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Regularized Determinants for Quantum Field Theories with Fermions

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Abstract. A new type of regularized determinant for the ratio of two Dirac operators is presented. Some of its properties with application to the chiral anomaly are given.

1. Introduction

The Fredholm determinants have been a tool for formulating quantum field theories involving fermions since they appeared in the Matthews and Salam formulae which express the Green's functions of a Yukawa theory [1]. In order to avoid the divergences of the determinants in those formulae, some regularization procedure is needed [2–4].

Meanwhile, Fujikawa argued, in the formalism of quantum field theories using the integration on a Grassman algebra [5], about a certain kind of regularization necessary to get the chiral anomaly [6] correctly [7].

The purpose of this paper is to present a definition of a new type of regularized determinant for the ratio of two Dirac operators in which Fujikawa's idea is adopted. The intuitive idea of definition of our regularized determinant is the following. Suppose that we get an operator D_1 from another operator D_0 by performing successive infinitesimal transformations. Then the determinant of $D_1D_0^{-1}$ is the product of the Jacobians of all the infinitesimal transformations. Our regularized determinant of $D_1D_0^{-1}$ is obtained by replacing these Jacobians with their regularized ones introduced in [7, 8].

We explain this procedure more explicitly. Let D be a suitable operator-valued map defined on the interval $[0,1]$ which connects D_0 and D_1, i.e. $D(1)=D_1$, $D(0)=D_0$. If the operator $(dD(s)/ds)D(s)^{-1}$ is trace class for every $s\in I$, then we can get for the Fredholm determinant of $D_1D_0^{-1}$ the following expression

$$\det D_1D_0^{-1}=\exp\left(\operatorname{Tr}\int_0^1\frac{dD(s)}{ds}D(s)^{-1}\,ds\right). \tag{1.1}$$

If the right-hand side of (1.1) is not well-defined,

$$\exp\left[\operatorname{Tr}\int_0^1 \frac{dD(s)}{ds} D(s)^{-1} \exp(D(s)^2/M^2)ds\right] \tag{1.2}$$

with M an arbitrary positive constant, may have a meaning, so that (1.2) may be regarded as a regularized determinant of $D_1 D_0^{-1}$. A crucial point is that the regularized determinant defined in this way depends only on D_1 and D_0 but not on the choice of an operator-valued map D connecting D_1 and D_0.

In Sect. 2 a precise definition of our regularized determinant is given together with its properties. The main theorem concerns independence of the choice of an operator-valued map. The proof is given in Sect. 3. Section 4 gives an application to derive chiral anomaly by differentiating our regularized determinant. In the course of this derivation, the connection of our regularized determinant with the discussion in [7] becomes clear.

2. Definition of the Regularized Determinant

In this section we define a regularized determinant for the ratio of two Dirac operators and study its properties.

We consider the Hilbert space

$$H = \mathbb{C}^N \otimes \mathbb{C}^4 \otimes L^2(\mathbb{R}^4) \tag{2.1}$$

with an arbitrary fixed positive integer N, and the Banach spaces $\mathscr{L}(H)$, $\mathscr{I}_\infty(H)$ and $\mathscr{I}_1(H)$. $\mathscr{L}(H)$ is the Banach space of bounded linear operators on H with operator norm $\|\cdot\|$. $\mathscr{I}_\infty(H)[\mathscr{I}_1(H)]$ is the Banach space of the compact [trace class] operators on H with operator norm $\|\cdot\|$ [trace norm $\|\cdot\|_1$]. We also consider the Hilbert spaces

$$H_k = \mathbb{C}^N \otimes \mathbb{C}^4 \otimes H^k(\mathbb{R}^4), \qquad k = 0, 1, 2, \ldots \tag{2.2}$$

and the Banach spaces $\mathscr{L}(H_k)$, $k = 0, 1, 2, \ldots$, where $H^k(\mathbb{R}^4)$ is the k^{th} Sobolev space on $\mathbb{R}^4$. Note $H = H_0$.

By $\partial\!\!\!/$ we denote the anti-selfadjoint operator

$$\sum_{\mu=0}^{3} 1 \otimes \gamma_\mu \otimes \frac{\partial}{\partial x_\mu} \tag{2.3}$$

in H with domain H_1, where γ_μ, $\mu = 0, 1, 2, 3$, are Hermitian matrices with

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \tag{2.4}$$

Throughout this paper, we use m as a fixed positive constant called *mass*.

For each positive integer n, we consider a pair (A, B) of bounded operators on H satisfying the following condition (P_n):

$(P_n.1)$ $\partial\!\!\!/ + m + A$ and $\partial\!\!\!/ + m + B$ have bounded inverses;
$(P_n.2)$ $A(\partial\!\!\!/ + m)^{-1}$, $B(\partial\!\!\!/ + m)^{-1} \in \mathscr{I}_\infty(H)$;
$(P_n.3)$ $(A - B)(\partial\!\!\!/ + m)^{-n} \in \mathscr{I}_1(H)$;
$(P_n.4)$ $AH_k \subset H_k$ and $BH_k \subset H_k$ for $k = 0, 1, \ldots, n-1$.

We note that the closed graph theorem combined with the condition $(P_n.4)$ and the boundedness of A and B on H implies that $A \upharpoonright H_k$ and $B \upharpoonright H_k \in \mathscr{L}(H_k)$ for $k = 0, 1, \ldots, n-1$.

Given the pair (A, B), satisfying the condition (P_n), we want to find a continuous and piecewise continuously differentiable map

$$\alpha: I = [0, 1] \to \mathscr{L}(H)$$

which satisfies the following condition (Q_n):

$(Q_n.0)$ $\alpha(0) = B$, $\alpha(1) = A$;
$(Q_n.1)$ $\not{\partial} + m + \alpha(s)$ has a bounded inverse for every $s \in I$;
$(Q_n.2)$ $\alpha(s)(\not{\partial} + m)^{-1}$ is compact and, as a map: $I \to \mathscr{I}_\infty(H)$, continuous and piecewise continuously differentiable;
$(Q_n.3)$ $\alpha'(s)(\not{\partial} + m)^{-n}$ is trace class and, as a map: $I \to \mathscr{I}_1(H)$, piecewise continuous;
$(Q_n.4)$ $\alpha(s)H_k \subset H_k$ for every s and $\alpha(\cdot) \upharpoonright H_k$ is, as a map: $I \to \mathscr{L}(H_k)$, continuous for $k = 0, 1, \ldots, n-1$.

Here the piecewise continuously differentiability of α means that there exists a finite subset N of I, depending on α, such that α is continuously differentiable on $I \backslash N$. N is the set of the points of discontinuity of the first kind for the derivative α' of α.

Now given (A, B) with the property (P_n), a map α with the property (Q_n) as stated above always exists. In fact, the map $f: z \to (B + z(A - B))(\not{\partial} + m)^{-1}$ is an analytic operator-valued function in $\mathbb{C}$ such that $f(z)$ is a compact operator for each $z \in \mathbb{C}$. Since

$$(1 + f(z))^{-1} = 1 - (B + z(A - B))(\not{\partial} + m + B + z(A - B))^{-1},$$

$(1 + f(z))^{-1}$ exists at $z = 0$ and $z = 1$. The analytic Fredholm theorem [9] shows that $(1 + f(z))^{-1}$ is analytic in $\mathbb{C} \backslash S$ where S is a discrete subset of $\mathbb{C}$, and that $0, 1 \notin S$. Therefore we can find a smooth function $\zeta: I \to \mathbb{C} \backslash S$ with $\zeta(0) = 0$, $\zeta(1) = 1$. Define a map $\alpha: I \to \mathscr{L}(H)$ by $\alpha(s) = B + \zeta(s)(A - B)$. This map satisfies the condition (Q_n).

Due to the condition on α, it is easy to see that for each fixed $s \in I$, the operator $(\not{\partial} + m + \alpha(s))^2$ generates a strongly continuous semigroup $\exp t(\not{\partial} + m + \alpha(s))^2$, $t \geqq 0$. For further details, we refer to Sect. 3.

To define our regularized determinant we need the following theorem.

Theorem 1. *Let $\alpha: I \to \mathscr{L}(H)$ be a continuous and piecewise continuously differentiable map with property (Q_n) for a pair (A, B) satisfying the condition (P_n). Then*

$$\mathrm{L_M}(\alpha) \equiv \int_0^1 ds\, \alpha'(s)(\not{\partial} + m + \alpha(s))^{-1} \exp[(\not{\partial} + m + \alpha(s))^2/M^2] \tag{2.5}$$

belongs to $\mathscr{I}_1(H)$ for every $M > 0$. If $\beta: I \to \mathscr{L}(H)$ is another continuous and piecewise continuously differentiable map with property (Q_n) for the same (A, B), then there exists an integer l independent of M such that

$$\mathrm{Tr}\mathrm{L_M}(\alpha) - \mathrm{Tr}\mathrm{L_M}(\beta) = 2\pi l i. \tag{2.6}$$

Remark 1. When fermions are coupled with regularized fields as in gauge theories or a Yukawa theory, there arises the case that A and B are finite sums of operators on H of the form $T\otimes\Gamma\otimes f$. Here T and Γ are $N\times N$ and 4×4 matrices, respectively. f is a multiplication operator by a function $f(x)$ in $\mathscr{S}(\mathbb{R}^4)$, the Schwartz space of C^∞ functions of rapid decrease. In this case the pair (A,B) satisfies $(P_5.2)\sim(P_5.4)$ [10]. Moreover if A and B are anti-selfadjoint, then $(P_5.1)$ is also satisfied.

Remark 2. The condition $(P_n.4)$ and the corresponding condition $(Q_n.4)$ are imposed in order to make the operator $(\not\partial+m)^n(\not\partial+m+\alpha(s))^{-n+1}(\lambda-\not\partial-\alpha(s))^{-1}$ bounded on H for every $(s,\lambda)\in I\times\Lambda_{\sigma,\varepsilon}$. This is necessary in the proof of Theorem 1. The proof as well as the definition of $\Lambda_{\sigma,\varepsilon}$ are given in Sect. 3.

Now we define our regularized determinant for $(\not\partial+m+A)(\not\partial+m+B)^{-1}$ by

$$\mathrm{D_M}(\not\partial+m+A;\not\partial+m+B)=\exp\operatorname{Tr}\mathrm{L_M}(\alpha). \tag{2.7}$$

Here (A,B) is a pair of operators on H satisfying the condition (P_n), and $\alpha: I\to\mathscr{L}(H)$ is a continuous and piecewise continuously differentiable map which satisfies the condition (Q_n) for the pair (A,B).

The above definition makes sense because Theorem 1 guarantees that the right-hand side of Eq. (2.7) is finite and independent of the choice of α. As stated in the following Theorem 2, if (A,B) satisfies the condition (P_1), then the right-hand side of Eq. (2.7) coverges to the Fredholm determinant of $(\not\partial+m+A)(\not\partial+m+B)^{-1}$ as $M\to\infty$. So we can regard $\mathrm{D_M}(\not\partial+m+A;\not\partial+m+B)$ as a regularized determinant, although it depends, for M fixed, both on $\not\partial+m+A$ and $\not\partial+m+B$ rather than only on $(\not\partial+m+A)(\not\partial+m+B)^{-1}$.

Theorem 2. (1) *If (A,B) and $(B,C)\in\mathscr{L}(H)\times\mathscr{L}(H)$ satisfy the condition (P_n), so does (A,C), and*

$$\begin{aligned}&\mathrm{D_M}(\not\partial+m+A;\not\partial+m+B)\mathrm{D_M}(\not\partial+m+B;\not\partial+m+C)\\&\quad=\mathrm{D_M}(\not\partial+m+A;\not\partial+m+C).\end{aligned} \tag{2.8}$$

(2) *If $(A,A)\in\mathscr{L}(H)\times\mathscr{L}(H)$ satisfies the condition (P_n), then*

$$\mathrm{D_M}(\not\partial+m+A;\not\partial+m+A)=1. \tag{2.9}$$

(3) *If $(A,B)\in\mathscr{L}(H)\times\mathscr{L}(H)$ satisfies the condition (P_n), so does (B,A), and*

$$\mathrm{D_M}(\not\partial+m+B;\not\partial+m+A)=\mathrm{D_M}(\not\partial+m+A;\not\partial+m+B)^{-1}. \tag{2.10}$$

(4) *If $(A,B)\in\mathscr{L}(H)\times\mathscr{L}(H)$ satisfies the condition (P_1), then*

$$\lim_{M\to\infty}\mathrm{D_M}(\not\partial+m+A;\not\partial+m+B)=\det[(\not\partial+m+A)(\not\partial+m+B)^{-1}]. \tag{2.11}$$

Here det *denotes the Fredholm determinant* [11].
(5) *If $A(\cdot)$ is a continuously differentiable $\mathscr{L}(H)$-valued map on a neighborhood of* 0 *in* $\mathbb{R}$ *and $(A(t),B)\in\mathscr{L}(H)\times\mathscr{L}(H)$ satisfies the condition (P_n) for every t in the*

neighborhood of 0, *then*

$$\frac{d}{dt} \mathrm{D_M}(\not{\partial} + m + A(t); \not{\partial} + m + B)|_{t=0}$$

$$= \mathrm{Tr}\{A'(0)(\not{\partial} + m + A(0))^{-1} \exp[(\not{\partial} + m + A(0))^2/M^2]\}$$

$$\cdot \mathrm{D_M}(\not{\partial} + m + A(0); \not{\partial} + m + B). \tag{2.12}$$

3. Proofs of Theorems 1 and 2

3.1. Proof of Theorem 1. Let us begin the proof of Theorem 1 with the following lemma.

Lemma 1. *Let* $\alpha, \beta: I \to \mathscr{L}(H)$ *be the same as in Theorem 1 with* $n = 1$. *Then there exist positive constants* σ *and* ε *such that*

$$\Lambda_{\sigma,\varepsilon} = \{\lambda \in \mathbb{C} \,||\, \mathrm{Re}\,\lambda| \geqq \sigma\} \cup \{\lambda \in \mathbb{C} \,||\, \lambda + m \,| \leqq \varepsilon\}$$

is included in the resolvent set $\rho(\not{\partial} + \alpha(s))$ *of* $\not{\partial} + \alpha(s)$ *for every* $s \in I$, *and*

$$\int_0^1 \alpha'(s)(\lambda - \not{\partial} - \alpha(s))^{-1} ds, \quad \int_0^1 \beta'(s)(\lambda - \not{\partial} - \beta(s))^{-1} ds$$

belong to $\mathscr{I}_1(H)$ *for every* $\lambda \in \Lambda_{\sigma,\varepsilon}$. *For each* $\lambda \in \Lambda_{\sigma,\varepsilon}$, *there exists an integer* l *such that*

$$\mathrm{Tr} \int_0^1 \alpha'(s)(\lambda - \not{\partial} - \alpha(s))^{-1} ds - \mathrm{Tr} \int_0^1 \beta'(s)(\lambda - \not{\partial} - \beta(s))^{-1} ds = 2\pi l i. \tag{3.1}$$

Here l, *regarded as a function of* λ, *is constant on each connected component of* $\Lambda_{\sigma,\varepsilon}$.

Proof. By assumption the operator $1 + \alpha(s)(\not{\partial} + m)^{-1}$ has the bounded inverse $1 - \alpha(s)(\not{\partial} + m + \alpha(s))^{-1}$ for each $s \in I$, and $\alpha(s)(\not{\partial} + m)^{-1}$ is uniformly continuous in $s \in I$. Hence if we choose $\varepsilon > 0$ sufficiently small, $1 + (\alpha(s) - \lambda - m)(\not{\partial} + m)^{-1}$ has a bounded inverse for each $s \in I$ and for each λ with $|\lambda + m| \leqq \varepsilon$. The operator $\lambda - \not{\partial} - \alpha(s)$ has the bounded inverse $-(\not{\partial} + m)^{-1}(1 + (\alpha(s) - \lambda - m)(\not{\partial} + m)^{-1})^{-1}$ for such s and λ. Moreover assume that $\varepsilon < m$. Then

$$\{\lambda \in \mathbb{C} \,||\lambda + m| \leqq \varepsilon\} \subset \rho(\not{\partial}) = \mathbb{C} \backslash i\mathbb{R}$$

and $1 - \alpha(s)(\lambda - \not{\partial})^{-1}$ has the bounded inverse $1 + \alpha(s)(\lambda - \not{\partial} - \alpha(s))^{-1}$ for each $s \in I$. By decreasing $\varepsilon > 0$ if necessary, the same argument also applies to β.

Next let $\sigma > \sup_{s \in I} \|\alpha(s)\| + \sup_{s \in I} \|\beta(s)\|$. Then for λ with $|\mathrm{Re}\,\lambda| \geqq \sigma$, we have $\lambda \in \rho(\not{\partial})$ and $\|\alpha(s)(\lambda - \not{\partial})^{-1}\| < 1$ for each $s \in I$ because $\|(\lambda - \not{\partial})^{-1}\| = |\mathrm{Re}\,\lambda|^{-1}$. Therefore $1 - \alpha(s)(\lambda - \not{\partial})^{-1}$ has bounded inverse, and so does $\lambda - \not{\partial} - \alpha(s)$. The same is true for β. Thus for every $\lambda \in \Lambda_{\sigma,\varepsilon}$ and for every $s \in I$, $\lambda - \not{\partial} - \alpha(s)$, $\lambda - \not{\partial}$, and $1 - \alpha(s)(\lambda - \not{\partial})^{-1}$ have bounded inverses.

Using the identity

$$\alpha(s)(\lambda - \not{\partial})^{-1} = -\alpha(s)(\not{\partial} + m)^{-1} + (\lambda + m)\alpha(s)(\not{\partial} + m)^{-1}(\lambda - \not{\partial})^{-1}$$

with the assumption $(Q_1.2)$, we see that $\alpha(\cdot)(\lambda-\not{p})^{-1}$ is, as a map: $I\to\mathscr{I}_\infty(H)$, continuous and piecewise continuously differentiable for each $\lambda\in\Lambda_{\sigma,\varepsilon}$.

Similarly, $\alpha'(\cdot)(\lambda-\not{p})^{-1}$ is, as a map: $I\to\mathscr{I}_1(H)$, piecewise continuous for each $\lambda\in\Lambda_{\sigma,\varepsilon}$.

Hence

$$\int_0^1 \alpha'(s)(\lambda-\not{p}-\alpha(s))^{-1}\,ds=\int_0^1 \alpha'(s)(\lambda-\not{p})^{-1}(1-\alpha(s)(\lambda-\not{p})^{-1})^{-1}\,ds$$

belongs to $\mathscr{I}_1(H)$. The same arguments can be applied to β.

If it can be shown for each $\lambda\in\Lambda_{\sigma,\varepsilon}$ that Eq. (3.1) holds with some integer l, then the statement in Lemma 1 concerning the λ-dependence of l is a consequence of the continuity of the left-hand side of Eq. (3.1) as a function of λ.

Set

$$K_1(s)=\alpha(s)(\lambda-\not{p})^{-1},$$
$$K_2(s)=\beta(s)(\lambda-\not{p})^{-1},$$

then we have $K_1(1)=K_2(1)$ and $K_1(0)=K_2(0)$. Thus for the proof of Eq. (3.1), it is sufficient to prove

$$\det[(1+K(1))(1+K(0))^{-1}]=\exp\operatorname{Tr}\int_0^1 K'(s)(1+K(s))^{-1}\,ds, \tag{3.2}$$

for $K=K_1,K_2$.

In fact, by the assumptions for α and β we may assume that $K:I\to\mathscr{I}_\infty(H)$ is a continuous and piecewise continuously differentiable map, that $K':I\to\mathscr{I}_1(H)$ is a piecewise continuous map, and that $1+K(s)$ has bounded inverse for each $s\in I$. Let K be continuously differentiable on the open intervals (s_{j-1},s_j), $j=1,2,\dots,J$, where $0=s_0<s_1<\dots<s_J=1$. By virtue of the fundamental property of determinants [3, 11]

$$\det[(1+K(1))(1+K(0))^{-1}]=\prod_{j=1}^{J}\det[(1+K(s_j))(1+K(s_{j-1}))^{-1}],$$

we need only to show Eq. (3.2) in the case where K is continuously differentiable on the open interval $(0,1)$. Let Δ be a partition of $I:0=\xi_0<\xi_1<\dots<\xi_L=1$. Then

$$\det[(1+K(1))(1+K(0))^{-1}]=\prod_{j=1}^{L}\det[1+(K(\xi_j)-K(\xi_{j-1}))(1+K(\xi_{j-1}))^{-1}]. \tag{3.3}$$

We may assume that

$\|\Delta\|=\max_{1\le j\le L}|\xi_j-\xi_{j-1}|$ is small enough to satisfy $\max_{1\le j\le L}\|(K(\xi_j)-K(\xi_{j-1}))(1+K(\xi_{j-1}))^{-1}\|_1<1$, since K is uniformly continuous on I so that $(1+K(\cdot))^{-1}$ is uniformly bounded on I. Then the right-hand side of Eq. (3.3) is equal to

$$\exp\operatorname{Tr}\sum_{j=1}^{L}\log[1+(K(\xi_j)-K(\xi_{j-1}))(1+K(\xi_{j-1}))^{-1}], \tag{3.4}$$

where $\log(1+X)=\sum_{k=1}^{\infty}((-1)^{k-1}/k)X^k$. As $L\to\infty$ and $\|\Delta\|\to 0$, (3.4) converges to $\exp \mathrm{Tr}\left[\int_0^1 K'(s)(1+K(s))^{-1}ds\right]$ because K is continuously differentiable. On the other hand the left-hand side of Eq. (3.3) does not depend on the partition Δ. ■

Using the Hille–Yosida–Phillips theorem it is now obvious that $(\not{\partial}+m+\alpha(s))^2$ generates a strongly continuous semigroup. However we describe the operator $\exp[(\not{\partial}+m+\alpha(s))^2/M^2]$ using the Dunford functional calculus [12]. With $\theta\in(\pi/4,\pi/2)$ fixed, we define the three contours $\Gamma_{\sigma,1}$, $\Gamma_{\sigma,2}$, Γ_ε as follows. $\Gamma_{\sigma,1}$ is the union of the half line $-m+\rho(\sigma+2m)(1-i\tan\theta)$ with ρ running from $-\infty$ to -1, the interval $-m-(\sigma+2m)(1+i\rho\tan\theta)$ with ρ running from -1 to 1, and the half line $-m-\rho(\sigma+2m)(1+i\tan\theta)$ with ρ running from 1 to $+\infty$. $\Gamma_{\sigma,2}$ is the union of the half line $-m-\rho(\sigma+2m)(1-i\tan\theta)$ with ρ running from $-\infty$ to -1, the interval $-m+(\sigma+2m)(1+i\rho\tan\theta)$ with ρ running from -1 to 1, and the half line $-m+\rho(\sigma+2m)(1+i\tan\theta)$ with ρ running from 1 to $+\infty$. Γ_ε is the circle $\{\lambda\,|\,|\lambda+m|=\varepsilon/2\}$ directed clockwise. Set $\Gamma_\sigma=\Gamma_{\sigma,1}\cup\Gamma_{\sigma,2}$ and $\Gamma_{\sigma,\varepsilon}=\Gamma_\sigma\cup\Gamma_\varepsilon$. Note $\Gamma_{\sigma,\varepsilon}\subset\Lambda_{\sigma,\varepsilon}$. Let σ and ε be those in Lemma 1. Then we have

$$\exp[(\not{\partial}+m+\alpha(s))^2/M^2]=(2\pi i)^{-1}\int_{\Gamma_\sigma} d\lambda \exp[(\lambda+m)^2/M^2](\lambda-\not{\partial}-\alpha(s))^{-1}, \tag{3.5}$$

where the integration on the right-hand side of Eq. (3.5) is norm convergent in $\mathscr{L}(H)$. Cauchy's integral theorem and the resolvent equation yield

$$(\not{\partial}+m+\alpha(s))^{-1}\exp[(\not{\partial}+m+\alpha(s))^2/M^2] = (2\pi i)^{-1}\int_{\Gamma_{\sigma,\varepsilon}} d\lambda(\lambda+m)^{-1}\exp[(\lambda+m)^2/M^2](\lambda-\not{\partial}-\alpha(s))^{-1}. \tag{3.6}$$

For $\alpha: I\to\mathscr{L}(H)$ with property (Q_1),

$$(s,\lambda)\to(\lambda+m)^{-1}\exp[(\lambda+m)^2/M^2]\alpha'(s)(\lambda-\not{\partial}-\alpha(s))^{-1} \tag{3.7}$$

can be considered as a map: $I\times\Gamma_{\sigma,\varepsilon}\to\mathscr{I}_1(H)$. This map is piecewise continuous in $s\in I$ and continuous in $\lambda\in\Gamma_{\sigma,\varepsilon}$. In the estimate

$$\begin{aligned}&\|(\lambda+m)^{-1}\exp[(\lambda+m)^2/M^2]\alpha'(s)(\lambda-\not{\partial}-\alpha(s))^{-1}\|_1\\ &\quad\leqq|(\lambda+m)^{-1}\exp[(\lambda+m)^2/M^2]|\,\|\alpha'(s)(\not{\partial}+m+\alpha(s))^{-1}\|_1\\ &\qquad\cdot\|-1+(\lambda+m)\lambda-\not{\partial}-\alpha(s))^{-1}\|,\end{aligned}$$

$|(\lambda+m)^{-1}\exp[(\lambda+m)^2/M^2]|$ decreases rapidly in the neighborhood of infinity on $\Gamma_{\sigma,\varepsilon}$.

$$\|\alpha'(s)(\not{\partial}+m+\alpha(s))^{-1}\|_1 \quad\text{and}\quad \|-1+(\lambda+m)(\lambda-\not{\partial}-\alpha(s))^{-1}\|$$

is bounded uniformly in $(s,\lambda)\in I\times\Gamma_{\sigma,\varepsilon}$.

Therefore the integration over $I\times\Gamma_{\sigma,\varepsilon}$ of the map (3.7) times $(2\pi i)^{-1}$ is convergent in $\mathscr{I}_1(H)$ and is equal to $\mathrm{L_M}(\alpha)$ in (2.5). Equation (2.6) in Theorem 1 with $n=1$ is a direct consequence of Lemma 1 and the following two lemmas.

Lemma 2. *Let $\alpha,\beta: I\to\mathscr{L}(H)$ have the property (Q_1). Then* $\mathrm{Tr}\mathrm{L_M}(\alpha)-\mathrm{Tr}\mathrm{L_M}(\beta)$ *is independent of* $M>0$.

Proof. This follows from the equation

$$\begin{aligned}\mathrm{TrL_M}(\alpha) - \mathrm{TrL_M}(\beta) &= (2\pi i)^{-1} \int_{\Gamma_{\sigma,\varepsilon}} d\lambda(\lambda + m)^{-1} \exp[(\lambda + m)^2/M^2] \\ &\quad \cdot \mathrm{Tr} \int_0^1 ds\, [\alpha'(s)(\lambda - \not{\partial} - \alpha(s))^{-1} - \beta'(s)(\lambda - \not{\partial} - \beta(s))^{-1}] \\ &= \sum_{j=1}^{2} \int_{\Gamma_{\sigma,j}} d\lambda(\lambda + m)^{-1} \exp[(\lambda + m)^2/M^2] l_j \\ &\quad + \int_{\Gamma_\varepsilon} d\lambda(\lambda + m)^{-1} \exp[(\lambda + m)^2/M^2] l_3 \\ &= 2\pi(l_1 - l_3)i + \pi(l_2 - l_1)i.\end{aligned}$$

In the first equality above we have used Eq. (3.6) and the dominated convergence theorem. The second equality follows from Lemma 1, where $l_j, j = 1, 2, 3$, are some integers. The third equality follows from an explicit calculation using Cauchy's integral formula. ■

Lemma 3. *Let* α *have property* (Q_1). *Then*

$$\begin{aligned}&\lim_{M\to\infty} \mathrm{Tr} \int_0^1 ds\, \alpha'(s)(\not{\partial} + m + \alpha(s))^{-1} \exp[(\not{\partial} + m + \alpha(s))^2/M^2] \\ &\quad = \mathrm{Tr} \int_0^1 ds\, \alpha'(s)(\not{\partial} + m + \alpha(s))^{-1}.\end{aligned}$$

Proof. By Eq. (3.5) and by Cauchy's integral formula,

$$\begin{aligned}&\exp[(\not{\partial} + m + \alpha(s))^2/M^2] - 1 \\ &\quad = (2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda(\exp[(\lambda + m)^2/M^2] - 1)[(\lambda - \not{\partial} - \alpha(s))^{-1} - (\lambda + m - \not{\partial})^{-1}] \\ &\qquad + (2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda \exp[(\lambda + m)^2/M^2](\lambda + m - \not{\partial})^{-1} - 1 \\ &\quad = (2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda \exp[(\lambda + m)^2/M^2](\lambda - \not{\partial} - \alpha(s))^{-1}(m + \alpha(s))(\lambda + m - \not{\partial})^{-1} \\ &\qquad + (\exp(\Delta/M^2) - 1),\end{aligned}$$

where Δ is the Laplacian. The first term of the above equation converges to 0 in $\mathscr{L}(H)$ as $M \to \infty$ by the dominated convergence theorem, since

$$\|(\lambda - \not{\partial} - \alpha(s))^{-1}\| + \|(\lambda + m - \not{\partial})^{-1}\| < \mathrm{const}\, |\mathrm{Re}\, \lambda|^{-1}, \quad \lambda \in \Gamma_\sigma.$$

It is easy to see that the second term converges strongly to 0. Thus

$$\operatorname*{s-lim}_{M\to\infty} \exp[(\not{\partial} + m + \alpha(s))^2/M^2] = 1.$$

Lemma 3 follows then from this fact and the following result of Grümm [3, 13]: for A_n, $A \in \mathscr{L}(H)$ which satisfy $\operatorname*{s-lim}_{n\to\infty} A_n = A$ and for $B \in \mathscr{I}_1(H)$, we have

$$\lim_{n\to\infty} \|A_n B - AB\|_1 = 0.$$

Therefore $\lim_{n\to\infty} \operatorname{Tr} A_n B = \operatorname{Tr} AB$ by the continuity of the trace in the $\|\cdot\|_1$ norm. Note $\operatorname{Tr} AB = \operatorname{TR} BA$ for $A \in \mathscr{L}(H)$, $B \in \mathscr{I}_1(H)$. ■

Now, we are in a position to prove Theorem 1 with $n > 1$. Let $\alpha : I \to \mathscr{L}(H)$ have the property (Q_n) with $n > 1$. Then we can find positive constants σ and ε such that for every $\lambda \in \Lambda_{\sigma,\varepsilon}$ and for every $s \in I$, $\lambda - \not{\partial} - \alpha(s)$, $\lambda - \not{\partial}$, and $1 - \alpha(s)(\lambda - \not{\partial})^{-1}$ have bounded inverses as in the proof of Lemma 1.

In the identity

$$(\lambda - \not{\partial} - \alpha(s))^{-1} = (\lambda - \not{\partial})^{-1}(1 + \alpha(s)(\lambda - \not{\partial} - \alpha(s))^{-1})$$

for $(s, \lambda) \in I \times \Lambda_{\sigma,\varepsilon}$, $(\lambda - \not{\partial})^{-1} : H \to H_1$ is a bounded linear operator and $1 + \alpha(s) \cdot (\lambda - \not{\partial} - \alpha(s))^{-1}$ belongs to $\mathscr{L}(H)$. Then $(\lambda - \not{\partial} - \alpha(s))^{-1} : H \to H_1$ is a bounded linear operator, and $(\lambda - \not{\partial} - \alpha(s))^{-1} \restriction H_1$ belongs to $\mathscr{L}(H_1)$. Applying similar arguments successively with $(Q_n \cdot 4)$, we get that $(\lambda - \not{\partial} - \alpha(s))^{-1} : H_k \to H_{k+1}$ is bounded and $(\lambda - \not{\partial} - \alpha(s))^{-1} \restriction H_{k+1}$ belongs to $\mathscr{L}(H_{k+1})$ for $k = 0, 1, \ldots, n-1$. Since $-m \in \Lambda_{\sigma,\varepsilon}$,

$$(\not{\partial} + m)^n (\not{\partial} + m + \alpha(s))^{-n+1} (\lambda - \not{\partial} - \alpha(s))^{-1} \text{ belongs to } \mathscr{L}(H)$$

for $\lambda \in \Lambda_{\sigma,\varepsilon}$. Using this fact and $(Q_n \cdot 3)$ with the Cauchy's integral theorem, and noting that $(\lambda - \not{\partial} - \alpha(s))^{-1}$ is bounded uniformly in $(s, \lambda) \in I \times \Lambda_{\sigma,\varepsilon}$, we get that

$$\begin{aligned} \mathrm{L_M}(\alpha) = \int_0^1 ds\, (2\pi i)^{-1} \int_{\Gamma_{\sigma,\varepsilon}} d\lambda\, (\lambda + m)^{n-2} \exp\left[(\lambda + m)^2/M^2\right] \\ \cdot \alpha'(s)(\not{\partial} + m)^{-n} (\not{\partial} + m)^n (\not{\partial} + m + \alpha(s))^{-n+1} (\lambda - \not{\partial} - \alpha(s))^{-1} \end{aligned} \tag{3.8}$$

belongs to $\mathscr{I}_1(H)$.

Define the map $\alpha_\kappa : I \to \mathscr{L}(H)$ by $\alpha_\kappa(\cdot) = \alpha(\cdot)(1 + \kappa(\not{\partial} + m))^{-n+1}$ for $\kappa > 0$. When $\kappa \to 0$, $\alpha_\kappa(s)(\not{\partial} + m)^{-1} - \alpha(s)(\not{\partial} + m)^{-1} \to 0$ in $\mathscr{L}(H)$ and uniformly in $s \in I$. So we can choose $\kappa_0 > 0$ so small that α_κ satisfies the condition $(Q_1 \cdot 1)$ as well as $(Q_1 \cdot 2) - (Q_1 \cdot 4)$ for every $\kappa \in (0, \kappa_0)$. Therefore to prove (2.6) for the case $n > 1$ we need only to show

$$\lim_{\kappa \to 0} \operatorname{Tr} \mathrm{L_M}(\alpha_\kappa) = \operatorname{Tr} \mathrm{L_M}(\alpha). \tag{3.9}$$

When we rewrite $\mathrm{L_M}(\alpha_\kappa) - \mathrm{L_M}(\alpha)$ using the representation of Eq. (3.8) and the definition of α_κ, we meet the operator

$$\begin{aligned} \alpha'(s)(\not{\partial} + m)^{-n}[(1 + \kappa(\not{\partial} + m))^{-n+1} (\not{\partial} + m)^n (\not{\partial} + m + \alpha_\kappa(s))^{-n+1} (\lambda - \not{\partial} - \alpha_\kappa(s))^{-1} \\ - (\not{\partial} + m)^n (\not{\partial} + m + \alpha(s))^{-n+1} (\lambda - \not{\partial} - \alpha(s))^{-1}]. \end{aligned} \tag{3.10}$$

By assumption, $\sup_{s \in I} \|\alpha'(s)(\not{\partial} + m)^{-n}\|_1 < \infty$. The operator in the bracket [ ] of (3.10) is a bounded operator on H which is bounded uniformly in $(s, \lambda, \kappa) \in I \times \Gamma_{\sigma,\varepsilon} \times (0, \kappa_0)$, and converges strongly to 0 as $\kappa \to 0$. Hence Eq. (3.9) is proved by the argument similar to the proof of Lemma 3. This completes the proof of Theorem 1.

3.2. Proof of Theorem 2. Theorem 2(4) follows from Lemma 3 and Eq. (3.2) in the proof of Lemma 1. The other parts of Theorem 2 are obvious.

4. Chiral Anomaly

In this section, we study the change of our regularized determinant under the infinitesimal local chiral transformation of the Dirac operator with gauge fields, i.e. chiral anomaly. We shall get essentially the same expression as in [7], thus providing an aposteriori justification for the algebraic manipulations performed there.

Set

$$A = -i\sum_{a,\mu} T_a \otimes \gamma_\mu \otimes A_\mu{}^a \equiv -i\not{A} \quad \text{and} \quad B=0,$$

where $A_\mu{}^a$ is the multiplication operator by the function $A_\mu{}^a(x)$ in $\mathscr{S}(\mathbb{R}^4;\mathbb{R})$ for each a and μ, and $\{T_a\}$ is a set of Hermitian matrices which are the N-dimensional representation of the generators of a certain compact Lie group. Note that A is anti-selfadjoint and of the form discussed in Remark 1.

Under the infinitesimal local chiral transformation, the Dirac operator $\not{\partial} + m - i\not{A}$ changes into $\not{\partial} + m - i\not{A} - i\varepsilon \sum_{\mu=0}^{3} \gamma_5\gamma_\mu(\partial_\mu\varphi) + 2i\varepsilon m\gamma_5\varphi$. Here we have used the following conventions: γ_μ, φ, and $(\partial_\mu\varphi)$ are shorthands of $1\otimes\gamma_\mu\otimes 1$, $1\otimes 1\otimes\varphi$, and $1\otimes 1\otimes(\partial_\mu\varphi)$ respectively with $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$; φ and $(\partial_\mu\varphi)$ are the multiplication operators by the function $\varphi(x)$ in $\mathscr{S}(\mathbb{R}^4;\mathbb{R})$ and its partial derivative $\partial\varphi(x)/\partial x_\mu$ respectively.

By Theorem 2(5), we get

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathrm{D_M}\left(\not{\partial} + m - i\not{A} - i\varepsilon\sum_\mu \gamma_5\gamma_\mu(\partial_\mu\varphi) + 2i\varepsilon m\gamma_5\varphi; \not{\partial}+m\right)\Big|_{\varepsilon=0}\\
&= \mathrm{Tr}\left\{\left(-i\sum_\mu \gamma_5\gamma_\mu(\partial_\mu\varphi) + 2im\gamma_5\varphi\right)(\not{\partial}+m-i\not{A})^{-1}\right.\\
&\quad\left.\cdot\exp[(\not{\partial}+m-i\not{A})^2/M^2]\right\}\mathrm{D_M}(\not{\partial}+m-i\not{A};\not{\partial}+m). \qquad (4.1)
\end{aligned}$$

The first factor on the right-hand side of Eq. (4.1) can be rewritten as follows.

$$\begin{aligned}
&\mathrm{Tr}\left\{\left(-i\sum_\mu \gamma_5\gamma_\mu(\partial_\mu\varphi) + 2im\gamma_5\varphi\right)(\not{\partial}+m-i\not{A})^{-1}\exp[(\not{\partial}+m-i\not{A})^2/M^2]\right\}\\
&= \mathrm{Tr}\left\{\exp[(\not{\partial}+m-i\not{A})^2/2M^2]\left(-i\sum_\mu \gamma_5\gamma_\mu(\partial_\mu\varphi) + 2im\gamma_5\varphi\right)\right.\\
&\quad\left.\cdot(\not{\partial}+m-i\not{A})^{-1}\exp[(\not{\partial}+m-i\not{A})^2/2M^2]\right\}\\
&= \mathrm{Tr}\{\exp[(\not{\partial}+m-i\not{A})^2/2M^2][(\not{\partial}+m-i\not{A})i\gamma_5\varphi\\
&\quad + i\gamma_5\varphi(\not{\partial}+m-i\not{A})](\not{\partial}+m-i\not{A})^{-1}\exp[(\not{\partial}+m-i\not{A})^2/2M^2]\}\\
&= 2i\,\mathrm{Tr}\{\gamma_5\varphi\exp[(\not{\partial}+m-i\not{A})^2/M^2]\}. \qquad (4.2)
\end{aligned}$$

We have used the equality

$$\exp[(\not{\partial}+m-i\not{A})^2/M^2] = \exp[(\not{\partial}+m-i\not{A})^2/2M^2]\exp[(\not{\partial}+m-i\not{A})^2/2M^2]$$

and the cyclic property of the trace in the first and third equalities. Apart from the appearance of m in the exponent this agrees with a result obtained in [7].

This shows that Fujikawa's idea of regularizing the Jacobian of the infinitesimal chiral transformations is incorporated into our regularized determinants. The appearance of m in Eq. (4.2) comes from the following requirement: The regularized determinant should be independent of the choice of the operator-valued map which connects two operators A and B.

The limit $M \to \infty$ of Eq. (4.2) exists and has a simple form, a fact already noticed in [7]. We show this by calculating the limit in a mathematically rigorous manner. Using the cyclic property of the trace, and $\gamma_5(\not\partial - i\not A) = -(\not\partial - i\not A)\gamma_5$, we get

$$\begin{aligned}
&2i\,\mathrm{Tr}\,\{\gamma_5\varphi \exp[(\not\partial + m - i\not A)^2/M^2]\} \\
&= 2i\,\mathrm{Tr}\left\{(2\pi i)^{-1} \int_{\Gamma_\sigma} \mathrm{d}\lambda \gamma_5 \varphi \exp[(\lambda+m)^2/M^2] \right. \\
&\quad \left. \cdot \tfrac{1}{2}[(\lambda - \not\partial + i\not A)^{-1} + (\lambda + \not\partial - i\not A)^{-1}]\right\} \\
&= 2i\,\mathrm{Tr}\left\{(2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda \gamma_5 \varphi \exp[(\lambda+m)^2/M^2] \right. \\
&\quad \left. \cdot \lambda\left[\lambda^2 - \sum_{\mu=0}^{3} (\partial_\mu - iA_\mu)^2 - \frac{1}{2}\sum_{\mu,\nu=0}^{3} \sigma_{\mu\nu}F_{\mu\nu}\right]^{-1}\right\},
\end{aligned} \tag{4.3}$$

where

$$A_\mu = \sum_a T^a \otimes 1 \otimes A_\mu{}^a, \quad \sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i,$$

and

$$F_{\mu\nu} = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) - i[A_\mu, A_\nu]$$

with

$$(\partial_\mu A_\nu) = \sum_a T^a \otimes 1 \otimes (\partial_\mu A_\nu{}^a).$$

We introduce the following shorthand notations:

$$\sum_{\mu=0}^{3} (\partial_\mu - iA_\mu)^2 = (\partial - iA)^2, \quad \sum_{\mu=0}^{3} A_\mu{}^2 = A^2, \quad \text{and}$$

$$\sum_{\mu,\nu=0}^{3} \sigma_{\mu\nu}F_{\mu\nu} = \sigma\cdot F \quad \text{and so on.}$$

By expanding the resolvent, the operator in the brace { } on the third member of Eq. (4.3) is equal to

$$\begin{aligned}
&(2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda\, \gamma_5 \varphi \exp[(\lambda+m)^2/M^2] \lambda[(\lambda^2 - (\partial - iA)^2)^{-1} \\
&\quad + (\lambda^2 - (\partial - iA)^2)^{-1} \tfrac{1}{2}\sigma\cdot F (\lambda^2 - (\partial - iA)^2)^{-1}] \\
&\quad + (2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda\, \gamma_5 \varphi \exp[(\lambda+m)^2/M^2] \\
&\quad \cdot \lambda[(\lambda^2 - (\partial - iA)^2)^{-1} \tfrac{1}{2}\sigma\cdot F]^2 (\lambda^2 - (\partial - iA)^2)^{-1} \\
&\quad + (2\pi i)^{-1} \int_{\Gamma_\sigma} d\lambda\, \gamma_5 \varphi \exp[(\lambda+m)^2/M^2] \lambda[(\lambda^2 - (\partial - iA)^2)^{-1} \tfrac{1}{2}\sigma\cdot F]^3 \\
&\quad \cdot (\lambda^2 - (\partial - iA)^2 - \tfrac{1}{2}\sigma\cdot F)^{-1}.
\end{aligned} \tag{4.4}$$

The last two terms of the operator (4.4) are trace class operators because $\varphi(\not{\partial}+m)^{-5}$ is a trace class operator and because

$$(\not{\partial}+m)^5\lambda[(\lambda^2-(\partial-iA)^2)^{-1}\tfrac{1}{2}\sigma\cdot F]^2(\lambda^2-(\partial-iA)^2)^{-1}$$

is a bounded operator on H and bounded uniformly in $\lambda\in\Gamma_\sigma$. Now the whole expression in (4.4) is trace class. Therefore the first term in (4.4) is trace class.

Notice that $\operatorname{Tr}\gamma_5=\operatorname{Tr}\gamma_5\sigma_{\mu\nu}=0$ on $\mathbb{C}^4$ and we see that the trace of the first term of (4.4) is 0.

We make a change of the integration variable $\lambda'=M^{-1}\lambda$, and then replace the contour $M^{-1}\Gamma_\sigma$ by the contour Γ_σ in the last term of (4.4). This is possible because the operators $-(\partial-iA)^2$ and $-(\partial-iA)^2-\frac{1}{2}\sigma\cdot F=(\not{\partial}-i\not{A})^*(\not{\partial}-i\not{A})$ are positive. Then this term is equal to

$$(2\pi i)^{-1}\int_{\Gamma_\sigma}d\lambda'\,\gamma_5\varphi\exp(\lambda'+m/M)^2M^2\lambda'[(M^2\lambda'^2-(\partial-iA)^2)^{-1}\tfrac{1}{2}\sigma\cdot F]^3$$
$$\cdot(M^2\lambda'^2-(\partial-iA)^2-\tfrac{1}{2}\sigma\cdot F)^{-1}.$$

Here

$$\gamma_5\varphi M\lambda'[(M^2\lambda'^2-(\partial-iA)^2)^{-1}\tfrac{1}{2}\sigma\cdot F]^3$$

has bounded trace norm uniformly in $M\lambda'$, and

$$\|M(M^2\lambda'^2-(\partial-iA)^2-\tfrac{1}{2}\sigma\cdot F)^{-1}\|=M^{-1}\lambda'^{-2}.$$

Thus this term converges to 0 in $\mathscr{I}_1(H)$ as $M\to\infty$ by the dominated convergence theorem.

Now we consider the second term of (4.4). Let us substitute the following equalities into this term:

$$(\lambda^2-(\partial-iA)^2)^{-1}$$
$$=(\lambda^2-\Delta)^{-1}+(\lambda^2-\Delta)^{-1}(-i\partial\cdot A-iA\cdot\partial-A^2)(\lambda^2-(\partial-iA)^2)^{-1},$$

and

$$(\lambda^2-\Delta)^{-1}\sigma\cdot F=\sigma\cdot F(\lambda^2-\Delta)^{-1}+(\lambda^2-\Delta)^{-1}\left[(\Delta\sigma\cdot F)+2\sum_\alpha(\partial_\alpha\sigma\cdot F)\partial_\alpha\right](\lambda^2-\Delta)^{-1},$$

and so on. Then the second term is equal to

$$(2\pi i)^{-1}\int_{\Gamma_\sigma}d\lambda\,\gamma_5\varphi\exp[(\lambda+m)^2/M^2][\tfrac{1}{4}(\sigma\cdot F)^2\lambda(\lambda^2-\Delta)^{-3}$$
$$+\tfrac{3}{2}\sigma\cdot F\sum_\alpha(\partial_\alpha\sigma\cdot F)\partial_\alpha\lambda(\lambda^2-\Delta)^{-4}-\tfrac{3}{4}i(\sigma\cdot F)^2A\cdot\partial\lambda(\lambda^2-\Delta)^{-4}]+R(M),\quad(4.5)$$

where $R(M)$ is a sum of terms which converge to 0 in $\mathscr{I}_1(H)$ as $M\to\infty$ as in the case of third of (4.4). For the trace of the operator (4.5), we use the following formulae.

$$\operatorname{Tr}A=\int_{\mathbb{R}^4}dx\,(\text{kernel of }A)(x,x)$$

for a trace class operator A on $L^2(\mathbb{R}^4)$ whose kernel is a continuous function, and

$$\operatorname{Tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta=4\varepsilon_{\mu\nu\alpha\beta}$$

for the trace on $\mathbb{C}^4$, where $\varepsilon_{\mu\nu\alpha\beta}$ is totally anti-symmetric in its indices with $\varepsilon_{0123} = 1$.

Notice that

$$(\text{kernel of } \partial_\mu(\lambda^2 - \Delta)^{-4})(x,x) = 0,$$

$$(\text{kernel of } (\lambda^2 - \Delta)^{-3})(x,x) = \frac{1}{32\pi^2\lambda^2}.$$

Then we get

$$2i\,\mathrm{Tr}\,\{\gamma_5\varphi \exp[(\not{\partial} + m - i\not{A})^2/M^2]\} \to -\frac{i}{16\pi^2} \sum_{\alpha,\beta,\mu,\nu} \varepsilon_{\mu\nu\alpha\beta} \int_{\mathbb{R}^4} dx\, \varphi(x)\,\mathrm{Tr}\,F_{\mu\nu}(x)F_{\alpha\beta}(x)$$

as $M \to \infty$. Here the trace on the right-hand side is taken as a matrix in $\mathbb{C}^N$.

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