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Intersections of Random Walks in Four Dimensions. II*

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Abstract. Let $f(n)$ be the probability that the paths of two simple random walks of length n starting at the origin in $\mathbb{Z}^4$ have no intersection. It has previously been shown that $f(n) \leqq c(\log n)^{-1/2}$. Here it is proved that for all $r > \frac{1}{2}$, $\lim_{n\to\infty} (\log n)^r f(n) = \infty$.

1. Introduction

Let $S_1(n,\omega)$ and $S_2(n,\omega)$ be independent simple random walks starting at the origin in $\mathbb{Z}^4$ (for definitions see [1]), and let Π_1, Π_2 denote the paths of the walks

$$\Pi_i(a,b) = \Pi_i(a,b,\omega) = \{S_i(n,\omega) : a < n < b\},$$
$$\Pi_i[a,b] = \Pi_i[a,b,\omega] = \{S_i(n,\omega) : a \leqq b \leqq b\},$$

and similarly for $\Pi_i(a,b]$ and $\Pi_i[a,b)$.

The probabilities that the paths Π_i intersect were studied in [1]. This paper follows up on that paper by giving a proof of a conjecture made. Let

$$f(n) = P\{\Pi_1[0,n] \cap \Pi_2(0,n] = \emptyset\}.$$

In [1] it was shown that there exist c_1, $c_2 > 0$ satisfying

$$c_1(\log n)^{-1} \leqq f(n) \leqq c_2(\log n)^{-1/2}, \tag{1.1}$$

and it was conjectured for $r > \frac{1}{2}$, that

$$\lim_{n\to\infty} (\log n)^r f(n) = \infty. \tag{1.2}$$

Here we prove (1.2).

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To give an idea of the technical problems involved in proving (1.2), we first sketch an argument similar to the one in [1] which led to the conjecture. If

$$A_n = \{\Pi_1[0,n] \cap \Pi_2(0,n] = \emptyset\},$$
$$B_n = \{\Pi_1[0,2n] \cap \Pi_2(n,2n] = \emptyset \text{ and } \Pi_1(n,2n] \cap \Pi_2(0,n] = \emptyset\},$$

then

$$P(A_{2n}) = P(A_n) P(B_n | A_n). \tag{1.3}$$

The methods of [1] allow one to calculate $P(B_n)$. However the set A_n has small probability and it is not clear how to compute $P(B_n|A_n)$, although it was expected that $P(B_n|A_n) \cong P(B_n)$. It was shown that if one could substitute $P(B_n)$ in (1.3), one could get the result.

The main technical step in this paper is a computation of such a conditional probability. We do not choose A_n and B_n exactly as above but instead use powers of the logarithm for scales.

Choose $\alpha > \gamma > \beta > 1$, and set

$$a_n = \left[\frac{n}{(\log n)^\alpha}\right], \quad b_n = \left[\frac{n}{(\log n)^\gamma}\right], \quad d_n = \left[\frac{n}{(\log n)^\beta}\right],$$

and consider the sets

$$A(a_n) = \{\Pi_1[0,a_n] \cap \Pi_2(0,a_n] = \emptyset\},$$
$$D(d_n, n) = \{\Pi_1[0,n] \cap \Pi_2(d_n,n] \neq \emptyset \text{ or } \Pi_1(d_n,n] \cap \Pi_2(0,n] \neq \emptyset\}.$$

In Theorem 2 we prove that for $\alpha - \beta > 7$,

$$P(D(d_n,n)) \cong P(D(d_n,n)|A(a_n)),$$

i.e. that $A(a_n)$ and $D(d_n,n)$ are asymptotically independent events.

It is easier to picture the idea of the proof if we consider S_1 and S_2 to be one "two-sided" random walk. Let Ω^n denote the set of two-sided walks of length $2n$, i.e. nearest neighbor walks $\omega(i)$, $-n \leqq i \leqq n$, with $\omega(0)=0$. We can define $A(a_n)$ and $D(d_n,n)$ as subsets of Ω^n. Let $\hat{P}$ denote the conditional measure on $A(a_n)$ derived from the usual measure P on Ω^n. Then we wish to estimate $\hat{P}(D(d_n,n))$.

We accomplish this by considering another measure on $A(a_n)$ which is close to $\hat{P}$. Let $\Omega = \Omega^{n+b_n}$ and P the usual measure. For each $\omega \in \Omega^n$, $i=1,\ldots,b_n$, we say ω is "a_n loop-free at step i" [or $I_i(\omega)=1$ in the notation of Sect. 3] if

$$\omega(j) \neq \omega(k), \qquad -a_n + i \leqq j \leqq i < k \leqq a_n + i,$$

that is, if ω is translated so that $\omega(i)$ becomes the origin, and is then cut off so that the translated walk is in Ω^n, the translated walk is in fact in $A(a_n)$. We can define a probability $\tilde{P}$ on $A(a_n)$ in the following fashion:

– choose $\omega \in \Omega^{n+b_n}$ (using P)

– consider all $i=1,\ldots,b_n$ such that $I_i(\omega)=1$, and randomly (i.e. with equal probability to each i) choose one such i

– translate the walk so that $\omega(i)$ is the origin.

What we prove is that $\tilde{P}$ is in fact close to $\hat{P}$. This can be proven as long as the number of "a_n loop-free" points for a particular ω is an almost constant random variable. This is true because the random variables I_i are $(2a_n)$-dependent, i.e. for $|i-j| \geqq 2a_n$, I_i and I_j are independent. If b_n is sufficiently larger than a_n $(\alpha-\gamma>7)$, we can prove the result.

We finally show, using the fact that b_n is small with respect to d_n, that $\tilde{P}(D(d_n, n)) \cong P(D(d_n, n))$. The details of the proof are worked out in Sect. 3.

In Sect. 2, it is shown how Theorem 2 can be used to prove (1.2). Essentially what is used is a logarithmic scale equivalent of (1.3).

2. The Main Theorem

Theorem 1. *If*

$$f(n) = P\{\Pi_1[0,n] \cap \Pi_2(0,n] = \emptyset\},$$

then for every $r > \frac{1}{2}$,

$$\lim_{n\to\infty} (\log n)^r f(n) = \infty .$$

For any $0<n<m$, define the sets

$$A_n = A(n) = \{\omega : \Pi_1[0, n, \omega] \cap \Pi_2(0, n, \omega] = \emptyset\},$$

$$D_{n,m} = D(n,m) = \{\omega : \Pi_1(n, m, \omega] \cap \Pi_2(0, m, \omega] \neq \emptyset \text{ or } \Pi_1[0, m, \omega] \cap \Pi_2(n, m, \omega] \neq \emptyset\}.$$

Then for $n<m$,

$$\begin{aligned} A_m &= A_n \cap (D_{n,m})^c, \\ P(A_m) &= P(A_n)\,[1 - P(D_{n,m}|A_n)]. \end{aligned} \tag{2.1}$$

A large portion of [1] is devoted to estimating $P(D_{n,m})$. Theorem 4.1 states that for $c>1$,

$$\lim_{n\to\infty} (\log n) P\{\Pi_1(n, cn] \cap \Pi_2(0,\infty) \neq \emptyset\} = \tfrac{1}{2}\log c. \tag{2.2}$$

Analysis of the proof shows that a similar argument will work if we replace c with $(\log n)^\beta$ for some $\beta>0$, giving

$$\lim_{n\to\infty} \frac{\log n}{\log\log n} P\{\Pi_1(n, n(\log n)^\beta) \cap \Pi_2(0,\infty) \neq \emptyset\} = \tfrac{1}{2}\beta. \tag{2.3}$$

The probability of the set in (2.3) differs from $P(D(n, n(\log n)^\beta))$ by at most

$$P\{\Pi_1[0,n] \cap \Pi_2(n, n(\log n)^\beta] \neq \emptyset\} + P\{\Pi_1[n, n(\log n)^\beta] \cap \Pi_2[n(\log n)^\beta, \infty) \neq \emptyset\}.$$

However, by Theorem 4.1, both of the above probabilities are $O\left(\frac{1}{\log n}\right)$. We can therefore conclude

$$\lim_{n\to\infty} \frac{\log n}{\log\log n} P(D(n, n(\log n)^\beta)) = \tfrac{1}{2}\beta. \tag{2.4}$$

In proving Theorem 1, we will use (2.1) and hence will need to estimate the conditional probability of $D(n, n(\log n)^{\beta})$ given A_n. Note that (2.4) only gives the unconditioned probability. The main independence result is contained in the following theorem which we prove in the next section.

Theorem 2. *Let* $1<\beta<\alpha-7<\infty$, *and let*

$$a_n=\left[\frac{n}{(\log n)^{\alpha}}\right], \qquad e_n=\left[\frac{n}{(\log n)^{\beta}}\right].$$

Then

$$\lim_{n\to\infty}\frac{\log n}{\log\log n}P(D(e_n,n)\,|\,A(a_n))=\lim_{n\to\infty}\frac{\log n}{\log\log n}P(D(e_n,n))=\tfrac{1}{2}\beta\,.$$

From Theorem 2 we can conclude a stronger independence result.

Theorem 3. *If* $\beta>7$ *and* $e_n=\left[\frac{n}{(\log n)^{\beta}}\right]$, *then*

$$\limsup_{n\to\infty}\frac{\log n}{\log\log n}P(D(e_n,n)\,|\,A(e_n))$$
$$\leqq\lim_{n\to\infty}\frac{\log n}{\log\log n}P(D(e_n,n))=\tfrac{1}{2}\beta\,.$$

Proof. For each n, choose $d_1,\dots,d_5$ (depending on n) by $d_5=n$ and for $i=1,\dots,4$,

$$d_i=\left[\frac{d_{i+1}}{(\log d_{i+1})^{\beta}}\right].$$

Then for $i=1,\dots,5$,

$$d_i=n(\log n)^{-\beta(5-i)}[1+o(1)]\,.$$

Therefore for $i=1,2,3$, Theorem 2 states that

$$\lim_{n\to\infty}\frac{\log n}{\log\log n}P(D(d_{i+1},d_{i+2})\,|\,A(d_i))=\tfrac{1}{2}\beta\,.$$

Fix $\varepsilon>0$, and suppose

$$\limsup_{n\to\infty}\frac{\log n}{\log\log n}P(D(d_4,d_5)\,|\,A(d_4))\geqq\tfrac{1}{2}\beta+2\varepsilon\,. \tag{2.5}$$

Choose N sufficiently large that for $n\geqq N$, $i=1,2,3$,

$$\frac{\log n}{\log\log n}P(D(d_{i+1},d_{i+2})\,|\,A(d_i))\leqq\tfrac{1}{2}\beta+\tfrac{1}{2}\varepsilon\,, \tag{2.6}$$

and choose $n>N$ such that

$$P(D(d_4,d_5)\,|\,A(d_4))\geqq(\tfrac{1}{2}\beta+\varepsilon)\frac{\log\log n}{\log n}\,. \tag{2.7}$$

Then

$$\frac{\log\log n}{\log n}\left(\tfrac{1}{2}\beta+\tfrac{1}{2}\varepsilon\right)\geqq P(D(d_4,d_5)\,|\,A(d_3))$$
$$\geqq P(A(d_4)\cap D(d_4,d_5)\,|\,A(d_3))$$
$$=P(A(d_4)\,|\,A(d_3))P(D(d_4,d_5)\,|\,A(d_4))$$
$$\geqq\left(\tfrac{1}{2}\beta+\varepsilon\right)\frac{\log\log n}{\log n}P(A(d_4)\,|\,A(d_3)),$$

or

$$P(D(d_3,d_4)\,|\,A(d_3))\geqq\frac{\varepsilon}{\beta+2\varepsilon}.$$

Doing a similar argument we get the estimate

$$P(A(d_3)\,|\,A(d_2))\leqq\frac{P(D(d_3,d_4)\,|\,A(d_3))}{P(D(d_4,d_5)\,|\,A(d_4))}\leqq\frac{\log\log n}{\log n}\frac{(\beta+\varepsilon)(\beta+2\varepsilon)}{\varepsilon},$$

and since this is less than $\frac{\varepsilon}{\beta+2\varepsilon}$ (for n sufficiently large), we can do this again and get

$$P(A(d_2)\,|\,A(d_1))\leqq\frac{\log\log n}{\log n}\frac{(\beta+\varepsilon)(\beta+2\varepsilon)}{\varepsilon}.$$

But

$$P(A(d_3))\leqq P(A(d_3)\,|\,A(d_1))$$
$$=P(A(d_3)\,|\,A(d_2))P(A(d_2)\,|\,A(d_1))$$
$$\leqq\left(\frac{\log\log n}{\log n}\right)^2\left(\frac{(\beta+\varepsilon)(\beta+2\varepsilon)}{\varepsilon}\right)^2. \tag{2.8}$$

However from (1.1) we know

$$P(A(d_3))=f(d_3)\geqq c_1(\log d_3)^{-1}\geqq c_1(\log n)^{-1}.$$

Hence (2.8) cannot hold for an infinite number of values of n and therefore neither can (2.7). This contradicts (2.5), which gives us the theorem.

Proof of Theorem 1. Fix $\beta>7$, and let e_j be an increasing sequence of integers satisfying

$$e_j=\left[\frac{e_{j+1}}{(\log e_{j+1})^{\beta}}\right].$$

Then by (2.1) and Theorem 3,

$$f(e_j)=P(A(e_j))=P(A(e_{j-1}))\left[1-P(D(e_{j-1},e_j)\,|\,A(e_j))\right]$$
$$\geqq f(e_{j-1})\left[1-\tfrac{1}{2}\beta\varrho_j\frac{\log\log e_j}{\log e_j}\right],$$

where ϱ_j is a sequence of numbers approaching 1. Fix $r>\frac{1}{2}$, and choose γ, $\frac{1}{2}<\gamma<r$.

Then for j sufficiently large

$$f(e_j) \geqq f(e_{j-1}) \left[1 - \beta\gamma \frac{\log\log e_j}{\log e_j} \right]. \tag{2.9}$$

Let $g(n) = (\log n)^{-r}$. Then

$$g(e_{j-1}) \geqq \left(\log \frac{e_j}{(\log e_j)^\beta} \right)^{-r} = (\log e_j - \beta \log\log e_j)^{-r}.$$

Hence

$$\frac{g(e_j)}{g(e_{j-1})} \leqq \left(\frac{\log e_j - \beta \log\log e_j}{\log e_j} \right)^r = \left(1 - \beta \frac{\log\log e_j}{\log e_j} \right)^r.$$

For j sufficiently large,

$$\left(1 - \beta \frac{\log\log e_j}{\log e_j} \right)^r \leqq 1 - \beta\gamma \frac{\log\log e_j}{\log e_j}. \tag{2.10}$$

Let J be an integer such that (2.9) and (2.10) hold for $j \geqq J$. Then

$$f(e_{j+1}) \geqq f(e_j) \frac{g(e_{j+1})}{g(e_j)}, \quad j \geqq J.$$

Hence for every i, by induction,

$$f(e_{J+1}) \geqq \frac{f(e_J)}{g(e_J)} g(e_{J+i}) = \frac{f(e_J)}{g(e_J)} (\log e_{J+i})^{-r}.$$

Hence there exists a $c_r > 0$ such that for all j, $f(e_j) \geqq c_r (\log e_j)^{-r}$. Now for an arbitrary integer n, choose j such that $e_j \leqq n < e_{j+1}$. Then

$$f(n) \geqq f(e_{j+1}) \geqq c_r (\log e_{j+1})^{-r} \geqq \tilde{c}_r (\log e_j)^{-r} \geqq \tilde{c}_r (\log n)^{-r}.$$

Since such an inequality holds for every $r > \frac{1}{2}$, we can conclude for $r > \frac{1}{2}$,

$$\lim_{n \to \infty} (\log n)^r f(n) = \infty.$$

3. Proof of Theorem 2

Let $1 < \beta < \alpha - 7 < \infty$ be fixed. Choose $\gamma > \beta$ with $\gamma < \alpha - 7$. For each n let

$$a_n = \left[\frac{n}{(\log n)^\alpha} \right], \quad b_n = \left[\frac{n}{(\log n)^\gamma} \right],$$

and c_n some number greater than $n + b_n$.

For each j, let Ω^j denote the set of two-sided simple random walks of length $2j$, i.e. the set of all nearest neighbor walks in Z^4, $\omega(i)$, $-j \leqq i \leqq j$, with $\omega(0) = 0$. We will use P to denote the usual simple random walk measure, i.e. the uniform probability measure on Ω^j. As in Sect. 2 we define the events

$$A(a_n)=\{\omega\in\Omega^{c_n}:\omega(j)\neq\omega(k),\ -a_n\leqq j\leqq 0<k\leqq a_n\},$$

$$D(b_n,n)=\{\omega\in\Omega^{c_n}:\omega(j)=\omega(k)\text{ for some }(j,k)\text{ with }$$
$$-n\leqq j<-b_n,\ 0<k\leqq n\text{ or }-n\leqq j\leqq 0,\ b_n<k\leqq n\}.$$

On Ω^{c_n} define for $1\leqq i\leqq b_n$,

$$I_i(\omega)=\text{indicator function of the set}$$
$$\{\omega:\omega(j)\neq\omega(k),\ i-a_n\leqq j\leqq i<k\leqq i+a_n\}.$$

Of course, I_i (as well as several other quantities defined below) depends on n. Then

$$E(I_i)=f(a_n). \tag{3.1}$$

Let

$$L(\omega)=\sum_{i=1}^{b_n} I_i(\omega);$$

then

$$E(L)=b_n f(a_n). \tag{3.2}$$

Also note that the $\{I_i\}$ are $(2a_n)$-dependent random variables, i.e. if $|i-j|\geqq 2a_n$, then I_i and I_j are independent.

Lemma 4. *If $X_1,\ldots,X_n$ are non-negative identically distributed m-dependent random variables with $X_i\leqq M$, then*

$$\operatorname{Var}(X_1+\ldots+X_n)\leqq 2nmM^2.$$

Proof.

$$E[(X_1+\ldots+X_n)^2]=\sum_{i=1}^{n}\sum_{j=1}^{n}E(X_iX_j)$$
$$\leqq\sum_{i=1}^{n}\sum_{j=1}^{n}E(X_i)E(X_j)+\sum_{|i-j|\leqq m}E(X_iX_j)$$
$$\leqq[E(X_1+\ldots+X_n)]^2+2nmM^2.$$

Lemma 5.
(a) $\operatorname{Var}(L)\leqq 2a_nb_n=2n^2(\log n)^{-(\alpha+\gamma)}$.
(b) *For some $c_3>0$, for every $\varepsilon>0$,*

$$P\left\{\left|\frac{L}{EL}-1\right|\geqq\varepsilon\right\}\leqq\frac{c_3}{\varepsilon^2}(\log n)^{2-(\alpha-\gamma)}.$$

Proof. Lemma 4 immediately implies (a) since $I_i\leqq 1$. Chebyshev's Inequality on (a) gives

$$P\{|L-EL|\geqq\varepsilon(EL)\}\leqq\frac{\operatorname{Var}L}{\varepsilon^2(EL)^2}$$
$$\leqq[\varepsilon(EL)]^{-2}2(\log n)^{-(\alpha+\gamma)}n^2.$$

But by (3.2) and (1.1),

$$(EL)^2=(b_n f(a_n))^2 \geqq c_1^2 n^2 (\log n)^{-2-2\gamma}.$$

Therefore,

$$P\{|L-EL| \geqq \varepsilon(EL)\} \geqq \left(\frac{2}{c_1^2}\right) \varepsilon^{-2} (\log n)^{2-(\alpha-\gamma)}.$$

Let

$$\Lambda_n=\{\omega \in \Omega^n : \omega(i) \neq \omega(j),\ -a_n \leqq i \leqq 0 < j \leqq a_n\},$$

and let $\hat{P}$ denote the conditional probability measure on Λ_n induced by P, i.e. $\hat{P}(\omega) = \frac{1}{f(a_n)} P(\omega)$. We can restate Theorem 2 as

$$\lim_{n\to\infty} \frac{\log n}{\log\log n} \hat{P}\left(D\left(\frac{n}{(\log n)^\beta}, n\right)\right) = \tfrac{1}{2}\beta.$$

Unfortunately, the measure $\hat{P}$ is very difficult to work with. Instead we will replace it by a more tractable measure which we can show is close to $\hat{P}$. To set the framework for our strategy, we state an abstract lemma.

Lemma 6. *Let (Ω_1, P_1) and (Ω_2, P_2) be finite probability spaces and $T:\Omega_1 \to \Omega_2$. Suppose $\bar{\Omega}_1 \subset \Omega_1$, $\bar{\Omega}_2 \subset \Omega_2$ and for every $\omega_2 \in \bar{\Omega}_2$,*

$$\left|\frac{P_2(\omega_2)}{P_1(T^{-1}(\omega_2) \cap \bar{\Omega}_1)} - 1\right| \leqq \varepsilon.$$

Then if $F:\Omega_2 \to [0, M]$,

$$(1-\varepsilon)\left[E_{P_1}(F \circ T) - M\left[(1-P_1(\bar{\Omega}_1)) + \left(1 - \frac{1}{1+\varepsilon} P(\bar{\Omega}_2)\right)\right]\right]$$
$$\leqq E_{P_2}(F) \leqq (1+\varepsilon) E_{P_1}(F \circ T) + M[1-P_2(\bar{\Omega}_2)].$$

Proof. Let $U=\{\omega_1 : T\omega_1 \in \bar{\Omega}_2\}$. Then

$$P_1(U) \geqq \sum_{\omega_2 \in \bar{\Omega}_2} P_1(T^{-1}(\omega_2) \cap \bar{\Omega}_1) \geqq \sum_{\omega_2 \in \bar{\Omega}_2} \frac{1}{1+\varepsilon} P_2(\omega_2)$$
$$= \frac{1}{1+\varepsilon} P_2(\bar{\Omega}_2).$$

$$\begin{aligned}
E_{P_2}(F) &\geqq \sum_{\omega_2 \in \bar{\Omega}_2} F(\omega_2) P_2(\omega_2) \\
&\geqq \sum_{\omega_2 \in \bar{\Omega}_2} F(\omega_2)(1-\varepsilon) P_1(T^{-1}(\omega_2) \cap \bar{\Omega}_1) \\
&\geqq (1-\varepsilon)\left[\sum_{\omega_1 \in \bar{\Omega}_1} F(T\omega_1) P_1(\omega_1) - \sum_{\substack{\omega_1 \in \bar{\Omega}_1 \\ T\omega_1 \notin \bar{\Omega}_2}} F(T\omega_1) P_1(\omega_1)\right] \\
&\geqq (1-\varepsilon)\left[E_{P_1}(F \circ T) - \sum_{\omega_1 \in \bar{\Omega}_1} F(T\omega_1) P_1(\omega_1) - MP(U^c)\right] \\
&\geqq (1-\varepsilon)\left[E_{P_1}(F \circ T) - M\left[1 - P_1(\bar{\Omega}_1) + 1 - \frac{1}{1+\varepsilon} P_2(\bar{\Omega}_2)\right]\right].
\end{aligned}$$

Similarly,

$$\begin{aligned} E_{P_2}(F) &= \sum_{\omega_2 \in \bar{\Omega}_2} F(\omega_2)P_2(\omega_2) + \sum_{\omega_2 \in \bar{\Omega}_2^c} F(\omega_2)P_2(\omega_2) \\ &\leqq (1+\varepsilon) \sum_{\omega_1 \in \Omega_1} F(T\omega_1)P_1(\omega_1) + M(1-P_2(\bar{\Omega}_2)) \\ &\leqq (1+\varepsilon)E_{P_1}(F \circ T) + M(1-P_2(\bar{\Omega}_2)). \end{aligned}$$

We will now apply Lemma 6 to our particular case. Let (Ω_2, P_2) be $(\Lambda_n, \hat{P})$ as defined above. The function $F: \Omega_2 \to [0, 1]$ will be the indicator function of the set $D(n(\log n)^{-\beta}, n)$. The probability space (Ω_1, P_1) will be defined so that the measure TP_1 on Ω_2 will correspond to the measure $\tilde{P}$ as described in Sect. 1. Let

$$\bar{\Omega}_1 = \{(\omega, k): \omega \in \Omega^{c_n}, k \in \{1, \ldots, b_n\} \text{ with } I_k(\omega) = 1\}.$$

Define P_1 on $\bar{\Omega}_1$ by $P_1(\omega, k) = P(\omega)[L(\omega)]^{-1}$. That is, we take a point ω at random, using P, then randomly, according to a uniform distribution, choose an "a_n loop-free" point. Note that $P_1(\bar{\Omega}_1) = P\{L \geqq 1\}$. An easy estimate using $(2a_n)$-dependence gives

$$P\{L=0\} \leqq (1-f(a_n))^{b_n/2a_n} \leqq \left(1 - \frac{c_1}{\log n}\right)^{(\log n)^{\alpha} - \gamma/2} \leqq O\left(\frac{1}{n}\right).$$

We let $\Omega_1 = \bar{\Omega}_1 \cup \{*\}$, where $*$ is a dummy element with $P_1(*) = P\{L=0\}$.

Define $T: \bar{\Omega}_1 \to \Omega_2$ by

$$[T(\omega, k)](i) = \omega(i+k) - \omega(k), \qquad -n \leqq i \leqq n.$$

This is just a shift making $\omega(k)$ the origin. Since $I_k(\omega) = 1$ if $(\omega, k) \in \bar{\Omega}_1$, $T(\omega, k) \in \Omega_2$. We extend T to Ω_1 by defining $T(*)$ arbitrarily.

For $\omega_2 \in \Omega_2$, $1 \leqq i \leqq b_n$, let

$$J_i(\omega_2) = \sum_{j=1-i}^{b_n - i} I_j(\omega_2).$$

Then by definition of T,

$$\begin{aligned} P_1[T^{-1}(\omega_2) \cap \bar{\Omega}_1] &= P(\omega_2) \sum_{j=1}^{b_n} [J_j(\omega_2)]^{-1} \\ &= P_2(\omega_2) f(a_n) \sum_{j=1}^{b_n} [J_j(\omega_2)]^{-1}. \end{aligned} \tag{3.3}$$

It is difficult to analyze J_j directly because there is a dependence on the fact that $\omega_2 \in \Lambda_n$. Instead we define for $\omega \in \Omega^n$, $1 \leqq i \leqq b_n$,

$$H_i(\omega) = \sum_{\substack{j=1-i \\ |j| \geqq 2a_n}}^{b_n - 1} I_j(\omega).$$

Then for every ω,

$$H_i(\omega) \leqq J_i(\omega) \leqq H_i(\omega) + 4a_n, \tag{3.4}$$

and H_i is independent of the algebra of sets generated by $\{B_\eta\}$, $\eta \in \Omega^{a_n}$, where

$$B_\eta = \{\omega \in \Omega^n : \omega(i) = \eta(i),\ -a_n \leqq i \leqq a_n\}.$$

What we will show is for some $\theta > 1$, $\phi > 2$,

$$P\left\{\left| f(a_n) \sum_{j=1}^{b_n} (H_j)^{-1} - 1 \right| \geqq (\log n)^{-\theta}\right\} \leqq O((\log n)^{-\phi}), \tag{3.5}$$

$$P\left\{\left| f(a_n) \sum_{j=1}^{b_n} (H_j + 4a_n)^{-1} - 1 \right| > (\log n)^{-\theta}\right\} \leqq O((\log n)^{-\phi}). \tag{3.6}$$

However, since the H_i are independent of the sets $\{B_\eta\}$ we can replace P in the above inequalities with P_2. Then (3.4) gives

$$P_2\left\{\left| f(a_n) \sum_{j=1}^{b_n} (J_j)^{-1} - 1 \right| \geqq (\log n)^{-\theta}\right\} \leqq O((\log n)^{-\phi}). \tag{3.7}$$

In the notation of Lemma 6, let $\bar{\Omega}_2$ be the subset of Ω_2 given by

$$\bar{\Omega}_2 = \left\{\left| f(a_n) \sum_{j=1}^{b_n} (J_j)^{-1} - 1 \right| \leqq (\log n)^{-\theta}\right\}.$$

We now consider $E_{P_1}(F \circ T)$. Since the transformation T shifts the walk ω by at most b_n, we get that

$$\begin{aligned} &P\{\Pi_1[n(\log n)^{-\beta}, n - b_n] \cap \Pi_2[b_n, n] \neq \emptyset \text{ or} \\ &\quad \Pi_1[0, n - b_n] \cap \Pi_2[n(\log n)^{-\beta} + b_n, n] \neq \emptyset\} \\ &\quad \leqq E_{P_1}(F \circ T) \\ &\quad \leqq P\{(\Pi_1[0, n] \cup \Pi_2[0, b_n]) \cap \Pi_2[n(\log n)^{-\beta}, n + b_n] \neq \emptyset \text{ or} \\ &\qquad \Pi_1[n(\log n)^{-\beta} - b_n, n] \cap \Pi_2[n(\log n)^{-\beta}, n + b_n] \neq \emptyset\}. \end{aligned}$$

By (2.2) and (2.4), both the left- and right-hand sides of these inequalities equal $\frac{\log \log n}{\log n} (\frac{1}{2}\beta)\,(1 + o(1))$. Therefore

$$E_{P_1}(F \circ T) = \frac{\log \log n}{\log n} (\tfrac{1}{2}\beta)(1 + o(1)). \tag{3.8}$$

Plugging (3.7) and (3.8) into the result of Lemma 6, with $\varepsilon = (\log n)^{-\theta}$, we get

$$\frac{\log \log n}{\log n} (\tfrac{1}{2}\beta)(1 + o(1)) \leqq E_{P_2}(F) \leqq \frac{\log \log n}{\log n} (\tfrac{1}{2}\beta)(1 + o(1)),$$

which gives Theorem 2.

It remains to derive (3.5) and (3.6). First note that each H_j is a sum of $(2a_n)$-dependent random variables. The ideas of Lemma 4 and 5 can be applied to H_j giving (uniformly in j)

$$P\left\{\left| \frac{H_j}{EH_j} - 1 \right| \geqq \varepsilon\right\} \leqq \varepsilon^{-2} O((\log n)^{2 - (\alpha - \gamma)}),$$

or, in other words,

$$P\left\{\left|\frac{EH_j}{H_j}-1\right|\geqq\varepsilon\right\}\leqq\varepsilon^{-2}O((\log n)^{2-(\alpha-\gamma)}). \tag{3.9}$$

Let

$$L_1(\omega)=\sum_{i=2a_n}^{2a_n+\frac{1}{3}b_n} I_i(\omega), \qquad L_2(\omega)=\sum_{i=-2a_n\frac{1}{3}b_n}^{-2a_n} I_i(\omega).$$

The ideas of Lemmas 4 and 5 can again be applied to L_1 and L_2 giving

$$P\left\{\left|\frac{L_i}{EL_i}-1\right|\geqq\varepsilon\right\}\leqq\varepsilon^{-2}O((\log n)^{2-(\alpha-\gamma)}), \qquad i=1,2.$$

Since $EL_i\simeq\frac{1}{3}EL\geqq\frac{1}{3}EH_j$, and for each j, $H_j\geqq\min(L_1,L_2)$,

$$P\{H_j\geqq\tfrac{1}{8}EH_j \text{ for some } j=1,\dots,b_n\}\leqq O((\log n)^{2-(\alpha-\gamma)}). \tag{3.10}$$

Choose $1<\theta<\mu$ such that $\phi>2$, where $\phi=(\alpha-\gamma)-2-2\mu-\theta$. Let

$$\Gamma_j=\left\{\left|\frac{EH_j}{H_j}-1\right|\geqq(\log n)^{-\mu}\right\}.$$

Then (3.9) states that for some $c_4>0$, for all j, $P(\Gamma_j)\leqq c_4(\log n)^{2+2\mu-(\alpha-\gamma)}$.

Now let

$$\Delta=\left\{\left|\sum_{j=1}^{b_n}\frac{EH_j}{H_j}-b_n\right|\geqq b_n(\log n)^{-\theta}\right\}\cap\{H_j\geqq\tfrac{1}{8}EH_j \text{ all } j\}.$$

If $\omega\in\Delta$, since $\left|\frac{EH_i}{H_i}-1\right|\leqq 7$, we must have

$$7(\#\{j:\omega\in\Gamma_j\})+b_n(\log n)^{-\mu}\geqq b_n(\log n)^{-\theta},$$

or

$$\begin{aligned}\#\{j:\omega\in\Gamma_j\}&\geqq\tfrac{1}{7}b_n[(\log n)^{-\theta}-(\log n)^{-\mu}]\\&=b_nO((\log n)^{-\theta}).\end{aligned}$$

But

$$\begin{aligned}P\{\omega:\#\{j:\omega\in\Gamma_j\}\geqq R\}&\leqq\frac{1}{R}\sum_{i=1}^{b_n}P(\Gamma_j)\\&\leqq\frac{b_n}{R}c_4(\log n)^{2+2\mu-(\alpha-\gamma)}.\end{aligned}$$

Therefore,

$$P(\Delta)\leqq O((\log n)^{2+2\mu-(\alpha-\gamma)+\theta})=O((\log n)^{-\phi}). \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$P\left\{\left|\sum_{j=1}^{b_n}\frac{EH_j}{H_j}-b_n\right|\geqq b_n(\log n)^{-\theta}\right\}\leqq O((\log n)^{-\phi}).$$

In a very similar way one can show that

$$P\left\{\left|\sum_{j=1}^{b_n} \frac{E(H_j+4a_n)}{H_j+4a_n} - b_n\right| \geqq b_n(\log n)^{-\theta}\right\} \leqq O((\log n)^{-\phi}).$$

But, using (1.1) and (3.1),

$$EH_j = b_n f(a_n)\,[1+O((\log n)^{2-(\alpha-\gamma)})],$$
$$E(H_j+4a_n) = b_n f(a_n)\,[1+O((\log n)^{2-(\alpha-\gamma)}].$$

We therefore can conclude (3.5) and (3.6).

4. Remark

We have proven that $f(n)=F(n)(\log n)^{-1/2}$, where $F(n)\leqq c_2$, and for every $s>0$,

$$\lim_{n\to\infty} F(n)(\log n)^s = \infty.$$

It is still an open question whether or not

$$\lim_{n\to\infty} F(n) = 0,$$

i.e. does there exist a $c_6>0$ such that $c_6(\log n)^{-1/2} \leqq f(n)$?

References

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