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Remarks on a Paper by J. T. Beale, T. Kato, and A. Majda

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Abstract. We prove that the maximum norm of the deformation tensor controls the possible breakdown of smooth solutions for the 3-dimensional Euler equations. More precisely, the loss of regularity in a local smooth solution of the Euler equations implies the growth without bound of the deformation tensor as the critical time approaches; equivalently, if the deformation tensor remains bounded the existence of a smooth solution is guaranteed.

The motion of an ideal incompressible fluid is described by a system of partial differential equations known as Euler equations. In [1] J. T. Beale, T. Kato, and A. Majda have given a mathematically rigorous connection between the accumulation of vorticity and the development of singularities for the three-dimensional Euler equations. In fact, they have shown that the maximum norm of the vorticity alone controls the breakdown of smooth solution of these equations. Thus one may ask: Does the blow up of the solution imply also the blow up of the deformation tensor in the maximum norm? or, may it stay bounded for a longer time? In this note we answer these questions. More precisely, we obtain the same results as those in [1], when the vorticity is substituted by the deformation tensor.

Thus we consider the system

$$\begin{array}{ll} \text{(a)} & \left\{ \begin{array}{ll} u_t^k + u^j \cdot \partial_j u^k + \partial_k p = 0 & k = 1, 2, 3 \\ \operatorname{div} u = 0 & \end{array} \right. \\ \text{(b)} & \end{array} \tag{1}$$

where $x \in \mathbb{R}^3$, $t > 0$, $u = u(x, t) = (u^1, u^2, u^3)$ is the velocity field, and $p = p(x, t)$ is the pressure.

For this system the following local existence theorem is known: Given an initial velocity $u_0 \in H^s$, s integer, $s \geqq 3$ and $\operatorname{div} u_0 = 0$, there exists $T_0 = T_0(\|u_0\|_s)$ so that the system (1) has a unique solution $u \in C([0, T]: H^s) \cap C^1([0, T]: H^{s-1})$ at least for $T = T_0$. (See reference in [1]). $\square$

Here we denote by $H^s = H^s(\mathbb{R}^3)$ (s being a positive integer) the Sobolev space consisting of functions whose distributional derivatives up to order s belong to $L^2(\mathbb{R}^3)$, and by $\|u\|_s$ the norm of u in H^s. Also, we use $\omega = \nabla \times u$ for the vorticity and $T = (T_{ij})$ $i, j = 1, 2, 3$, where $T_{ij} = \partial_j u^i + \partial_i u^j$ for the deformation tensor.

Theorem 1. *Let* $u \in C([0, T_1]: H^s) \cap C^1([0, T_1]: H^{s-1})$ *be a solution of (1). Then the inequality*

$$\|u(t)\|_s \leqq \|u(0)\|_s \cdot e^{C_s \cdot \int_0^t |T_{ij}|_{L^\infty}(\tau) d\tau} \tag{2}$$

holds for all $t \in [0, T_1]$.

Corollary 1. *If the solution of (1) considered above exists in the time interval* $[0, T_2)$ *and cannot be extended beyond* T_2, *then*

$$\int_0^{T_2} |T_{ij}|_{L^\infty}(\tau)\, d\tau = \infty$$

and, in particular,

$$\limsup_{t \uparrow T_2} |T_{ij}|_{L^\infty}(t) = \infty.$$

Corollary 2. *If the solution of (1) considered above exists in the time interval* $[0, T_3]$, *and for some* $T_4 > T_3$ we have that

$$\int_0^{T_4} |T_{ij}|(\tau) d\tau < \infty,$$

then the solution can be extended to the interval $[0, T_4]$, *in which it remains of the same type.* □

Corollary 1 and Corollary 2 are immediate consequences of the local existence theorem and the estimate (2), and their proof will be omitted here.

Using classical energy estimates (see [1]) one can obtain the inequality

$$\|u(t)\|_s \leqq \|u(0)\|_s e^{C_s \int_0^t |\nabla u|_{L^\infty}(\tau) d\tau}$$

for all $t \in [0, T]$, where the solution of the type considered above exists. In [1] further estimates which involve the vorticity equation and the Biot-Savart law were used to find a bound of $|\nabla u|_L(t)$ as a function of $|\omega|_{L^\infty}(t)$. Here our method of proof of (2) is based only on a careful computation of the energy estimates.

Proof of Theorem 1. In the proof of this theorem we will use the following: If u is a solution of (1) and $v^1, v^2, v^3, w \in H^1(\mathbb{R}^3)$, then

$$\int u^j \partial_j v^k \cdot v^k = 0 \quad \text{and} \quad \int u^j \cdot \partial_j w = 0.$$

These facts follow from Eq. (1) (b) and integration by parts.

First we provide the proof for the case $s = 3$.

By the above $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ for all $t \in [0, T_1]$. Taking ∂^3_{ilm} derivatives of Eqs. (1) (a), multiplying the result by $\partial^3_{ilm} u^k$, adding in k, i, l, m and integrating, we obtain that

$$\frac{1}{2}\frac{d}{dt} \|\partial^3_{ilm} u^k(t)\|_{L^2} + \int \partial^3_{ilm}(u^j \partial_j u^k) \cdot \partial^3_{ilm} u^k = 0. \tag{3}$$

Since

$$\int u^j \cdot \partial^3_{ilm} \partial_j u^k \cdot \partial^3_{ilm} u^k = 0. \tag{4}$$

we only need to handle the remaining three terms of the integral in (3).

The first one

$$\int \partial_i u^j \cdot \partial^3_{jlm} u^k \cdot \partial^3_{ilm} u^k$$

can be written after summation in i and j as

$$\sum_{i \leqq j} \int T_{ij} \cdot \partial^3_{jlm} u^k \cdot \partial_{ilm} u^k$$

from which we obtain the estimate

$$|T_{ij}|_{L^\infty} \|D^3 u\|^2_{L^2} \tag{5}$$

for all l, m, k. The same applies to the term

$$\int \partial^3_{ilm} u^j \cdot \partial_j u^k \cdot \partial^3_{ilm} u^k$$

with k instead of i. The last term

$$\int \partial^2_{il} u^j \cdot \partial^2_{mj} u^k \cdot \partial^3_{ilm} u^k \tag{6}$$

can be bounded by

$$\|\partial^2_{il} u^j\|_{L^4} \cdot \|\partial_{mj} u^k\|_{L^4} \cdot \|\partial_{ilm} u^k\|_{L^2}.$$

In order to estimate $\|\partial^2_{il} u^j\|_{L^4}$ we write

$$\partial_k T_{kj} = \sum_{k=1}^{3} \partial_k(\partial_j u^k + \partial_k u^j) = \Delta u^j$$

and

$$\partial^2_{il} u^j = \partial^2_{il} \Delta^{-1}(\partial_k T_{kj}).$$

Thus by properties of the Riesz transform (see [2])

$$\|\partial^2_{il} u^j\|_{L^4} \leqq C \cdot \sum_{k=1}^{3} \|\partial_k T_{kj}\|_{L^4},$$

and by application of the Gagliardo–Nirenberg inequalities

$$\|\partial_k T_{kj}\|_{L^4} \leqq C \|D^2 T_{kj}\|^{1/2}_{L^2} \cdot |T_{kj}|^{1/2}_{L^\infty}.$$

Therefore (6) can be estimated by

$$|T_{ij}|_{L^\infty} \|D^2 T_{ij}\|_{L^2} \cdot \|D^3 u\|_{L^2} \leqq |T_{ij}|_{L^\infty} \cdot \|D^3 u\|^2_{L^2}. \tag{7}$$

Using (4), (5), (7) in (3), and then Gronwall's inequality (2) is proved for the case $s = 3$.

In (4) the use of $\partial^4 u$ can be justified by approximating $u(0)$ by smooth initial data of the same type, performing the above energy estimate, and then passing to the limit (see [3]).

Finally, we sketch the proof of (2) for general $s \geqq 3$. As before we can obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^{(\alpha)}u^k(t)\|_{L^2} + \int \partial^{(\alpha)}(u^j \cdot \partial_j u^k)\cdot \partial^{(\alpha)}u^k = 0, \tag{8}$$

here summation signs in $k, j = 1, 2, 3$ and $|\alpha| = s$ are omitted. First, the following three terms of the integral in (8) are considered:

$$\int u^j \cdot \partial^{(\alpha)}\partial_j u^k \cdot \partial^{(\alpha)}u^k = 0, \int \partial_l u^j \cdot \partial^{(\alpha')}\partial_j u^k \cdot \partial^{(\alpha)}u^k,$$

where $\alpha' = \alpha - e_l$ with $e_1 = (1, 0, 0)$ and so on. Summation in α and j gives us the bound

$$|T_{lj}|_{L^\infty} \cdot \|D^s u\|^2_{L^2},$$

and the term

$$\int \partial^{(\alpha)}u^j \cdot \partial_j u^k \cdot \partial^{(\alpha)}u^k,$$

for which the same technique given above applies, with l replaced by k.

The remaining terms of the integral in (8) can be estimated by

$$\|\partial^{(\alpha)}(u^j \partial_j u^k) - u^j \cdot \partial^{(\alpha)}\partial_j u^k - \partial_l u^j \cdot \partial^{(\alpha')}\partial_j u^k - \partial^{(\alpha)}u^j \cdot \partial_j u^k\|_{L^2} \cdot \|D^s u\|_{L^2}. \tag{9}$$

By calculus of inequalities (see [4], [5]) in the first factor above we bound (9) with

$$\|D^2 u\|_{L^q}\|D^{s-1}u\|_{L^r}\|D^s u\|_{L^2}. \tag{10}$$

where

$$q = 2(s-1) \quad \text{and} \quad r = \frac{2(s-1)}{s-2}.$$

Since $s \geqq 3$, using again $\partial^2_{lj}u^k = \partial^2_{lj}\Delta^{-1}\partial_m T_{mk}$, properties of the Riesz transform, and the Gagliardo–Nirenberg inequalities, it follows that

$$\|D^2 u\|_{L^q} \leqq C\|DT_{ij}\|_{L^q} \leqq C' \cdot \|D^{s-1}T_{ij}\|^{\alpha}_{L^2} \cdot |T_{ij}|^{1-\alpha}_{L^\infty},$$
$$\|D^{s-1}u\|_{L^r} \leqq C\|D^{s-2}T_{ij}\|_{L^r} \leqq C' \cdot \|D^{s-1}T_{ij}\|^{\beta}_{L^2} \cdot |T_{ij}|^{1-\beta}_{L^\infty},$$

where

$$\alpha = \frac{1}{s-1} \quad \text{and} \quad \beta = \frac{s-2}{s-1}.$$

Thus all the terms of the integral in (8) can be estimated by

$$C_s \cdot |T_{ij}|_{L^\infty}\|D^s u\|^2_{L^2}.$$

From this point on the proof is similar to that provided for that case $s = 3$.

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