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Construction of Canonical Coordinates on Polarized Coadjoint Orbits of Lie Groups

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Abstract. Construction of canonical coordinates on polarized coadjoint orbits of Lie groups is presented.

0. Introduction

Hamiltonian systems on orbits of coadjoint representation of Lie groups have been studied in numerous papers [1–8]. The general theory of such systems developed in refs. [1–4, 8] enables one to indicate cases of complete integrability. However, to explicitly describe a Hamiltonian system one should be able to introduce canonical coordinates on a requisite orbit. In the simplest case of the fourth-order matrices such coordinates were introduced in the paper by Symes [5] who used the M. Vergne algorithm [9]. This algorithm is applicable for any completely solvable Lie group but practically it turns out to be very cumbersome.

This paper is an extended version of the preceding note [7]. We will give in it a simple construction of canonical coordinates for orbits possessing polarization. Some polarizations are shown for graded Lie groups. Making use of this method one can explicitly parametrize the orbits of coadjoint representation of the Borel subgroups of the real split Lie groups and describe the corresponding Hamiltonians.

I. Polarizable Ad*-Orbits

In this section we construct canonical coordinates on polarizable orbits of coadjoint representation of Lie groups. Let us recall some definitions.

Let $\mathfrak{G}^*$ be a space dual to Lie algebra $\mathfrak{G}$, then for functions $f, g \in \mathscr{F} = C^\infty(\mathfrak{G}^*)$ the Poisson bracket

$$\{f, g\}(x) = \langle x, [\nabla f(x), \nabla g(x)]\rangle$$

is defined which continues the commutator in Lie algebra $\mathfrak{G}$ considered as a space of linear functions on $\mathfrak{G}^*$ ($\langle x, \xi\rangle$ stands for the value of functional $x \in \mathfrak{G}^*$ on element $\xi \in \mathfrak{G}$). The kernel of form $\langle x, [.,.]\rangle$ coincides with the isotropy subalgebra $\mathfrak{G}_x$ at all $x \in \mathfrak{G}^*$, that is why the Poisson bracket on the orbits of coadjoint representation is non-degenerate and defined by the symplectic 2-form ω, the Kirillov form

$$\omega(\mathrm{ad}^* \xi \cdot x, \mathrm{ad}^* \eta \cdot x) = \langle x, [\xi, \eta] \rangle ,$$

where $\mathrm{ad}^* \xi \cdot x$, $\mathrm{ad}^* \eta \cdot x$ are tangent vectors to the orbit at the point x [10, 11].

The polarization $\mathfrak{P}$ relative to $x \in \mathfrak{G}^*$ is called a Lie subalgebra in $\mathfrak{G}$ which is simultaneously the maximal isotropic subspace relative to form $\langle x, [., .] \rangle$, i.e. $\langle x, [\mathfrak{P}, \mathfrak{P}] \rangle = 0$ [11]. Evidently, the kernel $\mathfrak{G}_x$ of the form $\langle x, [., .] \rangle$ is contained in $\mathfrak{P}$.

Let G be a connected Lie group corresponding to the Lie algebra $\mathfrak{G}$. If $\mathfrak{P}$ is a polarization relative to x, then $\mathrm{Ad}_g \cdot \mathfrak{P}$ is a polarization relative $\mathrm{Ad}_g^* \cdot x$, i.e. all the points of the orbit $G \cdot x$ possess polarizations, and submanifolds

$$(\mathrm{Ad}_g^* \cdot x + (\mathrm{Ad}_g \cdot \mathfrak{P})^{\perp}) \cap G \cdot x \tag{1}$$

are fibers of the G-invariant fiber bundle of orbit $G \cdot x$ over G/P, where P is a closed subgroup in G corresponding to Lie algebra $\mathfrak{P}$. Note that inclusion $P \cdot x \subset x + \mathfrak{P}^{\perp}$ is always fulfilled. To fulfill the inverse inclusion the Pukanszky condition is necessary and sufficient,

$$x + \mathfrak{P}^{\perp} \subset G \cdot x . \tag{2}$$

Examples

1. Not each element $x \in \mathfrak{G}^*$ possesses polarization, for example, if $\mathfrak{G}$ is a semisimple compact Lie algebra, then polarization exists only relative to zero.

2. But if the Lie algebra is completely solvable, polarization does exist relative to any element $x \in \mathfrak{G}^*$ and it can be constructed by means of the M. Vergne construction [10, 11].

3. If the Lie algebra is complex, polarization exists relative to any regular element in $\mathfrak{G}^*$ (i.e., relative to an element belonging to any orbit of the maximal dimension [11]).

Lemma. (on canonical coordinates). *Let $\mathfrak{P}$ be a polarization relative to $x \in \mathfrak{G}^*$ and Q be a lagrangian submanifold of orbit $G \cdot x$ intersecting each fiber of lagrangian fiber bundle (1) determined by polarization $\mathfrak{P}$, no more than at a single point. If q^k are coordinates on Q, $q = q(q^k) \in Q$, then from relation*

$$x = q(q^k) + p_k \, \mathrm{ad}^*(\nabla q^k) \cdot q \tag{3}$$

canonical coordinates $q^k(x)$, $p_k(x)$ on the orbit can be determined unambiguously.

Remark. If polarization satisfies the Pukanszky condition coordinates p_k take any values.

Proof. Functions q^k are constants on the fibers (1), therefore $\nabla q^k(y) \in \mathfrak{G}/\mathfrak{G}_y$, and expressions $\mathrm{ad}^* \nabla q^k(y) \cdot y$ are correctly determined for any $y \in (x + \mathfrak{P}^{\perp}) \cap G \cdot x$. On each fiber (1) $\mathrm{ad}^* \nabla q^k(y) \cdot y$ is constant,

$$\mathrm{ad}^* \nabla q^k(y) \cdot y = \mathrm{ad}^* \nabla q^k(x) \cdot x \tag{4}$$

for $y \in (x+\mathfrak{P}^{\perp}) \cap G \cdot x$. Indeed, for any $\xi \in \mathfrak{G}$

$$\begin{aligned}
&\langle \mathrm{ad}^* \nabla q^k(y) \cdot y, \xi \rangle \\
&= -\langle y, [\nabla q^k(y), \xi] \rangle \\
&= -\langle \mathrm{ad}^* \xi \cdot y, \nabla q^k(y) \rangle = -\left.\frac{d}{dt}\right|_{t=0} q^k(\mathrm{Ad}^*_{\exp t\xi} \cdot y) \\
&= -\left.\frac{d}{dt}\right|_{t=0} q^k(\mathrm{Ad}^*_{\exp t\xi} \cdot x) = \langle \mathrm{ad}^* \nabla q^k(x) \cdot x, \xi \rangle .
\end{aligned}$$

The conditions $\{q^i, q^j\}=0$ follow from isotropy of the fibers (1) since $\nabla q^k(x) \in \mathfrak{P}/\mathfrak{G}_x$. Accounting for these relations, we have the equivalence

$$\begin{aligned}
\{p_i, q^j\} = \delta_i^j \;\Leftrightarrow\; \frac{\partial}{\partial p_i}(x) = \mathrm{ad}^* \nabla q^i(x) \cdot x , \\
\delta_i^j = \langle x, [\nabla p_i(x), \nabla q^j(x)] \rangle = \langle \mathrm{ad}^* \nabla q^j(x) \cdot x, \nabla p_i(x) \rangle , \qquad (5) \\
0 = \langle x, [\nabla q^i(x), \nabla q^j(x)] \rangle = \langle \mathrm{ad}^* \nabla q^j(x) \cdot x, \nabla q^i(x) \rangle .
\end{aligned}$$

Recall that each fiber (1) is an open submanifold in the plane $\mathrm{Ad}^*_g \cdot x + (\mathrm{Ad}_g \cdot \mathfrak{P})^{\perp}$ and thus relations (3) determine, by virtue of (4) and (5), such coordinates q^k, p_k on that $\{p_i, q^j\} = \delta_i^j$, coordinates p_i being affine on each fiber.

Let $\{p_i, p_j\} = \omega_{ij}$, then

$$\omega = dp_k \wedge dq^k + \omega_{ij} dq^i \wedge dq^j$$

and relations $\dfrac{\partial}{\partial p_k}\omega_{ij}=0$ follow from closeness of the form ω. The isotropy of submanifold Q implies that $\omega_{ij} dq^i \wedge dq^j = \omega|_Q = 0$ or $\omega_{ij}(q) \equiv 0$, hence the relations $\{p_i, p_j\} = 0$.

Remark. One can sometimes reject the isotropy of submanifold Q. In this case coordinates q^k, p_k defined from relation (3) satisfy the conditions

$$\{q^i, q^j\} = 0 , \qquad \{p_i, q^j\} = \delta_i^j , \qquad \{p_i, p_j\} = \omega_{ij}(q) ,$$

and the symplectic Kirillov form ω is

$$\omega = dp_k \wedge dq^k + \omega_{ij}(q) dq^i \wedge dq^j .$$

Suppose, polarization $\mathfrak{P}$ satisfies the Pukanszky condition (2). Then there exists the smooth cross section

$$s : G/P \to G \cdot x$$

of the affine lagrangian fiber bundle

$$G \cdot x \xrightarrow{x+\mathfrak{P}^{\perp}} G/P , \qquad (6)$$

which follows from contractability of the fiber $x + \mathfrak{P}^{\perp}$ [12]. Let Q be an image of cross section s in $G \cdot x$. Relation (3) gives the diffeomorphism

$$\sigma : T^*Q \to G \cdot x \subset \mathfrak{G}^*$$

of the cotangent fiber bundle T^*Q on the Ad*-orbit $G \cdot x$, which is affine on each fiber and independent of the choice of coordinate q^k. Indeed, let $q^i = q^i(\bar{q}^j)$. Then $dq^i = \frac{\partial q^i}{\partial \bar{q}^j} d\bar{q}^j$ and $\bar{p}_j = \frac{\partial q^i}{\partial \bar{q}^j} p_i$ on the manifold T^*Q and $\nabla q^i = \frac{\partial q^i}{\partial \bar{q}^j} \nabla \bar{q}^j$ on the orbit $G \cdot x$. Hence

$$q + p_k \mathrm{ad}^* \nabla q^k \cdot q = q + \bar{p}_k \mathrm{ad}^* \nabla \bar{q}^k \cdot q .$$

Now, from the remark to the lemma it follows

Theorem. *Let the orbit $G \cdot x$ to permit polarization $\mathfrak{P}$ satisfying the Pukanszky condition (2) and $Q \subset G \cdot x$ be a global cross section of the fiber bundle (6). Denote the Kirillov form on the orbit $G \cdot x$ through ω and the standard symplectic form of the cotangent fiber bundle $\pi : T^*Q \to Q$ through Ω. Then relation (3) determines the symplectic diffeomorphism*

$$\sigma : (T^*Q, \Omega + \pi^*(\omega|_Q)) \to (G \cdot x, \omega)$$

affine on each fiber and independent of the choice of coordinates q^k on submanifold Q.

Consequence. In order for the orbit $G \cdot x$ to permit introducing the structure of a G-invariant cotangent fiber bundle in a way that inclusion $G \cdot x \subset \mathfrak{G}^*$ would be affine on each fiber, it is necessary and sufficient for the orbit $G \cdot x$ to possess a polarization satisfying the Pukanszky condition and for the Kirillov form to be exact.

Proof. Necessity. The existence of polarization follows from G-invariance of the fiber bundle (1). The other conditions are evident.

Sufficiency. Let us construct the lagrangian cross section. From the exactness of the Kirillov form there follows the existence of such a form ϑ on Q that $\omega|_Q = d\vartheta$, and the theorem gives the equality

$$\omega = d(\Theta + \pi^* \vartheta),$$

where Θ is the standard 1-form of the cotangent fiber bundle, $\Theta = p_k dq^k$. Evidently, the equation $\theta + \pi^* \vartheta = 0$ gives the required lagrangian cross section.

Remark. Novikov and Schmeltzer [6] have constructed the global (ambiguous) "almost canonical" coordinates on orbits of the coadjoint representation of three-dimensional Euclidean space motion groups diffeomorphic to T^*S^2, with nonexact symplectic form $\Omega + \pi^*(\omega|S^2)$.

Construction of coordinates on the orbit $G \cdot x$ of coadjoint representation can be simplified if there exists such a closed subgroup K of group G, that $G = K \cdot P$ and $K \cap P \subset G_x$ (P is a subgroup corresponding to polarization $\mathfrak{P}$). Then the submanifold $K \cdot x$ can be taken as cross section Q and in order for the cross section to be lagrangian it is necessary and sufficient that the Lie subalgebra $\mathfrak{K}$ corresponding to subgroup K would be isotropic relative to the form $\langle x, [.\,,.]\rangle$. If, in addition, polarization $\mathfrak{P}$ satisfies the Pukanszky condition, we obtain the structure of the cotangent fiber bundle $T^*(K \cdot x)$ (with standard symplectic form) on the orbit $G \cdot x$.

2. Polarization of Graded Lie Algebras

Construction of polarization of the coadjoint representation orbit, if any, may be a very complicated problem even in the cases when a definite algorithm exists. A very simple construction may be indicated for special orbits of $\mathbb{Z}_+$-graded Lie algebras.

Let $\mathfrak{G} = \sum_{k \geqq 0} \mathfrak{G}_k$ be a $\mathbb{Z}_+$-graded Lie algebra and $\mathfrak{G}^* = \sum_{k \geqq 0} \mathfrak{G}^*_{-k}$ be a dual space with dual grading $\mathfrak{G}^*_{-k} = \left(\sum_{i \neq k} \mathfrak{G}_i\right)^{\perp}$,

$$[\mathfrak{G}_i, \mathfrak{G}_j] \subset \mathfrak{G}_{i+j}, \qquad \mathrm{ad}^* \mathfrak{G}_i \cdot \mathfrak{G}^*_{-j} \subset \mathfrak{G}^*_{i-j}.$$

Hereafter we shall consider special orbits containing the elements $x \in \mathfrak{G}^*_{-k}$. Evidently, isotropy subalgebras for such elements are graded,

$$\mathfrak{G}_x = \sum_{i \geqq 0} \mathfrak{G}_{x,i}, \qquad \mathfrak{G}_{x,i} = \mathfrak{G}_x \cap \mathfrak{G}_i,$$

and at even k subspace $\mathfrak{G}_{k/2}$ is orthogonal to its complement $\sum_{i \neq k/2} \mathfrak{G}_i$ relative to the form $\langle x, [\cdot, \cdot] \rangle$.

Theorem. $\mathfrak{G} = \sum_{i \geqq 0} \mathfrak{G}_i$ *is the* $\mathbb{Z}_+$*-graded Lie algebra and* $x \in \mathfrak{G}^*_{-k}$, $k > 0$. *Suppose that at even k there exists a* $\mathfrak{G}_{x,0}$*-invariant maximal subspace* $\mathfrak{G}'_{k/2}$ *of space* $\mathfrak{G}_{k/2}$ *isotropic relative to the form* $\langle x, [\cdot, \cdot] \rangle$; *at odd k* $\mathfrak{G}'_{k/2} = 0$.

Then

$$\mathfrak{P} = \mathfrak{G}_x \cap \sum_{i < k/2} \mathfrak{G}_i + \mathfrak{G}'_{k/2} + \sum_{j > k/2} \mathfrak{G}_j \tag{7}$$

is the polarization relative to x satisfying the Pukanszky condition and at odd k for any $\xi \in \mathfrak{P}$

$$\mathrm{Ad}^*_{\exp \xi} \cdot x = x + \mathrm{ad}^* \xi \cdot x. \tag{8}$$

Remarks. 1. If $x \in \mathfrak{G}^*_0$, then $\mathfrak{G}^+ = \sum_{i > 0} \mathfrak{G}_i \subset \mathfrak{G}_x$, and in order for polarization in $\mathfrak{G}$ relative to x to exist, it is necessary and sufficient for polarization $\mathfrak{P}_0$ in $\mathfrak{G}_0$ relative to $x|\mathfrak{G}_0$ to exist. $\mathfrak{P} \cap \mathfrak{G}_0$ can be taken as $\mathfrak{P}_0$ and $\mathfrak{P}_0 + \mathfrak{G}^+$ as $\mathfrak{P}$. The Pukanszky conditions for $\mathfrak{P}$ and $\mathfrak{P}_0$ fulfilled or not simultaneously.

2. It is evident that the dimension of the orbit $G \cdot x$ is

$$2 \dim \mathfrak{G}_{k/2} / \mathfrak{G}'_{k/2} + 2 \sum_{k/2 < j \leqq k} \dim \mathrm{ad}^* \mathfrak{G}_j \cdot x .$$

3. If Lie algebra is completely solvable and its graduation agrees with filtration of the algebra with its derivative series, polarization (7) can be constructed by means of the Vergne construction described in [10, 11].

Proof. Let k be odd. Then

$$\mathfrak{G} = \sum_{i < k/2} \mathfrak{G}_i + \sum_{j > k/2} \mathfrak{G}_j$$

is the sum of two isotropic subspaces and $\sum_{j>k/2} \mathfrak{G}_j$ is an ideal in $\mathfrak{G}$, hence

$$\mathfrak{P} = \mathfrak{G}_x \cap \sum_{i<k/2} \mathfrak{G}_i + \sum_{j>k/2} \mathfrak{G}_j$$

is a subalgebra and, consequently, polarization. Besides, for $\xi \in \mathfrak{P}$

$$(\mathrm{ad}_\xi^*)^2 \cdot x \in \sum_{i,j>k/2} \mathfrak{G}_{i+j-k}^* = 0,$$

and hence Eq. (8) and the Pukanszky condition follow.

2. Let k be even and $\mathfrak{G}_{k/2}''$ be an isotropic subspace in $\mathfrak{G}_{k/2}$ complementary to $\mathfrak{G}_{k/2}'$, $\mathfrak{G}_{k/2} = \mathfrak{G}_{k/2}' + \mathfrak{G}_{k/2}''$. Then $\mathfrak{G}$ is the sum of two isotropic subspaces,

$$\mathfrak{G} = \left(\sum_{i<k/2} \mathfrak{G}_i + \mathfrak{G}_{k/2}''\right) + \left(\mathfrak{G}_{k/2}' + \sum_{j>k/2} \mathfrak{G}_j\right).$$

By assumption $[\mathfrak{G}_{x,0}, \mathfrak{G}_{k/2}'] \subset \mathfrak{G}_{k/2}'$. Consequently

$$\mathfrak{P} = \mathfrak{G}_x \cap \sum_{i<k/2} \mathfrak{G}_i + \mathfrak{G}_{k/2}' + \sum_{j>k/2} \mathfrak{G}_j$$

is the polarization. Taking into account the grading of polarization, $\mathfrak{P} = \sum_{i \geqq 0} \mathfrak{P}_i$, we get

$$\mathfrak{P}^\perp = \sum_{i \leqq k/2} \mathfrak{P}_i^\perp, \qquad \mathfrak{P}_i^\perp = \mathrm{ad}^* \mathfrak{P}_{k-i} \cdot x,$$

and for the subspace $\mathfrak{N} = \sum_{i>k/2} \mathfrak{P}_i$,

$$\mathrm{Ad}_{\exp \mathfrak{N}}^* \cdot x = x + \sum_{i<k/2} \mathfrak{P}_i^\perp,$$

since for $\xi \in \sum_{i>k/2} \mathfrak{G}_i$, $\mathrm{Ad}_{\exp \xi}^* \cdot x = x + \mathrm{ad}^* \xi \cdot x$. Let $\xi \in \mathfrak{P}_{k/2}$, then

$$\mathrm{Ad}_{\exp \xi}^* \cdot x = x + \mathrm{ad}^* \xi \cdot x + \tfrac{1}{2}(\mathrm{ad}^* \xi)^2 \cdot x \in x + \mathfrak{P}_{k/2}^\perp + \mathfrak{P}_0^\perp,$$

and there exists such $\eta(\xi) \in \mathfrak{P}_k$ that $\mathrm{ad}^* \eta(\xi) \cdot x = \frac{1}{2}(\mathrm{ad}^* \xi)^2 \cdot x$. Consequently

$$\mathrm{Ad}_{\exp(\xi - \eta(\xi))}^* \cdot x = x + \mathrm{ad}^* \xi \cdot x.$$

Finally we get: for $\xi \in \mathfrak{P}_{k/2}$ and $\zeta \in \sum_{i>k/2} \mathfrak{P}_i$,

$$\mathrm{Ad}_{\exp(\xi - \eta(\xi) + \zeta)}^* \cdot x = x + \mathrm{ad}^* \xi \cdot x + \mathrm{ad}^* \zeta \cdot x,$$

hence the Pukanszky condition follows.

The theorem conditions are fulfilled for the Borel subalgebras of semisimple split Lie algebras. Indeed, the following holds

Proposition. *Let $\mathfrak{G} = \sum \mathfrak{G}_k$ be a split semisimple Lie algebra graded by the height of roots. Then for any k, subspace $\mathfrak{G}_k$ can be decomposed into a linear sum of commutative subalgebras $\mathfrak{G}_k'$ and $\mathfrak{G}_k''$, $\mathfrak{G}_k = \mathfrak{G}_k' + \mathfrak{G}_k''$ spanned on the root subspaces and consequently $\mathfrak{G}_0$-invariant.*

This proposition can be proved by accounting for all the root systems.

3. The van Moerbeke Theorem

In this section we will construct canonical coordinates and Hamiltonian systems on some orbits of coadjoint representation of Borel subalgebras of the semisimple Lie algebras. The construction of Hamiltonian systems with the set of the first integrals in involution is based on the van Moerbeke theorem [1].

Recall first some facts on semisimple Lie algebras [13]. Let $\mathfrak{G}$ be a Lie algebra. The form

$$(x, y) = \mathrm{tr}(\mathrm{ad}\, x \cdot \mathrm{ad}\, y) \tag{9}$$

on $\mathfrak{G}$ is called the Killing form; Lie algebra $\mathfrak{G}$ is semisimple if its Killing form is nondegenerate. We shall consider split semisimple Lie algebras. For them there is the decomposition

$$\mathfrak{G} = \mathfrak{G}^- + \mathfrak{G}^0 + \mathfrak{G}^+ \tag{10}$$

connected with the root system. Subalgebra $\mathfrak{G}^0$ (the maximal commutative subalgebra since the operators $\mathrm{ad} \cdot x$ are semisimple for all of its elements x) is the split Cartan subalgebra, its dimension is called rank $rk\mathfrak{G}$ of algebra $\mathfrak{G}$.

The Lie algebra $\mathfrak{G}$ is completely determined by its system of roots Δ – by the set of nonzero linear functionals α on the Cartan subalgebra $\mathfrak{G}^0$, such that

$$[h, x] = \langle \alpha, h \rangle x \tag{11}$$

for any $h \in \mathfrak{G}^0$ and any $x \in \mathfrak{G}^\alpha$, where $\mathfrak{G}^\alpha$ is the root subspace – the nonzero subspace of all elements $x \in \mathfrak{G}$ satisfying relation (11). Let us mention the properties of the root system Δ

$$[\mathfrak{G}^\alpha, \mathfrak{G}^\beta] = \mathfrak{G}^{\alpha+\beta} \tag{12}$$

if $\alpha, \beta, \alpha+\beta \in \Delta$; $0 \neq [\mathfrak{G}^\alpha, \mathfrak{G}^{-\alpha}] \subset \mathfrak{G}^0$ if $\alpha \in \Delta$; the dimension of each root subspace $\mathfrak{G}^\alpha$ is equal to 1, $\dim \mathfrak{G}^\alpha = 1$. The root system can be represented as a union of positive Δ^+ and negative Δ^- roots, $\Delta = \Delta^+ \cup \Delta^-$, and $\Delta^- = -\Delta^+$; in the set of positive roots Δ^+ one can choose the basis $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, $r = rk\mathfrak{G}$, from simple roots α_i since any root $\alpha \in \Delta^\pm$ can be unambiguously expanded over simple roots with integer coefficients, positive or negative simultaneously,

$$\alpha = \pm \sum m_i \alpha_i, \qquad m_i \in \mathbb{Z}_+ . \tag{13}$$

This Killing form (9) restricted on $\mathfrak{G}^0$ is nondegenerate, $(\mathfrak{G}^\alpha, \mathfrak{G}^\beta) = 0$, if $\alpha + \beta \neq 0$; hence $(\mathfrak{G}^\alpha, \mathfrak{G}^{-\alpha}) \neq 0$ for $\alpha \in \Delta$. There is a classification for semisimple Lie algebras [13]. Let us finally define decomposition (10): subspaces

$\mathfrak{G}^\pm = \sum\limits_{\alpha \in \Delta^\pm} \mathfrak{G}^\alpha$ are nilpotent subalgebras spanned on the root subspaces corresponding to positive and negative roots. The solvable subalgebras $\mathfrak{G}_\pm = \mathfrak{G}^0 + \mathfrak{G}^\pm$ are called the Borel ones. Let us determine the height $|\alpha| = \pm \sum m_i$ for any root $\alpha \in \Delta^\pm$. Integer numbers m_i are coefficients from expression (13), $|\alpha| > 0$, if $\alpha \in \Delta^+$ and $|\alpha| < 0$ if $\alpha \in \Delta^-$. It is seen from relations (11) and (12) that the height of the root gives in algebra $\mathfrak{G}$ the grading

$$\mathfrak{G} = \sum \mathfrak{G}_k$$

where $\mathfrak{G}_0 = \mathfrak{G}^0$ and $\mathfrak{G}_k = \sum_{|\alpha|=k} \mathfrak{G}^\alpha$ at $k \neq 0$; the Killing form gives nondegenerate pairing of subspaces $\mathfrak{G}_k$ and $\mathfrak{G}_{-k}$.

There exists an involutive antiautomorphism τ of algebra $\mathfrak{G}$ such that $\tau\mathfrak{G}^\alpha = \mathfrak{G}^{-\alpha}$, algebra $\mathfrak{G}$ is represented as the sum $\mathfrak{G} = \mathfrak{G}_s + \mathfrak{G}_a$ of subspace $\mathfrak{G}_s$, such that $\tau|\mathfrak{G}_s = 1$ and subalgebras $\mathfrak{G}_a$ such that $\tau|\mathfrak{G}_a = -1$ which are orthogonal, $(\mathfrak{G}_s, \mathfrak{G}_a) = 0$.

Evidently, polynomials $I_k(x) = \mathrm{tr}(\mathrm{ad}\cdot x)^k$ are invariants of adjoint representation which is equivalent to the coadjoint one as a consequence of nondegeneracy of the Killing form. According to the Chevalley theorem, there are exactly r independent polynomials, $r = rk\mathfrak{G}$, in algebra $I(\mathfrak{G})$ of invariant polynomials.

Example. The Lie algebra $\mathfrak{G} = \mathfrak{sl}(n, \mathbb{R})$ of matrices $n \times n$ over $\mathbb{R}$ with nonzero trace.

The Cartan subalgebra $\mathfrak{G}^0$ is the subalgebra of diagonal matrices, $rk\mathfrak{G} = n-1$. The root system: $\Delta^+ = \{\alpha_{ij}\}$ and $\Delta^- = \{\alpha_{ji}\}$, $1 \leqq i < j \leqq n$, where $\langle \alpha_{em}, h\rangle = h_{ee} - h_{mm}$, $h \in \mathfrak{G}^0$, $\alpha_k = \alpha_{k,k+1}$, $1 \leqq k < n$ are simple roots; the height $|\alpha_{em}| = m - l$. The root space $\mathfrak{G}^\alpha$, $\alpha = \alpha_{ij}$ is one-dimensional subspace of the matrices x all the matrix elements x_{em} of which, except for x_{ij}, are zero. Subalgebras $\mathfrak{G}_\pm(\mathfrak{G}^\pm)$ are subalgebras of (exactly) upper-triangular and lower-triangular matrices. $\mathfrak{G}_k$ is the subspace of the matrices, all the elements of which, except for those on the k^{th} diagonal, are zero. The Killing form $(x, y) = 2n \cdot \mathrm{tr}(xy)$. The involutive antiautomorphism τ is the matrix transposition, $\mathfrak{G}_s$ is the space of symmetric matrices and $\mathfrak{G}_a$ is the algebra of skew-symmetric matrices. $I_k(x) = \mathrm{tr}(x^k)$, $2 \leqq k \leqq n$ generate the algebra of invariant polynomials $I(\mathfrak{G})$.

Let us consider the Borel subalgebra $\mathfrak{G}_-$ and identify the dual space $\mathfrak{G}^*_-$ with the subalgebra $\mathfrak{G}_+$ by means of the Killing form. Then the coadjoint representation is given by

$$\mathrm{ad}^*\xi \cdot x = [\xi, x]_+ ,$$

where $x \in \mathfrak{G}_+$ and $\xi \in \mathfrak{G}_-$.

Recall that canonical equations on $\mathfrak{G}^*$ have the form

$$\dot{x} = -\mathrm{ad}^* \nabla H(x) \cdot x ,$$

where H is Hamiltonian on $\mathfrak{G}^*$. For $\mathfrak{G}^*_- = \mathfrak{G}_+$ we get

$$\dot{x} = -[\nabla H(x), x]_+ .$$

Let $x = x_+ = x^0 + x^+$, then $x + \tau(x^+) \in \mathfrak{G}_s$, and determine the Hamiltonian

$$H_I(x) = I(x + \tau(x^+))$$

for $I \in I(\mathfrak{G})$ on the space $\mathfrak{G}_+$.

It follows from the Kostant-Symes theorem [2, 4]

Theorem (P. van Moerbeke [1]).

1. *Canonical system*

$$\dot{x} = -[\nabla H_I(x), x]_+ , \qquad x \in \mathfrak{G}_+ \tag{14}$$

is equivalent to the Lax equation

$$\dot{L}=[M,L], \tag{15}$$

where $L=x+\tau(x^+)$ *and* $M=\nabla I(L)^+-\nabla I(L)^-$.

2. *Functions* H_J, $J\in I(\mathfrak{G})$ *are the first integrals of the system (14) being pairwise in involution.*

3. *Statements 1 and 2 hold on subspace* $\mathfrak{M}^\perp\cap\mathfrak{G}_+$ *for arbitrary ideal* $\mathfrak{M}$ *in* $\mathfrak{G}_-$, *which is natural to identify with* $(\mathfrak{G}_-/\mathfrak{M})^*$.

Remark. Goodman and Wallach [8] have shown that if integrals H_J are restricted on orbit $G_-\cdot x_+$ of coadjoint representation, the number of functionally independent integrals is equal to the dimension of projection of the orbit on $\mathfrak{G}^0$ parallel to $\mathfrak{G}^+$.

Let us present the proof of the theorem following [1].

I. Expand L and ∇I according to (10),

$$L=L^-+L^0+L^+, \qquad \nabla I=\nabla I^-+\nabla I^0+\nabla I^+.$$

Then for any $\bar{x}\in\mathfrak{G}_+$, $\bar{L}=\bar{x}+\tau(\bar{x}^+)$,

$$\begin{aligned}\langle\nabla H_I(x),\bar{x}\rangle&=(\nabla I(L),\bar{L})=(\nabla I^0,\bar{L}^0)+(\nabla I^-,\bar{L}^+)+(\nabla I^+,\bar{L}^-)\\&=(\nabla I^0,\bar{L}^0)+2(\nabla I^-,\bar{L}^+)=(\nabla I^0+2\nabla I^-,\bar{x}).\end{aligned}$$

Hence

$$\nabla H_I(x)=\nabla I(L)^0+2\nabla I(L)^-.$$

Since I is invariant of the adjoint representation,

$$[\nabla I(L),\,L]=0 \quad\text{or}\quad [\nabla I^--\nabla I^+,L]=[\nabla I^0+2\nabla I^-,L].$$

Making use of symmetry $[\nabla I^--\nabla I^+,L]$, we get the first statement of the theorem, since

$$[\nabla I^0+2\nabla I^-,L]_+=[\nabla I^0+2\nabla I^-,x]_+.$$

2. Let J', $J''\in I(\mathfrak{G})$. Then

$$\begin{aligned}\{H_{J'},H_{J''}\}(x)&=(x,[\nabla H_{J'},\nabla H_{J''}])\\&=([x,\nabla H_{J'}],\nabla H_{J''})=([L,M']_+,\nabla H_{J''})\\&=([L,M'],\nabla H_{J''})=([L,M'],\nabla J''),\end{aligned}$$

where the last equality follows by symmetry $[L,M']$ and $\nabla J''(L)$, $M'=\nabla J'^+-\nabla J'^-$, and the last term is $(M',[\nabla J''(L),L])$.

3. The invariance of subspace $\mathfrak{M}^\perp$ relative to the action of algebra $\mathfrak{G}_-$ implies the last statement of the theorem.

4. Examples. Toda Systems

This section is a direct continuation of the preceding one. We will calculate the series of Hamiltonian systems on orbits of the coadjoint representation of Borel subalgebras of the split simple Lie algebras. Let us make some preliminary remarks.

We shall consider only such orbits of the Borel subalgebra $\mathfrak{G}_-$ which are not orbits of some Borel subalgebra $\mathfrak{A}_-$, $\mathfrak{A} \neq \mathfrak{G}$, with the natural identification. Such orbits will be called eigenorbits. As was shown by Goodman and Wallach [8] these are those and only those orbits on which the number of independent functions in involution of the form $H_k(x) = \mathrm{tr}(\mathrm{ad}\,L)^k$ (from the van Moerbeke theorem) is maximal and equal to the rank of algebra $\mathfrak{G}$. For simplicity we will assume that the orbits cross only one root subspace $\mathfrak{G}^\alpha$, $\alpha \in \Delta^+$, and will call such orbits elementary. The simple criterion of the eigenvalue of an elementary orbit which crosses $\mathfrak{G}^\alpha$ is contained in the paper by Goodman and Wallach [8]: in expansion $\alpha = \sum m_k \alpha_k$ over simple roots α_k, $1 \leqq k \leqq rk\mathfrak{G}$, all $m_k > 0$.

Examples of elementary eigenorbits are given by maximal roots for A- and C-series (and only for them), the dimension of orbit being equal to $2rk\mathfrak{G}$ and the Hamiltonian systems according to the van Moerbeke theorem being completely integrable. Eigenorbits of dimension $2rk\mathfrak{G}$ according to Goodman and Wallach [8] will be called Toda orbits.

Goodman and Wallach [8] have found a method of getting elementary Toda orbits: in Lie algebras of type A, D, E, all roots of which are of an equal length, the Toda orbit is generated by the root $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r$, while in the other Lie algebras of type B, C, G_2, F_4 the Toda orbits are generated by the (short) root $\alpha_s = \alpha_1 + \alpha_2 + \ldots + \alpha_r$, and by the (long) root

$$\alpha_e = (\check{\alpha}_1 + \check{\alpha}_2 + \ldots + \check{\alpha}_r)^\vee,$$

where the mapping $^\vee : \mathfrak{G}_0^* \to \mathfrak{G}_0$, so that $(\alpha, h) = 2\dfrac{\langle \alpha, h\rangle}{(\alpha, \alpha)}$, $h \in \mathfrak{G}_0$, establishes the duality between roots and coroots; the inverse mapping is denoted by the same symbol. The Goodman-Wallach mechanism has exactly one exclusion: in the exceptional Lie algebra G_2, except for the roots $\alpha_s = \alpha_1 + \alpha_2$ and $\alpha_e = 3\alpha_1 + \alpha_2$ the Toda orbit is generated by the root $2\alpha_1 + \alpha_2 = \frac{1}{2}(\alpha_s + \alpha_e)$ (in all the other algebras $\frac{1}{2}(\alpha_s + \alpha_e)$ either coincides with $\alpha_s = \alpha_e$ or is not a root). There is the following

Proposition. *$\mathfrak{G}_-$ is the Borel subalgebra of the simple algebra $\mathfrak{G}$. The number of elementary Toda orbits of the algebra $\mathfrak{G}_-$ is equal to the doubled multiplicity of higher coupling within the Dynkin scheme of a root system of Lie algebra $\mathfrak{G}$.*

Let us make a calculation in the remaining part of this section. Using the theorem of Sect. 1 on elementary Toda orbits let us construct canonical coordinates; sometimes we shall use polarizations different from those constructed in the theorem of Sect. 2 to make Hamiltonians more natural.

We will fix some notations. For the root vector x_α from $\mathfrak{G}^\alpha$, $\alpha \in \Delta^+$, denote by $\mathfrak{P}$ the polarization in $\mathfrak{G}_-$ relative to x_α (it should be emphasized that for all polarizations the Pukanszky condition will be fulfilled), by $\mathfrak{P}'$ the (commutative) subalgebra in $\mathfrak{P}$ supplementary to the isotropy subalgebra of the element x_α, by $\mathfrak{K}$ the subalgebra in $\mathfrak{G}_-$ isotropic relative to $(x_\alpha, [\cdot, \cdot])$ supplementary to the polarization $\mathfrak{P}$, and by $Q = \exp \mathfrak{K} \cdot x_\alpha$ the lagrangian section of the orbit $G_- \cdot x_\alpha$ as a lagrangian bundle given by the polarization $\mathfrak{P}$. Here $\mathbb{R}^+$ stands for the set of positive real numbers; indices (i, j), provided summation limits are not indicated, take the values from 1 to r. The notations of roots and root vectors are taken from Bourbaki [13, 14].

Let us consider the parametrization of orbits.

Series A. The root $\alpha_1+\ldots+\alpha_r$ corresponds to the root vector $E_{1,r+1}$. As an essential part $\mathfrak{P}'$ of polarization $\mathfrak{P}$ we take the commutative ideal of dimension r in the algebra $\mathfrak{G}_-$

$$\mathfrak{P}'=\sum_i \mathbb{R}E_{r+1,i}\,.$$

Polarization $\mathfrak{P}$ is supplemented by the isotropic subalgebra

$$\mathfrak{K}=\mathbb{R}(E_{1,1}-E_{r+1,r+1})+\sum_{k=2}^{r}\mathbb{R}E_{k,1}\,.$$

Let us calculate the lagrangian section $Q=\exp\mathfrak{K}\cdot E_{1,r+1}$

$$\exp\mathbb{R}(E_{1,1}-E_{r+1,r+1})\cdot E_{1,r+1}=\mathbb{R}^+E_{1,r+1}\,,$$

$$\begin{aligned}Q&=\mathbb{R}^+E_{1,r+1}+\sum_{k=2}^{r}[\mathbb{R}E_{k,1}\mathbb{R}^+E_{1,r+1}]\\&=\mathbb{R}^+E_{1,r+1}+\sum_{k=2}^{r}\mathbb{R}E_{k,r+1}\,.\end{aligned}$$

Parametrize this section

$$q=\sum_i q_iE_{i,r+1}\,,\qquad q_1>0\,,$$

and determine the gradients

$$\nabla q_i(q)\equiv E_{r+1,i}\,,$$

the latter being determined correctly since they belong to the ideal $\mathfrak{P}'$ and thus belong to polarization relative to any element q of section Q. Coordinates q_i are really constant on fibers of the bundle given by polarization $\mathfrak{P}$. We find the element of the orbit

$$x_A=\sum_i q_iE_{i,r+1}+\sum_{i\leq j}p_jq_i[E_{r+1,j},E_{i,r+1}]\,,\qquad q_1>0$$

or

$$x_A=\sum_i q_iE_{i,r+1}-\sum_{i\leq j}q_ip_jE_{ij}+\sum q_ip_iE_{r+1,r+1}\,,\qquad q_1>0\,. \tag{16}$$

Series C. For the short root $\alpha_s=\alpha_1+\ldots+\alpha_r$ (the root vector $E_{1,-r}+E_{r,-1}$) calculating analogously to the preceding procedure, we obtain

$$\mathfrak{P}'=\sum_i \mathbb{R}(E_{-i,r}+E_{-r,i})\,,\qquad \mathfrak{K}=\sum_i \mathbb{R}(E_{i,1}-E_{-1,-i})\,,$$

$$\begin{aligned}x_{C,s}&=\frac{1}{2}\sum_{k=1}^{r-1}q_k(E_{k,-r}+E_{r,-k})+\frac{1}{\sqrt{2}}q_rE_{r,-r}\\&-\frac{1}{\sqrt{2}}\sum_{k=1}^{r-1}q_kp_r(E_{k,r}-E_{-r,-k})\\&-\left(\frac{1}{2}\sum_{k=1}^{r-1}q_kp_k+q_rp_r\right)(E_{r,r}-E_{-r,-r})\\&-\frac{1}{2}\sum_{l\leq m<r}q_ep_m(E_{e,m}-E_{-m,-l})\,,\qquad q_1>0\,.\end{aligned} \tag{17}$$

The manifold Q is reconstructed by the element of orbit x_C when all $p_i=0$.

The long root $\alpha_l=2\alpha_1+\ldots+2\alpha_{r-1}+\alpha_r$ corresponds to the root vector $E_{1,-1}$,

$$\mathfrak{P}'=\sum_i \mathbb{R}(E_{-i,1}+E_{-1,i}), \qquad \mathfrak{K}=\sum_i \mathbb{R}(E_{i,1}-E_{-1,-i}).$$

Let us parametrize the lagrangian section Q:

$$\frac{1}{\sqrt{2}} q_1^2 E_{1,-1},$$

$$\sum_{k=2}^r \left[\frac{q_k}{q_1}(E_{k,1}-E_{-1,-k}), \frac{q_1^2}{\sqrt{2}} E_{1,-1}\right]=\frac{1}{\sqrt{2}}\sum_{k=2}^r q_1q_k(E_{k,-1}+E_{1,-k}),$$

$$\frac{1}{2}\sum_{i=2}^r\left[\frac{q_i}{q_1}(E_{i,1}-E_{-1,-i}), \frac{1}{\sqrt{2}}\sum_{j=2}^r q_1q_j(E_{j,-1}+E_{1,-j})\right]$$

$$=\frac{1}{2\sqrt{2}}\sum_{i,j>1} q_iq_j(E_{i,-j}+E_{j,-i}),$$

$$\frac{1}{3!}\sum_{k=2}^r\left[\frac{q_r}{q_1}(E_{k,1}-E_{-1,-k}), \frac{1}{2\sqrt{2}}\sum_{i,j>1} q_iq_j(E_{i,-j}+E_{j,-i})\right]=0.$$

Summing up three nonzero lines we get the element of section

$$q=\frac{1}{2\sqrt{2}}\sum_{i,j} q_iq_j(E_{i,-j}+E_{j,-i}), \qquad q_1>0.$$

Determine gradients

$$\nabla q_i(q)=\frac{1}{\sqrt{2}q_i} E_{-i,i}$$

$\left(\text{belonging to ideal } \sum_{i,j}\mathbb{R}(E_{-i,j}+E_{-j,i}) \text{ and belonging to polarization } \mathfrak{P}\right)$ and obtain the element of orbit

$$x_{C,l}=\frac{1}{2\sqrt{2}}\sum_{i,j} q_iq_j(E_{i,-j}+E_{j,-i})-\frac{1}{2}\sum_{i\leq j} q_ip_j(E_{i,j}-E_{-j,-i}), \qquad q_1>0. \tag{18}$$

Note that the gradients ∇q_i have singularities but the result obtained is nevertheless correct. One should choose gradients lying in $\mathfrak{P}'$, then calculations would be a bit longer.

Series B. For the short root $\alpha_s=\alpha_1+\ldots+\alpha_r$ (the root vector $2E_{1,0}+E_{0,1}$:

$$\mathfrak{P}'=\sum_i \mathbb{R}(E_{0,i}+2E_{-i,0}), \qquad \mathfrak{K}=\sum_i \mathbb{R}(E_{i,1}-E_{-1,-i}),$$

$$x_{B,s}=\frac{1}{\sqrt{10}}\sum q_i(2E_{i,0}+E_{0,-i})-\frac{1}{2}\sum_{i\leq j} q_ip_j(E_{i,j}-E_{-j,-i}), \qquad q_1>0. \tag{19}$$

For the long root $\alpha_e=\alpha_1+\ldots+\alpha_{r-1}+2\alpha_r$ (the root vector $E_{1,-r}-E_{r,-1}$,

$$\mathfrak{P}'=\mathbb{R}(E_{0,1}+2E_{-1,0})+\sum_{k=1}^{r-1}\mathbb{R}(E_{-r,k}-E_{-k,r}),$$

$$\mathfrak{R}=\mathbb{R}(E_{0,r}+2E_{-r,0})+\sum_{k=1}^{r-1}\mathbb{R}(E_{k,1}-E_{-1,-k}),$$

$$\begin{aligned}x_{B,l}=&\frac{1}{2}\sum_{k=1}^{r-1}q_k(E_{k,-r}-E_{r,-k})+\frac{1}{\sqrt{10}}q_r(2E_{r,0}+E_{0,-r})\\&-\frac{1}{2}\sum_{l\leqq m<r}q_ep_m(E_{e,m}-E_{-m,-l})-\frac{\sqrt{10}}{8}\sum_{k=1}^{r-1}q_kp_k(2E_{k,0}+E_{0,-k})\\&-\frac{1}{2}\sum_i q_ip_i(E_{r,r}-E_{-r,-r}),\qquad q_1>0.\end{aligned}\tag{20}$$

In addition to Toda orbits let us calculate the orbit passing through vector $E_{1,-2}-E_{2,-1}$ of the maximal root

$$\mathfrak{P}'=\mathbb{R}(2E_{-1,0}+E_{0,1})+\sum_{m=1}^{2}\sum_{k=m+1}^{r}\mathbb{R}(E_{-m,k}-E_{-k,m}),$$

$$\mathfrak{R}=\mathbb{R}(2E_{-2,0}+E_{0,2})+\sum_{m=1}^{2}\sum_{k=3}^{r}\mathbb{R}(E_{k,m}-E_{-m,-k})+\mathbb{R}(E_{1,1}-E_{-1,-1}),$$

$$\begin{aligned}x_{B,\max}=&-\frac{1}{4}q_1p_1(E_{1,1}+E_{2,2}-E_{-1,-1}-E_{-2,-2})\\&-\frac{1}{4}\left(q_0p_0+\sum_{k=3}^{r}(q_kp_k-q_k'p_k')\right)(E_{1,1}-E_{2,2}-E_{-1,-1}+E_{-2,-2})\\&+\frac{1}{4}\left(q_0^2+2\sum_{k=3}^{r}q_kp_k'\right)(E_{1,2}-E_{-2,-1})+\frac{q_0q_1}{2}(2E_{1,0}+E_{0,-1})\\&+\frac{p_0q_1}{2}(2E_{2,0}+E_{0,-2})+q_1^2(E_{1,-2}-E_{2,-1})\\&+q_1\sum_{k=3}^{r}\Big\{q_k(E_{1,-k}-E_{k,-1})-q_k'(E_{2,-k}-E_{k,-2})\\&-\frac{p_k'}{2}(E_{1,k}-E_{-k,-1})-\frac{p_k}{2}(E_{2,k}-E_{-k,-2})\Big\}\\&-\frac{1}{2}\sum_{3\leqq i\leqq j}(q_ip_j+q_i'p_j')(E_{i,j}-E_{-j,-i})\\&+\sum_{3\leqq i<j}(q_i'q_j-q_iq_j')(E_{i,-j}-E_{j,-i})\\&+\frac{1}{2}\sum_{k=3}^{r}(q_0q_k'+q_kp_0)(2E_{k,0}+E_{0,-k}),\qquad q_1>0.\end{aligned}\tag{21}$$

Series D. The Toda orbit for the root $\alpha_1+\ldots+\alpha_r$ (the root vector $E_{1,1-r}-E_{r-1,1}$) is

$$\mathfrak{P}' = \mathbb{R}(E_{r,r-1} - E_{1-r,-r}) + \sum_{i \neq r-1} \mathbb{R}(E_{-i,r-1} - E_{1-r,i}),$$

$$\mathfrak{R} = \mathbb{R}(E_{-r,1} - E_{-1,r}) + \sum_{i \neq r-1} \mathbb{R}(E_{i,1} - E_{-1,-i}),$$

$$\begin{aligned} x_D = {} & \frac{1}{2} \sum_{i \neq r-1} q_i(E_{i,1-r} - E_{r-1,-i}) + \frac{1}{2} q_{r-1}(E_{r-1,r} - E_{-r,1-r}) \\ & - \frac{1}{2} \sum_{i \leqq j} q_i p_j (E_{i,j} - E_{-j,-i}) - \frac{1}{2} \sum_i q_i p_i (E_{r-1,r-1} - E_{1-r,1-r}) \\ & - \frac{1}{2} \sum_{i \neq r-1} q_i p_{r-1}(E_{r,-i} - E_{i,-r}) \\ & + \frac{1}{2} q_{r-1} p_{r-1}(E_{r,r} - E_{-r,-r}), \qquad q_1 > 0. \end{aligned} \tag{22}$$

The orbit passing through the root vector $E_{1,-2} - E_{2,-1}$ of the maximal root $\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ is obtained from parametrization of the corresponding orbit of the Lie algebra of series B at $q_0 = 0$, $p_0 = 0$.

Let us consider the Hamiltonians

$$H_A(x) = \tfrac{1}{2}(L, L), \qquad H_{B(C,D)}(x) = (L, L)$$

on the Toda orbits (16)–(20), (22), and recall that $L = x + \tau(x^+)$. Since $(L, L) = (L^0, L^0) + 2(L^-, L^+)$, then to calculate (L, L) it suffices to sum up the matrix element squares of the element x of the orbit: diagonal – with coefficient 1, nondiagonal – with coefficient 2. Omitting trivial calculations, we obtain the Hamiltonians

$$H_A = \sum_{i \leqq j} q_i^2 p_j^2 + \sum_{i<j} q_i q_j p_i p_j + q^2,$$

$$H_{B,s} = \frac{1}{2} \sum_i q_i^2 p_i^2 + \sum_{i<j} q_i^2 p_j^2 + q^2,$$

$$H_{B,l} = \sum_{i \leqq j} q_i^2 p_j^2 + \sum_{i<j} q_i q_j p_i p_j + \frac{1}{16}(9q^2 - 17q_r^2) p_r^2 + q^2,$$

$$H_{C,s} = \sum_{i \leqq j} q_i^2 p_j^2 + \sum_{i<j} q_i q_j p_i p_j + \sum_{k=1}^{r-1} q_k q_r p_k p_r + q^2 p_r^2 + q^2,$$

$$H_{C,l} = \frac{1}{2} \sum_i q_i^2 p_i^2 + \sum_{i<j} q_i^2 p_j^2 + q^4, \qquad q^4 = (q^2)^2,$$

$$H_D = \frac{1}{2} \sum_i q_i^2 p_j^2 + \sum_{i<j} q_i^2 p_j^2 + q_r^2 p_{r-1}^2 - q_{r-1}^2 p_r^2 - q_{r-1} q_r p_{r-1} p_r + q^2.$$

All of these systems are completely integrable. These Hamiltonians can be considered not on one orbit at $q_1 > 0$, but on the closure of both orbits, $q_1 > 0$ and $q_1 < 0$, and we obtain the Hamiltonian systems in $\mathbb{R}^{2r}$. The systems can be naturally interpreted as oscillators on Riemannian manifolds (kinetic energy is positively determined at $q_1 \neq 0$).

Let us present the Hamiltonian for the orbit (21) passing through the root space of the maximal root in orthogonal series B and D (in H_D, $q_0=p_0=0$),

$$
\begin{aligned}
H_{B(D)\max}=&\frac{1}{2}\sum_{k=3}^{r}(q_kp_k+q_k'p_k')^2+\sum_{3\leqq i<j}(q_ip_j+q_i'p_j')^2\\
&+q_1^2\left(\frac{5}{2}p_0^2+\frac{p_1^2}{4}+\sum_{k=3}^{r}(p_k^2+p_k'^2)\right)\\
&+\frac{5}{2}\left(p_0^2\sum_{k=3}^{r}q_k^2+2q_0p_0\sum_{k=3}^{r}q_kq_k'\right)\\
&+\frac{1}{4}\left(q_0p_0+\sum_{k=3}^{r}(q_kp_k-q_k'p_k')\right)^2\\
&+\left(q_0^2\sum_{k=3}^{r}q_kp_k'+\left(\sum_{k=3}^{r}q_kp_k'\right)^2\right)\\
&+4\left(\frac{5}{8}q_0^2+q_1^2+\sum_{k=3}^{r}q_k^2\right)\left(q_1^2+\sum_{k=3}^{r}q_k'^2\right)\\
&-4\left(\sum_{k=3}^{r}q_kq_k'\right)^2+\frac{1}{4}q_0^4\,.
\end{aligned}
$$

Let us give the results of the calculations for exceptional algebras. All the calculations were made in the Chevalley basis [13] up to the signs of structure constants $N_{\alpha,\beta}$, $\alpha+\beta\neq 0$, and this was enough to calculate the Hamiltonians,

$$H_{G_2(1,1)}=q_1^2p_1^2+3q_2^2p_2^2+3q_1q_2p_1p_2+3q_1^2p_2^2+q^2\,,$$

$$H_{G_2(2,1)}=q_1^2(p_1^2+q_1^2)+4q_1^2(p_2^2+q_2^2)+3q_2^2(p_2^2+q_2^2)\,,$$

$$H_{G_2(3,1)}=q_1^2p_1^2+q_2^2p_2^2-q_1q_2p_1p_2+3q_1^2p_2^2+q^6-\tfrac{3}{4}q_2^6\,,$$

$$H_{F_4,s}=\sum_i q_i^2p_i^2+2\sum_{i<j\leqq 3}q_i^2p_j^2+\sum_{i=1}^{3}q_i^2p_4^2+\sum_{i=1}^{3}q_iq_4p_ip_4+q^2\,,$$

$$
\begin{aligned}
H_{F_4,l}=&\sum_{i=1}^{3}q_i^2p_i^2+3q_4^2p_4^2+2(q_2p_2+q_3p_3)q_4p_4\\
&+2\sum_{i<j\leqq 3}q_i^2p_j^2+4(q_1^2+2q_2^2+2q_3^2)p_4^2\\
&+4(q_2^2+q_3^2)\frac{p_4^2}{q_1^2}+\left(\sum_{i=1}^{3}q_i^2\right)^2+q_1^2q_4^2\,,
\end{aligned}
$$

$$
\begin{aligned}
H_E=&\frac{3}{4}\sum_i q_i^2p_i^2-\frac{1}{2}q_\omega^2p_\omega^2+\frac{1}{2}\sum_{\omega<i<j}q_iq_jp_ip_j\\
&-\sum_{i=\omega+1}^{5}q_iq_8p_ip_8+q_\omega^2p_8^2+\sum_{i<j<8}q_i^2p_j^2\\
&+\sum_{i=2}^{5}q_i^2q_8^2\frac{p_6^2+p_7^2}{q_\omega^2}+q_1^2q_8^2+q_\omega^2\sum_{i=1}^{7}q_i^2+\sum_{i=2}^{5}q_i^2q_8^2\,,
\end{aligned}
$$

where for E_6 (E_7, E_8) $\omega=3$, $q_1=q_2=0$ ($\omega=2$, $q_1=0$; $\omega=1$, respectively).

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