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Dynamical Supersymmetry of the Magnetic Monopole and the $1/r^2$-Potential*

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Abstract. We examine the recently discovered dynamical OSp(1, 1) supersymmetry of the Pauli Hamiltonian for a spin $\frac{1}{2}$ particle with gyromagnetic ratio 2, in the presence of a Dirac magnetic monopole. Using this symmetry and algebraic methods only, we construct the spectrum and obtain the wave functions. At all but the lowest angular momenta, the states transform under a single irreducible representation of OSp(1, 1). On the lowest angular momentum states, it is impossible to define self-adjoint supercharges, and the states transform under an irreducible representation of SO(2, 1) only. The Hamiltonian is not self-adjoint in the s-wave sector, but admits a one parameter family of self-adjoint extensions. The full SO(2, 1) algebra can be realized only for two specific values of the parameter.

The Pauli Hamiltonian is generalized to accommodate a λ^2/r^2 potential. The new Hamiltonian exhibits a dynamical OSp(2, 1) supersymmetry. The spectrum and the wave functions are obtained. The states at all but the lowest angular momenta transform under the sum of two irreducible representations of OSp(2, 1). These two representations are distinguished by the "chirality" of their ground state. On the lowest angular momentum states, the OSp(2, 1) group is still realized, since the supercharges can all be rendered self-adjoint simultaneously, but the states only transform according to a single irreducible representation of OSp(2, 1). The chirality of the ground state for this representation is related to the signs of λ and eg. The Hamiltonian is not self-adjoint in the s-wave sector when $|\lambda| < \frac{3}{2}$. Only one of its self-adjoint extensions supports the OSp(2, 1) supersymmetry, and yields the wave functions obtained

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from the group theoretic approach. The supersymmetry is always spontaneously broken as there exists no normalizable zero energy states.

The massless Dirac Hamiltonian in the presence of a magnetic monopole and a λ/r potential is related to a generator of an OSp(2, 1) superalgebra which also contains the Pauli Hamiltonian. This symmetry is used to generate the complete spectrum of the Dirac Hamiltonian.

I. Introduction

The possibility of resolving an equation by means of algebraic methods alone is related to the existence of large symmetries. In quantum mechanics, many systems are known for which the presence of a symmetry greatly simplifies the wave equation. For example, the Schrödinger equation for a spherically symmetric interaction potential separates into an angular part—which can be solved by algebraic methods only—and a radial part. In the special cases of the harmonic oscillator, the Coulomb [1, 2], $1/r^2$ [3] or magnetic monopole [4] potentials, also the radial equation may be solved by means of algebraic methods, because additional symmetries exist. The symmetries that occur in such systems were always generated by Lie algebras.

Recently, the authors have discovered that a physically interesting quantum mechanical system can be solved completely because it possesses a large *supersymmetry* invariance [5]. This symmetry is generated by a super-algebra instead of a Lie algebra. It was shown that the Pauli Hamiltonian

$$H_0 = \frac{1}{2M}(p_i - eA_i)^2 - \frac{e}{M}B_iS_i, \qquad S_i = \frac{\sigma_i}{2} \quad i = 1, 2, 3 \tag{1.1}$$

for a spin $\frac{1}{2}$ particle with gyromagnetic ratio equal to 2, in the presence of a Dirac magnetic monopole exhibits a dynamical OSp(1, 1) supersymmetry. The symmetry algebra now contains bosonic and fermionic generators and the invariance algebra OSp(1, 1) is a superalgebra.

In the present paper, we shall show that the spectrum of the Hamiltonian H_0 can be obtained using algebraic methods only. The Pauli Hamiltonian H_0 is then generalized to accommodate a λ^2/r^2 potential while preserving supersymmetry. This can be achieved with Dirac γ matrices instead of Pauli matrices and the new system exhibits an OSp(2, 1) supersymmetry with 4 bosonic and 4 fermionic generators. The new Hamiltonian is

$$H = \frac{1}{2M}(p_i - eA_i)^2 - \frac{e}{M}B_iS_i + \frac{\lambda^2}{2Mr^2} + \frac{\lambda}{2Mr^3}r_i\gamma^i\gamma^0,$$

$$2S_i = \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad i = 1, 2, 3. \tag{1.2}$$

With the help of its dynamical OSp(2, 1) supersymmetry, H will be solved completely using algebraic methods only. In the absence of the magnetic monopole, the Hamiltonian H also provides a new supersymmetric extension of the $1/r^2$ potential, which can be generalized to arbitrary dimensions.

The Dirac Hamiltonian for a spin $\frac{1}{2}$ particle in the presence of a magnetic monopole and a $1/r$ potential

$$h = \alpha_i(p_i - eA_i) + \frac{\lambda}{r}\gamma^0\gamma^5 \tag{1.3}$$

can be related to one of the supercharges of the OSp(2, 1) algebra of H so that OSp(2, 1) is a spectrum generating algebra for h. Knowing the spectrum of H, we deduce the spectrum of the Dirac Hamiltonian.

It is the object of the present paper to establish the symmetries of the Hamiltonians H_0, H and h and to obtain their spectra using algebraic methods only. The states will be obtained using group theory, whereas to find the wave functions, we shall have to integrate linear first order equations only. Special importance will also be attached to the questions of self-adjointness of the different group theoretic operators. When the Hamiltonian is not self-adjoint, it always admits a family of self-adjoint extensions. To preserve the symmetries of the Hamiltonian, the Hermitian generator corresponding to that symmetry should be well defined and self-adjoint on the self-adjoint extension of the Hamiltonian. When the full symmetry is enforced, the results from this analytic procedure always precisely coincides with those of the algebraic procedure.

We conclude the Introduction with the summary of our results.

In Sect. II, we examine the supersymmetry of the Pauli Hamiltonian in the presence of a Dirac magnetic monopole and use it to construct all the states and obtain the wave functions. It is found that all fixed angular momentum states (but the lowest) transform under a single irreducible representation of OSp(1, 1), and can be labelled by energy and "fermion number." In the lowest angular momentum states (or s-waves), fermion number is not defined and the charges of the OSp(1, 1) algebra are no longer self-adjoint. The supercharges admit no self-adjoint extensions, and cannot be implemented as legitimate quantum operators in the s-wave sector. The Hamiltonian admits a one parameter family of self-adjoint extensions, but in order to properly define the other bosonic generator of the SO(2, 1) algebra, one must restrict to either of two special values of the extension parameter. The states in the s-wave sector then transform under a single irreducible representation of the SO(2, 1) algebra, and are labelled by energy only.

In Sect. III, we consider a spin $\frac{1}{2}$ particle with gyromagnetic ratio 2, in the presence of a Dirac magnetic monopole suitably modified to include a λ^2/r^2 potential as in (1.2). We exhibit the OSp(2, 1) supersymmetry, compute the structure equations for this algebra and find the representations, the states and the wave functions. At fixed angular momentum, except for the lowest, states are labelled by energy, fermion number and chirality. This representation is actually reducible into two irreducible representations, each of which is labelled by the chirality of the ground state. At lowest angular momentum, the charges are self-adjoint only provided $|\lambda| \geqq \frac{3}{2}$, in which case OSp(2, 1) supersymmetry is realized without any further complication. When $|\lambda| < \frac{3}{2}$, the charges are not self-adjoint, but for certain values of the extension parameters, all charges may be rendered self-adjoint simultaneously so that the full OSp(2, 1) can still be realized in this sector. However, fermion number coincides with chirality, and states transform under a single irreducible representation of OSp(2, 1) and are labelled by energy and chirality.

In Sect. IV, the Dirac Hamiltonian h of (1.3) is related to one of the supercharges of OSp(2, 1), and using the spectrum of the Pauli Hamiltonian H, the spectrum of the Dirac Hamiltonian is constructed.

In Sect. V, we address the question of self-adjointness and extensions. The Pauli Hamiltonian H_0 of (1.1) admits a one parameter family of extensions, none of which supports supersymmetry. The full SO(2, 1) algebra can be realized for two special values of the extension parameter. The Pauli Hamiltonian H of (1.2) is not self-adjoint for $|\lambda| < \frac{3}{2}$. For $\frac{3}{2} > |\lambda| \geqq \frac{1}{2}$, H admits a one parameter family of extensions, and only for a single value of this parameter does the extension support the OSp(2, 1) supersymmetry. When $\frac{1}{2} > |\lambda| > 0$, a 4 parameter family of extensions exists, but only for one value of these parameters does the extension support OSp(2, 1) supersymmetry. At $\lambda = 0$, H is the direct sum of two H_0 Hamiltonian for left and right chiralities, and H still admits a four parameter family of extensions, which typically mix the chiralities. OSp(2, 1) supersymmetry can be realized for two well determined values of the extension parameter.

Finally in Sect. VI, we show that H has no normalizable zero energy states so that supersymmetry is spontaneously broken and Witten's index vanishes. We also compute $\mathrm{Tr}(-1)^F e^{-\beta H}$, which takes the value $-eg \operatorname{sign} \lambda$, and is fractional when eg is a half integer. The spectrum is bounded by zero, but this value is not attained by any (continuum) normalizable state.

In Appendix A, we show how, in the absence of the magnetic monopole, the Hamiltonian H can be generalized to a supersymmetric system in arbitrary dimension. In Appendix B, we present some definitions and results in the theory of self-adjoint extensions.

II. Dynamical Supersymmetry of the Pauli Equation in the Presence of a Dirac Magnetic Monopole

In this section, we establish the OSp(1, 1) supersymmetry of the Pauli Hamiltonian H_0 for a spin $\frac{1}{2}$ particle, with gyromagnetic ratio equal to 2, in the presence of a Dirac magnetic monopole field[1]. We then use this symmetry to solve for the complete spectrum and obtain the wave functions using algebraic methods only. The Hamiltonian H_0 is given by [5]

$$H_0 = \frac{1}{2M}(p_i - eA_i)^2 - \frac{e}{M} B_i S_i, \quad i = 1, 2, 3. \tag{2.1}$$

Here M is the mass of the spin $\frac{1}{2}$ particle, e its electric charge, $B_i = g(r_i/r^3)$ is the monopole magnetic field strength and A_i the corresponding vector potential, which can be defined by patches so as to avoid string singularities [6]. It will always be

1 The existence of the OSp(1, 1) supersymmetry was first discussed in ref. [5] where several of the results of this section were announced

understood that the wave functions are similarly defined by patches. Electric and magnetic charges of course obey the Dirac quantization condition

$$eg = \tfrac{1}{2}\text{integer} \tag{2.2}$$

A. Symmetries.

To establish the different symmetries of H_0, it is most convenient to consider the associated Lagrangian,

$$L_0 = L_{\text{KIN}} + L_{\text{INT}}, \tag{2.3}$$

$$L_{\text{KIN}} = \frac{1}{2}M\dot{r}_1^2 + \frac{i}{2}\psi_i\dot{\psi}_i, \tag{2.4}$$

$$L_{\text{INT}} = eA_i\dot{r}_i + \frac{e}{M}B_iS_i, \tag{2.5}$$

Here the ψ_i are real generators of a Grassmann algebra describing the spin degrees of freedom of a classical particle [7] with position r_i. They satisfy

$$\psi_i\psi_j + \psi_j\psi_i = 0, \tag{2.6}$$

and we have

$$S_i = -\frac{i}{2}\varepsilon_{ijk}\psi_j\psi_k. \tag{2.7}$$

The Lagrangian L_0 is invariant under spatial rotations if both r_i and ψ_i transform as vectors.

$$\delta_J r_i = \varepsilon_{ijk}\omega_j r_k, \quad \delta_J\psi_i = \varepsilon_{ijk}\omega_j\psi_k. \tag{2.8}$$

The Nöther charge is just total angular momentum

$$J_i = M\varepsilon_{ijk}r_j\dot{r}_k - eg\hat{r}_i + S_i, \tag{2.9}$$

so that S_i is correctly interpreted as the spin of the particle.

To find the dynamical symmetries of L_0, we first establish that the interaction Lagrangian L_{INT} is invariant under arbitrary reparametrizations of time[2]. Consistent with rotation symmetry, we make the following Ansatz for infinitesimal linear transformations of r_i and ψ_i:

$$\delta r_i = f\dot{r}_i + i\alpha\psi_i + gr_i, \tag{2.10a}$$
$$\delta\psi_i = f\dot{\psi}_i + i\beta\dot{r}_i + h\psi_i + \gamma r_i. \tag{2.10b}$$

Here f, g and h are ordinary functions of t, while α, β and γ are Grassmann algebra valued functions of t, which anticommute with each other and with the ψ's. The

2 The interaction term $A_i\dot{r}_i$ was shown to be reparametrization invariant in [4]

change in the interaction Lagrangian under the transformation (2.10) is given by

$$\delta L_{\mathrm{INT}} = \frac{d}{dt}\left(A_i \delta r_i + \frac{e}{M} f B_i S_i \right)$$
$$+ \frac{e}{M} \varepsilon_{ijk} \psi_k (\dot{r}_i B_j (i\alpha M - \beta) - i\psi_j B_i (\tfrac{1}{2}\dot{f} + g - h))$$
$$+ \frac{ie}{M} \partial_i B_i \alpha \psi_1 \psi_2 \psi_3. \tag{2.11}$$

Upon setting the third and fourth terms equal to zero, we obtain two relations between the transformation functions namely, $\beta = i\alpha M$ and $h = \frac{1}{2}\dot{f} + g$. The last term is proportional to $\partial_i B_i$, and vanishes due to Maxwell's equations for all magnetic fields, except for the magnetic monopole, where we have

$$\partial_i B_i = 4\pi g \delta^3(\mathbf{r}). \tag{2.12}$$

Hence, L_{INT} is invariant only if we excise the origin, where the monopole resides. We shall come back to this important question in more detail when we consider the quantum problem. With the help of the relations between transformation parameters, the interaction part of the Lagrangian is now invariant under the following arbitrary reparametrization of time

$$\delta r_i = f\dot{r}_i + i\alpha\psi_i + g r_i, \quad \delta\psi_i = f\dot{\psi}_i - \alpha M \dot{r}_i + (\tfrac{1}{2}\dot{f} + g)\psi_i + \gamma r_i. \tag{2.13}$$

The kinetic part of the Lagrangian is invariant only under a small subset of transformations, which we list below.

Time translations	$\delta_{H_0} r_i = \dot{r}_i$,	$\delta_{H_0} \psi_i = \dot{\psi}_i$,	(2.14a)
Dilations	$\delta_D r_i = t\dot{r}_i - \frac{1}{2} r_i$,	$\delta_D \psi_i = t\dot{\psi}_i$,	(2.14b)
Conformal	$\delta_K r_i = t^2 \dot{r}_i - t r_i$,	$\delta_K \psi_i = t^2 \dot{\psi}_i$,	(2.14c)
	$\delta_Q r_i = i\alpha\psi_i$,	$\delta_Q \psi_i = -\alpha M \dot{r}_i$,	(2.14d)
Supersymmetry	$\delta_S r_i = i\alpha t \psi_i$,	$\delta_S \psi_i = \alpha M (t\dot{r}_i - r_i)$,	(2.14e)

with α constant Grassmann parameter.

The first three transformation laws are familiar from the conformal invariance of the Schrödinger equation in the presence of the magnetic monopole and close by themselves on an SO(2, 1) algebra [4]. The supertransformations δ_Q and δ_S are new, and (2.14) closes under composition yielding an OSp(1, 1) superconformal algebra. (The structure equations will be given in subsection B). The associated Nöther charges are

$$H_0 = \tfrac{1}{2} M \dot{r}_i^2 - \frac{e}{M} B_i S_i, \tag{2.15a}$$

$$D = tH_0 - \frac{M}{4}(r_i \dot{r}_i + \dot{r}_i r_i), \tag{2.15b}$$

$$K = -t^2 H_0 + 2tD + \frac{M}{2} r_i^2, \tag{2.15c}$$

$$Q = \sqrt{M}\dot{r}_i\psi_i, \tag{2.15d}$$

$$S = -tQ + \sqrt{M}r_i\psi_i. \tag{2.15e}$$

We have thus found the complete set of symmetries of the Lagrangian L_0.

B. *Quantization and Quantum Numbers*

Canonical quantization is straightforward; the momentum conjugate to r_i is given by

$$p_i = \frac{\partial L_0}{\partial \dot{r}_i} = M\dot{r}_i + eA_i, \tag{2.16}$$

and the canonical variables satisfy

$$[r_i, p_j] = i\delta_{ij}, \tag{2.17}$$

$$\{\psi_i, \psi_j\} = \delta_{ij}. \tag{2.18}$$

The canonical anticommutation relations for ψ_i define a real (or Hermitian) Clifford algebra, whose lowest dimensional representation is given in terms of the Pauli matrices

$$\psi_i = \frac{\sigma_i}{\sqrt{2}}, \tag{2.19}$$

so that

$$S_i = \frac{\sigma_i}{2}, \tag{2.20}$$

as expected. With the help of (2.19), it is easily shown that J_i obeys the standard angular momentum algebra.

$$[J_i, J_j] = i\varepsilon_{ijk}J_k. \tag{2.21}$$

Similarly, the charges H_0, D, K, Q, and S of (2.15) are all Hermitian (i.e. symmetric) operators and obey the structure equations of an OSp(1, 1) superalgebra[3]:

$$[H_0, D] = iH_0, \quad [H_0, K] = 2iD, \quad [D, K] = iK, \tag{2.22a}$$

$$\{Q, Q\} = 2H_0, \quad \{Q, S\} = -2D, \quad \{S, S\} = 2K, \tag{2.22b}$$

$$[H_0, Q] = 0, \qquad [K, S] = 0, \tag{2.22c}$$

$$[H_0, S] = -iQ, \quad [K, Q] = iS, \tag{2.22d}$$

$$[D, Q] = -\frac{i}{2}Q, \quad [D, S] = \frac{i}{2}S. \tag{2.22e}$$

Relations (2.22a) define the SO(2, 1) subalgebra. Naturally, (2.22) only holds

3 The OSp(1.1) superalgebra was studied in [8] and has been found to play a crucial role in conformally invariant supersymmetric two dimensional field theories (see e.g., [9])

provided $\partial_i B_i = 0$. At the level of the quantum mechanical states, this suggests that the supersymmetry can be realized only on the states whose wave function vanishes at the location of the monopole. By construction, the charges H_0, D, K, Q and S are rotation invariant and commute with J_i. Thus, the full invariance of H_0 is specified by the group

$$G_{H_0} = \mathrm{SO}(3)_{\mathrm{rotations}} \times \mathrm{OSp}(1,1)_{\mathrm{superconformal}}. \tag{2.23}$$

The OSp(1, 1) symmetry is *dynamical*: the charges D, K and S do not commute with H_0 because they explicitly depend on time, but their total time derivative vanishes. For example we have

$$\dot{S} = \frac{\partial S}{\partial t} + i[H_0, S] = 0. \tag{2.24}$$

Henceforth we shall consider the charges at $t = 0$ only, without loss of generality since their time evolution is given by (2.24). Because G_{H_0} is a dynamical symmetry of H_0, the states must span a representation of this algebra. The spectrum of H_0 is found by looking for the action of H_0 on this representation space. The representations of O(3) are labelled by the eigenvalues of J^2 and J_z, which we shall denote by $j(j+1)$ and m. Their range is

$$j = j_0, j_0 + 1, j_0 + 2, \ldots, \tag{2.25a}$$

$$m = -j, -j+1, \ldots, j-1, j, \tag{2.25b}$$

$$j_0 = |eg| - \tfrac{1}{2}. \tag{2.26}$$

In classifying the unitary (infinite dimensional) representations of OSp(1, 1), it will turn out to be convenient to label the states by the Casimirs of the canonical chain of maximal subgroups:

$$\mathrm{OSp}(1,1) \supset \mathrm{O}(2,1) \supset \mathrm{O}(2). \tag{2.27}$$

In the case of OSp(1, 1), this is equivalent to specifying the representation by its highest (or lowest) weight. The Casimir of the O(2) algebra is just the compact generator R of SO(2, 1):

$$R = \frac{1}{2a^2} K + \frac{a^2}{2} H_0. \tag{2.28}$$

Here a is an arbitrary parameter which fixes the scale. The Casimir of SO(2, 1) is well known [10] and is given by

$$C_0 = \tfrac{1}{2}(H_0 K + K H_0) - D^2. \tag{2.29}$$

It commutes with H_0, D and K, but does not commute with Q or S. It is convenient to express C_0 in terms of the coordinate representation,

$$C_0 = \tfrac{1}{4}(J^2 - J_k \sigma_k - eg\hat{r}_k \sigma_k - e^2 g^2). \tag{2.30}$$

It is easy to see that within OSp(1, 1) there exists another SO(2, 1) invariant operator (we shall call it a Casimir as well) which is expressed in terms of Q and S solely:

$$A_0 = i[Q, S] - \tfrac{1}{2}. \tag{2.31}$$

The operator A commutes with H_0, D and K but anticommutes with Q or S, so that it plays the role of "fermion number." In terms of the coordinate representation, we find

$$A_0 = J_k\sigma_k + eg\hat{r}_k\sigma_k - \tfrac{1}{2}. \tag{2.32}$$

Clearly then, the combination

$$C_1 = C_0 + \tfrac{1}{4}A_0 + \tfrac{3}{16} = \tfrac{1}{2}\{H_0, K\} - D^2 + \frac{i}{4}[Q, S] + \tfrac{1}{16}, \tag{2.33}$$

which in coordinate representation yields

$$C_1 = \tfrac{1}{4}(J^2 - e^2g^2 + \tfrac{1}{4}), \tag{2.34}$$

is a Casimir of the full OSp(1, 1) algebra, since H_0, D, K, Q and S are rotation invariant. Now, we see that the Casimir C_1 is completely specified by the quantum numbers of the rotation group[4]. Furthermore, we have

$$C_1 = \tfrac{1}{4}A_0^2, \tag{2.35}$$

so that the norm of the eigenvalue of A_0 is fixed. [We shall see in the next subsection that the eigenvalue of A_0 can actually vanish so that then sign A_0 is ill-defined.] A complete set of labels is given by eigenvalues of the remaining Casimirs

$$J^2, J_z, \text{sign}\, A_0 \text{ and } R, \tag{2.36}$$

which clearly commute. All representations of G_{H_0} for our problem will be specified by the eigenvalues of the Casimirs. Instead of R, we could have diagonalized H_0. The reason for considering R first is that it arises naturally in representation theory. The eigenstates and wave functions in the energy representations are obtained in Subsect. D through further group theoretic reasoning.

C. States and Wave Functions

We now derive the spectrum of R by means of algebraic methods, and determine the wave functions by solving linear first order differential equations only. To do so, we construct the representations of G_{H_0}. We need to know the eigenvalues of the Casimirs given in (2.36). The eigenvalue of sign A_0 is denoted by α, and takes on the values of ± 1. The eigenvalues of R are $r_n = \delta_{j,\alpha} + n$ for a representation[5] of the bosonic subgroup SO(2, 1) with the following eigenvalue for C_0 [4, 5, 10],

$$C_0 = \delta_{j,\alpha}(\delta_{j,\alpha} - 1). \tag{2.37}$$

4 Strictly speaking, this means that the product of O(3) and OSp(1, 1) in (2.23) is not direct

5 Please note that here $\delta_{j,\alpha}$ does not stand for the Kronecker delta symbol

With the help of (2.33) and (2.34), we get an expression for $\delta_{j,\alpha}$ in terms of j and α:

$$\delta_{j,\alpha}(\delta_{j,\alpha}-1)=\tfrac{1}{4}d_j^2-\tfrac{1}{4}\alpha d_j-\tfrac{3}{16}, \tag{2.38}$$

where

$$d_j=\sqrt{j(j+1)-j_0(j_0+1)},\quad j_0=|eg|-\tfrac{1}{2}. \tag{2.39}$$

Since R is a positive operator, we only need to consider $\delta_{j,\alpha}>0$ and we obtain

$$\delta_{j,\alpha}=\tfrac{1}{2}d_j-\frac{\alpha}{4}+\frac{1}{2}. \tag{2.40}$$

We can now list the quantum states by their quantum numbers:

$$\begin{aligned}
J^2|j,m;\alpha,n\rangle &= j(j+1)|j,m;\alpha,n\rangle,\\
J_z|j,m;\alpha,n\rangle &= m|j,m;\alpha,n\rangle,\\
A_0|j,m;\alpha,n\rangle &= \alpha d_j|j,m;\alpha,n\rangle,\\
R|j,m;\alpha,n\rangle &= (\delta_{j,\alpha}+n)|j,m;\alpha,n\rangle,\quad n\geqq 0
\end{aligned} \tag{2.41}$$

All these states are connected to each other through the action of the OSp(1, 1) group at fixed j and m. Raising and lowering operators are defined in the Cartan basis [5] where the basic generators are R (the compact diagonal generator), and

$$B_\pm=\frac{1}{2a^2}K-\frac{a^2}{2}H_0\pm iD, \tag{2.42}$$

$$F_\pm=\frac{1}{2a}S\mp\frac{ia}{2}Q. \tag{2.43}$$

The structure equations (2.22) in the Cartan basis are:

$$[R,B_\pm]=\pm B_\pm,\qquad [R,F_\pm]=\pm\tfrac{1}{2}F_\pm, \tag{2.44a}$$

$$[B_+,B_-]=-2R,\qquad \{F_+,F_-\}=R, \tag{2.44b}$$

$$\{F_\pm,F_\pm\}=B_\pm,\qquad [B_\pm,F_\mp]=\mp F_\pm. \tag{2.44c}$$

With the help of the expressions of C_0 and C_1 in terms of J^2 and A_0, one can easily obtain the action of raising and lowering operators on the states:

$$J_\pm|j,m;\alpha,n\rangle=\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1;\alpha,n\rangle, \tag{2.45a}$$

$$B_\pm|j,m;\alpha,n\rangle=\sqrt{(\delta_{j,\alpha}+n)(\delta_{j,\alpha}+n\pm 1)-\delta_{j,\alpha}(\delta_{j,\alpha}-1)}\,|j,m;\alpha,n\pm 1\rangle \tag{2.45}$$

$$F_\pm|j,m;\alpha,n\rangle=\sqrt{\tfrac{1}{2}(\delta_{j,\alpha}+n)\pm\tfrac{1}{8}\pm\tfrac{1}{4}\alpha d_j}\,|j,m;-\alpha,n-\tfrac{1}{2}\alpha\pm\tfrac{1}{2}\rangle. \tag{2.45c}$$

We now examine the case where $j=j_0$, and $d_{j_0}=0$. On those states, A_0 vanishes and the quantum number α is not defined. To better understand why this happens, let us recall that in the s-wave states, the wave function overlaps with the origin, so that the complete superalgebra (2.22) is not realized. Mathematically the reason is that on the s-wave states, the supercharges Q and S are not self-adjoint, and admit no self-adjoint extensions. (Note that Q is proportional to the helicity operator, which is known to possess this property [11]; see Sect. V.) Thus Q and

S are not legitimate quantum operators. The action of the non-self-adjoint charges Q and S on s-wave states is ill-defined, and so is fermion number. This is consistent with the fact that A_0 vanishes on s-wave states so that fermion number cannot be defined and J^2, J_z and R are the only Casimirs. The bosonic charges H_0, D, K and R are not self-adjoint either on the s-wave states, but H_0 admits a one parameter family of self-adjoint extensions. Only for two specific values of this extension parameter ($\alpha = \pm 1$) can the SO(2, 1) algebra be defined. In each case, the states of the s-wave sector transform under a single irreducible representation of SO(2, 1) (see again Sect. V).

Returning to the case where $j \neq j_0$, we learn from (2.45) that for each angular momentum, two states are annihilated by B_-:

$$B_- |j, m; \alpha, 0\rangle = 0, \tag{2.46}$$

of which one is also annihilated by F_-:

$$F_- |j, m; 1, 0\rangle = 0. \tag{2.47}$$

This state defines a ground state, from which all higher states are constructed by applying F_+ or B_+. First we have

$$|j, m; -1, 0\rangle = (\tfrac{1}{2} d_j + \tfrac{1}{4})^{-1/2} F_+ |j, m; 1, 0\rangle, \tag{2.48}$$

which allows us to compute

$$|j, m; \alpha, n\rangle = \left(\frac{\Gamma(2\delta_{j,\alpha})}{n! \Gamma(2\delta_{j,\alpha} + n)} \right)^{1/2} B_+^n |j, m; \alpha, 0\rangle. \tag{2.49}$$

We see that at fixed angular momentum, all states of the spectrum can be obtained by applying F_+ or B_+ and thus transform according to a *single* irreducible representation of OSp(1, 1).

When $j = j_0$, there still exist two states annihilated by B_- and labelled by α:

$$B_- |j_0, m; 0\rangle_\alpha = 0, \tag{2.50}$$

and

$$|j_0, m; n\rangle_\alpha = \left(\frac{\Gamma(2\delta_{j_0,\alpha})}{n! \Gamma(2\delta_{j_0,\alpha} + n)} \right)^{1/2} B_+^n |j_0, m; 0\rangle_\alpha, \tag{2.51}$$

with $\alpha = \pm 1$, but now α is determined by the self-adjoint extension on which H_0, B_+, B_- and R are properly defined so that H_0 and R are self-adjoint and B_+ is the adjoint of B_-. Of course, only one single choice of a self-adjoint extension can be made once and forever, so in this case α is fixed as part of the definition of the operators.

Now that we have constructed all states, we also wish to obtain the wave functions. We first restrict to the case where $j \neq j_0$, so that sign A_0 is a good quantum number and start by diagonalizing the Casimirs J^2, J_z, sign A_0 and R simultaneously. The quantum numbers j, m and α fix the angular dependence of the wave function, whereas n pertains to the radial dependence. In a basis where $L^2 = (J - S)^2$ is diagonal, the wave functions are just the standard monopole harmonics [6]

$$L^2|l,l_z\rangle = l(l+1)|l,l_z\rangle, \tag{2.52a}$$

$$L_z|l,l_z\rangle = l_z|l,l_z\rangle, \tag{2.52b}$$

$$\langle\theta,\phi|l,l_z\rangle = Y_{eg,l,l_z}(\theta,\phi). \tag{2.53}$$

Standard Clebsch–Gordan coefficients relate the states in which L^2, L_z and S_z are diagonal to the ones where J^2, J_z and L^2 are diagonal

$$|j,m;l\rangle = (C_+\delta_{l+1/2,j} - C_-\delta_{l-1/2,j})|l,m-\tfrac{1}{2}> \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + (C_-\delta_{l+1/2,j} + C_+\delta_{l-1/2,j})|l,m+\tfrac{1}{2}> \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle, \tag{2.54a}$$

with

$$C_\pm = \sqrt{\frac{l+\frac{1}{2}\pm m}{2l+1}}. \tag{2.54b}$$

In this basis the operator A_0 is not diagonal. To diagonalize A_0, we first express it in terms of angular momentum operators:

$$A_0 = J^2 - L^2 + eg\hat{r}_i\sigma_i + \tfrac{1}{4}. \tag{2.55}$$

Both J^2 and L^2 are diagonal in the basis $|j,m,l\rangle$, so it suffices to evaluate $\hat{r}_i\sigma^i$. Since A_0 commutes with J_i, we can restrict to the case $m=-j$ and using the explicit expressions for the monopole harmonics [6, 11] we find:

$$\langle j,-j;j-\sigma|\sigma^i\hat{r}_i|j,-j;j-\sigma'\rangle = \begin{pmatrix} S_1 & -S_2 \\ -S_2 & -S_1 \end{pmatrix}_{\sigma\sigma'}, \tag{2.56a}$$

with

$$S_1 = \frac{-2eg}{2j+1}, \quad S_2 = \frac{2d_j}{2j+1}. \tag{2.56b}$$

It is then straightforward to evaluate A_0 in this basis and diagonalize it to find the normalized eigenvectors:

$$|j,-j;\alpha\rangle = N_{j,\alpha}|j+\tfrac{1}{2},-j-\tfrac{1}{2}> \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + N_{j,\alpha}\left(-\frac{1}{\sqrt{2j+1}}|j+\tfrac{1}{2},-j+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle + M^j_\alpha|j-\tfrac{1}{2},-j+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right), \tag{2.57}$$

where

$$N^j_\alpha = \sqrt{\frac{j+\frac{1}{2}-\alpha d_j}{2j+2}}, \tag{2.58a}$$

$$M_{j,\alpha} = \frac{\alpha}{eg}\sqrt{\frac{2j+2}{2j+1}}\quad (j+\tfrac{1}{2}+\alpha d_j). \tag{2.58b}$$

The angular part of the wave functions ($\eta_{j,m,\alpha}(\theta,\phi)$) is now easy to obtain:

$$\eta_{j,-j;\alpha}(\theta,\phi)=\langle\theta,\phi,\sigma|j,-j,\alpha\rangle$$
$$=N_{j,\alpha}\begin{pmatrix} Y_{eg,j+1/2,-j-1/2}(\theta,\phi) \\ -\dfrac{1}{\sqrt{2j+1}}Y_{eg,j+1/2,-j+1/2}(\theta,\phi)+M_\alpha^j Y_{eg,j-1/2,-j+1/2}(\theta,\phi)\end{pmatrix}_\sigma \tag{2.59}$$

The wave functions with $m\neq -j$ are obtained by applying J_+^{j+m}. Let us recall that in this basis, we have

$$A_0|j,m;\alpha\rangle=\alpha d_j|j,m;\alpha\rangle, \tag{2.60a}$$
$$\sigma^i\hat{r}_i|j,m;\alpha\rangle=-|j,m;-\alpha\rangle. \tag{2.60b}$$

A similar analysis yields the wave functions for the lowest angular momentum states where $j=j_0$. It is found that

$$\eta_{j_0,-j_0,\alpha}(\theta,\phi)=\langle\theta,\phi,\sigma|j_0,-j_0\rangle$$
$$=\frac{1}{\sqrt{1+e^2g^2}}\begin{pmatrix} Y_{eg,|eg|,-|eg|}(\theta,\phi) \\ -\dfrac{1}{|eg|}Y_{eg,|eg|,-|eg|+1}(\theta,\phi)\end{pmatrix}_\sigma \tag{2.61}$$

In this basis, L^2 is diagonal and we have

$$\sigma_i\hat{r}_i|j_0,m\rangle=\frac{eg}{|eg|}|j_0,m\rangle \tag{2.62}$$

The radial dependence of the wave function is factorized, so that

$$\langle r,\theta,\phi,\sigma|j,m;\alpha,n\rangle=\Phi_{n,\alpha}(r)\langle\theta,\phi,\sigma|j,m;\alpha\rangle. \tag{2.63}$$

We could now diagonalize R by using its expression in a basis with fixed angular momentum (see ref. [4] and [5]). Instead, we prefer to construct the ground state wave function first, after which we obtain the wave function for an arbitrary state by applying F_+ or B_+. From (2.47), we see that the ground state wave function (for $j\neq j_0$) is determined by

$$\langle r,\theta,\phi,\sigma|F_-|j,m;1,0\rangle=0. \tag{2.64}$$

With the help of (2.31), and the structure equations, we obtain a useful form for F_-:

$$F_\pm=\frac{1}{2a}S\pm\frac{a}{K}\left(\frac{3}{8}S+\frac{1}{4}SA_0+\frac{i}{2}DS\right). \tag{2.65}$$

The coordinate representation of D is given by [4]

$$D=\frac{i}{2}\left(r\frac{\partial}{\partial r}+\frac{3}{2}\right). \tag{2.66}$$

The action of S on the ground state is simple

$$S|j,m;1,0\rangle = \sqrt{\frac{M}{2}}r|j,m;-1,0\rangle, \tag{2.67}$$

and (2.65) leads to the desired equation for $\Phi_{0,1}(r)$:

$$\frac{1}{r}\frac{d}{dr}r\Phi_{0,1}(r) + \left(\frac{Mr}{a^2} - \frac{d_j}{r}\right)\Phi_{0,1}(r) = 0, \tag{2.68}$$

whose normalized solution is

$$\Phi_{0,1}(r) = \left(\frac{2}{r^3\Gamma(2\delta_{j,1})}\right)^{1/2}\left(\frac{Mr^2}{a^2}\right)^{\delta_{j,1}} e^{-Mr^2/2a^2}. \tag{2.69}$$

Using (2.48) and the definitions of $F_\pm$ in (2.43), it is easy to compute the radial wave function for $|j,m;-1,0\rangle$:

$$\langle r,\theta,\phi,\sigma|F_+|j,m;1,0\rangle = \frac{r}{a}\sqrt{\frac{M}{2}}\langle r,\theta,\phi,\sigma|j,m;-1,0\rangle, \tag{2.70}$$

from which we extract the radial dependence:

$$\Phi_{0,\alpha}(r) = \left(\frac{2}{r^3\Gamma(2\delta_{j,\alpha})}\right)^{1/2}\left(\frac{Mr^2}{a^2}\right)^{\delta_{j,\alpha}} e^{-Mr^2/2a^2}. \tag{2.71a}$$

The wave function for an arbitrary state is obtained by applying the raising operator B_+ and using Rodrigues' formula for the generalized Laguerre polynomials L_n^m. One finds:

$$\Phi_{n,\alpha}(r) = (-1)^n\left(\frac{2n!}{r^3\Gamma(2\delta_{j,\alpha}+n)}\right)^{1/2}\left(\frac{Mr^2}{a^2}\right)^{\delta_{j,\alpha}} \times e^{-Mr^2/2a^2} L_n^{2\delta_{j,\alpha}-1}\left(\frac{Mr^2}{a^2}\right). \tag{2.71b}$$

For $j=j_0$, the ground state wave functions are obtained by requiring

$$\langle r,\theta,\phi,\sigma|B_-|j,m;0\rangle_\alpha = 0. \tag{2.72}$$

Formally B_- can still be factorized as $2F_-^2$, and it is found that the wave functions are still given by (2.71), but α is uniquely determined by the boundary condition at the origin (i.e. by the self-adjoint extension for H_0 or R).

D. Diagonalization of the Hamiltonian

Instead of diagonalizing the compact generator R, we may also diagonalize the Hamiltonian. States are then labelled by the energy E, and we have

$$H_0|j,m;\alpha,E\rangle = E|j,m;\alpha,E\rangle. \tag{2.73}$$

To determine the wave functions in the energy realization, we can of course separate the Pauli equation in angular and radial dependence, and solve directly. However, we prefer to first compute the overlap between the states in the R basis $|\alpha, n\rangle_R$, and in the H_0 basis $|\alpha, E\rangle_{H_0}$ (we shall suppress the angular momentum labels). This can be achieved using algebraic methods only [4]. The operators of the OSp(1, 1) algebra can be realized in terms of Hermitian generators in an energy representation:

$$H_0 = E, \tag{2.74a}$$

$$D = -i\left(E\frac{d}{dE} + \frac{1}{2}\right), \tag{2.74b}$$

$$K = -E\frac{d^2}{dE^2} - \frac{d}{dE} + \frac{1}{E}\left(\tfrac{1}{2}d_j - \frac{\sigma^3}{4}\right)^2, \tag{2.74c}$$

$$Q = \sqrt{E}\,\sigma^1, \tag{2.74d}$$

$$S = i\left(\sqrt{E}\frac{d}{dE} + \frac{1}{4\sqrt{E}}\right)\sigma^1 - \frac{d_j}{2\sqrt{E}}\sigma^2. \tag{2.74e}$$

Then, we calculate the overlap ${}_{H_0}\langle 1, E|1, 0\rangle_R$ using the fact that ${}_{H_0}\langle 1, E|F_-|1, 0\rangle_R = 0$. We find

$${}_{H_0}\langle 1, E|1, 0\rangle_R = \left(\frac{1}{E\Gamma(2\delta_{j,1})}\right)^{1/2}(2a^2E)^{\delta_{j,1}}e^{-a^2E}. \tag{2.75}$$

The overlaps of the higher states are obtained by application of raising operators, and we find that

$${}_{H_0}\langle \alpha, E|\alpha, n\rangle_R = \alpha\left(\frac{\alpha n!}{E\Gamma(2\delta_{j,\alpha}+n)}\right)^{1/2}(2a^2E)^{\delta_{j,\alpha}} \times e^{-a^2E}L_n^{2\delta_{j,\alpha}-1}(2a^2E). \tag{2.76}$$

The wave functions in the basis where energy is diagonal are then obtained by

$$\langle \mathbf{r}, \sigma|\alpha, E\rangle_{H_0} = \sum_n \langle \mathbf{r}, \sigma|\alpha, n\rangle_{RR}\langle \alpha, n|\alpha, E\rangle_{H_0}. \tag{2.77}$$

With the help of formula (2.71) for the wave functions in the R-basis, we get

$$\begin{aligned}\Phi_{\alpha,E}(r) &= \langle r|\alpha, E\rangle_{H_0} \\ &= \alpha\left(\frac{2\alpha}{Er^3}\right)^{1/2}(2MEr^2)^{\delta_{j,\alpha}}e^{-Mr^2/2a^2-a^2E} \\ &\quad\times\sum_n\frac{(-1)^n n!}{\Gamma(2\delta_{j,\alpha}+n)}L_n^{2\delta_{j,\alpha}-1}\left(\frac{Mr^2}{a^2}\right)L_n^{2\delta_{j,\alpha}-1}(2a^2E),\end{aligned} \tag{2.78}$$

and after performing the sum over n [4], we obtain the properly normalized wave function:

$$\Phi_{\alpha,E}(r) = \alpha\left(\frac{2M\alpha}{r}\right)^{1/2}J_{2\delta_{j,\alpha}-1}(\sqrt{2ME}\,r). \tag{2.79}$$

This construction of course only holds for the $j \neq j_0$. When $j = j_0$, only the generators of SO(2, 1) can be used. The states are determined by the condition that

$B_-\langle r|0\rangle_\alpha = 0$, which yields two inequivalent self-adjoint extensions of R (see V) and which has two independent solutions: $\Phi_{\alpha,0}$ given by (2.70). The overlaps can be constructed in an analogous fashion, and we find that the wave functions in the energy representation for the two different extensions are still given by (2.77).

III. Generalization to Include the r^{-2} Potential

We shall now prove that the Hamiltonian H of (1.2) describing the dynamics of a spinning particle in the presence of a magnetic monopole and a λ^2/r^2 potential has a dynamical OSp(2, 1) supersymmetry. Using this symmetry, the spectrum will be constructed with the help of purely algebraic methods only. For brevity, we shall immediately introduce the quantum charges, instead of deriving them from the Lagrangian and the invariance transformations. We shall briefly return to the Lagrangian formulation in Subsect. C.

A. Hamiltonian and Charges

The simplest generalization of the Pauli equation for the Dirac monopole which includes a λ^2/r^2 potential preserving the existing symmetries is realized with Dirac γ matrices instead of the Pauli matrices. This Hamiltonian is

$$H = \frac{1}{2M}(p_i - eA_i)^2 - \frac{e}{2M}B_i\Sigma_i + \frac{\lambda^2}{2Mr^2} + \frac{\lambda}{2Mr^3}\gamma^i\gamma^0 r_i, \tag{3.1a}$$

with

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \Sigma_i = \frac{i}{4}\varepsilon_{ijk}[\gamma_j, \gamma_k], \tag{3.1b}$$

and λ a real parameter. Please note that in a basis where γ^5 is diagonal, H is block diagonal. When $\lambda = 0$, we recover the problem of the magnetic monopole with more spin degrees of freedom, which can be reduced to the case examined in the previous section. The dynamical symmetries are[6] dilations (D), conformal transformation (K), parity (Y) and four supersymmetries (Q_1, Q_2, S_1 and S_2), and we have

$$D = tH - \tfrac{1}{4}(p_i r_i + r_i p_i), \tag{3.2a}$$

$$K = -t^2 H + 2tD + \tfrac{1}{2}Mr^2, \tag{3.2b}$$

$$Y = \tfrac{1}{2}\gamma^5(\Sigma_i L_i + \tfrac{3}{2} + \lambda\gamma^0\gamma^i\hat{r}_i), \tag{3.2c}$$

$$Q_1 = \frac{1}{\sqrt{2M}}\gamma^5\left(\gamma^i(p_i - eA_i) + i\gamma^0\frac{\lambda}{r}\right), \tag{3.2d}$$

$$Q_2 = -i\gamma^5 Q_1, \tag{3.2e}$$

6 For all charges, except for the Hamiltonian, we use the same symbols as in Sect. II, even though their expression in coordinate representation contains additional terms

$$S_1 = -tQ_1 + \sqrt{\frac{M}{2}}\gamma^5\gamma^i r_i, \tag{3.2f}$$

$$S_2 = -i\gamma^5 S_1. \tag{3.2g}$$

The charges H, D, K and Y are bosonic, while Q_α and $S_\alpha (\alpha = 1,2)$ are fermionic. The structure equations are

$$[H,D] = iH, \quad [H,K] = 2iD, \quad [D,K] = iK, \tag{3.3a}$$

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H, \quad \{S_\alpha, S_\beta\} = 2\delta_{\alpha\beta}K,$$

$$\{Q_\alpha, S_\beta\} = -2\delta_{\alpha\beta}D + 2\varepsilon_{\alpha\beta}Y, \tag{3.3b}$$

$$[H,Q_\alpha] = 0, \quad [K,S_\alpha] = 0, \tag{3.3c}$$

$$[H,S_\alpha] = -iQ_\alpha, \quad [K,Q_\alpha] = iS_\alpha, \tag{3.3d}$$

$$[D,Q_\alpha] = -\frac{i}{2}Q_\alpha, \quad [D,S_\alpha] = \frac{i}{2}S_\alpha, \tag{3.3e}$$

$$[Y,H] = 0, \quad [Y,D] = 0, \quad [Y,K] = 0, \tag{3.3f}$$

$$[Y,Q_\alpha] = \frac{i}{2}\varepsilon_{\alpha\beta}Q_\beta, \quad [Y,S_\alpha] = \frac{i}{2}\varepsilon_{\alpha\beta}S_\beta. \tag{3.3g}$$

This superalgebra[7] is known as Spl(2, 1) or OSp(2, 1) [8] and is of rank 2. This is clear from the fact that the generator Y commutes with any other bosonic generator. Thus the total invariance algebra at this stage is

$$G_H = \mathrm{SO}(3)_{\text{rotations}} \times \mathrm{OSp}(2,1)_{\text{superconformal}}.$$

In fact this result only holds for $|\lambda| \neq |eg|$. We shall see in Subsect. C that $\lambda = \pm eg$, the invariance algebra can be enlarged to include an SO(3) group which leaves the spatial coordinates r_i alone, but rotates the spinning degrees of freedom. We shall also see that it introduces no new quantum numbers. Thus at $\lambda = \pm eg$, we have

$$G_H|_{\lambda=\pm eg} = \mathrm{SO}(3)_{\text{rotations}} \times \mathrm{SO}(3)_{\text{spinning}} \times \mathrm{OSp}(2,1)_{\text{superconformal}}$$

B. Casimir Operators, Quantum Numbers and States

The representations of OSp(2, 1) have been studied in the literature [8]. Rather than making use of the general theory of representations for this algebra, we shall start from scratch.

Let us consider the canonical chain of maximal subgroups.

$$\underset{C_2,C_3}{\mathrm{OSp}(2,1)} \supset \underset{C_0}{\mathrm{SO}(2,1)} \times \underset{Y}{\mathrm{O}(2)} \supset \mathrm{O}(2) \times \underset{R}{\mathrm{O}(2)}. \tag{3.4}$$

7 An alternative supersymmetric extension of the λ^2/r^2 potential has recently been proposed [12], when there is no magnetic monopole. The Hamiltonian of [12] is different from ours, its invariance group however is also OSp(2, 1)

We have listed the different Casimir operators, and OSp(2, 1) has in principle a quadratic *and* a cubic Casimir since its rank is 2. The Casimirs are given by

$$R = \frac{1}{2a^2} K + \frac{a^2}{2} H, \tag{3.5a}$$

$$C_0 = \tfrac{1}{2}(HK + KH) - D^2, \tag{3.5b}$$

$$C_2 = C_0 + \frac{i}{4}[Q_1, S_1] + \frac{i}{4}[Q_2, S_2] - Y^2, \tag{3.5c}$$

$$C_3 = (Y - \tfrac{1}{4}\gamma^5) C_2. \tag{3.5d}$$

The expression of the cubic Casimir is special to the form of the supercharges. The general expression was given in [8], and (3.5d) is obtained by using $Q_2 = -i\gamma^5 Q_1$ and $S_2 = -i\gamma^5 S_1$. In terms of the coordinate representation, we find

$$C_0 = \tfrac{1}{4}(J^2 - J_k \Sigma_k - eg\hat{r}_k \Sigma_k - \lambda\gamma^0\gamma^k \hat{r}_k - e^2 g^2 + \lambda^2), \tag{3.6a}$$
$$C_2 = C_3 = 0. \tag{3.6b}$$

We see that neither C_2, nor C_3 label the states, and the representations of G_H are so-called non-typical [8]. The general theory states that such representations of G_H are not completely determined by the values of the Casimirs.

Instead of C_0, a new Casimir A may be introduced for SO(2, 1):

$$A = i[Q_1, S_1] - \tfrac{1}{2} = i[Q_2, S_2] - \tfrac{1}{2}. \tag{3.7}$$

The Casimir A commutes with H, D, K and Y, but anticommutes with Q_α and S_α, and can be interpreted as "fermion number." In coordinate representation, we have

$$A = A_0 + \lambda\gamma^0\gamma^k \hat{r}_k, \tag{3.8a}$$
$$A_0 = J_k \Sigma_k + eg\Sigma_k \hat{r}_k - \tfrac{1}{2}, \tag{3.8b}$$

and the square of A is determined in terms of angular momentum,

$$A^2 = J^2 - e^2 g^2 + \tfrac{1}{4} + \lambda^2, \tag{3.9}$$

which never vanishes when $\lambda \neq 0$.

Instead of Y, we can again use the slightly more convenient operator γ^5, or "chirality." Chirality commutes with all bosonic charges and rotates the supercharges, so that it can play the role of Y:

$$[\gamma^5, Q_\alpha] = 2i\varepsilon_{\alpha\beta} Q_\beta, \tag{3.9a}$$
$$[\gamma^5, S_\beta] = 2i\varepsilon_{\alpha\beta} S_\beta. \tag{3.9b}$$

Thus, all states can be labelled by the eigenvalues of angular momentum (J^2, J_z), fermion number (sign A), chirality (γ^5) and R:

$$\begin{aligned}
J^2|j,m;\alpha,\chi,n\rangle &= j(j+1)|j,m;\alpha,\chi,n\rangle,\\
J_z|j,m;\alpha,\chi,n\rangle &= m|j,m;\alpha,\chi,n\rangle,\\
A|j,m;\alpha,\chi,n\rangle &= \alpha D_j|j,m;\alpha,\chi,n\rangle,\\
\gamma^5|j,m;\alpha,\chi,n\rangle &= \chi|j,m;\alpha,\chi,n\rangle,\\
R|j,m;\alpha,\chi,n\rangle &= (\Delta_{j,\alpha}+n)|j,m;\alpha,\chi,n\rangle, \quad n\geqq 0,
\end{aligned} \tag{3.10}$$

with

$$D_j=\sqrt{j(j+1)-j_0(j_0+1)+\lambda^2}, \tag{3.11a}$$

$$\Delta_{j,\alpha}=\frac{1}{2}D_j-\frac{\alpha}{4}+\frac{1}{2}. \tag{3.11b}$$

and

$$\alpha^2=\chi^2=1.$$

For $j=j_0$, there are some restrictions, which we discuss later on.

When $j\neq j_0$, the states $|j,m;\alpha,\chi,n\rangle$ form a reducible representation of OSp(2, 1). To see this we notice that there exists a Casimir operator of OSp(2, 1) which does not vanish. Clearly γ^5 and A both commute with all bosonic charges, but anticommute with supercharges, so that their product

$$\Omega=\gamma^5 A \tag{3.12a}$$

commutes with all generators of OSp(2, 1) and is a Casimir. Using the expressions for Y and A, it is easy to show that

$$\Omega=2(Y-\tfrac{1}{4}\gamma^5)=2C_3/C_2. \tag{3.12b}$$

Thus, even though the set of all "canonical" Casimirs does not specify the representations uniquely there are in fact further Casimirs in the problem which allow for complete reduction of the representation. The eigenvalues of Ω are

$$\Omega|j,m;\alpha,\chi,n\rangle=\omega D_j|j,m;\alpha,\chi,n\rangle \qquad \text{with } \omega=\chi\alpha. \tag{3.13}$$

Thus, α is an independent quantum number, and the representations labelled by α and χ decompose into two irreducible representations labelled by $|j,m;\chi,\chi,n\rangle$ and $|j,m;-\chi,\chi,n\rangle$.

When $j=j_0$, another accident occurs and we have

$$A_0|j_0,m;\alpha,\chi,n\rangle=0. \tag{3.14}$$

Combined with the fact that L^2 is diagonal for $j=j_0$, we find, exactly as for (2.55) that

$$\hat{r}_i\Sigma_i|j_0,m;\alpha,\chi,n\rangle=\frac{eg}{|eg|}|j_0,m;\alpha,\chi,n\rangle. \tag{3.15}$$

With the help of the fact that $\gamma^0\gamma^i=-\gamma^5\Sigma^i$, it is straightforward to evaluate A

on these states:

$$A|j_0, m; \alpha, \chi, n\rangle = \alpha D_j|j_0, m; \alpha, \chi, n\rangle, \quad \alpha = \chi\omega_0, \quad \text{and} \quad \omega_0 = \frac{-\lambda eg}{|\lambda eg|}. \tag{3.16}$$

When $\lambda \neq 0$, A is proportional to the chirality on s-wave states, and ω_0 is completely determined by the parameters of the problem. Only a single irreducible representation of OSp(2, 1) occurs, and α is related to the chirality. When $\lambda = 0$, $D_j = 0$, and A vanishes on the s-wave states, so that α is not defined. Contrary to the case of the fermion monopole coupling discussed in Sect. II, the supercharges now admit self-adjoint extensions, and the full algebra can be realized for two values of the extension parameters, corresponding to $\omega_0 = \chi\alpha = \pm 1$. This situation may also be thought of as the limiting cases when $\lambda \to 0$, which can happen either with $\lambda < 0$ or $\lambda > 0$, yielding $\omega_0 = \pm eg/|eg|$ respectively. The cases with $j \neq j_0$ and $j = j_0$ can be treated in parallel, if we remember that in the latter case we have the restriction $\alpha = \chi\omega_0$.

Now that we have identified the representations and the states, we shall show that at fixed j, m and ω, these states are all connected to each other through the action of OSp(2, 1). Raising and lowering operators are again defined in the Cartan basis where R and Y are simultaneously diagonal, and we find

$$B_\pm = \frac{1}{2a^2}K - \frac{a^2}{2}H \pm iD, \tag{3.17a}$$

$$F_\pm^{L,R} = \frac{1}{2a}S^{L,R} \mp \frac{ia}{2}Q^{L,R}, \tag{3.17b}$$

$$Q^{L,R} = \frac{1}{\sqrt{2}}(Q_1 \mp iQ_2), \tag{3.17c}$$

$$S^{L,R} = \frac{1}{\sqrt{2}}(S_1 \mp iS_2). \tag{3.17d}$$

Here $F_\pm^{L(R)}$ are the purely left (right) creation and annihilation operators, and we have

$$(F_\pm^L)^\dagger = F_\mp^R, \tag{3.18}$$

where $\dagger$ stands for the adjoint of the operator. The Cartan equations are very simple in this case:

$$[R, B_\pm] = \pm B_\pm, \quad [B_+, B_-] = -2R, \tag{3.19a}$$

$$\{F^{L,R}, F^{L,R}\} = 0, \quad \{F_\pm^L, F_\pm^R\} = B_\pm, \quad \{F_\pm^L, F_\mp^R\} = R \pm Y, \tag{3.19b}$$

$$[R, F_\pm^{L,R}] = \pm\tfrac{1}{2}F_\pm^{L,R}; \quad [Y, F_\pm^L] = -\tfrac{1}{2}F_\pm^L; \quad [Y, F_\pm^R] = \tfrac{1}{2}F_\pm^R, \tag{3.19c}$$

$$[B_\pm, F_\pm^{L,R}] = 0, \quad [B_\pm, F_\mp^{L,R}] = \mp F_\pm^{L,R}. \tag{3.19d}$$

Next, we determine how raising and lowering operators act on the states. The action on the quantum numbers α and χ is obtained from the fact that fermion number and chirality anticommute with F's and commute with B's. Normalization of this action is easily deduced with the help of some further commutation relations.

$$[F_+^L, F_-^R] = \tfrac{1}{2}(-A - \tfrac{1}{2} - 2R\gamma^5), \tag{3.20a}$$
$$[F_+^R, F_-^L] = \tfrac{1}{2}(-A - \tfrac{1}{2} + 2R\gamma^5). \tag{3.20b}$$

One finds

$$B_\pm |j, m; \alpha, \chi, n\rangle = \sqrt{(\Delta_{j,\alpha} + n)(\Delta_{j,\alpha} + n \pm 1) - \Delta_{j,\alpha}(\Delta_{j,\alpha} - 1)}\, |j, m; \alpha, \chi, n \pm 1\rangle,$$
$$F_\pm^L |j, m; \alpha, \chi, n\rangle = (1 + \chi)^{1/2} \sqrt{\frac{1}{2}(\Delta_{j,\alpha} + n) \pm \frac{1}{8} \pm \frac{\alpha}{4} D^j}\, \left|j, m; -\alpha, -\chi, n - \frac{\alpha}{2} \pm \frac{1}{2}\right\rangle,$$
$$F_\pm^R |j, m; \alpha, \chi, n\rangle = (1 - \chi)^{1/2} \sqrt{\frac{1}{2}(\Delta_{j,\alpha} + n) \pm \frac{1}{8} \pm \frac{\alpha}{4} D_j}\, \left|j, m; -\alpha, -\chi, n - \frac{\alpha}{2} \pm \frac{1}{2}\right\rangle, \tag{3.21}$$

The only fundamental difference between these relations and (2.45) is the appearance of the factors depending on the chirality. Their presence implies that an operator of given chirality annihilates all states with that chirality. From (3.21), we also learn that there are two basis states for each angular momentum, and they are distinguished by their chirality.

$$F_-^L |j, m; 1, 1, 0\rangle = 0, \tag{3.22a}$$
$$F_-^R |j, m; 1, -1, 0\rangle = 0. \tag{3.22b}$$

Of course, for $j = j_0$, only one of these states actually exists. The two irreducible representations of OSp(2, 1) for any given j, m are then obtained by applying a definite chain of raising operators:

$$|j, m; \alpha, 1, n\rangle = \left(\frac{\Gamma(2\Delta_{j,\alpha})}{n!\,\Gamma(2\Delta_{j,\alpha} + n)}\right)^{1/2} F_+^R F_+^L \ldots F_+^R F_+^L |j, m; \alpha, 1, 0\rangle, \tag{3.23a}$$

$$|j, m; \alpha, -1, n\rangle = \left(\frac{\Gamma(2\Delta_{j,\alpha})}{n!\,\Gamma(n + 2\Delta_{j,\alpha})}\right)^{1/2} F_+^L F_+^R \ldots F_+^L F_+^R |j, m; \alpha, -1, 0\rangle. \tag{3.23b}$$

This concludes our construction of the states.

The wave functions for each of these states may be easily constructed, in exactly the same fashion as for the case of the monopole alone as discussed in Sect. II. Of course, as the spin algebra is realized in terms of the Dirac matrices instead of the Pauli matrices, the wave functions will be four component Dirac spinors. The angular basis for these spinors is obtained by direct product of the angular basis for two component spinors (the $\eta_{j,m,\alpha}$ of (2.59)) and the quantum number chirality (χ),

$$\Psi_{j,m;\alpha,\chi,n}(r, \theta, \phi) = \begin{pmatrix} \varphi^+_{n,\alpha,\chi}(r)\eta_{j,m,\alpha}(\theta, \phi) \\ \varphi^-_{n,\alpha,\chi}(r)\eta_{j,m,\alpha}(\theta, \phi) \end{pmatrix}. \tag{3.24}$$

It is clear that in a basis where chirality is diagonal, we also have $\varphi^+_{n,\alpha,-1} = \varphi^-_{n,\alpha,+1} = 0$. The ground state wave function is obtained by resolving the differential equations,

$$\langle r,\theta,\phi,\sigma|F_-^{L,R}|j,m;1,\omega,0\rangle=0. \tag{3.25}$$

With the help of the expressions

$$F_\pm^{L,R}=\frac{1}{2a}S^{L,R}\mp\frac{a}{K}\left(\frac{3}{8}S^{L,R}+\frac{1}{4}S^{L,R}A+\frac{i}{2}DS^{L,R}\right), \tag{3.26}$$

the equation (3.25) is readily reduced to an equation for the radial functions $\varphi^+_{0,1,+1}$ and $\varphi^-_{0,1,-1}$:

$$\frac{1}{r}\frac{d}{dr}r\varphi^\pm_{0,1,\pm1}(r)+\left(\frac{Mr}{a^2}-\frac{D_j}{r}\right)\varphi^\pm_{0,1,\pm1}(r)=0, \tag{3.27}$$

whose normalized solutions are

$$\varphi^\pm_{0,1,\pm1}(r)=\left(\frac{2}{r^3\Gamma(2\Delta_{j,1})}\right)^{1/2}\left(\frac{Mr^2}{a^2}\right)^{\Delta_{j,1}}e^{-Mr^2/2a^2},$$
$$\varphi^\pm_{0,1,\mp1}(r)=0. \tag{3.28}$$

where D^j and $\Delta_{j,\alpha}$ are defined in (3.11). The radial functions for both chiralities are thus the same, reflecting the perfect symmetry between them. The general radial solution may be obtained by application of $F_+^{L(R)}$ and B_+, and we find

$$\varphi^\pm_{n,\alpha,\pm1}(r)=(-1)^n\left(\frac{2n!}{r^3\Gamma(2\Delta_{j,\alpha}+n)}\right)^{1/2}\left(\frac{Mr^2}{a^2}\right)^{\Delta_{j,\alpha}}e^{-Mr^2/2a^2}L_n^{2\Delta_{j,\alpha}-1}\left(\frac{Mr^2}{a^2}\right),$$
$$\varphi^\pm_{n,\alpha,\mp1}(r)=0. \tag{3.29}$$

We have now constructed the wave functions for all states. Please remember that for $j=j_0$, there is just one ground state with chirality $\omega_0=-\lambda eg/|\lambda eg|$, so that $\alpha=\omega_0\chi$. With this restriction, the wave functions are given by (3.29).

So far we have diagonalized the compact generator R. With the same group theoretic techniques as in Sect. II, we may also construct the wave functions in the representation where H is diagonal. The overlap functions are

$$_H\langle\alpha,\chi,E|\alpha,\chi,n\rangle_R=\alpha\left(\frac{\alpha n!}{E\Gamma(2\Delta_{j,\alpha}+n)}\right)^{1/2}(2a^2E)^{\Delta_{j,\alpha}}e^{-a^2E}L_n^{2\Delta_{j,\alpha}-1}(2a^2E). \tag{3.30}$$

The phase between the two irreducible representations (labelled by $\omega=\alpha\chi$) is arbitrary, and set to zero by definition. The wave functions in the energy representation are now easily obtained:

$$\varphi^\pm_{E,\alpha,\pm1}(r)=\alpha\left(\frac{2\alpha M}{r}\right)^{1/2}J_{2\Delta_{j,\alpha}-1}(\sqrt{2ME}\,r),$$
$$\varphi^\pm_{E,\alpha,\mp1}(r)=0. \tag{3.31}$$

It should be stressed that the present construction holds for $j=j_0$ included.

It may seem surprising that, nowhere in the erection of the s-wave states, the question has arisen as to whether the Hermitian charges, utilized to derive the representations are truly self-adjoint so that the representations are truly unitary.

The problem of self-adjointness will be addressed in full in Sect. V, but we shall here list the results of the analysis.

For $|\lambda| \geqq \frac{3}{2}$, the operator R (or H) is self-adjoint, all wave functions vanish at the origin, and all Hermitian charges of OSp(2, 1) are self-adjoint.

For $\frac{1}{2} \leqq |\lambda| < \frac{3}{2}$, the operator R (or H) is not self-adjoint, but admits a one parameter family of self-adjoint extensions. The operators Q_α are well defined only for two specific values of the extension parameter, and the remaining charges are defined and self-adjoint only when a single value is chosen for this parameter.

For $0 \leqq |\lambda| < \frac{1}{2}$, the operator R (or H) is not self-adjoint and admits a four parameter family of extensions. The charge Q_1 is self-adjoint only on a one parameter subfamily. To make also Q_2 self-adjoint, the choice is further restricted, and only two values of the latter parameter are retained. Self-adjointness of the remaining charges fixes the parameter uniquely for $\lambda \neq 0$, while for $\lambda = 0$, two different values are allowed.

C. The Lagrangian and Points of Higher Symmetry

As the Hamiltonian H is expressed in terms of Dirac γ matrices, it is not hard to see that at the Lagrangian level four Grassmannian spinning coordinates are needed. It turns out that these can be chosen to be real and we obtain the following Lagrangian:

$$L = \tfrac{1}{2} M \dot{r}_i^2 - eA_i \dot{r}_i - \frac{\lambda^2}{2Mr^2} + L_{\text{spin}},$$

$$L_{\text{spin}} = \frac{i}{2} \psi_\mu \dot{\psi}_\mu - \frac{ie}{2M} B_i \varepsilon_{ijk} \psi_j \psi_k + \frac{i\lambda}{2Mr^3} \psi_i \psi_0 r_i, \qquad \mu = 0, 1, 2, 3. \tag{3.32}$$

Canonical quantization is performed by requiring

$$[r_i, p_j] = i\delta_{ij}, \qquad \{\psi^\mu, \psi^\nu\} = \delta^{\mu\nu}, \tag{3.33}$$

and the Clifford algebra is realized in terms of the (Minkowski space) Dirac matrices

$$\psi^0 = \frac{1}{\sqrt{2}} \gamma^0, \qquad \psi^i = \frac{i}{\sqrt{2}} \gamma^i. \tag{3.34}$$

From (3.32), it is manifest that the kinetic term of ψ^μ is O(4) invariant, and we shall now see that this O(4) symmetry can be shared by the interaction terms when $\lambda = \pm eg$. To do so, we remark that $\psi_\mu \psi_\nu$ is an antisymmetric tensor, which can be decomposed into its self-dual and anti-self-dual parts with the help of the 't Hooft η symbols [13]. In this notation, the spin part of the Lagrangian becomes:

$$L_{\text{spin}} = \frac{i}{2} \psi_\mu \dot{\psi}_\mu - \frac{i}{4Mg} B_i (\lambda + eg) \eta^i_{\mu\nu} \psi_\mu \psi_\nu + \frac{i}{4Mg} B_i (\lambda - eg) \bar{\eta}^i_{\mu\nu} \psi_\mu \psi_\nu. \tag{3.35}$$

Under an O(4) transformation, given in terms of its infinitesimal generators

$$\omega_{\mu\nu} = \eta^a_{\mu\nu}\omega^a + \bar{\eta}^a_{\mu\nu}\bar{\omega}^a, \tag{3.36}$$

the η and $\bar{\eta}$ symbols transform as O(3) vectors:

$$\begin{aligned} \delta_\omega \eta^a_{\mu\nu} &= -\varepsilon^{abc}\omega^b\eta^c_{\mu\nu}, \\ \delta_\omega \bar{\eta}^a_{\mu\nu} &= -\varepsilon^{abc}\bar{\omega}^b\bar{\eta}^c_{\mu\nu}. \end{aligned} \tag{3.37}$$

With $\lambda \neq \pm eg$, rotation on the spatial coordinate r_i cannot compensate for a general O(4) rotation, and the largest invariance is obtained for $\bar{\omega}^a = \omega^a$, i.e. ordinary rotation on ψ_i and ψ_0 left alone. However, when for example $\lambda = eg$, $\bar{\omega}$ transformations do not act on L_{spin} at all, and ω transformations can be compensated by ordinary rotations. At the points $\lambda = \pm eg$, the full invariance group is thus

$$G_{H|\lambda=\pm eg} = \mathrm{SO}(3)_{\text{rotations}} \times \mathrm{SO}(3)_{\text{spinning}} \times \mathrm{OSp}(2,1)_{\text{superconformal}}.$$

All Casimir operators associated with this additional O(3) spinning are easily shown to be fixed, so that no new quantum numbers are needed at these points.

IV. Generating the Spectrum of the Dirac Hamiltonian with the OSp(2, 1) Algebra

So far, we have considered only dynamical or Lagrangian symmetries. These are characterized by the fact that under the corresponding symmetry transformation, the Lagrangian changes by a total time derivative, so that the equations of motion are left invariant under the transformation. At the Hamiltonian level, dynamical symmetries give rise to Nöther charges which are, in general, time dependent and which, together with the Hamiltonian, form an algebra that closes. Thus, the total time derivative of the charges vanishes. When the invariance algebra is a superalgebra, the Hamiltonian should be an even or bosonic charge, so that time translation is given by Schrödinger's equation.

Spectrum generating symmetries are rather different in character. They do not correspond to symmetries of the Lagrangian or of the equations of motion. Rather, it is first established that the spectrum of the Hamiltonian (say H) coincides, as a whole or in part, with the spectrum of a different Hamiltonian (say H'). When H' possesses a dynamical symmetry G, then its spectrum transforms under a unitary representation of G, and so does (part or the whole of) the spectrum of H. In this sense, a spectrum generating symmetry is an "on shell" symmetry which is not a symmetry of the equations of motion. The famous example is the Coulomb problem, whose spectrum transforms under a representation of O(4, 2) [2].

In this section, we generate the spectrum of the Dirac Hamiltonian h for a spin $\frac{1}{2}$ particle in the presence of a Dirac magnetic monopole and a scale invariant mass with

$$h = \alpha_i(p_i - eA_i) + \frac{\lambda}{r}\beta, \tag{4.1}$$

and A_i the monopole field. This Hamiltonian is clearly related to the supercharge Q_2 of the OSp(2, 1) algebra given in (3.2). In fact, h is obtained by replacing in $\sqrt{2M}\,Q_2$ the Dirac matrices $-i\gamma^i$ by α^i, which simply amounts to a change in the representation of the γ matrices. The complete correspondence is then easily derived, and we have

$$\begin{aligned} &\gamma^0 \to \gamma^0, \\ &\gamma^i \to i\alpha^i = i\gamma^0\gamma^i, \\ &\gamma^5 \to i\gamma^0\gamma^5. \end{aligned} \tag{4.2}$$

After this substitution is made, we may identify h with Q_2 of the OSp(2, 1) algebra. So h is in fact part of the superalgebra. The reason that OSp(2, 1) is not a dynamical symmetry of h, is that h does not obey commutation relations, but anticommutation relations. Thus there is no sense in which h generates time translation which is necessarily a bosonic operation. Nevertheless, the spectrum of h (or Q_2) is readily derived from the spectrum of H, which was obtained in Sect. III. Working at fixed angular momentum, we have

$$H|\alpha, \chi, E\rangle = E|\alpha, \chi, E\rangle, \tag{4.3}$$

where the chirality χ and fermion number α can take values ± 1 independently, except in the s-wave where they are related (see Sect. III).

The action of Q_2 is also known:

$$Q_2|\alpha, \chi, E\rangle = \sqrt{E}\,|-\alpha, -\chi, E\rangle, \tag{4.4}$$

so that Q_2 is easily diagonalized.

$$\begin{aligned} &Q_2(|1,1,E\rangle \pm |-1,-1,E\rangle) = \pm\sqrt{E}(|1,1,E\rangle \pm |-1,-1,E\rangle), \\ &Q_2(|1,-1,E\rangle \pm |-1,1,E\rangle) = \pm\sqrt{E}(|1,-1,E\rangle \pm |-1,1,E\rangle). \end{aligned} \tag{4.5}$$

In the s-wave sector two of the four states defined above are dependent. It is clear that (4.5) provides the general solution to the eigenvalue problem for Q_2, and thus for h.

V. Self-Adjointness in the s-Wave Sector

For states not belonging to the s-wave sector, the centrifugal barrier is strong enough to make the wave functions vanish at the origin. All operators, encountered in our analysis, are then self-adjoint on this restricted set of wave functions [11]. They are legitimate quantum operators, and the states in this sector fall into unitary representations of the corresponding algebra.

In the s-wave sector on the other hand, the situation is quite different. It has long been known that the magnetic monopole has an attractive potential, strong enough to cancel the centrifugal barrier which normally exists for orbiting spin $\frac{1}{2}$ particles [11, 14]. As a consequence, the particle can travel down to the center of the monopole and can actually be absorbed by it. A different way of expressing this property is to say that a net current can flow into the monopole; the evolution operator is not unitary, the Hamiltonian is not self-adjoint, and the quantum mechanical problem is ill-defined [11]. Physically, one expects new physics to set in at very short distances (as compared to the inverse mass of the spinning particle). A complete description of the monopole spin $\frac{1}{2}$ particle system is expected to be given in terms of a self-adjoint Hamiltonian and a unitary evolution operator. For monopoles, such a framework is very plausible within the context of non-Abelian gauge theories [15] and it has been shown [16] that physics at the scale of the full theory imposes definite boundary conditions at the center of the monopole which prevent the spinning particle from collapsing. Mathematically, when an operator is not self-adjoint, it may under certain conditions be extended to a self-adjoint operator which then correctly describes the physics.[8]

In the present section, we shall analyze the question of the self-adjointness of the different operators encountered in Sects. II and III in Subsect. A and B respectively. Interestingly, and unlike most of the cases treated in standard mathematics literature, we shall primarily deal with the simultaneous self-adjoint extension of several operators. Indeed, to justify the group theoretic manipulation which lie at the heart of our approach, it is necessary to deal with operators which are self-adjoint on the *same* space of representations. It is a remarkable fact that the results obtained from the algebraic, group theoretic manipulations on the one hand and from the analytic theory of *simultaneous* self-adjoint extensions on the other hand are always in precise agreement. Group theory knows about self-adjointness and extensions. We shall directly restrict to the s-wave sector and indicate this restriction by appending a subscript "(s)" to the operator under consideration. Also a factor of $1/r$ is extracted from the wave functions, so that the problems are reduced to those of a one dimensional quantum mechanical system restricted to the half line.[9] In Appendix B, we recall a few definitions and facts from the theory of self-adjoint extensions.

A. Spin $\frac{1}{2}$ Particle in the Presence of the Monopole

The Hamiltonian

$$H_{0(s)} = -\frac{d^2}{dr^2}, \qquad r \geqq 0, \tag{5.1}$$

is Hermitian on the set of wave functions that vanish at the origin, but it is not self-adjoint and has deficiency indices $n_+ = n_- = 1$. Thus $H_{0(s)}$ admits a one parameter family of self-adjoint extensions [17, 11], labelled by the real parameter c:

$$\psi'(0) = -c\psi(0). \tag{5.2}$$

The normalized eigenfunctions satisfying $H_{0(s)}\psi_E = E\psi_E$ are

$$\psi_E(r) = (\pi^2 E)^{-1/4} \cos(\sqrt{E}\, r + \phi_E), \qquad E \geqq 0, \tag{5.3}$$

$$\phi_E = \text{Arctg}\frac{c}{\sqrt{E}}. \tag{5.4}$$

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8 Excellent expositions of the problem are given in [17]

9 All properties discussed in this section are independent of the value of M and a, so we set $M = \frac{1}{2}$ and $a = 1$

For $c>0$, there also exists a negative energy bound state, whose wave function is given by

$$\psi_{-c^2}(r)=\sqrt{2c}\,e^{-cr}\theta(+c), \tag{5.5}$$

and they form a complete set, for all values of c:

$$\int_0^\infty dE\,\psi_E(r)\psi_E(r')+\theta(+c)\psi_{-c^2}(r)\psi_{-c^2}(r')=\delta(r-r'). \tag{5.6}$$

By Von Neumann's theory, $H_{0(s)}$ is now self-adjoint on this Hilbert space $\mathscr{H}_0$; its new deficiency indices are both zero. (See App. B.). The operator $R_{(s)}$ is self-adjoint on $\mathscr{H}_0$, but D is not. Even though D satisfies

$$(\psi_1, D\psi_2)=(D\psi_1,\psi_2) \tag{5.7}$$

for all ψ_1, $\psi_2 \in \mathscr{H}_0$, the operator does not respect the boundary condition (5.2) for any general value of c. Indeed, if ψ satisfies (5.2), and we can expand it in a Taylor series, we have

$$\begin{aligned}\psi(r)&=\psi(0)+r\psi'(0)+O(r^2),\\ \tilde{\psi}(r)&=D\psi(r)=\frac{i}{4}\psi(0)+\frac{3i}{4}r\psi'(0)+O(r^2),\end{aligned} \tag{5.8}$$

so that

$$\tilde{\psi}'(0)=-3c\tilde{\psi}(0), \tag{5.9}$$

which is compatible with (5.2) only if

$$\psi'(0)=0 \tag{5.10a}$$

or

$$\psi(0)=0. \tag{5.10b}$$

The fact that the dilatation generator does not respect the boundary conditions (5.2) for *finite* c is to be expected since c introduces a length scale in the theory, breaking dilatation invariance. It is easily seen that all charges now map regular functions of $\mathscr{H}_0$, into $\mathscr{H}_0$ and that H_0, R, D, and K are self-adjoint and B_+ is the adjoint of B_-. The two boundary conditions (5.10a) and (5.10b) correspond to the case $\alpha=+1$ and $\alpha=-1$ of the algebraic procedure, as is clear from (2.71).

Of course, one may wonder why H_0 was rendered self-adjoint first, instead of, say R. To see what happens in this case, we start by obtaining the self-adjoint extensions of R.

$$R_{(s)}=\frac{1}{2}\frac{d^2}{dr^2}+\frac{1}{8}r^2. \tag{5.11}$$

The deficiency indices are $n_+=n_-=1$, and the extensions are again given by the boundary conditions (5.2). The eigenfunctions satisfying $R_{(s)}\psi_z=z\psi_z$ are so-called Whittaker functions [18].

$$\psi_{z_n}(r)=N_{z_n}\frac{1}{\sqrt{r}}W_{z_n,-1/4}(\tfrac{1}{2}r^2) \tag{5.12}$$

These functions satisfy the boundary condition (5.2) provided z_n is a solution to the equation

$$\Gamma(\tfrac{3}{4}-z_n)=\frac{c}{\sqrt{2}}\Gamma(\tfrac{1}{4}-z_n), \tag{5.13}$$

where Γ denotes the gamma function [18]. This equation can be easily solved for $c=0$ or $c=\infty$ and we get

$$c=0, \qquad z_n=\tfrac{1}{4}+n, \tag{5.14a}$$
$$c=\infty, \qquad z_n=\tfrac{3}{4}+n, \tag{5.14b}$$

corresponding to the symmetric (respectively antisymmetric) eigenfunctions of $R_{(s)}$. We see that in both cases the levels are spaced by an integer. When c is neither 0 nor ∞, the equation must be resolved approximately and the following qualitative features appear

i) for $c>0$ there exists one negative eigenvalue z_{-1};

ii) the spectrum lies entirely on the real axis, is discrete, and interpolates between the spectra at $c=0$ and $c=\pm\infty$;

iii) consecutive eigenvalues are *not separated by integers.*

The last fact (iii) is in contradiction with the results derived from group theory. If indeed B_+ and B_- were each other's adjoint on the domain on which R is defined and self-adjoint, then the eigenstates of R would be generated by successive applications of B_+, and the eigenvalues of R are separated by integers. The operators B_+ and B_- are not, however, adjoints of each other on the Hilbert space defined by (5.2) since the operator D respects the boundary condition (5.2) only if (5.10) is realized. Thus, we see that simultaneously requiring R to be self-adjoint and the Cartan generators, B_+ and B_- to be adjoints admits only two possible and consistent solutions, given by (5.14). These two solutions were indeed obtained group theoretically in Sect. II; (5.14a) corresponds to $\alpha=1$, while (5.14b) corresponds to $\alpha=-1$. It is easily shown that all other Hermitian charges are now also self-adjoint. In both cases we have

$$z_{n,\alpha}=\frac{1}{2}-\frac{\alpha}{4}+n, \tag{5.15}$$

and the Whittaker function (5.12) reduces to

$$\psi_{z_n,\alpha}(r)=N_{z_n,\alpha}r^{(1-\alpha)/2}e^{-r^2/4}L_n^{-\alpha/2}\left(\frac{r^2}{2}\right), \tag{5.16}$$

in agreement with formula (3.29), obtained by algebraic methods only.

Let us remark that the operator $R_{(s)}$ of (5.11) is also the radial part of the Hamiltonian for a three dimensional harmonic oscillator, and one may wonder what the above discussed extensions correspond to. The answer is that $\psi(0)=0$ is the only viable solution of the eigenvalue problem, as functions with $\psi(0)=b\neq 0$ produce δ functions at the origin. The extensions are the result of an artificial singularity at $r=0$ of spherical coordinates. In the case of the monopole, the origin (where the monopole resides) is a physical singularity, and the different self-adjoint extensions are *not* artifacts. As was shown in ref. [16], different self-adjoint extensions are the result of different short distance physics.

Now that we have settled the self-adjointness properties of the bosonic SO(2, 1) subalgebra, we turn to the adjointness properties of the supercharges. Concentrating on the Cartan supercharges $F_{\pm}$, we have

$$F_{\pm(s)} = \mp\frac{1}{2}\frac{d}{dr} + \frac{1}{4}r. \tag{5.17}$$

Clearly, F satisfies the hermiticity conjugation condition

$$(\psi_1, F_{\pm}\psi_2) = (F_{\mp}\psi_1, \psi_2) \qquad \text{for all } \psi_1, \psi_2 \in \mathscr{H}_0,$$

provided $c = \infty(\psi(0) = 0)$ only. However, even on this space, $F_{\pm}$ will not respect the boundary condition $\psi(0) = 0$, and F_+ is not the adjoint of F_-. Alternatively, the supercharge Q

$$Q_{(s)} = -i\frac{d}{dr} \tag{5.18}$$

is Hermitian on $\mathscr{H}_0$, but does not respect the boundary condition $\psi(0) = 0$. Hence, for no self-adjoint extension of R (or H), is it possible to render Q self-adjoint. Consequently, on the s-wave sector, only the bosonic subalgebra SO(2, 1) can be unitarily implemented, whereas the superalgebra OSp(1, 1) has no *unitary* implementation.[10]

One could have started by attempting to render Q self-adjoint first, and then implement the bosonic generators. However, the deficiency indices of $Q_{(s)}$ are $n_+ = 1$, $n_- = 0$ [11], and from the theory of self-adjoint extensions, we know that for this operator, no self-adjoint extensions exist [18]. The fact that the supercharge Q is maximally Hermitian [17] implies that the Hamiltonian always has an extension for which there exists a negative energy bound state, so that supersymmetry is not realized.

B. Spin $\frac{1}{2}$ Particle with Two Spin Degrees of Freedom in the Presence of the Monopole and the λ^2/r^2 Potential

Following the philosophy of the previous subsection, we shall analyze the self-adjointness properties of the generators of the OSp(2, 1) superalgebra (defined in (3.1) and (3.2)) in the s-wave sector. We start again by analyzing the Hamiltonian of the system,

$$H_{(s)} = -\frac{d^2}{dr^2} + \frac{\lambda^2 + \lambda\sigma^3}{r^2}. \tag{5.19}$$

Here σ_3 is the chirality operator with eigenvalue χ. The deficiency indices are easily

10 The infinitesimal generator of a unitary operator must be (anti) self-adjoint; (anti-) Hermiticity is not sufficient. [17]

found from

$$H_{(s)}\psi^{\pm} = \pm i\psi^{\pm}, \tag{5.20}$$

and we have the following solution, converging to zero exponentially fast as $r \to \infty$,

$$\psi_\chi^+(r) = \sqrt{r}H_{\nu_\chi}^{(1)}(e^{i\pi/4}r), \tag{5.21a}$$

$$\psi_\chi^-(r) = \sqrt{r}H_{\nu_\chi}^{(1)}(e^{i3\pi/4}r). \tag{5.21b}$$

Here $H^{(1)}$ denotes the Hankel function [18], and we have used the shorthand

$$\nu_\chi = \left|\lambda + \frac{\chi}{2}\right|. \tag{5.22}$$

The functions will be normalizable only if they are square integrable at $r = 0$, so that

$$\nu_\chi < 1. \tag{5.23}$$

We deduce the deficiency indices and the number of parameters in the extension:

$$N_\pm(\lambda) = \begin{cases} 0 & \frac{3}{2} \leqq |\lambda| & \text{Case I : self-adjoint,} \\ 1 & \frac{1}{2} \leqq |\lambda| < \frac{3}{2} & \text{Case II : 1 parameter family,} \\ 2 & |\lambda| < \frac{1}{2} & \text{Case III: 4 parameter family.} \end{cases} \tag{5.24}$$

For Case I, the eigenfunctions of $H_{(s)}$ are Bessel functions which vanish at the origin, and all Hermitian charges of OSp(2, 1) are self-adjoint (see also [19]). The algebra is realized in terms of self-adjoint operators and the representations are unitary.

For Case II, the Hamiltonian is self-adjoint in the sector of one chirality, but not in the sector of the opposite chirality. The extensions in this sector are labelled by one real parameter c and the normalized eigenfunctions are

$$\psi_E^\chi(r) = \left(\frac{r}{2(A_\chi^2 + B_\chi^2)}\right)^{1/2} (A_\chi J_{\nu_\chi}(\sqrt{E}r) + B_\chi Y_{\nu_\chi}(\sqrt{E}r)),$$

$$\nu_\chi < 1 \begin{cases} A_\chi = 1 - \dfrac{E^{\nu_\chi}}{c}\cos \pi\nu_\chi \\ B_\chi = +\dfrac{E^{\nu_\chi}}{c}\sin \pi\nu_\chi \end{cases},$$

$$\nu_\chi \geqq 1 \begin{cases} A_\chi = 1 \\ B_\chi = 0 \end{cases}. \tag{5.25}$$

When $c > 0$, there also exists a negative energy bound state at $E_0 = (c)^{1/\nu_\chi}$ and its normalized wave function is given by

$$\psi_{-E_0}(r) = \left(\frac{2r \sin \pi\nu_\chi}{\pi\nu_\chi} E_0\right)^{1/2} K_{\nu_\chi}(\sqrt{E_0}r), \quad \text{for } \nu_\chi < 1, \tag{5.26a}$$

$$\psi_{-E_0}^\chi(r) = 0, \qquad \text{for } \nu_\chi \geqq 1. \tag{5.26b}$$

With the help of some contour integrals, one can show that this set of eigenfunctions is orthonormal[11] and complete:

$$\int_0^\infty dr\,\psi_E^\chi(r)\psi_{E'}^\chi(r) = \delta(E-E') \tag{5.27}$$

$$\int_0^\infty dE\,\psi_E^\chi(r)\psi_E^\chi(r') + \theta(c)\psi_{-E_0}^\chi(r)\psi_{-E_0}^\chi(r') = \delta(r-r'). \tag{5.28}$$

Following the same arguments as we presented in the previous subsection, we see that R is well defined, self-adjoint and respects the boundary condition implied by (5.25) for all c. The Hermitian operator D however, fails to respect the boundary condition implied by (5.25) except when $c=0$ or $c=\infty$. We now discuss the adjointness properties of the supercharges:

$$Q_{1(s)} = -i\sigma^1\frac{d}{dr} - \sigma^2\frac{\lambda}{r},$$

and

$$Q_{2(s)} = -i\sigma^2\frac{d}{dr} + \sigma^1\frac{\lambda}{r}. \tag{5.29}$$

Since both Q_1 and Q_2 exchange the chirality, it is clear that the asymptotic behavior for small r of the chirality for which $\nu_\chi < 1$, is completely determined by the behavior of the opposite chirality (for which $\nu_{-\chi} \geqq 1$). As the wave function for $\nu_{-\chi} \geqq 1$ vanishes faster than $r^{3/2}$ as $r \to 0$, the wave function of ψ_E^χ must vanish faster than $r^{1/2}$, so that $B_\chi = 0$ and we must choose $c=\infty$. Alternatively, if one starts by analyzing the supercharges Q_1 and Q_2, one finds that both are self-adjoint. Thus, of the one parameter family of extensions of H, supersymmetry is supported for only one value of the parameter, in agreement with the results from group theory.

For Case III, the Hamiltonian is not self-adjoint in either chirality sector and the extensions are labelled by four real parameters. The dependence of the wave functions on these parameters is quite involved, and we shall not exhibit them here. Instead, we examine the self-adjointness properties of the supercharge $Q_{1(s)}$ first. Its deficiency indices are $n_+ = n_- = 1$, and $Q_{1(s)}$ admits a one parameter family of self-adjoint extensions, labeled by the real parameter c. The normalized eigenfunctions of Q_1,

$$Q_{1(s)}\psi = \sqrt{E}\,\psi, \quad E \geqq 0, \tag{5.30}$$

are given by

$$\psi_+(r) = \left(\frac{r}{(A^2+B^2)}\right)^{1/2}\left(AJ_{\nu_+}(\sqrt{E}r) + BY_{\nu_+}(\sqrt{E}r)\right),$$

$$i\psi_-(r) = \left(\frac{r}{(A^2+B^2)}\right)^{1/2}\Big((A\cos\pi\nu_- + B\sin\pi\nu_-)J_{\nu_-}(\sqrt{E}r)$$
$$+(-A\sin\pi\nu_- + B\cos\pi\nu_-)Y_{\nu_-}(\sqrt{E}r)\Big), \tag{5.31}$$

11 It is convenient to prove that the left-hand side is the identity operator in a basis which one knows to be complete, like $J_{\nu_+}(\sqrt{E}r)$ with $E>0$

with

$$A(\sqrt{E}) = \frac{C}{E^{\lambda}} + \cos \pi\nu_+ , \quad B(\sqrt{E}) = \sin \pi\nu_+ . \tag{5.32}$$

There are no bound states, and the wave functions satisfy

$$\int_0^\infty dr(\psi_+^*(\sqrt{E}r)\psi_+(\sqrt{E'}r) + \psi_-^*(\sqrt{E}r)\psi_-(\sqrt{E'}r)) = \delta(E-E'). \tag{5.33}$$

The Hamiltonian is of course self-adjoint on this set of functions. We consider the supercharge $Q_{2(s)}$, and request that $Q_{2(s)}$ be Hermitian on the space of functions defined by (5.31). This can be realized only when the coefficient of either Y_{ν_+} or Y_{ν_-} vanishes in (5.31) so that[12] $C=0$ or $C=\infty$. For $\lambda \neq 0$, the remaining supercharges are defined and self-adjoint provided $C=\infty$, whereas for $\lambda = 0$, there are no further restrictions. The wave functions of (5.31) then (for $\lambda \neq 0$) reduce to

$$\psi_+(r) = \sqrt{r} J_{\nu_+}(\sqrt{E}r), \quad i\psi_-(r) = \sqrt{r} J_{-\nu_-}(\sqrt{E}r). \tag{5.34}$$

Keeping in mind that α is related to the chirality, we see that the result (3.31) of the algebraic method is recovered in this case (a factor of $1/r$ has always been absorbed in the wave functions in this section).

VI. Supersymmetry Breaking and the Witten Index

Supersymmetry can be broken down spontaneously even in quantum mechanics. This happens when the Heisenberg equations of motion are invariant under supersymmetry, but no state exists which is annihilated by the supercharges. For the problem at hand, this definition should be slightly generalized, since the supercharges satisfy the structure equation only for $r \neq 0$. Thus we shall require that the superalgebra be realized only on the states, i.e. in a weak sense.

In the case of the magnetic monopole alone, discussed in Sect. II, supersymmetry is not even realized in this weaker sense, since no self-adjoint supercharges can be defined. Thus, the full Hamiltonian and the full quantum mechanical system is not actually supersymmetric, and there is no point in discussing breakdown.

In the case of the quantum mechanical system described by the Hamiltonian H, supersymmetry is realized in the weak sense, and we shall now examine when it is spontaneously broken. We always assume that an extension has been chosen for which the full algebra is realized on the states. The states of interest are the ones of zero energy.

$$H\psi = 0. \tag{6.1}$$

From conformal invariance, it is already clear that no states exist with normalizable wave functions, and supersymmetry is always broken. This is the familiar situation of a system whose spectrum is continuous and extends down to zero, but zero energy is not actually attained by a state of the spectrum [3].[13]

12 The fact that the algebra spanned by H, Q_1 and Q_2 can be realized only for two values of the extension parameter was noted in [20]

13 Such systems have also recently been shown to exist in quantum field theory [21]

The Witten index [22]

$$\Delta = n^B_{E=0} - n^F_{E=0} \tag{6.2}$$

vanishes, as there exist no states at zero energy at all. Since the spectrum of the Hamiltonian is continuous and extends down to zero energy one may also wish to consider the index $\mathrm{Tr}(-1)^F$. It is easily realized that there are two choices for $(-1)^F$. Indeed, the only requirement on $(-1)^F$ is that it commutes (respectively anti-) with all bosonic (respectively fermionic) charges. Two such operators are known: sign A, which we have so far called fermion number and γ^5, which is the chirality. On the self-adjoint extensions on which the full algebra is defined, sign A and γ^5 anticommute with Q_α and S_α. In the case $\lambda = 0$, A vanishes on s-wave states, and sign A is defined through the self-adjoint extension or as a limit of $\lambda \to 0^\pm$, so both operators are always properly defined. We shall see that they are inequivalent. Before evaluating $\mathrm{Tr}(-1)^F$ in the energy representation it is instructive to evaluate $\mathrm{Tr}(-1)^F$ first in the representation where R is diagonal. [23].

$$\Delta^{\mathrm{sign}\,A}_R \equiv \mathrm{Tr}_R\, \mathrm{sign}\, A = \sum_{j,m,\alpha,[\omega],n} \langle j, m; \alpha, \omega\alpha, n | \mathrm{sign}\, A | j, m; \alpha, \omega\alpha, n \rangle. \tag{6.3}$$

The summation over ω is denoted by $[\omega]$ because it is always understood in the following sense: when $j \neq j_0$, $\omega = \pm 1$ is summed over, whereas when $j = j_0$, ω is set to the fixed value ω_0 determined in (3.16). Clearly we have

$$\Delta^{\mathrm{sign}\,A}_R = 0, \tag{6.4a}$$

and similarly

$$\Delta^{\gamma^5}_R = 0, \tag{6.4b}$$

in agreement with (6.2). The evaluation and interpretation of the index $\mathrm{Tr}(-1)^F$ in energy representation is complicated by the fact that the spectrum is continuous and extends down to zero. Denoting by $(-1)^F$ either of the operators sign A or γ^5, we have

$$\begin{aligned}\Delta^{(-1)^F}_{H,\beta} &\equiv \mathrm{Tr}_H (-1)^F e^{-\beta H} \\ &= \int_0^\infty dE\, e^{-\beta E} \sum_{j,m} \sum_{\alpha,[\omega]} \langle j, m; \alpha, \omega\alpha, E | (-1)^F | j, m; \alpha, \omega\alpha, E \rangle. \end{aligned} \tag{6.5}$$

Clearly $(-1)^F$ is equal to either α (for sign A) or $\alpha\omega$ (for γ^5), and the energy integral may be evaluated with the help of the overlap functions given in (3.30):

$$\begin{aligned}\mathrm{Del}_{j,\alpha,\omega} &= \int_0^\infty dE\, e^{-\beta E} \langle j, m; \alpha, \omega\alpha, E | j, m; \alpha, \omega\alpha, E \rangle \\ &= \sum_{n=0}^\infty \int_0^\infty dE\, e^{-\beta E} |\langle j, m; \alpha, \omega\alpha, E | j, m; \alpha, \omega\alpha, E \rangle|^2 \\ &= \sum_{n=0}^\infty \left(\frac{\beta - 2a^2}{\beta + 2a^2} \right)^n (1 + \beta/2a^2)^{-2\Delta_{j,\alpha}} P_n^{(2\Delta_{j,\alpha} - 1, 0)} \left(\frac{\beta^2 + 4a^4}{\beta^2 - 4a^4} \right), \end{aligned} \tag{6.6}$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials [18]. With the help of the generating function for P's and introducing a regulator ε, we get

$$\mathrm{Del}_{j,\alpha,\omega} = \frac{1}{\varepsilon}\frac{1}{1+\beta/2a^2} + \frac{1}{2}(1-2\Delta_{j,\alpha}) + O(\varepsilon). \tag{6.7}$$

Thus

$$\Delta_{H,\beta}^{\mathrm{sign}\,A} = \sum_{j=j_0}^{\infty} (2j+1)\sum_{[\omega]} \tfrac{1}{2}. \tag{6.8}$$

The sum over ω gets a contribution from all angular momenta and the total sum diverges: $\Delta_{H,\beta}^{\mathrm{sign}\,A} = \infty$. This is of course to be expected, since on the ground state (for fixed j and m) α is always equal to 1, and bosonic states are favored. On the other hand,

$$\Delta_{H,\beta}^{\gamma^5} = \sum_{j=j_0}^{\infty} (2j+1)\sum_{[\omega]} \tfrac{1}{2}\omega = (2j_0+1)\tfrac{1}{2}\omega_0, \tag{6.9}$$

and we find that this index is finite and non-zero:

$$\Delta_{H,\beta}^{\gamma^5} = \omega_0|eg|, \quad \omega_0 = \frac{-\lambda eg}{|\lambda eg|}. \tag{6.10}$$

This result was obtained in ref. [20] for the special case $\lambda = 0$ and when $|eg|$ is half integer $\Delta_{H,\beta}^{\gamma^5}$ is fractionalized. The β dependence has completely disappeared, but $\Delta_{H,\beta}^{\gamma^5}$ does depend on the sign of λ, and is discontinuous at $\lambda = 0$.

Of course, the OSp(2, 1) conformal supersymmetry encountered in this analysis is rather different from the usual supersymmetries; in particular all supercharges are rotation invariant. Thus one may be interested in answering different questions about the way in which supersymmetry is realized on the states. For example, what is the symmetry realized on the ground states for fixed j and m, i.e., which generators of OSp(2,1) annihilate the ground states $|j,m;1,\omega,0\rangle$? The state $|j,m;1,1,0\rangle$ is annihilated by the charges F_-^L, F_-^R, F_+^R, B_- and $R-Y$, and the broken generators are F_+^L, B_+ and $R+Y$. Similarly the ground states $|j,m;1,-1,0\rangle$ are annihilated by F_-^L, F_-^R, F_+^L, B_- and $R+Y$. These results are independent of λ.

Appendix A Supersymmetry of the $1/r^2$ potential in the higher dimensions

Setting the monopole field to zero in the Hamiltonian H and in the charges D, K, Y, Q_α and S_α, we see that little is special about three dimensions.

We can define a supersymmetric extension of the $1/r^2$ potential for any dimension d in the following fashion:

$$H = \frac{1}{2M}\left(p_a p^a + 2\Gamma^a\Gamma^0\frac{\lambda r_a}{r^3} + \frac{\lambda^2}{r^2}\right),$$

$$\{\Gamma^0,\Gamma^0\} = 2, \quad \{\Gamma^0,\Gamma^a\} = 0, \quad \{\Gamma^a,\Gamma^b\} = 2\delta^{ab}, \quad a,b = 1,2,\dots d. \tag{A.1}$$

Clearly, H is invariant under the rotation group SO(d), as well as under the previously discussed SO(2, 1) algebra, with generators

$$D = tH - \tfrac{1}{4}(p_a r^a + r_a p^a), \tag{A.2}$$

$$K = -t^2 H + 2tD + \frac{M}{2} r^2, \tag{A.3}$$

It is also clear that there are two supercharges

$$Q_2 = \frac{1}{\sqrt{2M}}\left(-i\Gamma^a p_a + \Gamma^0 \frac{\lambda}{r}\right), \tag{A.4}$$

$$S_2 = -tQ_2 - i\sqrt{\frac{M}{2}}\Gamma^a r_a. \tag{A.5}$$

Together with H, D and K, these form an OSp(1, 1) superalgebra. To construct further supersymmetry charges, it suffices to find charges which anti-commute with Q_2 and S_2. When d is odd, there exists a matrix,

$$\Gamma^{d+1} = \Gamma^0 \Gamma^1 \ldots \Gamma^d, \tag{A.6}$$

which has this property, and consequently we also have the supercharges

$$Q_1 = i\Gamma^{d+1} Q_2,$$
$$S_1 = i\Gamma^{d+1} S_2. \tag{A.7}$$

The charges H, D, K, $Y = \frac{1}{2}\{Q_1, S_2\}$, Q_α and S_α then satisfy the OSp(2, 1) superalgebra of (3.3). When d is even, no such matrix exists in the irreducible representation, and the supersymmetry is really OSp(1, 1). The Casimir eigenvalue of the OSp(1, 1) group (which is also a Casimir of OSp(2, 1) in the case of odd dimensions) is completely determined by the quadratic Casimir eigenvalue of SO(d) ($L_{ab}L^{ab}$)

$$C_1 = \tfrac{1}{2}(HK + KH) - D^2 + \frac{i}{4}[Q_2, S_2] = \tfrac{1}{4}(\tfrac{1}{2}L_{ab}L^{ab} - \tfrac{1}{4}(d^2 - 3d) + \lambda^2), \tag{A.8}$$

and the representations of OSp(1, 1) are labelled by the eigenvalues of R and $A = i[Q_2, S_2] - \frac{1}{2}$ as before.

It is remarkable that the supersymmetric generalization of the $1/r^2$ potential we have discovered here is distinct from the one discussed by S. Fubini and E. Rabinovici [12].

Appendix B. Definitions and results on self-adjoint extensions

We collect a few definitions and results in the theory of self-adjoint extensions [17]. Let $\mathscr{H}$ be a Hilbert space and T a Hermitian operator with adjoint $T^\dagger$.

The operator T is said to be self-adjoint if $D(T) = D(T^\dagger)$. The space $K(z)$ defined by

$$K(z) = \mathrm{Ker}\,(T^\dagger - z), \tag{B.1}$$

has constant dimension throughout the upper (respectively lower) half planes, and we assume these dimensions to be finite. The deficiency subspaces are then isomorphic to

$$K_{\pm} = \mathrm{Ker}\,(T^{\dagger} \mp i), \tag{B.2}$$

and the dimensions are the deficiency indices of T:

$$n_{\pm} = \dim K_{\pm}. \tag{B.3}$$

From the definition of $T^{\dagger}$, we have the canonical decomposition

$$D(T^{\dagger}) = D(T) \oplus K_{+} \oplus K_{-}. \tag{B.4}$$

Clearly when $n_{+} = n_{-} = 0$, T is self-adjoint. If $n_{+} = n_{-} = n \neq 0$, there exist unitary transformations parametrized by $U(n)$ which map K_{+} into K_{-}, extending T to a self-adjoint operator. For this extended version of T, the new deficiency indices vanish by Von Neumann's theorem [17]. If $n_{+} = 0, n_{-} \neq 0$ (or $n_{+} \neq 0, n_{-} = 0$), T has no self-adjoint extensions and T is said to be maximally Hermitian.

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Note added in proof. Since this paper was submitted we have supplemented the results of section II by providing a superspace formulation of the supersymmetries of the magnetic monopole. See

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