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The Low Density Limit for an N-Level System Interacting with a Free Bose or Fermi Gas

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Abstract. It is proved that the reduced dynamics of an N-level system coupled to a free quantum gas converges to a quantum dynamical semigroup in the low density limit. The proof uses a perturbation series of the quantum BBGKY-hierarchy, and the analysis of this series is based on scattering theory. The limiting semigroup contains the full scattering cross section, but it does not depend on the statistics of the reservoir. The dynamics of the semigroup is discussed.

1. Introduction

In recent years there has been considerable progress in the rigorous derivation of the Boltzmann equation from the microscopic dynamics of a classical many particle system with short range forces. Using ideas of Grad [1], Lanford [2, 3] proved the convergence of the hierarchy of correlation functions for a hard sphere gas in the Boltzmann–Grad limit for sufficiently short times. This proof was extended by King to positive potentials of finite range [4]. The limiting dynamics preserves factorisation of the correlation functions, and the evolution of the one particle distribution is governed by the non-linear Boltzmann equation.

The test particle problem was studied by Spohn [5] and Lebowitz, Spohn [6]. One considers the motion of a single particle through an environment of randomly placed, infinitely heavy scatterers (Lorentz gas). In the Boltzmann–Grad limit successive collisions become independent and the position and velocity distribution of the particle, when averaged over the positions of the scatterers, converges to the solution of the linear Boltzmann equation. In fact, also multi-time correlations converge, and the convergence holds even for a typical fixed environment [7].

The rigorous derivation of a quantum Boltzmann equation is an open problem. Physically one expects (e.g. [8–16]) that the evolution of a quantum gas at low density should be described by the Boltzmann equation with the classical differential cross section replaced by the quantum mechanical cross section. The Boltzmann distribution function should then be understood as the low density limit of the one-particle Wigner function. On a non-rigorous level, the author regards the derivation by Wittwer [17] as the most convincing one.

In this paper the certainly simpler quantum mechanical test particle problem at low density is studied. The system is taken to be an N-level system and the reservoir to be an ideal Bose or Fermi gas. The total Hamiltonian is then formally given by

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_I, \tag{1.1}$$

where $H_S = \sum_{n=1}^{N} \omega_n |n\rangle\langle n|$ is the Hamiltonian of the system and $H_B = \int d^3k\, k^2/2\, a^+(\underline{k})a(\underline{k})$ is the free Hamiltonian of the bath. The interaction is of the form $H_I = Q \otimes \sum_j \alpha_j\, a^+(f_j)a(f_j)$, where Q is a self adjoint operator of the system and $\sum_j \alpha_j\, a^+(f_j)a(f_j)$ describes the scattering of the bath particles. This form of the interaction implies that no particles are created or annihilated. It will be proved that, in the limit, when the particle density of the reservoir converges to zero and time is appropriately speeded up, such that the collision rate stays constant, the reduced dynamics converges to a quantum dynamical semigroup.

Since the interaction between gas particles and the N-level system is strong, the dynamical semigroup involves the full differential cross section. Let $T_{nn'}(\underline{k}, \underline{k}')$ denote the matrix element $\langle n\underline{k}|T|n'\underline{k}'\rangle$ of the T matrix for the scattering process of one reservoir particle with the system. For $\omega \in \mathrm{Sp}(L_S)$, where $L_S \cdot = [H_S, \cdot]$ denotes the Liouvillian of the system, one defines $T_\omega(\underline{k}, \underline{k}') = \sum_{\omega_m - \omega_n = \omega} T_{mn}(\underline{k}, \underline{k}')|m\rangle\langle n|$. At low density the rescaled particle density of reservoir particles with momentum $\underline{k}$ is $R^0(k) = n(2\pi/\beta)^{3/2} e^{-\beta k^2/2}$, where n is the rescaled density and β the inverse temperature. The dissipative part of the generator of the limiting semigroup may be written in the form

$$\begin{aligned} K_D^{\#}\rho = 2\pi \sum_{\omega \in \mathrm{Sp}(L_S)} \int d\underline{k} \int d\underline{k}'\, \delta(k'^2/2 - k^2/2 + \omega) R^0(\underline{k}) \\ \cdot \{ T_\omega(\underline{k}', k)\rho T_\omega^*(\underline{k}', \underline{k}) - \tfrac{1}{2}(T_\omega^*(\underline{k}', \underline{k}) T_\omega(k', \underline{k})\rho \\ + \rho T_\omega^*(\underline{k}', \underline{k}) T_\omega(k', k)) \}. \end{aligned} \tag{1.2}$$

The equation of motion formed with the generator (1.2) is a fully quantum mechanical generalisation of the classical linear Boltzmann equation. Note, as expected, any information on the statistics of the gas is lost, and $R^0(\underline{k})$ is simply the Maxwell distribution.

A preliminary analysis of the problem considered was given by Palmer [18]. However, Palmer truncated the interaction and, in the limit, obtained a semigroup which contains only the second Born approximation instead of the full scattering cross section.

The methods used in this paper are completely different from those used in the analysis of the weak and singular coupling limits based on the Dyson series [19–22]. Here the starting point is the quantum BBGKY-hierarchy and its associated perturbation series. The analysis of this series is based on multiparticle scattering theory.

The limiting semigroup differs in two respects from the semigroup obtained in the weak coupling limit. Clearly, in the weak coupling limit the scattering cross section appears only in its Born approximation, reflecting the fact that the interaction becomes weak in the limit. In the low density limit all information on the reservoir statistics is lost, whereas in the weak coupling limit some information on the reservoir statistics is retained in the two point correlation function of the reservoir.

A quantum Boltzmann equation should also be obtained in the weak coupling limit of an interacting quantum gas. Here one expects a similar structure. The equation will contain the differential cross section in the Born approximation and a quartic collision term giving information on the statistics. Hugenholtz considered recently the weak coupling limit for a Fermi gas on the lattice [23]. He succeeded in identifying the terms of the Dyson series contributing at weak coupling, however, without proving the convergence of the series. In the limit the set of quasi-free translation invariant states is preserved by the dynamics, and the time evolution of the two point function is governed by the quantum Boltzmann equation.

For a classical system one obtains the Landau equation in the weak coupling limit, which describes a diffusion process. In contrast, the Boltzmann equation describes a jump process in momentum space. This reflects the wave nature of quantum scattering. For a weak potential in most scattering events the particle is not deflected at all, but when it is eventually scattered, there is a finite probability for a large angle deflection.

The paper is organized as follows. In Sect. 2 the dynamics of the system is defined in the algebraic framework of quantum statistical mechanics, and the main theorem of this paper is stated. In Sect. 3 the perturbation series for the BBGKY-hierarchy is introduced. First the equivalence of the unitary dynamics in Fock space with the dynamics given by the BBGKY-hierarchy equations for a reservoir in a finite volume is proved. Then the infinite volume limit is performed, thereby proving the equivalence of the dynamics defined in Sect. 2 with the time evolution given by the perturbation series of the BBGKY-hierarchy. Section 4 introduces the scaling for the low density limit and outlines the strategy of the proof. The proof is presented in Sect. 5. In Sect. 5.1 some auxiliary results are proved. In Sects. 5.2 and 5.3 the theorem is proved in two steps. In the first step an intermediate approximation is established, where the reservoir statistics is still retained. In the second step it is proved that the contributions from the statistics vanish in the limit. The semigroup generator obtained does not preserve positivity, in general. In Sect. 6 an averaged generator is defined, which has the required positivity properties. The relaxation properties of this semigroup are studied. In Sect. 7 some modifications and generalisations of the result are discussed.

2. The Model and Results

In this section the model with Hamiltonian (1.1) is described in detail and the infinite volume dynamics is constructed. Then the main theorem of this paper is stated. The analysis is carried through for a Fermi gas. The modifications to be made for a Bose gas are discussed in Sect. 7.

For a Hilbert space $\mathscr{H}$ let $\mathscr{B}(\mathscr{H})$ denote the Banach space of bounded operators on $\mathscr{H}$ with norm $\|\cdot\|$ and $\mathscr{T}(\mathscr{H})$ the Banach space of trace class operators with norm $\|\cdot\|_1$.

The Hilbert space $\mathscr{H}_S$ of the system is assumed to be finite dimensional. The single particle space of the reservoir is $\mathscr{H}_e = L^2(\mathbb{R}^3)$. The system Hamiltonian $H_S \in \mathscr{B}(\mathscr{H}_S)$ is self adjoint. The bath Hamiltonian is formally given by $H_B = \int d^3\underline{k}\, k^2/2\; a^+(\underline{k})a(\underline{k})$, where $a^+(\underline{k})$ and $a(\underline{k})$ denote Fermion creation and annihilation operators of momentum $\underline{k}$. The interaction Hamiltonian is given by $H_I = Q \otimes F$, where $Q \in \mathscr{B}(\mathscr{H}_S)$ is self adjoint and $F = \sum_i \alpha_i a^+(f_i)a(f_i)$. $\{f_i\}$ is an orthonormal system on $\mathscr{H}_e$ and $\alpha_i \in \mathbb{R}$, $\sum_i |\alpha_i| < \infty$. H_I preserves the particle number of the bath, and therefore the bath particles are only scattered and not created or destroyed. The interaction is a generalisation of the interaction used by Davies [19] in the analysis of the weak coupling limit and of the interaction used by Palmer [18] in the treatment of the low density limit.

The Hamiltonian for the coupled motion of the system and one bath particle is

$$H_1 = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_e + Q \otimes A \tag{2.1}$$

on the Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_e \cdot H_e = -1/2\Delta$ is the Hamiltonian for the free evolution of one bath particle and $A = \sum_i \alpha_i |f_i\rangle\langle f_i|$. A is in the trace class of $\mathscr{H}_e$ and from $\dim \mathscr{H}_S < \infty$ follows $Q \otimes A \in \mathscr{T}(\mathscr{H}_S \otimes \mathscr{H}_e)$. Let $H_0 = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_e$ denote the free Hamiltonian. From Kato–Birman theory [24] follows the completeness of the scattering system (H_0, H_1). For the following one needs the conditions:

(E) The spectral subspace belonging to the point spectrum of H_1 is finite dimensional.

(F) For a dense set $\mathscr{D}$ of vectors $\phi \in \mathscr{D}$, $\int_{-\infty}^{\infty} \|Ae^{-iH_e t}\phi\|\, dt < \infty$.

These conditions will be discussed in the appendix.

The initial state of the composite system is $\omega = \omega_\rho \otimes \omega_{\eta,\beta}$. $\omega_\rho(\cdot) = \operatorname{tr} \rho \cdot$ is a state of the system determined by an arbitrary state operator $\rho \in \mathscr{T}(\mathscr{H}_S)$, $\rho \geqq 0$, $\operatorname{tr} \rho = 1$. $\omega_{n,\beta}$ is the thermal equilibrium state of the bath with particle density n and inverse temperature β with respect to the free dynamics.

The dynamics of the infinite system is defined in the algebraic framework of quantum statistical mechanics by an automorphism group on the algebra of quasi-local observables. Let $\mathscr{A}_S = \mathscr{B}(\mathscr{H}_S)$ be the algebra of bounded observables of the system and $\mathscr{A}_B = \mathrm{CAR}(\mathscr{H}_e)$ the algebra of the canonical anticommutation relations over $\mathscr{H}_e$. The algebra of the composite system is $\mathscr{A} = \mathscr{A}_S \otimes \mathscr{A}_B$. The automorphism group $\alpha_0(t)$ of the free evolution is defined by the relation

$$\alpha_0(t) X \otimes a^+(f) = e^{iH_S t} X e^{-H_S t} \otimes a^+(e^{iH_e t} f). \tag{2.2}$$

$L_I \cdot = [H_I, \cdot]$ is a bounded derivation on $\mathscr{A}$. Therefore the interacting dynamics $\alpha(t)$ corresponding to the formal Hamiltonian (1.1) may be obtained by the Dyson series

$$\alpha(t)Y = \sum_{n=0}^{\infty} i^n \int_{0 \leqq t_1 \leqq \dots \leqq t_n \leqq t} dt_1 \dots dt_n \alpha_0(t - t_n) L_I \dots L_I \alpha_0(t_1) Y \tag{2.3}$$

for all $Y \in \mathscr{A}$. The series converges in norm for all $t \geqq 0$. For $t < 0$ one puts $\alpha(t) = \alpha^{-1}(-t)$.

Expectation values of system observables are completely determined by the reduced dynamics $t \mapsto T(t)$, which is implicitly defined by the relation $\operatorname{tr} X\, T(t)\rho = \omega(\alpha(t)X \otimes \mathbb{1})$ for all $X \in \mathscr{B}(\mathscr{H}_S)$.

The scaling for the low density limit is as follows. The particle density of the reservoir is scaled as $n_\varepsilon = \varepsilon n$. For small ε, collisions of bath particles with the system become very infrequent. To obtain a non-trivial limit, time is rescaled such that the collision rate is kept constant. As the mean free path is $(\sigma n_\varepsilon)^{-1}$, where σ denotes the total scattering cross section, one should scale time as $t_\varepsilon = \varepsilon^{-1} t$. The scaled reduced dynamics is determined by the relation

$$\operatorname{tr} X\, T_\varepsilon(t)\rho = \omega_\rho \otimes \omega_{n_\varepsilon,\beta}(\alpha(\varepsilon^{-1}t)X \otimes \mathbb{1}) \tag{2.4}$$

for all $X \in \mathscr{B}(\mathscr{H}_S)$.

In addition to the dissipative part (1.2) the generator of the asymptotic semigroup has also a Hamiltonian part, and the total generator reads

$$K^{\#}\rho = -i\left[\sum_{\substack{n,n' \\ \omega_n = \omega_{n'}}} \int d\underline{k}\, R^0(\underline{k})(T_{nn'}(\underline{k},\underline{k}) + \overline{T_{nn'}(\underline{k},\underline{k})})|n\rangle\langle n'|, \rho\right] + K^{\#}_D \rho. \tag{2.5}$$

The asymptotic semigroup is given by $T^{\#}_\varepsilon(t) = \exp(-i\varepsilon^{-1}L_S + K^{\#})$, where $L_S\cdot = [H_S, \cdot]$ denotes the system Liouvillian.

The main result of this paper is

Theorem 2.1. *Assume that* (E) *and* (F) *hold. Then there is a finite time* $T \geqq 0$, *such that for* $t \in [0, T)$

$$\lim_{\varepsilon \downarrow 0} \| T_\varepsilon(t)\rho - T^{\#}_\varepsilon(t)\rho \|_1 = 0 \tag{2.6}$$

holds for all $\rho \in \mathscr{T}(\mathscr{H}_S)$.

To prove the theorem, the series representation (2.3) of the dynamics is not appropriate. Instead of (2.3) a perturbation series of the BBGKY-hierarchy is used. The limitation $t \in [0, T)$ comes from the limited radius of convergence of this perturbation series, and is probably an artefact of the method used. The relevance of the restrictions of the particular model are discussed in Sect. 7, where also some possible generalisations are indicated.

3. The Quantum BBGKY-Hierarchy

3.1. BBGKY-Hierarchy on Fock Space

The series (2.3) is not suitable to perform the low density limit. Therefore one introduces reduced density matrices, for which the time evolution is expressed by a perturbation series for the BBGKY-hierarchy. It is convenient to perform the necessary manipulations for states on Fock space. In the next subsection the thermodynamic limit is taken. In this section quantities are not scaled. The scaled objects are introduced in Sect. 4.

The joint Hilbert space of the system and n bath particles is denoted by $\mathscr{H}_n = \mathscr{H}_S \otimes \bigotimes_1^n \mathscr{H}_e$ and the subspace of $\mathscr{H}_n$, which is totally antisymmetric in the bath components, by $(\mathscr{H}_n)_- = \mathscr{H}_S \otimes \left(\bigotimes_1^n \mathscr{H}_e \right)_-$. For $\pi \in S(n)$, where $S(n)$ is the group of permutations of n elements, one defines the unitary operator $U_\pi: \mathscr{H}_n \to \mathscr{H}_n$ by $U_\pi f_0 \otimes f_1 \otimes \cdots \otimes f_n = f_0 \otimes f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$. For $\phi \in (\mathscr{H}_n)_-$ holds $U_\pi \phi = \operatorname{sgn} \pi \phi$. The Fock space of the total system is denoted by $\mathscr{F} = \bigoplus_{n=0}^{\infty} (\mathscr{H}_n)_-$.

Let $\rho \in \mathscr{T}(\mathscr{F})$ be a state operator commuting with the number operator N. Then ρ may be represented in the form

$$\rho = \bigoplus_{n=0}^{\infty} \rho_n \tag{3.1}$$

with $\rho_n \in \mathscr{T}((\mathscr{H}_n)_-)$, $\sum_{n=0}^{\infty} \|\rho_n\|_1 = 1$. The reduced density matrices R_n, $n = 0, 1, 2, \ldots$, are implicitly defined by the relation

$$\begin{aligned} &\operatorname{tr}_{\mathscr{F}} \rho |f_0\rangle\langle g_0| \otimes a^+(f_n) \ldots a^+(f_1) a(g_1) \ldots a(g_n) \\ &= (g_0 \otimes g_1 \otimes \cdots \otimes g_n, R_n f_0 \otimes f_1 \otimes \cdots \otimes f_n). \end{aligned} \tag{3.2}$$

Using the representation (3.1) one obtains the well known expression [25]

$$R_n = \sum_{m=0}^{\infty} (n+m)!/m! \operatorname{tr}_{[n+1,n+m]} \rho_{n+m}. \tag{3.3}$$

In this formula ρ_{n+m} is regarded as an operator on $\mathscr{H}_{n+m}$ with $\rho_{n+m}\psi = 0$ for $\psi \in (\mathscr{H}_{n+m})_-^\perp$. Then $\operatorname{tr}_{[n+1,n+m]}$ denotes the partial trace over the components $n+1, \ldots, n+m$ of $\mathscr{H}_{n+m}$, and no problems arise because of the antisymmetrisation.

The sum in (3.3) converges, if the following condition holds:
(A) There is $0 < q < 1$ and $C > 0$ such that $\|\rho_n\|_1 < Cq^n$ for all $n \in \mathbb{N}$.

Lemma 3.1. *If* (A) *holds* R_n, $n \in \mathbb{N}$, *is a bounded operator on* H_n *and the estimate*

$$\|R_n\| \leqq \frac{C}{1-q} n! \left(\frac{q}{1-q} \right)^n \tag{3.4}$$

holds for all $n \in \mathbb{N}$.

Proof. Using $\|\operatorname{tr}_{[n+1,n+m]} \rho_{n+m}\| \leqq \|\operatorname{tr}_{[n+1,n+m]} \rho_{n+m}\|_1 \leqq \|\rho_{n+m}\|_1$, one obtains from (3.3) the bound

$$\|R_n\| \leqq \sum_{m=0}^{\infty} \frac{(n+m)!}{m!} C q^{n+m}. \tag{3.5}$$

For $0 < q < 1$ the right-hand side is convergent. This proves the boundedness of R_n. Evaluating the right-hand side of (3.5) one obtains the bound (3.4). ☒

Let $H_{Bj} = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes H_e \otimes \cdots \otimes \mathbb{1}$ denote the free Hamiltonian of the j^{th} bath particle. The Hamiltonian of the interaction of the j^{th} bath particle with the system is $H_{Ij} = Q \otimes \mathbb{1} \otimes \cdots \otimes A \otimes \cdots \otimes \mathbb{1}$. The total Hamiltonian on $(\mathscr{H}_n)_-$ is

$$H_n = H_S + \sum_{j=1}^{n} (H_{Bj} + H_{Ij}), \tag{3.6}$$

where, for simplicity of notation, the continuation of H_S on $\mathscr{H}_n$ is also denoted by H_S. The time evolution of ρ_m is given by

$$\rho_m(t) = e^{-iH_m t}\rho_m e^{iH_m t}. \tag{3.7}$$

If (A) holds, the reduced density matrix at time t is the bounded operator

$$R_n(t) = \sum_{m=0}^{\infty} \frac{(n+m)!}{m!} \operatorname{tr}_{[n+1,n+m]}\rho_{n+m}(t). \tag{3.8}$$

It will be proved that the reduced density matrices satisfy the integral equations

$$R_n(t) = \underline{U}_n(t)R_n(0) + \int_0^t ds\, \underline{U}_n(t-s)\underline{C}_{nn+1}R_{n+1}(s),$$

where

$$n = 0, 1, 2, \ldots, \tag{3.9}$$

$$\underline{U}_n(t)R_n = e^{-iH_n t}R_n e^{iH_n t}$$

and

$$\underline{C}_{nn+1}R_{n+1} = -i\operatorname{tr}_{n+1}[H_{In+1}, R_{n+1}].$$

tr_{n+1} denotes the partial trace over the $n+1^{\text{th}}$ component of $\mathscr{H}_{n+1}$. The differential form of (3.9) is

$$i\frac{d}{dt}R_n(t) = [H_n, R_n(t)] + \operatorname{tr}_{n+1}[H_{In+1}, R_{n+1}]. \tag{3.10}$$

This is the quantum analogue of the classical BBGKY-hierarchy equations. To avoid domain problems with the unbounded operators H_n, $n \in \mathbb{N}$, it is convenient to work with the integral equation (3.9) instead of (3.10).

Lemma 3.2. *$\underline{C}_{nn+1}$ is a bounded operator from $\mathscr{B}(\mathscr{H}_{n+1})$ in $\mathscr{B}(\mathscr{H}_n)$ with the bound $\|\underline{C}_{nn+1}\| \leqq 2\|Q\|\,\|A\|_1$.*

Proof. One defines $\underline{C}_{nn+1}R_{n+1}$ by the relation

$$\operatorname{tr}_{[0,n]}X_n\underline{C}_{nn+1}R_{n+1} = -i\operatorname{tr}_{[0,n+1]}[X_n \otimes \mathbb{1}, H_{In+1}]R_{n+1}$$

for all $X_n \in \mathscr{T}(\mathscr{H}_n)$. From the estimate

$$\begin{aligned}|\operatorname{tr}_{[0,n]}X_n\underline{C}_{nn+1}R_{n+1}| &\leqq \|[X_n \otimes \mathbb{1}, Q \otimes \mathbb{1}^{(n)} \otimes A]\|_1\,\|R_{n+1}\| \\ &\leqq 2\|X_n\|_1\|Q\|\,\|A\|_1\|R_{n+1}\|\end{aligned}$$

follows the boundedness of $\underline{C}_{nn+1}$ and the bound stated. ⊠

Theorem 3.3. *On condition* (A) *the integral equation* (3.9) *holds.*

Proof. One writes $H_{n+m} = H^0_{n+m} + H^1_{n+m}$, where $H^0_{n+m} = H_n + \sum_{j=n+1}^{n+m} H_{Bj}$, and $H^1_{n+m} = \sum_{j=n+1}^{n+m} H_{Ij}$. Considering H^1_{n+m} as a perturbation in the Liouville equation, $i(d/dt)\rho_{n+m}(t) = [H_{n+m}, \rho_{n+m}(t)]$, one obtains the integral equation,

$$\rho_{n+m}(t) = \underline{U}^0_{n+m}(t)\rho_{n+m}(0) - i\int_0^t ds\,\underline{U}^0_{n+m}(t-s)[H^1_{n+m}, \rho_{n+m}(s)], \tag{3.11}$$

where $\underline{U}^0_{n+m}(t)\cdot = e^{-iH^0_{n+m}t}\cdot e^{iH^0_{n+m}t}$. The partial trace of (3.11) gives

$$\begin{aligned}\mathrm{tr}_{[n+1,n+m]}\rho_{n+m}(t) &= \underline{U}_n(t)\mathrm{tr}_{[n+1,n+m]}\rho_{n+m}(0)\\ &\quad - i\int_0^t ds\,\mathrm{tr}_{[n+1,n+m]}\underline{U}^0_{n+m}(t-s)[H^1_{n+m}, \rho_{n+m}(s)].\end{aligned} \tag{3.12}$$

Using the fact that H_n commutes with $\sum_{j=n+1}^{n+m} H_{Bj}$, and that the latter operator acts only on the components $n+1,\dots,n+m$, one obtains the identity

$$\mathrm{tr}_{[n+1,n+m]}\,\underline{U}^0_{n+m}(t)\cdot = \underline{U}_n(t)\mathrm{tr}_{[n+1,n+m]}\cdot\,.$$

Exploiting the symmetry of $\rho_{n+m}(s)$, one gets from (3.12)

$$\begin{aligned}\mathrm{tr}_{[n+1,n+m]}\rho_{n+m}(t) &= \underline{U}_n(t)\,\mathrm{tr}_{[n+1,n+m]}\rho_{n+m}(0)\\ &\quad + m\int_0^t ds\,\underline{U}_n(t-s)\underline{C}_{nn+1}\,\mathrm{tr}_{[n+2,n+m]}\rho_{n+m}(s).\end{aligned} \tag{3.13}$$

Equation (3.9) follows by inserting (3.13) in (3.8). ☒

The integral equation (3.9) may be iterated and leads to a perturbation series for $R_n(t)$. The following theorem shows that $R_n(t)$ is indeed represented by the perturbation series for short times.

To simplify the notation of the integrals, one uses the abbreviations $\Delta(t,n,t') := \{(t_1,\dots,t_n)\in\mathbb{R}^n | t \leqq t_n \leqq \cdots \leqq t_1 \leqq t'\}$ and $\underline{dt} = dt_1 \dots dt_n$.

Theorem 3.4. *Assume that* (A) *holds. For* $t\in[0,T)$ *with* $T = (1-q)/(2q\|Q\|\,\|A\|_1)$, *the series*

$$\sum_{m=0}^{\infty}\int_{\Delta(0,m,t)} \underline{dt}\,\underline{U}_n(t-t_1)\underline{C}_{nn+1}\cdots\underline{C}_{n+m-1\,n+m}\underline{U}_{n+m}(t_m)R_{n+m}(0) \tag{3.14}$$

converges in norm to $R_n(t)$.

Proof. The m^{th} term of the series has the bound

$$\int_{\Delta(0,m,t)} \underline{dt}\dots \leqq \frac{t^m}{m!}(2\|Q\|\,\|A\|_1)^m \frac{C}{1-q}(n+m)!\left(\frac{q}{1-q}\right)^{n+m} =: B_m.$$

The series $\sum_m B_m$ converges if $2\|Q\|\,\|A\|_1 t(q/1-q) \leqq 1$, which proves the convergence of (3.14).

The k-fold iteration of (3.9) leads to

$$\begin{aligned}R_n(t) &= \sum_{m=0}^{k}\int_{\Delta(0,m,t)} \underline{dt}\,\underline{U}_n(t-t_1)\underline{C}_{nn+1}\cdots\underline{C}_{n+m-1\,n+m}\underline{U}_{n+m}(t_m)R_{n+m}(0)\\ &\quad + \int_{\Delta(0,k+1,t)} \underline{dt}\,\underline{U}_n(t-t_1)\underline{C}_{nn+1}\cdots\underline{C}_{n+k\,n+k+1}R_{n+k+1}(t_{k+1}).\end{aligned}$$

To conclude the proof of the theorem one has to show

$$\lim_{k\to\infty}\left\| \int_{\Delta(0,k+1,t)} dt\, \underline{U}_n(t-t_1)\underline{C}_{n\,n+1}\cdots \underline{C}_{n+k\,n+k+1}R_{n+k+1}(t_{k+1})\right\| = 0.$$

From the bound (3.4) for $R_{n+k+1}(t)$, one obtains

$$\left\| \int_{\Delta(0,k+1,t)} dt\, \underline{U}_n(t-t_1)\underline{C}_{n\,n+1}\cdots \underline{C}_{n+k\,n+k+1}R_{n+k+1}(t_{k+1})\right\|$$
$$\leqq \frac{t^{k+1}}{(k+1)!}(2\|Q\|\,\|A\|_1)^{k+1}\frac{C}{1-q}(n+k+1)!\left(\frac{q}{1-q}\right)^{n+k+1}.$$

For $k\to\infty$ the right-hand side converges to zero. ☒

3.2. Thermodynamic Limit of the BBGKY-*Hierarchy*

Let Λ be a bounded region in R^3 and $\omega^\Lambda = \omega_\rho \otimes \omega^\Lambda_{n,\beta}$, where $\omega^\Lambda_{n,\beta}$ is the equilibrium state for a Fermi system with density n and temperature β^{-1} in the volume Λ with respect to the free dynamics with Dirichlet boundary conditions. For $\Lambda \uparrow \mathbb{R}^3$ the Fock states ω^Λ converge to the initial state $\omega = \omega_\rho \otimes \omega_{n,\beta}$.

In Lemma 3.6 it is shown that ω^Λ satisfies the following improvement of condition (A):

(A_0) For all $0<q<1$ there is a $C>0$ such that $\|\rho^\Lambda_n\|_1 \leqq Cq^n$ for all $n\in\mathbb{N}$.

The following two conditions are imposed on the reduced density matrices R_n of $\omega^\Lambda_{n,\beta}$:

(B) There are constants $a>0$ and $C_1>0$ such that for all $n\in\mathbb{N}$ and all bounded $\Lambda\subset\mathbb{R}^3$ $\|R_n\|\leqq C_1 n!\, a^n$ holds.

(C) For all $A\in\mathscr{T}(\mathscr{H}_n)$ $\lim_{\Lambda\uparrow\mathbb{R}^3} \mathrm{tr}_{\mathscr{H}_n} AR^\Lambda_n = \mathrm{tr}_{\mathscr{H}_n} AR_n$.

Theorem 3.5. *Assume that* (A_0), (B) *and* (C) *hold. Then for* $t\in[0,T)$ *with* $T=(2\|Q\|\,\|A\|_1 a)^{-1}$,

$$\omega(\alpha(t)\{|f_0\rangle\langle g_0|\otimes a^+(f_n)\dots a^+(f_1)a(g_1)\dots a(g_n)\})$$
$$=(g_0\otimes g_1\otimes\cdots\otimes g_n, R_n(t)f_0\otimes f_1\otimes\cdots\otimes f_n), \tag{3.15}$$

where

$$R_n(t)=\sum_{m=0}^{\infty}\int_{\Delta(0,m,t)} dt\,\underline{U}_n(t-t_1)\underline{C}_{n\,n+1}\cdots\underline{C}_{n+m-1\,n+m}\underline{U}_{n+m}(t_m)R_{n+m}. \tag{3.16}$$

Proof. From Theorem 3.4 one knows that

$$\omega^\Lambda(\alpha(t)\{|f_0\rangle\langle g_0|\otimes a^+(f_n)\dots a^+(f_1)a(g_1)\dots a(g_n)\})$$
$$=(g_0\otimes g_1\otimes\cdots\otimes g_n, R^\Lambda_n(t)f_0\otimes f_1\otimes\cdots\otimes f_n), \tag{3.17}$$

where

$$R^\Lambda_n(t)=\sum_{m=0}^{\infty}\int_{\Delta(0,m,t)} dt\,\underline{U}_n(t-t_1)\underline{C}_{n\,n+1}\cdots\underline{C}_{n+m-1\,n+m}\underline{U}_{n+m}(t_m)R^\Lambda_{n+m}. \tag{3.18}$$

(A_0) implies the convergence of (3.18) for all $t>0$. ω^Λ converges to ω in the weak-* topology. Therefore the left-hand side of (3.17) converges to the left-hand side of

(3.15) for $\Lambda\uparrow\mathbb{R}^3$. To prove the theorem it is sufficient to show the convergence of $R_n^\Lambda(t)$ to $R_n(t)$ in the weak-* topology of $\mathscr{B}(\mathscr{H}_n)$. The convergence of the series (3.18) to the series (3.16) is proved, if one finds a majorant for (3.18) uniformly in Λ, and if each term of the series converges.

From condition (B) one obtains the majorant $\sum_{m=0}^{\infty}(t^m/m!)(2\|Q\|\,\|A\|_1)^m C_1(n+m)!\ a^{n+m}$, which converges for $2\|Q\|\,\|A\|_1 at<1$. Weak-* convergence is denoted by $\overset{*}{\to}$. $R_n^\Lambda \overset{*}{\to} R_n$ implies $\underline{U}_n(t)R_n^\Lambda \overset{*}{\to} \underline{U}_n(t)R_n$. From the identity $(\psi, C_{nn+1}(R_{n+1}^\Lambda - R_{n+1})\phi) = -i\,\mathrm{tr}_{[0,n+1]}[|\phi\rangle\langle\psi|\otimes\mathbb{1},\ H_{In+1}](R_{n+1}^\Lambda - R_{n+1})$ and the fact that $[|\phi\rangle\langle\psi|\otimes\mathbb{1}, H_{In+1}]\in\mathscr{T}(\mathscr{H}_{n+1})$, one concludes that $R_{n+1}^\Lambda \overset{*}{\to} R_{n+1}$ implies $\underline{C}_{nn+1}R_{n+1}^\Lambda \overset{*}{\to} \underline{C}_{nn+1}R_{n+1}$. One obtains thus the pointwise convergence of the integrand in each term of (3.18). The convergence of the integral follows by dominated convergence. ☒

The following lemma shows that the conditions (A_0), (B) and (C) hold for the initial states $\omega^\Lambda = \omega_\rho \otimes \omega_{n,\beta}^\Lambda$.

Lemma 3.6. *ω^Λ satisfies (A_0). The reduced density matrices are given by*

$$R_n^\Lambda = \rho \otimes \sum_{\pi\in S(n)} \operatorname{sgn}\pi\, U_\pi R^\Lambda \otimes \cdots \otimes R^\Lambda, \tag{3.19}$$

where

$$R^\Lambda = (\exp(\beta(H_e^\Lambda - \mu)) + 1)^{-1}. \tag{3.20}$$

$H_e^\Lambda|_{L^2(\Lambda)}$ denotes the operator $-1/2\,\Delta$ on Λ with Dirichlet boundary condition and $H_e^\Lambda|_{L^2(\mathbb{R}^3\setminus\Lambda)}=0$. (B) holds with $a=1$. (C) holds and

$$R_n = \rho \otimes \sum_{\pi\in S(n)} \operatorname{sgn}\pi\, U_\pi R \otimes \cdots \otimes R, \tag{3.21}$$

where

$$R = (\exp(\beta(H_e - \mu)) + 1)^{-1}. \tag{3.22}$$

Proof. To prove A_0 it is sufficient to look at the bath component. The grand canonical partition function $Z_\Lambda(\beta,\mu) = \mathrm{tr}\exp(-\beta(H^\Lambda - \mu N))$ is finite for all chemical potentials μ, and therefore for all $\alpha>0$,

$$\langle e^{\alpha N}\rangle_{\beta,\mu,\Lambda} = Z_\Lambda(\beta, \mu+\beta^{-1}\alpha)/Z_\Lambda(\beta,\mu) < \infty.$$

As $\langle e^{\alpha N}\rangle_{\beta,\mu,\Lambda} = \sum_{n=0}^{\infty} e^{\alpha n}\|\rho_{\beta,\mu\,n}^\Lambda\|_1$ and the series is convergent, there is a $C>0$ such that $\|\rho_{\beta,\mu\,n}^\Lambda\|_1 \leqq Ce^{-\alpha n}$. As α is arbitrary, one obtains an estimate of this kind for all $q=e^{-\alpha}\in(0,1)$.

The explicit representations for R_n^Λ and R_n are obtained from the well known formula

$$\omega(a^+(f_1)\dots a^+(f_n)a(g_n)\dots a(g_1)) = \det\{(g_i, Rf_j)\}$$

for quasi-free states, where $0\leqq R\leqq 1$ is the defining operator. Using the explicit formulas for R_n^Λ and R_n, it is straightforward to verify that (B) and (C) hold. ☒

4. The Low Density Limit

In this section the scaling for the low density limit is introduced. Instead of rescaling time as $t_\varepsilon = \varepsilon^{-1} t$, one may alternatively rescale the Hamiltonian as $H^\varepsilon = \varepsilon^{-1}(H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_I)$. This point of view is taken in the following.

The reduced density matrices of the scaled initial state $\omega^\varepsilon = \omega_\rho \otimes \omega_{\varepsilon n,\beta}$ are

$$R_n^\varepsilon = \rho \otimes \sum_{\pi \in S(n)} \operatorname{sgn} \pi \, U_\pi R^\varepsilon \otimes \cdots \otimes R^\varepsilon, \tag{4.1}$$

where $R^\varepsilon = (\exp(\beta(H_e - \mu_\varepsilon)) + 1)^{-1}$. The chemical potential μ_ε is determined by the relation

$$\int d\underline{k} (\exp(\beta(k^2/2 - \mu_\varepsilon)) + 1)^{-1} = n_\varepsilon .$$

The scaled reduced density matrix at the rescaled time is denoted by $R_n^\varepsilon(t)$. It is given by the perturbation series

$$R_n^\varepsilon(t) = \sum_{m=0}^{\infty} \varepsilon^{-m} \int_{\Delta(0,m,t)} \underline{dt}\, \underline{U}_n^\varepsilon(t - t_1) \underline{C}_{n n+1} \underline{U}_{n+1}^\varepsilon(t_1 - t_2)$$
$$\ldots \underline{C}_{n+m-1\, n+m} \underline{U}_{n+m}^\varepsilon(t_m) R_{n+m}^\varepsilon, \tag{4.2}$$

where $\underline{U}_n^\varepsilon(t) = \underline{U}_n(\varepsilon^{-1} t)$. For $n = 0$ one obtains the reduced dynamics of the system: $T_\varepsilon(t)\rho = R_0^\varepsilon(t)$.

As in the weak coupling problem [20], Theorem 2.1 is proved in two steps. In the first step $R_0^\varepsilon(t)$ is approximated by $\exp\{(-i\varepsilon^{-1} L_S + K)t\}\rho$. The generator $-i\varepsilon^{-1} L_S + K$ will not preserve positivity, in general, and therefore an averaged generator $K^\#$ is introduced in a second approximation, which leads to a quantum dynamical semigroup in the sense of Lindblad [26]. In this section the first approximation is formulated (Theorem 4.1), and the strategy of the proof is outlined. The details of the proof are given in Sect. 5. This approximation involves all the essential difficulties. The averaging procedure, which is rather standard, is considered in Sect. 6.

The semigroup $\exp\{(-i\varepsilon^{-1} L_S + K)t\}$ is given in form of a perturbation series, which is defined as follows. Putting $U_{0n}^\varepsilon(t) = \exp\left\{-i\left(H_S + \sum_{j=1}^n H_{Bj}\right)\varepsilon^{-1} t\right\}$ and $U_{1n}^\varepsilon(t) = \exp\left\{-i\left(H_S + \sum_{j=1}^n H_{Bj} + H_{In}\right)\varepsilon^{-1} t\right\}$, one obtains the scattering operator $\Omega_n = s - \lim_{\varepsilon \downarrow 0} U_{1n}^\varepsilon(t) U_{0n}^\varepsilon(-t)$, $t > 0$. The corresponding operators acting on reduced density matrices are:

$$\underline{U}_{0n}^\varepsilon(t)\cdot = U_{0n}^\varepsilon(t) \cdot U_{0n}^\varepsilon(-t), \tag{4.3}$$

$$\underline{U}_{1n}^\varepsilon(t)\cdot = U_{1n}^\varepsilon(t) \cdot U_{1n}^\varepsilon(-t), \tag{4.4}$$

$$\underline{\Omega}_n \cdot = \Omega_n \cdot \Omega_n^* . \tag{4.5}$$

With these definitions the perturbation series may be written in the form

$$R_0^0(t) = \sum_{m=0}^{\infty} \int_{\Delta(0,m,t)} \underline{dt}\, \underline{U}_{00}^\varepsilon(t - t_1) \underline{C}_{01} \underline{\Omega}_1 \underline{U}_{01}^\varepsilon(t_1 - t_2) \ldots \underline{C}_{m-1\, m} \underline{\Omega}_m \underline{U}_{0m}^\varepsilon(t_m) R_m^0 . \tag{4.6}$$

The initial state is given by

$$R_m^0 = \rho \otimes R^0 \otimes \cdots \otimes R^0,$$

where $R^0 = n(2\pi/\beta)^{3/2} e^{-\beta H_e}$. As R^0 commutes with the free evolution of the bath, (4.6) may be written equivalently as

$$R_0^0(t) = \sum_{m=0}^{\infty} \int_{\Delta(0,m,t)} d\underline{t}\, \underline{U}_S^\varepsilon(t-t_1)\underline{K}\,\underline{U}_S^\varepsilon(t_1-t_2)\underline{K}\ldots\underline{K}\,\underline{U}_S^\varepsilon(t_m)\rho, \tag{4.7}$$

where

$$\underline{U}_S^\varepsilon(t)\rho = \exp(-iH_S\varepsilon^{-1}t)\,\rho\,\exp(iH_S\varepsilon^{-1}t) \tag{4.8}$$

and

$$\underline{K}\rho = -i\,\mathrm{tr}_1 \underline{L}_{I1}\underline{\Omega}_1\rho \otimes R^0. \tag{4.9}$$

Clearly, from (4.7) one obtains $R_0^0(t) = \exp\{(-i\varepsilon^{-1}L_S + K)t\}$.

Theorem 4.1. *Assume that conditions* (E) *and* (F) *hold and put* $T = (2\|Q\|\,\|A\|_1\|R^0\|)^{-1}$. *For* $t \in [0, T)$,

$$\lim_{\varepsilon\downarrow 0} \|R_0^\varepsilon(t) - R_0^0(t)\|_1 = 0.$$

Strategy of the Proof. First a uniform majorant will be found for the series (4.2) (Theorem 4.3). With this the proof is reduced to studying the convergence of the individual terms of the series. In Sect. 5 this is done in two steps. In the first step one proves the convergence of the terms of the series for $R_0^\varepsilon(t)$ to the terms of the series

$$R_0^0(t) = \sum_{m=0}^{\infty} \int_{\Delta(0,m,t)} d\underline{t}\, \underline{U}_{00}^\varepsilon(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}_{01}^\varepsilon(t_1-t_2)\ldots\underline{C}_{m-1\,m}\underline{\Omega}_m\underline{U}_{0m}^\varepsilon(t_m)\tilde{R}_m^0, \tag{4.10}$$

with the initial state

$$\tilde{R}_m^0 = \rho \otimes \sum_{\pi\in S(m)} \mathrm{sgn}\,\pi\, U_\pi R^0 \otimes \cdots \otimes R^0. \tag{4.11}$$

Here the statistics is still retained in the initial state. In the second step one proves that the terms of the series (4.10) converge to the terms of the series (4.6). ☒

To prove the existence of a uniform majorant for the series (4.2), one needs

Lemma 4.2. *Let* $R^0 = n(2\pi/\beta)^{3/2} e^{-\beta H_e}$. *Then*

$$\lim_{\varepsilon\downarrow 0} \|\varepsilon^{-1}R^\varepsilon - R^0\| = 0. \tag{4.11}$$

Proof. Let $z_\varepsilon = \exp(\beta\mu_\varepsilon)$. From an asymptotic expansion of the Fermi function follows $\lim_{\varepsilon\downarrow 0} \varepsilon z_\varepsilon^{-1} = n^{-1}(2\pi/\beta)^{3/2}$. It is straightforward to show that

$$\lim_{\varepsilon\downarrow 0}\sup_{k\in\mathbb{R}^3} |(\varepsilon^{-1}(z_\varepsilon^{-1}e^{\beta k^2/2}+1)^{-1} - n(2\pi/\beta)^{3/2}e^{-\beta k^2/2})| = 0.$$

The statement of the lemma follows from an application of the spectral calculus. ☒

Theorem 4.3. *For all* $t \in [0, T)$ *with* $T = (2\|Q\| \, \|A\|_1 \|R^0\|)^{-1}$, *there is* $\varepsilon_0 > 0$, *such that the series* (4.2) *converges in norm for* $\varepsilon < \varepsilon_0$. *It has a majorant not depending on* ε.

Proof. As in the proof of Theorem 3.5 one obtains the bound

$$\|R_n^\varepsilon(t)\| \leqq \sum_{m=0}^{\infty} (n+m)!/m!(2\varepsilon^{-1} t \|Q\| \, \|A\|_1)^m \|\rho\| \, \|R^\varepsilon\|^{n+m}.$$

The right-hand side converges for $2\|Q\| \, \|A\|_1 \|\varepsilon^{-1} R^\varepsilon\| t < 1$. As $\lim_{\varepsilon \downarrow 0} \|\varepsilon^{-1} R^\varepsilon\| = \|R^0\|$, for all $t < (2\|Q\| \, \|A\|_1 \|R^0\|)^{-1}$, there is $\varepsilon_0 > 0$, such that one has $2\|Q\| \times \|A\|_1 \times \|\varepsilon^{-1} R^\varepsilon\| t < 1$ for all $\varepsilon < \varepsilon_0$. This proves the theorem. ☒

5. Proof of the Theorem

5.1. Auxiliary Results

In this section some results are presented, which are used in the proof of Theorem 4.1.

The first is a cluster theorem for n-particle scattering (cf. also [27]).

The free evolution of j bath particles is denoted by

$$U_{Bj}^\varepsilon(t) = \exp\left(-i\left(\sum_{k=1}^{j} H_{Bk}\right)\varepsilon^{-1} t\right).$$

Theorem 5.1. *If* $\tau_1, \ldots, \tau_k > 0$ *or* $\tau_1, \ldots, \tau_k < 0$, *then*

$$s - \lim_{\varepsilon \downarrow 0} [U_k^\varepsilon(\tau_k) U_{B1}^\varepsilon(\tau_1) \ldots U_{Bk-1}^\varepsilon(\tau_{k-1}) - U_{1k}^\varepsilon(\tau_k) U_{B1}^\varepsilon(\tau_1) \ldots U_{Bk-1}^\varepsilon(\tau_{k-1})] = 0. \tag{5.1}$$

Proof. Let $\tau_1, \ldots, \tau_k > 0$. The proof for the second case is analogous. It is sufficient to prove the theorem for unit product vectors $\Phi = \phi_0 \otimes \phi_1 \otimes \cdots \otimes \phi_k$. Using the perturbation formula

$$U_k^\varepsilon(\tau_k) = U_{1k}^\varepsilon(\tau_k) - i\varepsilon^{-1} \int_0^{\tau_k} ds \, U_k^\varepsilon(\tau_k - s)\left(\sum_{j=1}^{k-1} H_{Ij}\right) U_{1k}^\varepsilon(s),$$

the proof is reduced to showing $\lim_{\varepsilon \downarrow 0} I_\varepsilon^j = 0$, where

$$I_\varepsilon^j = \left\| \varepsilon^{-1} \int_0^{\tau_k} ds \, U_k^\varepsilon(\tau_k - s) H_{Ij} U_{1k}^\varepsilon(s) U_{B1}^\varepsilon(\tau_1) \ldots U_{Bk-1}^\varepsilon(\tau_{k-1}) \Phi \right\|.$$

Exploiting the product structure one obtains

$$\begin{aligned} I_\varepsilon^j \leqq \varepsilon^{-1} \int_0^{\tau_k} ds \, \| & Q \exp(-iH_S \varepsilon^{-1} s) \phi_0 \\ & \otimes \exp\{-iH_{B1} \varepsilon^{-1}(\tau_1 + \cdots + \tau_{k-1} + s)\} \phi_1 \otimes \cdots \\ & \otimes A \exp\{-iH_{Bj} \varepsilon^{-1}(\tau_j + \cdots + \tau_{k-1} + s)\} \phi_j \otimes \cdots \\ & \otimes \exp\{-iH_{Bk-1} \varepsilon^{-1}(\tau_{k-1} + s)\} \phi_{k-1} \otimes \cdots \\ & \otimes \exp\{-i(H_{Bk} + H_{Ik}) \varepsilon^{-1} s\} \phi_k \| \end{aligned}$$

$$\leq \varepsilon^{-1} \int_0^{\tau_k} ds \|Q\| \, \|A \exp\{-iH_{Bj}\varepsilon^{-1}(\tau_j + \cdots + \tau_{k-1} + s)\phi_j\|$$

$$= \|Q\| \int_{\varepsilon^{-1}(\tau_j + \cdots + \tau_{k-1})}^{\varepsilon^{-1}(\tau_j + \cdots + \tau_k)} \|A \exp\{-iH_{Bj}s\}\phi_j\|.$$

By condition (F) the integral $\int_0^\infty ds \|Ae^{-iH_e s}\phi\|$ is convergent for $\phi \in \mathscr{D}$. Taking $\phi_1, \ldots, \phi_k \in \mathscr{D}$ one obtains $\lim_{\varepsilon \downarrow 0} I_\varepsilon^j = 0$. This holds also for finite linear combinations of such product vectors, which form a dense set in $\mathscr{H}_k$. A density argument completes the proof of the theorem. ☒

The next lemma deals with a trace class property of the tensor product $\mathscr{H}_k = \mathscr{H}_S \otimes \mathscr{H}_e \otimes \cdots \otimes \mathscr{H}_e$. Denote by $U_k: \mathscr{H}_k \to \mathscr{H}_k$, $U_k \phi_0 \otimes \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k = \phi_0 \otimes \phi_k \otimes \phi_2 \otimes \cdots \otimes \phi_1$ the unitary operator, which interchanges the first and the k^{th} component of the bath.

Lemma 5.2. *Let* $X \in \mathscr{T}(\mathscr{H}_{k-1})$ and $\rho \in \mathscr{T}(\mathscr{H}_1)$. *Then* $(U_k \rho \otimes \mathbb{1}^{(k-1)} U_k^{-1})(X \otimes \mathbb{1}) \in \mathscr{T}(\mathscr{H}_k)$ and $\|(U_k \rho \otimes \mathbb{1}^{(k-1)} U_k^{-1})(X \otimes \mathbb{1})\|_1 \leq \|\rho\|_1 \|X\|_1$.

Proof. It is sufficient to prove the lemma for $\rho = |\phi\rangle\langle\Psi|$, $\|\phi\| = \|\psi\| = 1$ and $X = |\Phi\rangle\langle\Psi|$, $\|\Phi\| = \|\Psi\| = 1$. For general ρ, X one uses the polar decompositions $\rho = \sum_i \alpha_i |\phi_i\rangle\langle\psi_i|$ and $X = \sum_i \beta_i |\Phi_i\rangle\langle\Psi_i|$. Let $(\chi_i)_i$ be a complete orthonormal system in $\mathscr{H}_S$. Using the representations

$$\Phi = \sum_k \chi_k \otimes \Phi_k, \qquad \Phi_k \in \bigotimes_1^{k-1} \mathscr{H}_e, \qquad \sum_k \|\Phi_k\|^2 = 1,$$

$$\Psi = \sum_k \chi_k \otimes \Psi_k, \qquad \Psi_k \in \bigotimes_1^{k-1} \mathscr{H}_e, \qquad \sum_k \|\Psi_k\|^2 = 1,$$

$$\phi = \sum_k \chi_k \otimes \phi_k, \qquad \phi_k \in \mathscr{H}_e, \qquad \sum_k \|\phi_k\|^2 = 1,$$

$$\psi = \sum_k \chi_k \otimes \psi_k, \qquad \psi_k \in \mathscr{H}_e, \qquad \sum_k \|\psi_k\|^2 = 1,$$

a straightforward calculation yields

$$U_\pi \rho \otimes \mathbb{1}^{(k-1)} U_\pi^{-1} X \otimes \mathbb{1} = \sum_m \left(\sum_k |\chi_k \otimes \Phi_m \otimes \phi_k\rangle \right) \langle \Psi \otimes \psi_m|.$$

The vectors $(|\chi_k \otimes \Phi_m \otimes \phi_k\rangle)_k$ form an orthonormal system, therefore $\left\| \sum_k |\chi_k \otimes \Phi_m \otimes \phi_k\rangle \right\|^2 = \|\Phi_m\|^2$. The estimate

$$\left\| \sum_m \left(\sum_k |\chi_k \otimes \Phi_m \otimes \phi_k\rangle \right) \langle \Psi \otimes \psi_m| \right\|_1 \leq \sum_m \left\| \left(\sum_k |\chi_k \otimes \Phi_m \otimes \phi_k\rangle \right) \langle \Psi \otimes \psi_m| \right\|_1$$

$$= \sum_m \|\Phi_m\| \, \|\psi_m\| \leq \left(\sum_m \|\Phi_m\|^2 \right)^{1/2} \left(\sum_m \|\psi_m\|^2 \right)^{1/2} = 1$$

completes the proof of the lemma. ☒

Lemma 5.3. *Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{K} \subset \mathscr{T}(\mathscr{H})$ a compact set. Let H be a self adjoint operator on $\mathscr{H}$ with purely absolutely continuous spectrum and denote by $U(t) = \exp(-iHt)$ the unitary group generated by H. Then*

$$\lim_{t\to\infty} \sup_{\rho\in\mathscr{K}} |\operatorname{tr} \rho\, U(t)| = 0.$$

Proof. The spectral theorem implies the weak convergence of $U(t)$ to zero. On bounded sets weak convergence and ultraweak convergence coincide. Therefore the boundedness of $\{U(t) | t \in \mathbb{R}\}$ implies $\lim_{t\to\infty} \operatorname{tr} \rho U(t) = 0$ for all $\rho \in \mathscr{T}(\mathscr{H})$.

Choose $\varepsilon > 0$. There are $\rho_1, \ldots, \rho_k \in \mathscr{T}(\mathscr{H})$ such that the balls $\{\rho \,|\, \|\rho - \rho_j\|_1 \leqq \varepsilon\}$ cover $\mathscr{K}$. Choose t_0 such that for all $t \geqq t_0$ $|\operatorname{tr} \rho_j U(t)| \leqq \varepsilon$ for all $j = 1, \ldots, k$. The estimate $|\operatorname{tr} \rho U(t)| \leqq \|\rho - \rho_j\|_1 + |\operatorname{tr} \rho_j U(t)| \leqq 2\varepsilon$ for all $t \geqq t_0$ shows that the convergence is indeed uniform on compact sets. □

5.2. *An Intermediate Approximation*

The first step in the proof of Theorem 4.1 is to show that $R_0^\varepsilon(t)$ approximates $\tilde{R}_0^0(t)$ given by the series (4.10).

Theorem 5.4. *For $t \in [0, T)$ with $T = (2\|Q\|\,\|A\|_1\,\|R^0\|)^{-1}$ holds*

$$\lim_{\varepsilon\downarrow 0} \|R_0^\varepsilon(t) - \tilde{R}_0^0(t)\|_1 = 0.$$

Proof. As $\dim \mathscr{H}_S < \infty$, it is sufficient to prove $\lim_{\varepsilon\downarrow 0} \operatorname{tr}_S X(R_0^\varepsilon(t) - \tilde{R}_0^0(t)) = 0$ for all $X \in \mathscr{T}(\mathscr{H}_S)$. From Theorem 4.3 one obtains a uniform majorant of the series (4.2), and therefore the theorem is proved if for all $m \in \mathbb{N}$

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \operatorname{tr}_S X\{\varepsilon^{-m} \int_{\Delta(0,m,t)} \underline{dt}\, \underline{U}_0^\varepsilon(t - t_1)\underline{C}_{01} \ldots \underline{C}_{m-1\,m} \underline{U}_m^\varepsilon(t_m) R_m^\varepsilon \\ - \int_{\Delta(0,m,t)} \underline{dt}\, \underline{U}_{00}^\varepsilon(t - t_1)\underline{C}_{01}\underline{\Omega}_1 \underline{U}_{01}^\varepsilon(t_1 - t_2) \ldots \underline{C}_{m-1\,m}\underline{\Omega}_m \underline{U}_{0m}^\varepsilon(t_m)\tilde{R}_m^0\} = 0.\end{aligned} \tag{5.2}$$

Using Lemma 4.2 one obtains immediately $\lim_{\varepsilon\to 0} \|\varepsilon^{-m} R_m^\varepsilon - \tilde{R}_m^0\| = 0$. The estimate

$$\begin{aligned}&\left\| \int_{\Delta(0,m,t)} \underline{dt}\, \underline{U}_0^\varepsilon(t - t_1)\underline{C}_{01} \ldots \underline{C}_{m-1\,m} \underline{U}_m^\varepsilon(t_m)(\varepsilon^{-m} R_m^\varepsilon - \tilde{R}_m^0) \right\| \\ &\leqq t^m/m!\,(2\|Q\|\,\|A\|_1)^m \|\varepsilon^{-m} R_m^\varepsilon - \tilde{R}_m^0\|\end{aligned}$$

shows that it is sufficient to prove

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \operatorname{tr}_S X\{\underline{U}_0^\varepsilon(t - t_1)\underline{C}_{01} \ldots \underline{C}_{m-1\,m} \underline{U}_m^\varepsilon(t_m)\tilde{R}_m^0 \\ - \underline{U}_{00}^\varepsilon(t - t_1)\underline{C}_{01}\underline{\Omega}_1 \underline{U}_{01}^\varepsilon(t_1 - t_2) \ldots \underline{C}_{m-1\,m}\underline{\Omega}_m \underline{U}_{0m}^\varepsilon(t_m)\tilde{R}_m^0\} = 0.\end{aligned} \tag{5.3}$$

By a straightforward estimate one sees that the proof of (5.3) may be reduced to prove

$$\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \mathrm{tr}_S X\{\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1-t_2)\dots$$
$$\cdots\underline{C}_{k-1\,k}[\underline{U}^{\varepsilon}_{k}(t_k-t_{k+1})-\underline{\Omega}_k\underline{U}^{\varepsilon}_{0k}(t_k-t_{k+1})]\underline{C}_{kk+1}$$
$$\cdot\underline{U}^{\varepsilon}_{k+1}(t_{k+1}-t_{k+2})\dots\underline{C}_{m-1\,m}\underline{U}^{\varepsilon}_{m}(t_m)\tilde{R}^0_m\}=0 \tag{5.4}$$

for $k=1,\dots,m$. Denote by $P_k\in\mathscr{B}(\mathscr{H}_k)$ the projection on the subspace associated to the continuous spectrum of $H_S+H_{Bk}+H_{Ik}$, and put $\underline{P}_k\cdot=P_k\cdot P_k$.

Equation (5.4) is proved in three steps:

Step 1.

$$\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \mathrm{tr}_S X\{\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1-t_2)\dots$$
$$\dots\underline{C}_{k-1\,k}[\underline{U}^{\varepsilon}_{k}(t_k-t_{k+1})-\underline{U}^{\varepsilon}_{1k}(t_k-t_{k+1})]\underline{C}_{kk+1}$$
$$\cdot\underline{U}^{\varepsilon}_{k+1}(t_{k+1}-t_{k+2})\dots\underline{C}_{m-1\,m}\underline{U}^{\varepsilon}_{m}(t_m)\tilde{R}^0_m\}=0. \tag{5.5}$$

Step 2.

$$\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \mathrm{tr}_S X\{\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1-t_2)\dots$$
$$\dots\underline{C}_{k-1\,k}\underline{U}^{\varepsilon}_{1k}(t_k-t_{k+1})[\mathbb{1}-\underline{P}_k]\underline{C}_{kk+1}$$
$$\cdot\underline{U}^{\varepsilon}_{k+1}(t_{k+1}-t_{k+2})\dots\underline{C}_{m-1\,m}\underline{U}^{\varepsilon}_{m}(t_m)\tilde{R}^0_m\}=0, \tag{5.6}$$

Step 3.

$$\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \mathrm{tr}_S X\{\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1-t_2)\dots$$
$$\dots\underline{C}_{k-1\,k}[\underline{U}^{\varepsilon}_{1k}(t_k-t_{k+1})\underline{P}_k-\underline{\Omega}_k\underline{U}^{\varepsilon}_{0k}(t_k-t_{k+1})]$$
$$\cdot\underline{C}_{kk+1}\underline{U}^{\varepsilon}_{k+1}(t_{k+1}-t_{k+2})\dots\underline{C}_{m-1\,m}\underline{U}^{\varepsilon}_{m}(t_m)\tilde{R}^0_m\}=0. \tag{5.7}$$

In the first step the fully interacting k-particle dynamics $\underline{U}^{\varepsilon}_k$ is replaced by the time evolution $\underline{U}^{\varepsilon}_{1k}$, where only the k^{th} bath particle interacts with the system. The second step shows that in the limit there is no contribution from bound states. Finally, in the third step the time evolution $\underline{U}^{\varepsilon}_{1k}$ is replaced by the free time evolution and the scattering operator.

Step 1 of the proof. One proves the convergence of the integrand for $0<t_m<\dots<t_1<t$. By the boundedness of the integrand, the convergence of the integral follows from the dominated convergence theorem. One defines $Y^{\varepsilon}_k(\underline{t}'_k)=\underline{C}_{kk+1}\underline{U}^{\varepsilon}_{k+1}(t_{k+1}-t_{k+2})\dots\underline{C}_{m-1m}\underline{U}^{\varepsilon}_m(t_m)\tilde{R}^0_m$, where $\underline{t}'_k=(t_{k+1},\dots,t_m)$. Putting $\underline{U}^{\varepsilon}_{0k}(t)=\underline{U}^{\varepsilon}_{00}(t)\underline{U}^{\varepsilon}_{Bk}(t)$ and using the fact that $\underline{U}^{\varepsilon}_{Bk}(t)$ commutes with $\underline{C}_{j-1\,j}$ and $\underline{\Omega}_j$ for $j>k$, one obtains

$$E_k=\mathrm{tr}_S X\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1-t_2)\dots\underline{C}_{k-1\,k}\underline{U}^{\varepsilon}_{k}(t_k-t_{k+1})Y^{\varepsilon}_k(\underline{t}'_k)$$
$$=\mathrm{tr}_S X\underline{U}^{\varepsilon}_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{00}(t_1-t_2)\dots\underline{C}_{k-1\,k}\underline{U}^{\varepsilon}_{B1}(t_1-t_2)\underline{U}^{\varepsilon}_{B2}(t_2-t_3)\dots$$
$$\dots\underline{U}^{\varepsilon}_{Bk-1}(t_{k-1}-t_k)\underline{U}^{\varepsilon}_{k}(t_k-t_{k+1})Y^{\varepsilon}_k(\underline{t}'_k).$$

One puts $\underline{\Omega}_j^* \cdot = \Omega_j^* \cdot \Omega_j$ and passes to the Heisenberg picture by repeated cyclic permutation under the trace:

$$E_k = i^k \operatorname{tr}_{\mathscr{H}_k}[\underline{U}_k^\varepsilon(t_{k+1} - t_k)\underline{U}_{Bk-1}^\varepsilon(t_k - t_{k-1}) \dots \underline{U}_{B1}^\varepsilon(t_2 - t_1)$$
$$\cdot \underline{L}_{Ik}\underline{U}_{00}^\varepsilon(t_k - t_{k-1})\underline{\Omega}_{k-1}^*\underline{L}_{Ik-1} \dots \underline{U}_{00}^\varepsilon(t_2 - t_1)\underline{\Omega}_1^*$$
$$\cdot \underline{L}_{I1}\underline{U}_{00}^\varepsilon(t_1 - t)X \otimes \mathbb{1}^{(k)}]\, Y_k^\varepsilon(\underline{t}_k').$$

In order to finish step 1 of the proof it is sufficient to show

$$\lim_{\varepsilon \downarrow 0} \| [\underline{U}_k^\varepsilon(t_{k+1} - t_k) - \underline{U}_{1k}^\varepsilon(t_{k+1} - t_k)]\underline{U}_{Bk-1}^\varepsilon(t_k - t_{k-1}) \dots$$
$$\dots \underline{U}_{B1}^\varepsilon(t_2 - t_1)X_k^\varepsilon(\underline{t}_k) \|_1 = 0,$$

where

$$X_k^\varepsilon(\underline{t}_k) = \underline{L}_{Ik}\underline{U}_{00}^\varepsilon(t_k - t_{k-1}) \dots \underline{U}_{00}^\varepsilon(t_2 - t_1)\underline{\Omega}_1^*$$
$$\cdot \underline{L}_{I1}\underline{U}_{00}^\varepsilon(t_1 - t)X \otimes \mathbb{1}^{(k)},$$

and $\underline{t}_k = (t_1, \dots, t_k)$.

For fixed $X_k \in \mathscr{T}(\mathscr{H}_k)$ instead of $X_k^\varepsilon(\underline{t}_k)$ this follows from Theorem 5.1. By an $\varepsilon/2$-argument this property holds uniformly for X_k on compact sets of $\mathscr{T}(\mathscr{H}_k)$. Therefore, if one can verify

(1) for $\varepsilon > 0 \quad X_k^\varepsilon(\underline{t}_k) \in \mathscr{T}(\mathscr{H}_k)$,
(2) $\overline{\{X_k^\varepsilon(\underline{t}_k) | \varepsilon \in (0,1]\}}$ is compact in $\mathscr{T}(\mathscr{H}_k)$,

the proof of step 1 is completed. These two properties are proven inductively.

First one notes that $X \in \mathscr{T}(\mathscr{H}_S) = \mathscr{T}(\mathscr{H}_0)$, and therefore also $\underline{U}_{00}^\varepsilon(t_1 - t)X \in \mathscr{T}(\mathscr{H}_0)$. As $A \in \mathscr{T}(\mathscr{H}_e)$ one obtains $X_1^\varepsilon(\underline{t}_1) = [Q \otimes A,\ \underline{U}_{00}^\varepsilon(t_1 - t)X \otimes \mathbb{1}] \in \mathscr{T}(\mathscr{H}_1)$. Assume $X_k^\varepsilon(\underline{t}_k) \in \mathscr{T}(\mathscr{H}_k)$. From the boundedness of Ω_k follows $\underline{\Omega}_k^* X_k(\underline{t}_k) \in \mathscr{T}(\mathscr{H}_k)$. The same arguments as for $k = 0$ show that $X_{k+1}^\varepsilon(\underline{t}_{k+1}) \in \mathscr{T}(\mathscr{H}_{k+1})$. This proves (1).

To prove (2) one notes that the set $\mathscr{U}(\mathscr{H}_S) = \{U \in \mathscr{B}(\mathscr{H}_S) | U^*U = UU^* = 1\}$ of unitary operators in $\mathscr{H}_S$ is compact, which follows from $\dim \mathscr{H}_S < \infty$. Let $\mathscr{T}_k \subset \mathscr{T}(\mathscr{H}_k)$ be compact. From the joint continuity of the maps $(U, X_k) \mapsto U \otimes \mathbb{1} X_k$ and $(U, X_k) \mapsto X_k U \otimes 1$ as maps from $\mathscr{B}(\mathscr{H}_S) \otimes \mathscr{T}(\mathscr{H}_k)$ to $\mathscr{T}(\mathscr{H}_k)$ follows the compactness of $\{U \otimes \mathbb{1} X_k | U \in \mathscr{U}(\mathscr{H}_S),\ X_k \in \mathscr{T}_k\}$ and $\{X_k U \otimes \mathbb{1} | U \in \mathscr{U}(\mathscr{H}_S),\ X_k \in \mathscr{H}_k\}$. The map $\mathscr{T}(\mathscr{H}_k) \to \mathscr{T}(\mathscr{H}_{k+1})$ $X_k \mapsto L_{Ik+1} X_k \otimes \mathbb{1}$ is also continuous, therefore $\{\underline{L}_{Ik+1} X_k \otimes \mathbb{1} | X_k \in \mathscr{T}_k\}$ is compact in $\mathscr{T}(\mathscr{H}_{k+1})$. Finally the map $X_k \mapsto \underline{\Omega}_k^* X_k$ is continuous as a map on $\mathscr{T}(\mathscr{H}_k)$ and consequently $\{\underline{\Omega}_k^* X_k | X_k \in \mathscr{T}_k\}$ is compact. Using these properties it is easy to prove (2) by induction.

Step 2 of the proof. Using the definition of $\underline{C}_{jj+1}$, the formula $\underline{U}_{0k-1}^\varepsilon(t_{k-1} - t_k)\operatorname{tr}_k \cdot = \operatorname{tr}_k \underline{U}_{0k}^\varepsilon(t_{k-1} - t_k) \cdot$ and the identity

$$-i \int_{t_{k+1}}^{t_{k-1}} dt_k\, \underline{U}_{0k}^\varepsilon(t_{k-1} - t_k)\underline{L}_{Ik}\underline{U}_{1k}^\varepsilon(t_k - t_{k+1})$$
$$= \varepsilon(\underline{U}_{1k}^\varepsilon(t_{k-1} - t_k) - \underline{U}_{0k}^\varepsilon(t_{k-1} - t_k)),$$

one writes (5.6) in the form

$$\lim_{\varepsilon\downarrow 0} \varepsilon \int\limits_{0\leqq t_m \leqq \dots \leqq t_{k+1} \leqq t_{k-1} \leqq \dots \leqq t_1 \leqq t} dt_1 \dots dt_{k-1}\, dt_{k+1} \dots dt_m$$

$$\cdot \mathrm{tr}_S X\{\underline{U}^\varepsilon_{00}(t-t_1)\underline{C}_{01}\underline{\Omega}_1 \underline{U}^\varepsilon_{01}(t_1-t_2)\dots \underline{C}_{k-2\,k-1}\underline{\Omega}_{k-1}$$

$$\cdot \mathrm{tr}_k(\underline{U}^\varepsilon_{1k}(t_{k-1}-t_k) - \underline{U}^\varepsilon_{0k}(t_{k-1}-t_k))(\mathbb{1} - \underline{P}_k)\underline{C}_{k\,k+1}$$

$$\underline{U}^\varepsilon_{k+1}(t_{k+1}-t_{k+2})\dots \underline{C}_{m-1\,m}\underline{U}^\varepsilon_m(t_m)\underline{\tilde{R}}^0_m\} = 0. \tag{5.8}$$

It is sufficient that the integrand is bounded by some constant C. Then the absolute value of the integral is bounded by $t^{m-1}/(m-1)!\ C$ and therefore the left-hand side of (5.8) converges to zero.

The integrand may be written in the form

$$\mathrm{tr}_{\mathscr{H}_k}\{(\mathbb{1} - \underline{P}_k)(\underline{U}^\varepsilon_{1k}(t_k - t_{k-1}) - \underline{U}^\varepsilon_{0k}(t_k - t_{k-1}))(\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1}\}$$

$$\underline{U}^\varepsilon_{B1}(t_1-t_2)\underline{U}^\varepsilon_{B2}(t_2-t_3)\dots \underline{U}^\varepsilon_{Bk-2}(t_{k-2}-t_{k-1})Y^\varepsilon_k(\underline{t}'_k).$$

$\underline{U}^\varepsilon_{B1}(t_1-t_2)\underline{U}^\varepsilon_{B2}(t_2-t_3)\dots \underline{U}^\varepsilon_{Bk-2}(t_{k-2}-t_{k-1})Y^\varepsilon_k(t'_k)\in\mathscr{B}(\mathscr{H}_k)$ is uniformly bounded in ε. Therefore it is sufficient to prove that

$$(\mathbb{1} - \underline{P}_k)(\underline{U}^\varepsilon_{1k}(t_k - t_{k-1}) - \underline{U}^\varepsilon_{0k}(t_k - t_{k-1}))(\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1} \in \mathscr{T}(\mathscr{H}_k),$$

and that this expression is uniformly bounded in ε.

To prove this, condition (E) is used. $P_\perp = \mathbb{1} - p_1$ is the projection on the eigenstates of H_1. Condition (E) is equivalent to $\|P_\perp\|_1 < \infty$.

The required estimates are proven separately for

$$(\mathbb{1} - \underline{P}_k)\underline{U}^\varepsilon_{1k}(t_k - t_{k-1})(\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1} \tag{5.9}$$

and

$$(\mathbb{1} - \underline{P}_k)\underline{U}^\varepsilon_{0k}(t_k - t_{k-1})(\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1}. \tag{5.10}$$

Defining $U^\varepsilon_{2k}(t) = \exp\{-i(H_S + H_{Bk} + H_{Ik})\varepsilon^{-1}t\}$ and $\underline{U}^\varepsilon_{2k}(t)\cdot = U^\varepsilon_{2k}(t)\cdot U^\varepsilon_{2k}(-t)$,

one obtains $\underline{U}^\varepsilon_{1k}(t) = \underline{U}^\varepsilon_{2k}(t)\underline{U}^\varepsilon_{Bk-1}(t)$. As $\underline{U}^\varepsilon_{2k}(t)$ and $(\mathbb{1} - \underline{P}_k)$ commute, the identity

$$\|(\mathbb{1} - \underline{P}_k)\underline{U}^\varepsilon_{1k}(t_k - t_{k-1})(\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1}\|_1$$

$$= \|(\mathbb{1} - \underline{P}_k)(\underline{U}^\varepsilon_{Bk-1}(t_k - t_{k-1})\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1}))\otimes \mathbb{1}\|_1$$

for (5.9) follows. $\tilde{X}^\varepsilon_{k-1}(\underline{t}_k) = \underline{U}^\varepsilon_{Bk-1}(t_k - t_{k-1})\underline{\Omega}^*_{k-1}X^\varepsilon_{k-1}(\underline{t}_{k-1})$ is in $\mathscr{T}(\mathscr{H}_{k-1})$, and there is a constant C_2 such that for all $\varepsilon > 0$ $\|\tilde{X}^\varepsilon_{k-1}(\underline{t}_k)\|_1 \leqq C_2$.

Let $X\in\mathscr{T}(\mathscr{H}_{k-1})$. Using the permutation operator U_k, one may write $\mathbb{1} - P_k = U_k P_\perp \otimes \mathbb{1}^{(k-1)} U_k$. With the identity $(\mathbb{1} - \underline{P}_k)(X\otimes\mathbb{1}) = P_k X\otimes\mathbb{1}(1-P_k) + (1-P_k)X\otimes\mathbb{1}$, the estimate

$$\|(\mathbb{1} - \underline{P}_k)X\otimes\mathbb{1}\|_1 \leqq 2\|P_\perp\|_1\|X\|_1 \tag{5.11}$$

follows from Lemma 5.2. In particular one obtains for (5.9) the estimate $\|(\mathbb{1} - \underline{P}_k)(\tilde{X}^\varepsilon_{k-1}(\underline{t}_k)\otimes\mathbb{1})\|_1 \leqq 2\|P_\perp\|_1 C_2$.

To estimate (5.10) one notes that $\underline{U}^{\varepsilon}_{0k}(t_k - t_{k-1})(\underline{\Omega}^{*}_{k-1}X^{\varepsilon}_{k-1}(\underline{t}_{k-1})) \otimes \mathbb{1} = (\underline{U}^{\varepsilon}_{0\,k-1}(t_k - t_{k-1})\underline{\Omega}^{*}_{k-1}X^{\varepsilon}_{k-1}(\underline{t}_{k-1})) \otimes \mathbb{1}$, and obtains with (5.11)

$$\|(\mathbb{1} - \underline{P}_k)\underline{U}^{\varepsilon}_{0k}(t_k - t_{k-1})(\underline{\Omega}^{*}_{k-1}X^{\varepsilon}_{k-1}(\underline{t}_{k-1})) \otimes \mathbb{1}\|_1$$
$$\leqq 2\|P_\perp\|_1 \|\underline{U}^{\varepsilon}_{0\,k-1}(t_k - t_{k-1})\underline{\Omega}^{*}_{k-1}X^{\varepsilon}_{k-1}(\underline{t}_{k-1})\|_1 .$$

The right-hand side of this inequality is bounded in ε.

Step 3 of the proof. Again it is sufficient to prove the convergence of the integrand. As $\underline{P}_k$ commutes with $\underline{U}^{\varepsilon}_{1k}(t_k - t_{k-1})$, the integrand of (5.7) may be written as

$$\operatorname{tr}_{\mathscr{H}_k}[(\underline{U}^{\varepsilon}_{0k}(t_k - t_{k+1})\underline{U}^{\varepsilon}_{1k}(t_{k+1} - t_k)\underline{P}_k - \underline{\Omega}^{*}_{k})X^{\varepsilon}_{k}(\underline{t}_k)]\underline{U}^{\varepsilon}_{B1}(t_1 - t_2)\ldots$$
$$\ldots \underline{U}^{\varepsilon}_{Bk-1}(t_{k-1} - t_k)\underline{U}^{\varepsilon}_{0k}(t_k - t_{k+1})Y^{\varepsilon}_{k}(\underline{t}'_k). \tag{5.12}$$

The existence and completeness of the wave operators implies $s - \lim_{\varepsilon\downarrow 0}(U^{\varepsilon}_{0k}(t_k) \times U^{\varepsilon}_{1k}(-t)P_k - \Omega^{*}_{k}) = 0$ for $t > 0$. The compactness argument used in step 1 of the proof shows that

$$\lim_{\varepsilon\downarrow 0} \|(\underline{U}^{\varepsilon}_{0k}(t_k - t_{k+1})\underline{U}^{\varepsilon}_{1k}(t_{k+1} - t_k)\underline{P}_k - \underline{\Omega}^{*}_{k})X^{\varepsilon}_{k}(\underline{t}_k)\|_1 = 0 .$$

Furthermore $\|\underline{U}^{\varepsilon}_{B1}(t_1 - t_2)\ldots \underline{U}^{\varepsilon}_{Bk-1}(t_{k-1} - t_k)Y^{\varepsilon}_{k}(\underline{t}'_k)\|$ is bounded uniformly in ε. Therefore (5.12) converges to zero in the limit $\varepsilon\downarrow 0$. ☒

5.3. *Contribution of the Statistics*

To complete the proof of Theorem 4.1 it remains to show that in the limit $\varepsilon\downarrow 0$ the contribution from the exchange terms in $\tilde{R}^0_0$ vanishes.

Theorem 5.5. *For* $t \in [0, T)$ *with* $T = (2\|Q\|\,\|A\|_1\,\|R^0\|)^{-1}$ *holds*

$$\lim_{\varepsilon\downarrow 0} \|\tilde{R}^0_0(t) - R^0_0(t)\|_1 = 0 .$$

Proof. Using the series (4.6) and (4.10) for $R^0_0(t)$ and $\tilde{R}^0_0(t)$, one sees that it is sufficient to prove for all $n \in \mathbb{N}$ and all non-trivial permutations $\pi \in S(m)$, $\pi \neq \mathrm{id}$,

$$\lim_{\varepsilon\downarrow 0} \int_{\Delta(0,m,t)} \underline{dt}\, \operatorname{tr}_S X \underline{U}^{\varepsilon}_{00}(t - t_1)\underline{C}_{01}\underline{\Omega}_1\underline{U}^{\varepsilon}_{01}(t_1 - t_2)\ldots$$
$$\ldots \underline{C}_{m-1\,m}\underline{\Omega}_m\underline{U}^{\varepsilon}_{0m}(t_m)\,\rho \otimes U_\pi R^0 \otimes \cdots \otimes R^0 = 0 .$$

One proves that the integrand converges to zero almost everywhere. Taking into account the boundedness of the integral, the result follows from the theorem on dominated convergence.

The integrand may be written in the form

$$I^{\varepsilon}_{m}(\underline{t}_m) = \operatorname{tr}_{\mathscr{H}_m}[\underline{U}^{\varepsilon}_{00}(-t_m)\underline{\Omega}^{*}_{m}X^{\varepsilon}_{m}(\underline{t}_m)]\underline{U}^{\varepsilon}_{B1}(t_1 - t_2)\ldots \underline{U}^{\varepsilon}_{Bm-1}(t_{m-1} - t_m)$$
$$\cdot \underline{U}_{Bm}(t_m)(U_\pi \rho \otimes R^0 \otimes \cdots \otimes R^0).$$

Using the invariance of R^0 under the free evolution one obtains

$$I^{\varepsilon}_{m}(\underline{t}_m) = \operatorname{tr}_{\mathscr{H}_m}[\underline{U}^{\varepsilon}_{00}(-t_m)\underline{\Omega}^{*}_{m}X^{\varepsilon}_{m}(\underline{t}_m)]\rho \otimes R^0 \otimes \cdots \otimes R^0 U_\pi V_m(\underline{t}_m),$$

where $V_m(\underline{t}_m) = U_\pi^{-1} U_{B1}^\varepsilon(t_1 - t_2) \dots U_{Bm}^\varepsilon(t_m) U_\pi U_{Bm}^\varepsilon(-t_m) \dots U_{B1}^\varepsilon(t_2 - t_1)$. A straightforward calculation gives $V_m(\underline{t}_m) = \exp(-i\varepsilon^{-1} \tilde{H}_m^\pi(\underline{t}_m))$, where $\tilde{H}_m^\pi(\underline{t}_m) = \sum_{j=1}^m H_{Bj}(t_{\pi(j)} - t_j)$.

If $\pi \neq \mathrm{id}$, then $\tilde{H}_m^\pi(\underline{t}_m) \neq 0$ on a set of full measure in $\Delta(0, m, t)$. As all H_{Bj}, $j = 1, \dots, m$ have purely absolutely continuous spectrum, $\tilde{H}_m^\pi(\underline{t}_m)$ has also purely absolutely continuous spectrum when it does not vanish. By the methods used in step 1 of the proof of Theorem 5.4, one proves the compactness of

$$\overline{\{[\underline{U}_{00}^\varepsilon(-t_m) \underline{\Omega}_m^* X_m^\varepsilon(\underline{t}_m)] \rho \otimes R^0 \otimes \cdots \otimes R^0 U_\pi \,|\, \varepsilon \in (0, 1]\}}$$

in $\mathscr{T}(\mathscr{H}_m)$. From Lemma 5.3 follows $\lim_{\varepsilon \downarrow 0} I_m^\varepsilon(\underline{t}_m) = 0$ for all $\underline{t}_m$ for which $\tilde{H}_m^\pi(\underline{t}_m) \neq 0$. ☒

6. The Averaged Generator

According to Theorem 4.1 the reduced dynamics is approximated by the semigroup $T_\varepsilon^0(t) = \exp\{(-i\varepsilon^{-1} L_S + K)t\}$, where the dissipative part of the generator is given by

$$K\rho = -i \operatorname{tr}_1 \underline{L}_{I1} \underline{\Omega}_1 \rho \otimes R^0. \tag{6.1}$$

In general, K will not be of the canonical form given by Lindblad, and therefore $T_\varepsilon^0(t)$ will not be a quantum dynamical semigroup. However, as in the weak coupling case [20], the semigroup $T_\varepsilon^\#(t) = \exp\{(-i\varepsilon^{-1} L_S + K^\#)t\}$ formed with the averaged generator

$$K^\# = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} dt \exp(-iL_S t) K \exp(iL_S t) \tag{6.2}$$

is a candidate for a quantum dynamical semigroup. The discreteness of the spectrum of H_S implies the existence of the limit in (6.2). The following approximation theorem holds:

Theorem 6.1. *For all $t \geqq 0$ and all $\rho \in \mathscr{T}(\mathscr{H}_S)$*

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \leqq s \leqq t} \| T_\varepsilon^\#(s)\rho - T_\varepsilon^0(s)\rho \|_1 = 0.$$

$T_\varepsilon^\#$ is a quantum dynamical semigroup.

Proof. The first part of the theorem is Theorem 1.4 of [20]. From Theorem 4.1 one has

$$\lim_{\varepsilon \downarrow 0} \| \exp\{i\varepsilon^{-1} L_S t\} R_0^\varepsilon(t) - \exp\{i\varepsilon^{-1} L_S t\} T_\varepsilon^\#(t)\rho \|_1 = 0$$

for sufficiently small t. $R_0^\varepsilon(t)$ is the reduced dynamics of a Hamiltonian time evolution, therefore $\rho \to \exp\{i\varepsilon^{-1} L_S t\} R_0^\varepsilon(t)$ is the dual of a completely positive map. As $[L_S, K^\#] = 0$, $\exp\{i\varepsilon^{-1} L_S t\} R_0^\varepsilon(t)$ has the weak limit $\exp\{i\varepsilon^{-1} L_S t\} \cdot T_\varepsilon^\#(t)\rho = \exp\{K^\# t\}\rho$. Therefore $\exp\{K^\# t\}$ is also the dual of a completely positive map, and consequently $T^{\#}_{\varepsilon}(t)=\exp\{-i\varepsilon^{-1}L_S t\}\exp\{K^{\#}t\}$ is a quantum dynamical semigroup. ☒

Using the spectral representation $H_S=\sum_n \omega_n|n\rangle\langle n|$ of the system Hamiltonian, one forms the operators $P_{mn}=|m\rangle\langle n|$, which are the eigenvectors of the system Liouvillian L_S. The operators $T=H_{I1}\Omega_1$ and $X=\Omega_1-\mathbb{1}$ may be represented as $T=\sum_{mn} P_{mn}\otimes T_{mn}$, $X=\sum_{mn} P_{mn}\otimes X_{mn}$.

Lemma 6.2. *The averaged generator is given explicitly by*

$$K^{\#}\rho=\sum_{m,n,m',n'} c_{mnm'n'}P_{mn}\rho P_{n'm'}+B^{\#}\rho+\rho B^{\#*}, \tag{6.3}$$

where

$$c_{mnm'n'}=-i\delta_{\omega_m-\omega_n,\omega_{m'}-\omega_{n'}}\operatorname{tr}(T_{mn}R^0X^*_{m'n'}-X_{mn}R^0T^*_{m'n'}) \tag{6.4}$$

and

$$B^{\#}=-i\sum_{\omega_n=\omega_{n'}} P_{nn'}\operatorname{tr}R^0T_{nn'}. \tag{6.5}$$

Proof. (6.1) may be written in the form

$$K\rho=\operatorname{tr}_1[-iT\rho\otimes R^0X^*+X\rho\otimes R^0(iT^*)]+\operatorname{tr}_1[-iT\rho\otimes R^0+\rho\otimes R^0(iT^*)].$$

Using the product representations of T and X, one obtains $K\rho=K_1\rho+K_2\rho$, where $K_1\rho=-i\sum_{mnm'n'} P_{mn}\rho P_{n'm'}\operatorname{tr}(T_{mn}R^0X^*_{m'n'}-X_{mn}R^0T^*_{m'n'})$ and $K_2\rho=B\rho+\rho B^*$, $B=-i\sum_{mn}P_{mn}\operatorname{tr}R^0T_{mn}$. Applying

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\exp(-iH_St)P_{mn}\exp(iH_St)\,\rho\exp(-iH_St)P_{n'm'}\exp(iH_St)$$
$$=\delta_{\omega_m-\omega_n,\omega_{m'}-\omega_{n'}}P_{mn}\rho P_{m'n'},$$

and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\exp(-iH_St)P_{mn}\exp(iH_St)=\delta_{\omega_m,\omega_n}P_{mn},$$

a straightforward calculation leads to (6.3). ☒

As $H_{I1}\in T(\mathscr{H}_S\otimes\mathscr{H}_e)$, the operator T is an integral operator with the momentum space representation $\langle m\otimes\phi|T|n\otimes\psi\rangle=\int d\underline{k}\int d\underline{k}'\,\hat{\phi}^*(\underline{k})T_{mn}(\underline{k},\underline{k}')\cdot\hat{\psi}(\underline{k}')$. If the kernel satisfies some additional regularity conditions, one obtains an integral representation for $K^{\#}$, from which the physical meaning of $K^{\#}$ becomes clear.

It is assumed:

(G) $\int_{-\infty}^{\infty} dt\,\|A\exp(-iH_et)A\|_1<\infty$,

(H) for all m, n, $T_{mn}(\underline{k}, \underline{k}')$ is jointly continuous and uniformly bounded in $\underline{k}$, $\underline{k}'$.

Theorem 6.3. *Assume* (G) *and* (H) *hold. Define* $T_\omega(\underline{k}, \underline{k}') = \sum_{\omega_m - \omega_n = \omega} T_{mn}(\underline{k}, \underline{k}') P_{mn}$. *Then*

$$\begin{aligned} K^\# \rho = &-i\left[\sum_{\omega_n = \omega_{n'}} P_{nn'} \operatorname{tr} R^0(T_{nn'} + T^*_{n'n}), \rho\right] \\ &+ 2\pi \sum_{\omega \in \mathrm{Sp}(L_S)} \int d\underline{k} \int d\underline{k}' R^0(\underline{k}) \delta(\underline{k}'^2/2 - \underline{k}^2/2 + \omega) \\ &\cdot \{T_\omega(\underline{k}', \underline{k}) \rho T^*_\omega(\underline{k}', \underline{k}) - \tfrac{1}{2}[T^*_\omega(\underline{k}', \underline{k}) T_\omega(\underline{k}', \underline{k}) \rho \\ &+ \rho T^*_\omega(\underline{k}', \underline{k}) T_\omega(\underline{k}', \underline{k})]\}. \end{aligned} \tag{6.6}$$

Proof. Let $\omega_m - \omega_n = \omega_{m'} - \omega_{n'} = \omega$. Using

$$\begin{aligned} X &= s - \lim_{t\to\infty} (e^{-iH_1 t} - e^{-iH_0 t}) e^{iH_0 t} \\ &= s - \lim_{t\to\infty} \int_0^t ds\, e^{-iH_0 s} [(-iH_{I1}) e^{-iH_1(t-s)} e^{iH_0(t-s)}] e^{iH_0 s}, \end{aligned}$$

one obtains

$$\begin{aligned} c_{mnm'n'} &= \operatorname{tr} R^0(X^*_{m'n'} T_{mn} - T^*_{m'n'} X_{mn}) \\ &= \lim_{t\to\infty} \int_0^t ds \operatorname{tr}\{|m'\rangle\langle n'| \otimes R^0 e^{-iH_0 s}[e^{-iH_0(t-s)} e^{iH_1(t-s)}(iH_{I1})] \\ &\cdot e^{iH_0 s}(\mathbb{1} \otimes T_{mn}) + |n\rangle\langle m| \otimes R^0(\mathbb{1} \otimes T^*_{m'n'}) \\ &\cdot e^{-iH_0 s}[iH_{I1} e^{-iH_1(t-s)} e^{iH_0(t-s)}] e^{iH_0 s}\}. \end{aligned} \tag{6.7}$$

The terms in square brackets converge ultraweakly to iT^* respectively iT for $t \to \infty$. The following estimate proves that they may be replaced by their limits:

$$\begin{aligned} &\int_0^t ds |\operatorname{tr}\{|m'\rangle\langle n'| \otimes R^0 e^{-iH_0 s} \\ &\quad \cdot [e^{-iH_0(t-s)} e^{iH_1(t-s)}(iH_{I1}) - iT^*] e^{iH_0 s} \mathbb{1} \otimes T_{mn}\}| \\ &\leqq \int_0^{t_0} ds |\operatorname{tr}\{|m'\rangle\langle n'| \otimes R^0 e^{-iH_0 s} \\ &\quad \cdot [e^{-iH_0(t-s)} e^{iH_1(t-s)}(iH_{I1}) - iT^*] e^{iH_0 s} \mathbb{1} \otimes T_{mn}\}| \\ &\quad + \int_{t_0}^\infty ds\, 2\|R^0\| \, \|Q\|_1 \, \|A e^{-iH_e s} A\|_1 \, \|Y^*_{nm}\|, \end{aligned}$$

where the product representation $\Omega_1 = \sum_{mn} P_{mn} \otimes Y_{mn}$ is used. By condition (G) the second term is smaller than any given $\varepsilon > 0$ for sufficiently large t_0. As $t \to \infty$, the integrand in the first integral converges to zero pointwise. It is bounded by the integrable function $s \mapsto 2\|R^0\| \, \|Q\|_1 \, \|A e^{-iH_e s} A\|_1 \, \|Y^*_{mn}\|$, therefore the integral vanishes for $t \to \infty$. A similar estimate holds for the second term in (6.7).

A straightforward calculation leads to

$$c_{mnm'n'} = \lim_{t\to\infty} \int_{-t}^{t} ds\, e^{i\omega s} \int d\underline{k} \int d\underline{k}' \exp\{-i(k^2/2 - k'^2/2)s\}$$
$$\cdot R^0(\underline{k}) \overline{T_{m'n'}(\underline{k}', \underline{k})}\, T_{mn}(\underline{k}', \underline{k})$$
$$= \lim_{\varepsilon\downarrow 0} \int d\underline{k} \int d\underline{k}' R^0(\underline{k}) 2\varepsilon/((k'^2/2 - k^2/2 + \omega) + \varepsilon^2)$$
$$\cdot \overline{T_{m'n'}(\underline{k}', \underline{k})}\, T_{mn}(\underline{k}', \underline{k}),$$

where an Abelian limit is used in the last step. By condition (H) one obtains finally

$$c_{mnm'n'} = 2\pi \int d\underline{k} \int d\underline{k}' R^0(\underline{k}) \delta(k'^2/2 - k^2/2 + \omega) \overline{T_{m'n'}(\underline{k}', \underline{k})}\, T_{mn}(\underline{k}', \underline{k}). \tag{6.8}$$

With this representation one has

$$\sum_{mnm'n'} c_{mnm'n'} P_{mn} \rho P_{m'n'} = \sum_{\omega\in \mathrm{Sp}(L_S)} 2\pi \int d\underline{k} \int d\underline{k}' \delta(k'^2/2 - k^2/2 + \omega)$$
$$\cdot R^0(\underline{k}) T_\omega(\underline{k}', \underline{k}) \rho T^*_\omega(\underline{k}', \underline{k}).$$

The remaining terms in (6.6) are obtained by writing $B^\#$ in the form $B^\# = B_1^\# + iB_2^\#$, where

$$B_1^\# = -i/2 \sum_{\omega_n = \omega_{n'}} P_{nn'} \operatorname{tr} R^0(T_{nn'} - T^*_{n'n}),$$

and

$$B_2^\# = -\sum_{\omega_n = \omega_{n'}} P_{nn'} \operatorname{tr} R^0(T_{nn'} + T^*_{n'n}).$$

The Hamiltonian term in (6.6) is formed with $B_2^\#$. $B_1^\#$ may be written in the form

$$B_1^\# = -\pi \sum_{\omega} \int d\underline{k} \int d\underline{k}' \delta(k'^2/2 - k^2/2 + \omega) R^0(\underline{k}) T^*_\omega(\underline{k}', \underline{k}) T_\omega(\underline{k}', \underline{k}), \tag{6.9}$$

which follows from the unitarity relation

$$T_{m'm}(\underline{k}', \underline{k}) - \overline{T_{mm'}(\underline{k}, \underline{k}')} = -2\pi i \sum_n \int d\underline{k}''\, \delta(k''^2/2 + \omega_n - k^2/2 - \omega_m)$$
$$\cdot \overline{T_{nm'}(\underline{k}'', \underline{k}')}\, T_{nm}(\underline{k}'', \underline{k}), \tag{6.10}$$

for $k^2/2 + \omega_m = k'^2/2 + \omega_{m'}$. Equation (6.10) is proved analogously to [24], Theorem XI.44. Using (6.9), one obtains the last two terms in (6.6). ☒

For each ω, $\underline{k}$, $\underline{k}'$

$$\rho \mapsto T_\omega(\underline{k}', \underline{k}) \rho T^*_\omega(\underline{k}', \underline{k}) - \tfrac{1}{2}[T^*_\omega(\underline{k}', \underline{k}) T_\omega(\underline{k}', \underline{k}) \rho + \rho T^*_\omega(\underline{k}', \underline{k}) T_\omega(\underline{k}', \underline{k})]$$

is of the canonical form given by Lindblad. The generator is obtained by a superposition of such terms with positive coefficients and is therefore also the generator of a quantum dynamical semigroup. The Hamiltonian term is a level shift induced by the interaction with the reservoir. The dissipative part describes the scattering process $|n\underline{k}\rangle \to |n'\underline{k}'\rangle$ of the system with the bath particles.

The δ-function accounts for the conservation of energy in the scattering process. The particle density $R^0(\underline{k})$ determines the rate at which bath particles are scattered.

The relaxation properties of the semigroup are stated in the following theorem.

Theorem 6.4. *Assume*:

(a) H_S *has non-degenerate spectrum,*

(b) *the microreversibility condition* $T_{mn}(\underline{k},\underline{k}') = T_{nm}(-\underline{k}', -\underline{k})$ *holds for* $k^2/2 + \omega_m = k'^2/2 + \omega_n$,

(c) *for each pair m, n there is a pair* $\underline{k}$, $\underline{k}'$ *with* $k^2/2 + \omega_m = k'^2/2 + \omega_n$ *satisfying* $T_{mn}(\underline{k},\underline{k}') \neq 0$.

Then $\rho_{eq} = \exp(-\beta H_S)/\mathrm{tr}\exp(-\beta H_S)$ *is a stationary state of* $T^{\#}_{\varepsilon}$ *and for all* $\rho \in \mathscr{T}(\mathscr{H}_S)$ *one has* $\lim_{t\to\infty} T^{\#}_{\varepsilon}(t)\rho = \rho_{eq}$.

Proof. $K^{\#}$ may be written as $K^{\#} = K^{\#}_H + K^{\#}_D$, where

$$K^{\#}_H \rho = -i\left[\sum_{\omega_n = \omega_{n'}} P_{nn'} \mathrm{tr} R^0 (T_{nn'} + T^*_{n'n}), \rho\right] \quad \text{and}$$

$$K^{\#}_D \rho = \sum_{mnm'n'} c_{mnm'n'} \{P_{mn}\rho P_{n'm'} - \tfrac{1}{2}(P_{n'm'}P_{mn}\rho + \rho P_{n'm'}P_{mn})\}.$$

As $K^{\#}_H$ and $K^{\#}_D$ commute and $K^{\#}_H \rho_{eq} = 0$ it is sufficient to prove $K^{\#}_D \rho_{eq} = 0$ and $\lim_{t\to\infty} \exp(K^{\#}_D t)\rho = \rho_{eq}$.

$\rho \in \mathscr{T}(\mathscr{H}_S)$ is represented as $\rho = \sum_{mn} \rho_{mn} P_{mn}$. For $\rho(t)$ evolving according to $(d/dt)\rho(t) = K^{\#}_D \rho(t)$, the coefficients $\rho_{mn}(t)$ satisfy the equations

$$\frac{d}{dt}\rho_{mn}(t) = \sum_{\substack{kl \\ \omega_k - \omega_l = \omega_m - \omega_n}} c_{mknl}\rho_{kl}(t) - \tfrac{1}{2}\sum_{k=1}^{N} (c_{kmkm} + c_{knkn})\rho_{mn}(t). \tag{6.11}$$

Let $I^{\omega} = \{m \mid \exists n : \omega_m - \omega_n = \omega\}$. In each class I^{ω} there is for every $m \in I^{\omega}$ a uniquely defined index $\hat{m}$ which satisfies $\omega_m - \omega_{\hat{m}} = \omega$. Each set $\{\rho_{m\hat{m}} \mid m \in I^{\omega}\}$ has a closed subdynamics.

First the diagonal elements are considered. The dynamics is given by

$$\frac{d}{dt}\rho_{mn}(t) = \sum_n c_{mnmn}\rho_{nn}(t) - \sum_n c_{nmnm}\rho_{mm}(t),$$

which is the master equation of a classical Markov process with transition rates $p_{m\to n} = c_{nmnm}$. From assumption (b) one obtains

$$c_{mnm'n'}\exp(-\beta\omega_n) = c_{nmn'm'}\exp(-\beta\omega_m), \tag{6.12}$$

from which follows the detailed balance condition

$$c_{mnmn}\exp(-\beta\omega_n) = c_{nmnm}\exp(-\beta\omega_m).$$

Hence ρ_{eq} is a stationary state. Furthermore, as (c) implies $c_{mnmn} > 0$ for all pairs m, n, $\lim_{t\to\infty} \rho_{nn}(t) = \rho_{eq\,nn}$ holds for all $\rho \in \mathscr{T}(\mathscr{H}_S)$.

It remains to show $\lim_{t\to\infty} \rho_{mn}(t) = 0$ for $m \neq n$. The closed subdynamics for any $\omega \neq 0$ is described by the equation of motion

$$\frac{d}{dt}\rho_{m\hat{m}}(t) = \sum_{k\in I^\omega} L^\omega_{mk}\rho_{k\hat{k}}(t), \tag{6.13}$$

where the coefficients L^ω_{mk} are determined by (6.11). Equation (6.13) is regarded as a differential equation in the Hilbert space $\mathscr{H}^\omega$ with the scalar product $(x,y) = \sum_{k\in I^\omega} x_k y_k \exp(\beta\omega_k)$. The proof is complete, if the matrix $L^\omega = (L^\omega_{mk})$ generates a strict contraction semigroup in $\mathscr{H}^\omega$.

The matrix L^ω is written as

$$L^\omega = \sum_{\substack{m,k\in I^\omega\\ m<k}} L^\omega_{(mk)} + L^\omega_{D1} + L^\omega_{D2},$$

$$L^\omega_{(mk)\,ij} = \begin{cases} c_{mk\hat{m}\hat{k}} & i=m, j=k\\ c_{km\hat{k}\hat{m}} & i=k, j=m\\ -\tfrac{1}{2}(c_{kmkm} + c_{\hat{k}\hat{m}\hat{k}\hat{m}}) & i=j=m\\ -\tfrac{1}{2}(c_{mkmk} + c_{\hat{m}\hat{k}\hat{m}\hat{k}}) & i=j=k\\ 0 & \text{elsewhere,}\end{cases}$$

$$L^\omega_{D1\,ij} = (c_{ii\hat{\imath}\hat{\imath}} - \tfrac{1}{2}(c_{iiii} + c_{\hat{\imath}\hat{\imath}\hat{\imath}\hat{\imath}}))\delta_{ij},$$

$$L^\omega_{D2\,ij} = -\frac{1}{2}\left(\sum_{k\in I^\omega_1} c_{kiki} + \sum_{k\in I^\omega_1} c_{k\hat{\imath}k\hat{\imath}}\right)\delta_{ij},$$

where $I^\omega_1 = \{1,\dots,N\}\backslash I^\omega$, $I^\omega_2 = \{1,\dots,N\}\backslash\{\hat{k}|k\in I^\omega\}$. Using the representation (6.8) of $c_{mnm'n'}$ and the relation (6.12), it is straightforward, but somewhat lengthy, to verify $\mathrm{Re}(x, L^\omega_{(mk)}x) \leqq 0$ for all $x\in\mathscr{H}^\omega$ and all $m<k$. Analogously one verifies $\mathrm{Re}(c_{mm\hat{m}\hat{m}} - \tfrac{1}{2}(c_{mmmm} + c_{\hat{m}\hat{m}\hat{m}\hat{m}})) \leqq 0$ for all $m\in I^\omega$, and consequently $\mathrm{Re}(x, L^\omega_{D1}x) \leqq 0$ for all $x\in\mathscr{H}^\omega$. Finally, as $c_{kmkm} > 0$ for all k, m and $I^\omega_1 \neq \emptyset$, the matrix L^ω_{D2} is strictly negative. ☒

7. Discussion

In this section some modifications and generalisations of the model are considered.

The Fermion reservoir may be replaced by a free Boson reservoir at some temperature $T>0$. For sufficiently low density one is above the transition temperature, and no condensate is present. Then the methods used in this paper apply. After performing the thermodynamic limit, one obtains the perturbation series (4.2) for the reduced density matrices, where the initial state is now given by

$$R^\varepsilon_n = \rho \otimes \sum_{\pi\in S(n)} U_\pi R^\varepsilon \otimes \cdots \otimes R^\varepsilon,$$

$$R^\varepsilon = (\exp(\beta(H_e - \mu_\varepsilon)) - 1)^{-1}.$$

As only the initial state changes, Theorem 4.1 also holds in the Boson case. At $T=0$, one remains in the condensed phase in the low density limit, and the methods used here are no longer applicable.

If the reservoir has Boltzmann statistics, the initial state in (4.2) is of the form $R^\varepsilon_n = \rho \otimes R^\varepsilon \otimes \cdots \otimes R^\varepsilon$. In this case the radius of convergence of (4.2) is infinite, and one obtains the convergence of $R_0^\varepsilon(t)$ for all times $t \geqq 0$. The finite radius of convergence of the series (4.2) for Fermion and Boson reservoirs is due to the statistics. As in the low density limit the reservoir has Boltzmann statistics, one would obtain convergence for all times, if one had a good *a priori* estimate of the contribution of the statistics.

It was assumed that the space dimension d of the reservoir is three. However, it is sufficient to assume $d \geqq 3$. For $d < 3$ condition (F) will not hold, in general.

The condition $\dim \mathscr{H}_S < \infty$ may be omitted. Then one assumes that H_S has completely discrete spectrum, which is a sufficient condition for the existence of the averaged generator. Instead of Theorem 4.1 one obtains the weaker result $\lim_{\varepsilon \downarrow 0} \operatorname{tr} X(R_0^\varepsilon(t) - R_0^0(t)) = 0$ for all $X \in \mathscr{T}(\mathscr{H}_S)$. If one takes $Q \in \mathscr{B}(\mathscr{H}_S)$, $Q \otimes A$ is no trace class operator, in general, and the existence and completeness of the wave operators does not follow from Kato–Birman theory. Therefore one has to state additional conditions to assure the existence and completeness of the wave operators.

The compactness argument for $X_k^\varepsilon(\underline{t}_k)$ used in the proof of Theorem 5.4 has to be modified. $\overline{\{X_k^\varepsilon(\underline{t}_k) | \varepsilon \in (0,1]\}}$ is no longer compact in $\mathscr{T}(\mathscr{H}_k)$. But for each $\delta > 0$ $X_k^\varepsilon(\underline{t}_k)$ may be decomposed in $X_k^\varepsilon(\underline{t}_k) = X_{k1}^\varepsilon(\underline{t}_k) + X_{k2}^\varepsilon(\underline{t}_k)$, such that $\|X_{k1}^\varepsilon(\underline{t}_k)\| \leqq \delta$ and $\overline{\{X_{k2}^\varepsilon(\underline{t}_k) | \varepsilon \in (0,1]\}}$ is compact. Using this decomposition, it is possible to modify the proof of Theorem 5.4 such that the result still holds.

The interaction may be generalized to $H_I = \sum_{i=1}^{r} Q_i \otimes F_i$, where $Q_i \in \mathscr{B}(\mathscr{H}_S)$, Q_i self adjoint and $F_i = \sum_{j \in \mathbb{N}} \alpha_{ij} a^+(f_{ij}) a(f_{ij})$, $\sum_j |\alpha_{ij}| < \infty$, $\alpha_{ij} \in \mathbb{R}$, $\|f_{ij}\| = 1$ for all $i = 1, \ldots, r$, $j \in \mathbb{N}$.

For times $t < 0$ a theorem similar to Theorem 4.1 may be derived, where the sign of the dissipative part K of the generator is reversed. This symmetric situation is due to the assumption of factorizing initial conditions at $t = 0$.

In this paper only the reduced density matrix $R_0^\varepsilon(t)$ of the system was studied. One would like also to consider the limit of the n-particle reduced density matrix $R_n^\varepsilon(t)$. For $\varepsilon \downarrow 0$ the motion of the bath becomes very fast, and the bath particles move out to infinity. Therefore $R_n^\varepsilon(t)$ will not have a limit, but in the interaction picture one may expect $\underline{U}_{0n}^\varepsilon(-t) R_n^\varepsilon(t)$ to have a limit. However, there is a problem. Technically, this problem is indicated in the fact that the cluster theorem, reducing the n-particle collision to a series of independent collisions, is no longer applicable. Physically, one should not expect the convergence to hold everywhere. This situation is familiar in the classical case. There one proves convergence only for those configurations, which have path histories without recollision events. The problem is to formulate a corresponding notion of convergence in the quantum case.

Appendix

Given an interaction Hamiltonian of the form $Q \otimes F$, it is in principle possible to check if conditions (E) and (F) are satisfied. The following theorem shows that the class of such interaction Hamiltonians is not empty.

Theorem A.1. *Let*

$$(1)\ \int_0^\infty \| |A|^{1/2} \exp(-iH_e t)|A|^{1/2} \| dt < \|Q\|^{-1},$$

$$(2)\ \int_{-\infty}^{\infty} \| |A|^{1/2} \exp(-iH_e t)\phi \| dt < \infty \quad \text{for } \phi \in \mathscr{D},$$

where $\mathscr{D}$ is a dense linear subspace in $\mathscr{H}_e$. Assume that $H_0 = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_e$ has purely absolutely continuous spectrum. Then $s\text{-}\lim_{t\to\infty} \exp(-iH_0 t)\exp(iH_1 t)\phi$ exists for all $\phi \in \mathscr{H}_1$, and $H_1 = H_0 + H_{I1}$ has purely absolutely continuous spectrum.

It is easy to find interaction Hamiltonians $H_{I1} = Q \otimes A$ satisfying (1) and (2). Let $A = \sum_{j=1}^{k} \alpha_j |f_j\rangle\langle f_j|$, where $\alpha_j \in \mathbb{R}$ and $f_j \in \mathscr{S}(\mathbb{R}^3)$. Then $t \mapsto (f_i, \exp(-iH_e t) f_j)$ is integrable for $i, j = 1, \ldots, k$, and therefore condition (1) is satisfied for sufficiently small $\|Q\|$. Condition (2) may also be satisfied, if one takes $\mathscr{D} = \mathscr{S}(\mathbb{R}^3)$.

If the theorem holds, H_1 has no bound states and therefore condition (E) is satisfied. The estimate $\|A \exp(-iH_e t)\phi\| \leq \| |A|^{1/2} \| \, \| |A|^{1/2} \exp(-iH_e t)\phi \|$ shows that (F) is also satisfied.

Proof of Theorem A.1. It is sufficient to prove the existence of $s\text{-}\lim_{t\to\infty} \exp(iH_0 t) \times \exp(-iH_1 t)\psi$ for all $\psi \in \mathscr{H}_1$. Then the scattering operator $s\text{-}\lim_{t\to\infty} \exp(iH_1 t) \times \exp(-iH_0 t)$ is unitary, and therefore H_1 has purely absolutely continuous spectrum.

One may restrict oneselves to proving the existence of $s\text{-}\lim_{t\to\infty} \exp(iH_0 t) \times \exp(-iH_1 t)\psi$ for all $\psi = \phi_S \otimes \phi$, $\phi_S \in \mathscr{H}_S$, $\phi \in \mathscr{D}$. The limit exists also for finite linear combinations of such vectors. The existence of the limit for all $\psi \in \mathscr{H}_1$ follows by a density argument.

To prove the existence of $s\text{-}\lim_{t\to\infty} \exp(iH_0 t)\exp(-iH_1 t)\psi$ one uses the perturbation series

$$\exp(iH_0 t)\exp(-iH_1 t)\psi = \sum_{n=0}^{\infty} (-i)^n \int_{\Delta(0,n,t)} \underline{dt}\, H_{I1}(t_n) \ldots H_{I1}(t_1)\psi, \tag{A.1}$$

where $H_{I1}(t) = \exp(-iH_0 t) H_{I1} \exp(iH_0 t)$. The n^{th} term of the series is estimated by

$$I_n = \left\| \int_{\Delta(0,n,t)} \underline{dt}\, H_{I1}(t_n) \ldots H_{I1}(t_1)\phi_S \otimes \phi \right\|$$

$$\leq \int_0^t ds_2 \ldots \int_0^t ds_n \int_0^{t-s_2-\cdots-s_n} ds_1 \|Q\|^n \|\phi_S\| \, \|A e^{-iH_e s_n} A \ldots A e^{-iH_e s_1} \phi\|,$$

where $s_1 = t_1$, $s_j = t_j - t_{j-1}$, $j = 2, \ldots, n$. Using the identity $A = |A|^{1/2} \operatorname{sgn} A \cdot |A|^{1/2}$, one obtains $I_n \leq ab^{n-1}$, where

$$a = \|\phi_S\| \, \|Q\| \, \| |A|^{1/2} \| \int_0^\infty ds \, \| |A|^{1/2} \exp(-iH_e s)\phi \|,$$

$$b = \|Q\| \int_0^\infty ds \, \| |A|^{1/2} \exp(-iH_e s) |A|^{1/2} \|.$$

Therefore the series (A.1) is majorized in norm by $1+\int_{n=1}^{\infty} I_n$, which is convergent for $b<1$.

To prove the convergence of the series (A.1) one still has to show that each term of the series has a limit. For this Cauchy's criterion is used. With the definitions $h(t)=\| |A|^{1/2}\exp(-iH_e t)|A|^{1/2}\|$, $h_1(t)=\| |A|^{1/2}\exp(-iH_e t)\phi\|$, one obtains the estimate

$$\left\| \left(\int_{\Delta(0,n,t+\tau)} \underline{dt} - \int_{\Delta(0,n,t)} \underline{dt} \right) H_{I1}(t_n)\dots H_{I1}(t_1)\psi \right\|$$

$$\leqq \int_t^{t+\tau} dt_n \int_0^{t_n} dt_{n-1}\dots\int_0^{t_2} dt_1 \|Q\|^n \|\phi_S\| \, \| |A|^{1/2}\| h(t_n-t_{n-1})\dots h(t_2-t_1)h_1(t_1)$$

$$\leqq \int_{\substack{s_1+s_2+\dots+s_n\geqq t \\ s_1,\dots,s_n\geqq 0}} ds_n \dots \int ds_1 \|Q\|^n \|\phi_S\| \, \| |A|^{1/2}\| h(s_n)\dots h(s_2)h_1(s_1).$$

$s_1,\dots,s_n \mapsto h_1(s_1)h(s_2)\dots h(s_n)$ is integrable on $([0,\infty))^n$, and the domain of integration decreases monotonically to the empty set for $t\to\infty$. Therefore the integral vanishes in the limit $t\to\infty$. □

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