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Borel-Le Roy Summability of the High Temperature Expansion for Classical Continuous Systems

Marco Maioli

Dipartimento di Matematica pura e applicata, Università di Modena, I-41100 Modena, Italy

Abstract. For classical gases with suitable pair interactions such that $\Phi(r) \sim (\ln r^{-1})^p$ as $r \to 0$ ($p \in \mathbb{N}$), the Taylor expansion in β of the correlation functions and the pressure are summable at $\beta = 0$ by the Borel-Le Roy method of order $p+1$.

I. Introduction

As it is known [5], for classical continuous systems with stable and regular pair potentials the correlation functions and the pressure admit a convergent power series expansion in the activity z, while the typical analyticity region in β ($\beta = (kT)^{-1}$) is the half plane $\mathrm{Re}\,\beta > 0$. As recently proved by Wagner [7], if the pair potential is bounded and absolutely integrable, the correlation functions and the pressure turn out to have Borel summable Taylor expansions at $\beta = 0$ (for Borel summability, see e.g. [4, 6]). Among other facts the proof uses analyticity for $\mathrm{Re}\,\beta > 0$ and the bound $\int |\Phi(x)|^n dx \leqq (\|\Phi\|_\infty)^{n-1} \|\Phi\|_1$.

Here the aim is to prove the Borel-Le Roy summability ([3, 2]) of these power series, under suitable hypotheses on the pair potential $\Phi(r)$. Hypotheses (1), (2), (3) below include, in particular, the asymptotic behaviour $\Phi(r) \sim (\ln r^{-1})^p$ as $r \to 0$ ($p \in \mathbb{N}$). These assumptions allow us to analytically continue the correlation functions beyond the right half plane, to a region containing $\left\{\beta / \mathrm{Re}\,\beta^{\frac{1}{1+p}} > 0\right\}$ on the Riemann surface of $\ln \beta$, which is suggested by the analytic structure of $\int (e^{-\beta\Phi(x)} - 1)dx$ in these cases (Proposition 2.1). Moreover the power series remainders are proved not to grow faster than $((p+1)n)!$, which is somehow suggested by bounds of the type $\int |\Phi(x)|^n dx \leqq c(pn)!$, and by a further factor $(n!)^2$ that can be expected in the estimates of n^{th} derivatives of correlation functions.

In the case $\nu = 2$, $p = 1$, conditions (1), (2), (3) include potentials exponentially decreasing as $r \to +\infty$ and with the asymptotic behaviour of two-dimensional Yukawa potentials (see e.g. [8, 1]) as $r \to 0$, although $\Phi(r) = e^{-ar}(\ln r^{-1})$ is not in this class owing to the technical requirement of the existence of an inverse function $r=\Psi(t)$ [Hypothesis (1)].

II. Notations

Let us assume the following hypotheses on the pair potential $\Phi(r)$, $r>0$:

(1) $\Phi(r)$ is the restriction to $r\in\mathbb{R}_+$ of a function analytic in some angular sector containing $\mathbb{R}_+$, which admits an inverse function $\Psi(t)$ analytic for $|\arg(t)|<p\pi/2$ (for some $p\in\mathbb{N}$);

(2) $\Phi(r)\sim c(\ln r^{-1})^p$ as $r\to 0$, for some $c>0$;

(3) $\Phi(r)\sim c'e^{-ar}r^n(\ln r)^m$ as $r\to+\infty$, for some $c',a>0$, $n,m\in\mathbb{Z}$. As a consequence, taking from now on $c=c'=1$, the inverse function $\Psi(t)$ admits the asymptotic behaviours:

$$\Psi(t)\sim\exp(-t^{1/p})\quad\text{as}\quad t\to\infty \tag{4}$$

$$\Psi(t)\sim a^{-1}\ln t^{-1}\quad\text{as}\quad t\to 0 \tag{5}$$

in the analyticity sector.

An example is provided by $\Phi(r)=e^{-r}(\ln r^{-1})^p(1-r)^{-p}$.

By these assumptions $\Phi(r)$ is a monotone and positive potential and it satisfies stability and regularity [5], with stability constant given by zero. Then, in order to represent the infinite volume correlation functions [5, Chap. 4.2], on the space E of sequences $\varphi=(\varphi(x)_n)_{n\in\mathbb{N}}$ of complex functions such that

$$\|\varphi\|=\sup_{n\geqq 1}\ \operatorname*{ess\,sup}_{(x)_n\in\mathbb{R}^{\nu n}}|\varphi(x)_n|<\infty\,,$$

we can define the operator Γ_β such that

$$(\Gamma_\beta\varphi)(x_1)=\sum_{n=1}^{\infty}(n!)^{-1}\int d(y)_n K_\beta(x_1,(y)_n)\varphi(y)_n\,, \tag{6}$$

$$(\Gamma_\beta\varphi)(x)_m=\varphi(x)'_{m-1}+\sum_{n=1}^{\infty}(n!)^{-1}\int d(y)_n K_\beta(x_1,(y)_n)\varphi((x)'_{m-1},(y)_n)\,, \tag{7}$$

where $(x)'_{m-1}=(x_2,x_3,\dots,x_m)$ and

$$K_\beta(x_1,(y)_n)=\prod_{j=1}^{n}(\exp(-\beta\Phi(x_1-y_j))-1)\,. \tag{8}$$

On the same space we can define

$$(\Delta_\beta\varphi)(x)_m=\exp(-\beta W^1(x)_m)\varphi(x)_m\,, \tag{9}$$

where

$$W^1(x)_m=0\quad\text{for}\quad m=1\,,\qquad W^1(x)_m=\sum_{j=2}^{m}\Phi(x_1-x_j)\quad\text{for}\quad m\geqq 2\,. \tag{10}$$

If $\operatorname{Re}\beta>0$, $\mathbb{K}_\beta=\Delta_\beta\Gamma_\beta$ is a product of bounded operators in E and $\|\mathbb{K}_\beta\|\leqq\exp(C(\beta))$, where $C(\beta)=\int|e^{-\beta\Phi(x)}-1|dx<\infty$ by regularity. For

$$|z|<\exp(-C(\beta))\,,\qquad \operatorname{Re}\beta>0\,,$$

the sequence of the infinite volume correlation functions belongs to E and can be written as:

$$\varrho(\beta, z) = (\mathbb{I} - z\mathbb{K}_\beta)^{-1} z\alpha \tag{11}$$

where $\alpha(x_1) = 1$, $\alpha(x)_m = 0$ for $m > 1$.

Under assumptions (1), (2), (3) we can consider the extended function $\tilde{C}(\beta)$ defined in the following proposition.

Proposition 2.1. *The integral* $\int_{\mathbb{R}^\nu} (e^{-\beta\Phi(x)} - 1)dx$ *admits an analytic extension* $\tilde{C}(\beta)$ *for* $\mathrm{Re}\,\beta^{(1+p)^{-1}} > 0$, *such that* $|\tilde{C}(\beta)| \leqq k|\beta|$ $(k > 0)$ *uniformly with respect to the phase of* β.

Proof. By assumption (1), for $\beta > 0$, the above integral is equal to

$$k_1 \int_0^\infty (e^{-\beta t} - 1)\Psi(t)^{\nu-1}\Psi'(t)dt = k_1 \int_0^\infty (\exp(-\beta^{(1+p)^{-1}}\tau) - 1)$$
$$\cdot \Psi(\tau\beta^{-p(1+p)^{-1}})^{\nu-1}\Psi'(\tau\beta^{-p(1+p)^{-1}})\beta^{-p(1+p)^{-1}}d\tau\,. \tag{12}$$

The last integral is absolutely convergent for $\mathrm{Re}\,\beta^{(1+p)^{-1}} > 0$. Indeed, setting $\beta = |\beta|e^{i\theta}$, $\tau = \sigma|\beta|^{p(p+1)^{-1}}$ and using assumption (1):

$$|\tilde{C}(\beta)| \leqq k_1 \int_0^\infty |(1 - \exp(-|\beta|\sigma e^{i\theta(p+1)^{-1}}))$$
$$\cdot \Psi(\sigma e^{-i\theta p(p+1)^{-1}})^{\nu-1}\Psi'(\sigma e^{-i\theta p(p+1)^{-1}})|d\sigma\,. \tag{13}$$

Since $e^{-z} - 1 = -ze^{-\varepsilon z}$ for some $\varepsilon = \varepsilon(z)$, $0 \leqq \varepsilon \leqq 1$, we have by (4), (5):

$$|\tilde{C}(\beta)| \leqq k_2 \int_0^1 |\beta|\sigma \exp\left(-\varepsilon|\beta|\sigma\cos\frac{\theta}{p+1}\right)|\ln\sigma^{-1}|^{\nu-1}\sigma^{-1}d\sigma$$
$$+ k_2 \int_1^\infty |\beta|\sigma \exp\left(-\varepsilon|\beta|\sigma\cos\frac{\theta}{p+1}\right)$$
$$\cdot \exp\left(-\nu\sigma^{p^{-1}}\cos\frac{\theta}{p+1}\right)\sigma^{p^{-1}-1}d\sigma \leqq k_3|\beta| \tag{14}$$

if $|\theta| < (p+1)\pi/2$, for some $k_2, k_3 > 0$. The uniformity of (14) with respect to θ can be checked by the equivalent substitution $t = (\beta e^{i\gamma})^{p(p+1)^{-1}}$ in (12), with γ real and small. Indeed the consideration of complex β leads to the estimate (14) with θ replaced by $\theta + \gamma$: whence the uniformity near $\theta = -(p+1)\pi/2$ and $\theta = (p+1)\pi/2$ by assuming $\gamma > 0$ and $\gamma < 0$ respectively, and the assertion is proved.

Let $z \in \mathbb{C}^\nu$: we say that $|\mathrm{Im}\,z| \leqq d$ if the imaginary part of each component is not larger than d. Let $S_\delta^n = \{(z)_n \in \mathbb{C}^{\nu n}/|\mathrm{Im}\,z_j| \leqq \delta$ for $j = 1, 2, \dots, n\}$ and let $T_\delta^n = \{(z)_n \in S_\delta^n/z_1 = z_j$ for some $j \neq 1\}$. We can consider the space F_δ of sequences of functions $\varphi(z)_n$ analytic [in each one of the νn components of $(z)_n$] at least on $S_\delta^n \setminus T_\delta^n$ and bounded on S_δ^n, such that

$$\|\varphi\|_\delta = \sup_{n \geqq 1} \sup_{(z)_n \in S_\delta^n} |\varphi(z)_n| < \infty\,. \tag{15}$$

Of course $\alpha \in \bigcap_{\delta > 0} F_\delta$ and $\|\alpha\|_\delta = 1$ for all δ. Moreover, by the properties of $e^{-\beta\Phi}$ on $\mathbb{C}^\nu$, $\varrho(\beta, z)$ belongs to these spaces for $\beta > 0$.

III. Analytic Continuation and Estimates

Proposition 3.1. *There is some $d>0$ such that if $\varphi \in F_{qd}$ $(q>1)$ the expressions* (6), (7) *admit analytic continuation $(\tilde{\Gamma}_\beta\varphi)(x)_m$ to $\operatorname{Re}\beta^{(1+p)^{-1}}>0$ such that $(\tilde{\Gamma}_\beta\varphi) \in F_{(q-1)d}$ and*

$$\|\tilde{\Gamma}_\beta\varphi\|_{(q-1)d} \leqq k_1 \exp(k_2|\tilde{C}(\beta)|)\,\|\varphi\|_{qd} \tag{16}$$

with k_1, k_2 independent of q.

Proof. It is sufficient to consider (7). For $\beta>0$:

$$\begin{aligned}(\Gamma_\beta\varphi)(x)_m &= \varphi(x)'_{m-1} + \sum_{n=1}^{\infty}(n!)^{-1} \\ &\cdot \int_{(\mathbb{R}_+)^n}\int_{T_n}\prod_{j=1}^{n}(e^{-\beta\Phi(r_j)}-1)(-1)^n r_j^{\nu-1}dr_j) \\ &\cdot \varphi((x)'_{m-1}, x_1 - r_1 f_1, \dots, x_1 - r_n f_n) d\mu_n .\end{aligned} \tag{17}$$

In these integrals $r_j = |x_1 - y_j|$ $(j=1,2,\dots,n)$, $x_1 - y_j = r_j f_j$, where f_j only depends on the angular part of the ν-dimensional polar coordinates, and $\int_{T_n} d\mu_n$ denotes the integration over such angular coordinates for all j. By the substitution $r_j = \Psi(\beta^{-p(1+p)^{-1}}t_j)$

$$\begin{aligned}(\Gamma_\beta\varphi)(x)_m &= \varphi(x)'_{m-1} + \sum_{n=1}^{\infty}(n!)^{-1}\int_{(\mathbb{R}_+)^n}\int_{T_n}\prod_{j=1}^{n}((1-\exp(t_j\beta^{(1+p)^{-1}})) \\ &\cdot \Psi(t_j\beta^{-p(1+p)^{-1}})^{\nu-1}\Psi'(t_j\beta^{-p(1+p)^{-1}})dt_j)\beta^{-pn(1+p)^{-1}} \\ &\cdot \varphi((x)'_{m-1}, x_1 - \Psi(t_1\beta^{-p(p+1)^{-1}})f_1, \dots, x_1 \\ &- \Psi(t_n\beta^{-p(1+p)^{-1}})f_n) d\mu_n ,\end{aligned} \tag{18}$$

where the f_j's are independent of β and t_j. Now, the right-hand-side of (18) makes sense as an analytic function of β for $\operatorname{Re}\beta^{(1+p)^{-1}}>0$. Indeed the integrand is analytic by assumption (1). Moreover, after the substitution $t_j = \tau_j|\beta|^{(1+p)^{-1}p}$ we have by (4), (5):

$$\Psi(\tau e^{-i\theta p(1+p)^{-1}}) \sim \exp(-\tau^{p^{-1}}e^{-i\theta(1+p)^{-1}}) \quad \text{as} \quad \tau\to\infty , \tag{19a}$$

$$\operatorname{Im}\Psi(\tau e^{-i\theta p(1+p)^{-1}}) \sim -a^{-1}ip\theta(1+p)^{-1} \quad \text{as} \quad \tau\to 0 . \tag{19b}$$

As a consequence:

$$|\operatorname{Im}\Psi(\tau e^{-i\theta p(1+p)^{-1}})| \leqq d \tag{20}$$

for some $d>0$, uniformly for $|\theta| \leqq (p+1)\pi/2$. On the other hand $|f_j| \leqq 1$, therefore:

$$|\varphi((x)'_{m-1}, x_1 - \Psi(\tau_1 e^{-i\theta p(1+p)^{-1}})f_1, \dots)| \leqq \|\varphi\|_{qd} \tag{21}$$

if $|\operatorname{Im} x_1| \leqq (q-1)d$. Comparing (18) and (21) with (12) we obtain (16) and the assertion is proved.

Proposition 3.2. *Let $\varphi \in F_{qd}$ (q, d as in Proposition 3.1). For fixed $R>0$ there are A_1, A_2 such that, for $\operatorname{Re}\beta^{(1+p)^{-1}}>0$, $|\beta|<R$,*

$$|D_\beta^s(\tilde{\Gamma}_\beta\varphi)(x)_m| \leqq A_1(A_2)^s((p+1)s)!\,\|\varphi\|_{qd} \tag{22}$$

for $(x)_m \in S^m_{(q-1)d}$, $s \in \mathbb{N}_0$.

Proof. It is sufficient to consider the $m>1$ cases and to bound the s^{th} derivative with respect to $|\beta|$. By the substitution $t_j=\tau_j|\beta|^{p(1+p)^{-1}}$ in (18), the only term depending on $|\beta|$ is

$$\prod_{j=1}^{n}(\exp(-|\beta|\tau_j e^{i\theta(1+p)^{-1}})-1).$$

Hence, by the same argument of Proposition 3.1 we have:

$$|D^s_{|\beta|}(\tilde{\Gamma}_\beta\varphi)(x)_m|\leqq\|\varphi\|_{qd}+\sum_{n=1}^{\infty}(n!)^{-1}\sum_{\substack{s_1,\ldots,s_n\geqq 0\\ s_1+\ldots+s_n=s}}\frac{s!}{s_1!\ldots s_n!}\cdot\|\varphi\|_{qd}$$
$$\cdot\prod_{j=1}^{n}\left(\int_{\mathbb{R}_+}|D^{s_j}_{|\beta|}\{\exp(-|\beta|\tau_j e^{i\theta(1+p)^{-1}})-1\}\right.$$
$$\left.\cdot\,\Psi(\tau_j e^{-i\theta p(1+p)^{-1}})^{\nu-1}\Psi'(\tau_j e^{-i\theta p(1+p)^{-1}})d\tau_j|\right). \tag{23}$$

Now, by (19) [compare with (14)],

$$\int_{\mathbb{R}_+}|D^s_{|\beta|}\{\exp(-|\beta|\tau e^{i\theta(1+p)^{-1}})-1\}\Psi(\tau e^{-i\theta p(1+p)^{-1}})^{\nu-1}\Psi'(\tau e^{-i\theta p(1+p)^{-1}})d\tau|$$
$$\leqq k_1\int_0^1|\beta|\tau^{s+1}|\ln\tau|^{\nu-1}\tau^{-1}d\tau+k_2\int_1^\infty|\beta|\tau^{s+1}$$
$$\exp\left(-\nu\tau^{p^{-1}}\cos\frac{\theta}{p+1}\right)\tau^{p^{-1}-1}d\tau\leqq k_3(ps)!(k_4)^s, \tag{24}$$

where the constants are independent of $|\beta|$ for $|\beta|<R$ and can be chosen independent of θ by the argument used in Proposition 2.1. By combining (23) and (24):

$$|D^s_{|\beta|}(\tilde{\Gamma}_\beta\varphi)(x)_m|\leqq\|\varphi\|_{qd}\sum_{n=0}^{\infty}(n!)^{-1}A^s(s!)(k_3)^n(ps)!(k_4)^s, \tag{25}$$

since $s_1+s_2+\ldots+s_n=s$, and the estimate (22) is proved.

Proposition 3.3. *There is a scale of spaces $F_{\delta,h}\subset F_{\delta,h-1}\subset\ldots\subset F_{\delta,0}=F_\delta$ (with norms $\|\cdot\|_{\delta,h}$, $\delta>0$, $h\in\mathbb{N}_0$) such that, if $|\beta|<R$,* (9), (10) *define a bounded operator Δ_β from $F_{\delta,h+1}$ to $F_{\delta,h}$ and*

$$\|(D^s_\beta\Delta_\beta)\varphi\|_{\delta,h}\leqq(A_3)^s(s!)\|\varphi\|_{\delta,h+1}, \tag{26}$$

uniformly for $s\in\mathbb{N}_0$, $\delta>0$, $h\in\mathbb{N}_0$, $|\beta|<R$.

Proof. We can simply consider the space $F_{\delta,h}$ of vectors $\varphi\in F_\delta$ such that

$$\|\varphi\|_{\delta,h}=\sup_{m\geqq 1}\sup_{(x)_m\in S^m_\delta}\exp(hR'|W^1(x)_m|)|\varphi(x)_m|<\infty, \tag{27}$$

where $R'>R$ and $W^1(x)_m$ is defined by (10). Then the first assertion is immediate.

Since

$$\begin{aligned}(D^s_\beta \Delta_\beta)\varphi(x)_m &= (-W^1(x)_m)^s \exp(-\beta W^1(x)_m)\varphi(x)_m\,,\\ |\exp(hR'|W^1(x)_m|)\,(D^s_\beta \Delta_\beta)\varphi(x)_m| &\leqq \exp((h+1)R'|W^1(x)_m|)\,|W^1(x)_m|^s\\ &\quad\cdot \exp((R-R')|W^1(x)_m|)\,|\varphi(x)_m|\\ &\leqq s!\,\|\varphi\|_{\delta,h+1}(A_3)^s\,,\end{aligned} \tag{28}$$

and the proposition is proved.

Lemma 3.4. *Let $R>0$ be fixed, $\operatorname{Re}\beta^{(1+p)^{-1}}>0$, $|\beta|<R$, $\tilde{\Gamma}_\beta$ and Δ_β as in Propositions 3.1 and 3.3. Then the product $\mathbb{K}_\beta=\Delta_\beta\tilde{\Gamma}_\beta$ is a bounded operator from $F_{qd,h+1}$ to $F_{(q-1)d,h}$ such that:*

$$\|D^r_\beta \mathbb{K}_\beta\|^{(q-1)d,h}_{qd,h+1} \leqq A_0 A^r((p+1)r)! \tag{29}$$

uniformly for $q>1$ and $h, r\in\mathbb{N}_0$.

Proof. By definition of the weighted norms (27) in $F_{\delta,h}$ ($\delta>0$, $h\in\mathbb{N}_0$) it obviously follows from Proposition 3.2 that

$$\|(D^s_\beta \tilde{\Gamma}_\beta)\varphi\|_{(q-1)d,h} \leqq A_1(A_2)^s((p+1)s)!\,\|\varphi\|_{qd,h} \tag{30}$$

for all $\varphi\in F_{qd,h}$. Thus by (30) and Proposition 3.3

$$\begin{aligned}\|D^r_\beta \Delta_\beta \tilde{\Gamma}_\beta\|^{(q-1)d,h}_{qd,h+1} &\leqq \sum_{s=0}^{r}\binom{r}{s}\|D^s_\beta\Delta_\beta\|^{(q-1)d,h}_{(q-1)d,h+1}\|D^{r-s}_\beta\tilde{\Gamma}_\beta\|^{(q-1)d,h+1}_{qd,h+1}\\ &\leqq \sum_{s=0}^{r} 2^r(A_3)^s s!\,A_1(A_2)^{r-s}((p+1)(r-s))!\\ &\leqq A_0A^r((p+1)r)!\end{aligned} \tag{31}$$

and the lemma is proved.

Lemma 3.5. *For any $R>0$ there is $R_1>0$ such that, for $|z|<R_1$, $\varrho(\beta,z)$ admits an analytic continuation $\tilde{\varrho}(\beta,z)$ in F_d to the region $\operatorname{Re}\beta^{(1+p)^{-1}}>0$, $|\beta|<R$. Moreover*

$$\|D^r_\beta\tilde{\varrho}(\beta,z)\|_d \leqq |z|B_0B^r((p+2)r)! \tag{32}$$

uniformly with respect to β, z.

Proof. Since $\alpha(x_1)=1$, $\alpha(x)_m=0$ for $m>1$, $\|\alpha\|_{\delta,h}=1$ for all δ and h. Hence the partial sums of the geometric series associated with (11) satisfy:

$$\begin{aligned}&\left\|\sum_{h=0}^{N}(z\mathbb{K}_\beta)^h z\alpha\right\|_d\\ &\leqq \sum_{h=0}^{N}|z|^{h+1}\|\mathbb{K}_\beta\|^{d,0}_{2d,1}\|\mathbb{K}_\beta\|^{2d,1}_{3d,2}\cdots\|\mathbb{K}_\beta\|^{hd,h-1}_{(h+1)d,h}\|\alpha\|_{(h+1)d,h}\\ &\leqq |z|(1-|z|A_0)^{-1}\end{aligned} \tag{33}$$

uniformly with respect to N, by Lemma 3.4. Thus, given $R>0$, there is $R_1=(A_0)^{-1}$ such that $\tilde{\varrho}(\beta,z)$ exists for $|\beta|<R$, $\operatorname{Re}\beta^{(1+p)^{-1}}>0$, $|z|<R_1$ as a uniform limit, in F_d, of analytic approximants.

Moreover, by Lemma 3.4:

$$\|D_\beta^r\tilde{\varrho}(\beta,z)\|_d = \left\| \sum_{h=0}^{\infty} z^{h+1} \sum_{\substack{r_1,\dots,r_h \geqq 0 \\ r_1+\dots+r_h=r}} \frac{r!}{r_1!\dots r_h!} (D_\beta^{r_1}\tilde{\mathbb{K}}_\beta)(D_\beta^{r_2}\tilde{\mathbb{K}}_\beta)\dots(D_\beta^{r_h}\tilde{\mathbb{K}}_\beta)\alpha \right\|_d$$

$$\leqq \sum_{h=0}^{\infty} |z|^{h+1} \sum_{\substack{r_1,\dots,r_h \geqq 0 \\ r_1+\dots+r_h=r}} \frac{r!}{r_1!\dots r_h!} \|D_\beta^{r_1}\tilde{\mathbb{K}}_\beta\|_{2d,1}^{d,0} \|D_\beta^{r_2}\tilde{\mathbb{K}}_\beta\|_{3d,2}^{2d,1} \dots \|D^{r_h}\tilde{\mathbb{K}}_\beta\|_{(h+1)d,h}^{hd,h-1}$$

$$\leqq \sum_{h=0}^{\infty} |z|^{h+1} \sum_{\substack{r_1,\dots,r_h \geqq 0 \\ r_1+\dots+r_h=r}} r!(A_0)^h A^{r_1}((p+1)r_1)!\dots A^{r_h}((p+1)r_h)!$$

$$\leqq |z|(B_1)^r(1-|z|A_0)^{-1}A^r((p+2)r)! \tag{34}$$

and (32) is proved.

A bound of the type (32) can be easily extended to the function $\beta p(\beta, z)$, where $p(\beta, z)$ is the thermodynamic limit of the pressure, as well as to $f(\tilde{\varrho}(\beta, z))$, where f is any linear functional defined on F_d (see [7]). As a consequence, the remainders of the Taylor expansions of such functions satisfy the criterion for Borel-Le Roy summability of order $p+1$, which is implicit in Watson-Nevanlinna theorem concerning Borel summability (see [2, 3, 6]).

Theorem 3.6. *If Φ satisfies assumptions* (1), (2), (3), *the power series expansion at $\beta=0$ of $\beta p(\beta, z)$ and $f(\tilde{\varrho}(\beta, z))$ (where f is any linear functional on F_d) admits a convergent Borel-Le Roy sum of order $p+1$.*

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