)}. \tag{1.130}$$

The sequence of cubes $\{\square_j\}$ is an admissible family of subsets satisfying (1.3), (1.4), $\square_k \subset \Omega_{k-1}$, $\square_j \subset \Omega_j$, $j < k$. Let us define

$$\Lambda_j' = \square_j^{(j)} \backslash \square_{j+1}^{(j)}, j = 1, \ldots, k-1, \Lambda_k' = \square_k^{(k)}, \Lambda_0' = T \backslash \square_1,$$
$$\mathfrak{C}_k = \bigcup_{j=0}^{k} \Lambda_j', \tag{1.131}$$

and let us define a configuration U_0'' as equal to U_0' on $\tilde{\square}$, and equal to 1 outside $\tilde{\square}$. It satisfies the conditions

$$U_0'' \in \mathfrak{U}_k(\{\square_j\}, L^3\alpha_0) \cap Ax_k(\mathfrak{C}_k, 1), \tag{1.132}$$

$$|\bar{U}_0''^j - 1| < 6dL^2 M\alpha_0 \quad \text{on} \quad \square_j^{(j)}, \quad j = 0, 1, \ldots, k, \tag{1.133}$$

by the construction of U_0'', and the inequality (1.130). If $7dL^2 M\alpha_0 \leq c_1$, then the assumptions of Theorem 4 are satisfied for the pair of configurations 1, U_0'', thus there exists a gauge transformation u such that $U_1 = U_0''^{u^{-1}}$ satisfies the conditions (1.36)–(1.39). Especially we have for $M = R_1 M_1$,

$$U_1 = e^{i\eta A}, |A|, |\nabla^\eta A| < 7dL^2 B_1 R_1 M_1 \alpha_0 \quad \text{on} \quad \square. \tag{1.134}$$

The configuration U_1 in a neighborhood of $\square$ is obtained from U_0 by a gauge transformation, hence for α_0 sufficiently small we have proved the regularity condition (3.35).

This implies that we can drop out this condition from the assumption (1.33). We formulate the more general result in

Proposition 6. *Let U_0, U_0', $\square$, $\tilde{\square}$ be as described above, and let $7dL^2 M\alpha_0 \leq c_1$. There exists a gauge transformation u defined on $\tilde{\square}$ and such, that*

$$U_0^{u^{-1}} = U_1 = e^{i\eta A} \quad \text{on} \quad \tilde{\square}, \tag{1.135}$$

$$L^j\eta|A|, (L^j\eta)^2|\nabla^\eta A|, (L^j\eta)^3|\partial^{\eta*}\partial^\eta A|, (L^j\eta)^3|\Delta^\eta A| < 7dL^2 B_1 M\alpha_0 \quad \text{on} \quad \square_j, \tag{1.136}$$

$$Q_k(\eta A) = \frac{1}{i}\log\overline{U_0'}^k \quad \text{on} \quad \square^{(k)}, \left|\frac{1}{i}\log\overline{U_0'}^k(x, x')\right| < |x - y|4\alpha_0 \leq 2dM\alpha_0$$

$$\text{for} \quad \langle x, x'\rangle \subset \square^{(k)}, y \text{ is a center of } \square, \tag{1.137}$$

$$R\partial^{\eta*}A = 0, \text{ the operator } R \text{ is determined by } \{\square_j\}. \tag{1.138}$$

The functions and derivatives above are defined without any external gauge field configuration (or the configuration is equal to 1).

G. An Inverse Theorem to Theorem 2

In this section we will prove a theorem which is, in a certain sense, inverse to Theorem 2. We assume that we are given a gauge field configuration U_0,

$$U_0 \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0), \tag{1.139}$$

and a Lie algebra valued configuration A satisfying

$$L^j\eta|A|, (L^j\eta)^2 |\nabla^\eta_{U_0} A|, (L^j\eta)^3 |D^{\eta*}_{U_0} D^\eta_{U_0} A| < \alpha_2 \quad \text{on} \quad \Omega_j. \tag{1.140}$$

We consider the configuration $U_1 U_0$, $U_1 = e^{i\eta A}$. We would like to prove bounds (1.7)–(1.9), i.e. to find spaces $\mathfrak{U}_k(\{\Omega_j\}, \alpha)$ to which this configuration belongs.

Let us start with estimates of $\partial U_1 U_0 - 1$. The identity (1.21), the bound (1.47) and the assumptions (1.139), (1.140) imply

$$|(U_1 U_0)(\partial p) - 1| \le |U_0(\partial p) - 1| + \eta^2 |(D^\eta_{U_0} A)(p)| + \tfrac{1}{2}\eta^2 (\partial|A|(p))^2$$
$$< (\alpha_0 + 2\alpha_2 + 8\alpha_2^2) L^{-2j} \quad \text{on} \quad \Omega_j. \tag{1.141}$$

The basic estimate (1.54) and the assumptions imply

$$|D^{\eta*}_{U_1 U_0} \partial U_1 U_0| < (\alpha_0 + \alpha_2 + 36d\alpha_2^2 + 50d\alpha_2^3 + 10d\alpha_0\alpha_2)(L^j\eta)^{-3}\eta^2. \tag{1.142}$$

To simplify formulations we assume that α_0, α_2 are so small that $\alpha_0 + 2\alpha_2 + 8\alpha_2^2 \le \alpha_0 + 3\alpha_2$, $\alpha_0 + \alpha_2 + 36d\alpha_2^2 + 50d\alpha_2^3 + 10d\alpha_0\alpha_2 \le \alpha_0 + 2\alpha_2$. The Proposition 4 from [3] gives also

$$|Q_j(U_0, \eta A)| < 2\alpha_2 \quad \text{on} \quad \Omega_j^{(j)}. \tag{1.143}$$

Let us take a gauge transformation u satisfying the conditions (1.29) and such that the configuration $U' = (U_1 U_0)^u U_0^{-1}$ satisfies the axial gauge conditions (1.19). This gauge transformation is determined uniquely. The above bounds imply the following

Proposition 7. *If the configurations U_0, A satisfy (1.139), (1.140), then for α_0, α_2 sufficiently small we have*

$$U' U_0 = (U_1 U_0)^u \in \mathfrak{U}_k(\{\Omega_j\}, \alpha_0 + 3\alpha_2) \cap Ax_k(\mathfrak{B}_k, U_0), \tag{1.144}$$

$$|\overline{(U' U_0)}^j - \bar{U}_0^j| = |\exp iQ_j(U_0, \eta A) - 1| < 2\alpha_2 \quad on \quad \Omega_j^{(j)}. \tag{1.145}$$

This theorem complements Theorem 2. The gauge transformations constructed there establish not only a mapping of the space (1.33)–(1.35) into the space (1.36)–(1.39), but a kind of isomorphism also, in the sense that the inverse mapping transforms the second space, defined by some constant α_2 instead of $B_1(\alpha_0 + \alpha_1)$, into the first one with properly chosen α_0, α_1.

H. Generalizations of the Main Results

Let us describe two possible generalizations. The first is connected with the fundamental gauge fixing condition (1.38) in the formulations of Theorems 2, 4. It is written with 0 on the right-hand side, but nothing prevents us to consider a more

general condition of the form

$$R(U_0)D_{U_0}^{\eta*}A = f, \tag{1.146}$$

where f is a function from the space $R(U_0)$, i.e. a Lie algebra valued function defined on Ω_0 and satisfying $R(U_0)f = f$. Of course we have to assume that f is in a sufficiently small neighborhood of 0, for example it is enough to assume that $|f|_{(-2)} < \gamma(\alpha_0 + \alpha_1)$ with a positive, not too big, constant γ, e.g. $\gamma = 1$. Inspecting the proofs of the theorems and propositions we can see easily that they work in this more general situation almost without any changes, only some constants change their numerical values. Thus we have the following generalization of Theorem 2.

Theorem 8. *There exist constants B_1, $B_2(\beta_0)$, c_1 such that for arbitrary U_0, U'_0 satisfying (1.33)–(1.35) with $\alpha_0 + \alpha_1 \leqq c_1$, and for an arbitrary function f from the space $R(U_0)$ satisfying the bound $|f|_{(-2)} < \gamma(\alpha_0 + \alpha_1)$, there exists exactly one gauge transformation u satisfying (1.29) and such, that the conditions (1.36), (1.37), (1.39), and (1.146) hold for the configuration $U_1 = U'^{u^{-1}}$. The constants B_1, $B_2(\beta_0)$ are as in Theorems 2, 4, the constant c_1 depends on d, L and γ.*

At one point in the next paper we will use this theorem.

A second generalization is connected with the second regularity condition (1.9). It is a strong assumption, and although it is a natural one because it is satisfied by solutions of the variational problem described in [1], it would be desirable to have a more general condition, and a more general result on Landau gauges. In fact it is possible to have such a generalization. Instead of condition (1.9), we assume the following, weaker Hölder regularity condition,

$$\frac{1}{|x-x'|^{\beta_0}}|R(U(\Gamma_{x,x'}))U(\partial p_{\mu\nu}(x')) - U(\partial p_{\mu\nu}(x))| < \alpha_0 L^{-2j}(L^j\eta)^{-\beta_0}$$

for

$$x, x' \in \Omega_j, (L^j\eta)^{-1}|x-x'| \leqq 1, \ 1 \leqq \mu < \nu \leqq d, \quad j = 0, 1, ..., k, \tag{1.147}$$

and for some positive β_0, $0 < \beta_0 \leqq 1$. With these conditions we can prove again an analog of Theorem 2, the only difference is that the configuration U_1, or rather $A = \frac{1}{i\eta}\log U_1$, satisfies conditions (1.36)–(1.38), with $\beta < \beta_0$, but of course not condition (1.39). Unfortunately a proof of this result is rather unelementary, and although it follows the main line of arguments of this paper, it makes use of operators, e.g. $G'^{\beta}(U_0)$, and their properties. It would be necessary to extend in an essential way our theory of Green's functions in [4], to include necessary results. We will not need the above generalization, so let us stop the discussion mentioning only the possibility of such results.

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