Renormalization of the Higgs Model: Minimizers, Propagators and the Stability of Mean Field Theory[*]

Tadeusz Bałaban[**], John Imbrie, and Arthur Jaffe

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Dedicated to the memory of Kurt Symanzik

Abstract. We study the effective actions $S^{(k)}$ obtained by k iterations of a renormalization transformation of the U(1) Higgs model in $d=2$ or 3 space-time dimensions. We identify a quadratic approximation $S_Q^{(k)}$ to $S^{(k)}$ which we call mean field theory, and which will serve as the starting point for a convergent expansion of the Green's functions, uniformly in the lattice spacing. Here we show how the approximations $S_Q^{(k)}$ arise and how to handle gauge fixing, necessary for the analysis of the continuum limit. We also establish stability bounds on $S_Q^{(k)}$, uniformly in k. This is an essential step toward proving the existence of a gap in the mass spectrum and exponential decay of gauge invariant correlations.

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1. The Model

We study the Higgs-type model of a U(1) gauge field u coupled to a complex scalar field ϕ with a quartic self interaction. The total action functional is given on the unit lattice as

$$S(u, \phi) = \sum_p e(\varepsilon)^{-2}(1 - \mathrm{Re}\, u(p)) + \tfrac{1}{2}\sum_b |(D_u\phi)_b|^2 + \sum_x P(\phi_x) + E. \qquad (1.1)$$

The relation between the ε-lattice and unit-lattice actions is described in the Cargèse lectures [1], as well as a general introduction to mass generation in this model. This approximation with $\mathrm{Re}\, u$ is called the Wilson approximation to the continuum action.

The lattice points $x \in Z^d$ are endpoints of directed, unit length lattice bonds $b = (b_-, b_+)$ which bound unit squares (plaquettes) denoted by p. The gauge field u: bonds $\to$ U(1), and

$$u(p) \equiv \prod_{b \in \partial p} u_b.$$

Furthermore, we assume $b^{-1} = (b_+, b_-)$ and $u_b^{-1} = u_{b^{-1}}$. The covariant derivative $D_u\phi$ is defined by $(D_u\phi)_b = u_b\phi_{b_+} - \phi_{b_-}$, and the coupling constant $e(\varepsilon)$ is $e(\varepsilon) = e\varepsilon^{(4-d)/2}$. We consider $d = 2$ or 3 space-time dimensions, in which case the self interaction is

$$\mathscr{P}(\xi) = \lambda(\varepsilon)\,|\xi|^4 - \tfrac{1}{4}(m^2 + \delta m^2)\varepsilon^2|\xi|^2, \qquad (1.2)$$

where $\lambda(\varepsilon) = \lambda\varepsilon^{4-d}$. The mass shift is consistent with perturbation theory with

$$\delta m^2(\varepsilon) = \begin{cases} O(\ln \varepsilon^{-1}), & d = 2, \\ O(\varepsilon^{-1}), & d = 3. \end{cases} \qquad (1.3)$$

The constant $E = E(e, \lambda, \varepsilon, \Lambda)$ is chosen to include vacuum energy normalization and renormalization counterterms from perturbation theory.

The continuum limit of the model with the action (1.1) has not yet been proved to exist, but our methods should lead to that result. The continuum limit of another action, the so-called Gaussian action, has been extensively studied. The Gaussian action is obtained by replacing $\mathrm{Re}\, u(p)$ in (1.1) with its quadratic

approximation in Lie algebra variables. In two dimensions the existence theorem for the Gaussian action was established in [2]. In three dimensions and with a massive gauge field, the fundamental stability bounds were proved in [3] and the existence theorems based on these bounds are deduced in [4]. The infinite volume limit in these papers was analyzed using correlation inequalities, a method which is well suited for establishing existence theorems but which has not yet led to results on the nature of the spectrum in the resulting model. A lattice cutoff version of our spectral results is given in [5], where we also discuss the connection between Gaussian and Wilson actions on a lattice.

Our program is to study the Wilson action using a cluster expansion method. In this way we expect to both prove the existence of the theory and to establish properties of the particle spectrum. We expect that the Wilson action yields the same limiting continuum field theory as the Gaussian action. Hence the resulting invariance properties of the ground state and the properties of the particle spectrum which are consequences of the cluster expansion will also extend to the Gaussian model constructed previously. The Wilson action furthermore gives a natural setting for a possible generalization of these results to a nonabelian gauge model.

The basic strategy to gain control over the spectrum is to establish exponential decay of gauge invariant correlations. We develop a method of steepest descent to study integrals of the form

$$\int \psi e^{-S} \mathscr{D}u\mathscr{D}\phi / \int e^{-S} \mathscr{D}u\mathscr{D}\phi, \tag{1.4}$$

where ψ is a gauge invariant functional of u and ϕ. Thus we need to define an expansion $S(u,\phi) = S(u_{cl}, \phi_{cl}) + \ldots$ in terms of a classical field (u_{cl}, ϕ_{cl}) and a fluctuation field $(uu_{cl}^{-1}, \phi - \phi_{cl})$. We obtain this expansion in a sequence of steps, by studying block spin renormalization transformations of $S(u,\phi)$.

For most of this paper, we are concerned with identifying $S(u_{cl}, \phi_{cl})$ and determining its properties. In particular, we find a quadratic action S_Q which is close to $S(u_{cl}, \phi_{cl})$. The transformation $S \to S_Q$ is an approximation to our renormalization transformation. We study the quadratic terms which arise by iteration of the renormalization transformation, yielding $S_{Q_1}, S_{Q_2}, \ldots, S_{Q_k}$, and establish uniform stability estimates (strictly positive lower bounds) on the S_Q's. These quadratic forms dominate the action in the "small field" region. In other papers we estimate the corrections due to large fields and show that they do not affect the basic picture. The end result is exponential decay of gauge invariant correlations, and a mass gap for both the Higgs particle and the photon.

Let us return to the action (1.1). The formal picture depends upon relating the action S to quadratic actions S_Q. Note that we can define the plaquette field

$$f_p = (ie(\varepsilon))^{-1} \ln u(p),$$

so for small f_p,

$$\sum_p e(\varepsilon)^{-2}(1 - \mathrm{Re}\, u(p)) = \sum_p \tfrac{1}{2} f_p^2 + O(e(\varepsilon)^2).$$

The error term vanishes pointwise as $\varepsilon \to 0$ for $d < 4$, and shows that the action (1.1) is the formally correctly scaled unit lattice action to study the continuum limit

$\varepsilon \to 0$. In this paper we establish stability properties of various quadratic approximations to S. We use these later to establish properties of integrals (1.4).

2. Block Fields and Scaling

We study block spin renormalization of the Wilson action. This involves defining averages of u and ϕ in lattice blocks, and performing the integral (1.4) conditioned by the block averages taking given values. In physics terminology we integrate high momentum degrees of freedom while fixing low momentum degrees of freedom in each block. If the blocks have side length L, a small positive integer, then we obtain functions on an LZ^d lattice. Repeating this process leads us to study lattices $Z^d, LZ^d, L^2Z^d, ..., L^kZ^d, ...$ Since after each step it is often convenient to scale the LZ^d lattice back to the unit lattice, we consider lattices with spacing a, with

$$a = \varepsilon, ..., L^{-k}, ..., L^{-1}, 1, L. \tag{2.1}$$

The natural L^2 inner product for spacing a comes from the norm (for scalar functions on the a-lattice),

$$\|\phi\|_a^2 = \sum_{i \in T_a} |\phi_i|^2 a^d. \tag{2.2}$$

Here T_a denotes a lattice with spacing a and periodic (toroidal) boundary conditions. Generally we take $a = 1$, L, or L^{-k}, the latter arising from k-fold scalings of an L-lattice to a unit lattice, following a block spin transformation. We also use lattices

$$T_a^{(k)} = T_{aL^{-k}} \cap aZ^d. \tag{2.3}$$

Averaging operators map functions on the T_a lattice to functions on the T_{La} lattice. We specify the map from the unit-lattice to the L-lattice, but this definition extends naturally from an a-lattice to an La-lattice. We assume this extension in dealing with powers of averaging operators, etc.

Consider the example of a scalar field. Imbed the unit lattice in an L-lattice of block $B(y)$. Here $y = Ln$ denotes a corner of a block, and $n \in Z^d$. Then $B(y)$ consists of points $x = (x_1, ..., x_d)$ such that

$$Ln_j \leq x_j < L(n_j + 1), \quad j = 1, 2, ..., d. \tag{2.4}$$

Let Γ denote a path (contour) composed of bonds, and define

$$u(\Gamma) = \prod_{b \in \Gamma} u_b. \tag{2.5}$$

Contours Γ_1, Γ_2 can be composed if the endpoint of Γ_2 coincides with the starting point of Γ_1, so $u(\Gamma_1 \circ \Gamma_2) = u(\Gamma_1)u(\Gamma_2)$.

We require a set of standard contours Γ_{yx} from corners y of blocks $B(y)$ to points $x \in B(y)$. Define Γ_{yx} as the path obtained by following the coordinate axis one, then axis two, ..., etc., in going from x to y. For example, if $x \in B(y)$, then

$$x - y = \sum_{i=1}^d n_i e_i,$$

where e_i denote unit vectors in the ith coordinate directions [and $n_i \geqq 0$ as $x \in B(y)$]. The path Γ_{yx} is the path of length n_1 in the direction $-e_1$ followed by n_2 bonds in direction $-e_2$, etc. Let Γ^{-1} denote the reverse of the path Γ. For a point y' which is a corner of an adjacent block, let $\Gamma_{yy'}$ denote the L-lattice bond from y to y'.

The average $Q\phi$ of ϕ is defined by

$$(Q\phi)_y = L^{-d} \sum_{x \in B(y)} u(\Gamma_{yx}) \phi_x. \tag{2.6}$$

An important property of the average is that Q commutes with gauge transformations. In other words, let h denote a map from the unit lattice to $U(1)$. Then h defines the gauge transformation

$$\phi_y \to h(y)\phi_y \equiv \phi_y^h, \qquad u_b \to h(b_-)h(b_+)^{-1}u_b = u_b^h. \tag{2.7}$$

Clearly

$$(Q\phi)^h = Q\phi^h. \tag{2.8}$$

A further property of Q is that

$$QQ^* = I, \tag{2.9}$$

where Q^* is the adjoint in the scalar product (2.2). Thus Q^* maps functions on the a-lattice into functions on the $L^{-1}a$ lattice. Furthermore Q^*Q is an orthogonal projection. In other words Q is a partial isometry.

One choice of gauge is especially good for algebraic operations, namely axial gauge. In axial gauge we choose h in order to set $u_b = 1$ for every b which occurs in some Γ_{yx}. In other words, we specify a maximal tree $T(y)$ in each block $B(y)$ composed of bonds in any Γ_{yx}, $x \in B(y)$.

In the axial gauge, $Q\phi$ reduces to the ordinary average of ϕ, and Q^*Q is the projection onto ϕ which are constant on blocks $B(y)$.

Let us next consider the average of a gauge field u. If $b' = yy'$ is a bond on the L-lattice, then $\Gamma_{yy'} = b'$ and we average over contours $\Gamma_{xx'}$ which are translates of $\Gamma_{yy'}$, i.e. $x - x' = y - y'$. More specifically, define Q on gauge fields on the unit lattice by

$$(Qu)_{yy'} = u(\Gamma_{yy'}) \exp\left[L^{-d} \sum_{x \in B(y)} \ln u(\Gamma_{yx} \circ \Gamma_{xx'} \circ \Gamma_{y'x'}^{-1} \circ \Gamma_{y'y}) \right]. \tag{2.10}$$

In (2.10) choose the logarithm so that

$$-\pi \leqq \arg \ln u < \pi. \tag{2.11}$$

With this choice, the discontinuity of $\ln u$ occurs at $u = -1$, namely the region of largest action. This is also the region of maximum suppression of probability, and hence the discontinuity will be seen not to affect integrals (1.4) in an important manner.

The average (2.10) is highly nonlinear. We also desire a linear average, defined on Lie algebra variables. Suppose that we can define

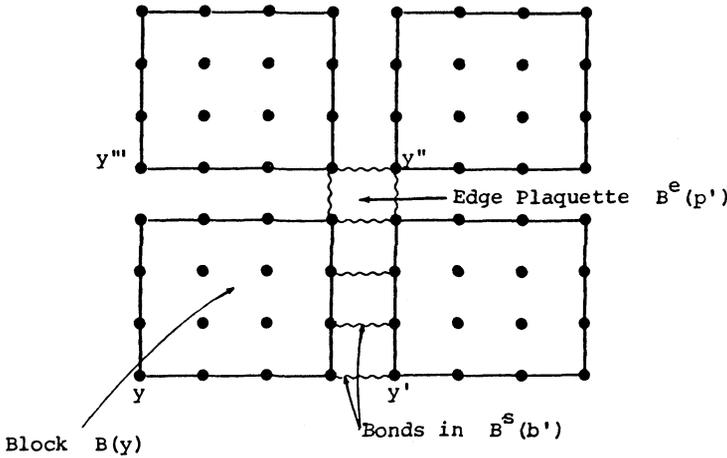$$A_b = (ie(\varepsilon))^{-1} \ln u_b. \tag{2.12}$$

Fig. 1. Block variables for gauge fields, drawn in $d=2$. Four blocks are illustrated. Let $B(y)$ denote a block with $L=4$. The L-lattice bonds illustrated are $b'=yy'$, $y'y''$, $y''y'''$, and $y'''y'$. These four L-bonds bound a plaquette p'. The corresponding edge plaquette $B^e(p')$ is indicated. The bond b' has associated surface bonds $B^s(b')$ which connect $B(y)$ and $B(y')$. A few bonds connecting distinct blocks are indicated by wavey lines

For $b'=(b'_-, b'_+)$ a (directed) L-lattice bond from b'_- to b'_+, we define the average

$$(QA)_{b'} = L^{-(d+1)} \sum_{x\in B(b'_-)} \sum_{b\in\Gamma_{xx'}} A_b, \tag{2.13}$$

where $\Gamma_{xx'}$ is the special contour from x to x' and xx' is the parallel transport of the bond b' to start at x. Thus the bonds b which enter the sum (2.13) range over the interior of the two L-blocks $B(b'_-)$ and $B(b'_+)$, as well as the (surface) bonds connecting these blocks.

The natural inner product on functions on bonds is

$$\langle A, B\rangle_a = \sum_b a^d A_b B_b. \tag{2.14}$$

We do not explicitly compute the adjoint Q^*. However, we now study related averages on surface bonds which connect adjacent blocks and on edge plaquettes which intersect four blocks.

Define the set of surface bonds $B^s(b')$ for an L-lattice bond b' as follows:

$$B^s(b') = \{b : b=(b_-, b_+), b_- \in B(b'_-), b_+ \in B(b'_+)\} \tag{2.15}$$

(see Fig. 1). Associated with the surface is the average

$$(Q^s A)_{b'} = L^{-(d-1)} \sum_{b\in B^s(b')} A_b. \tag{2.16}$$

Then

$$(Q^{s*}A)_b = \begin{cases} LA_{b'}, & \text{if } b\in B^s(b'), \\ 0, & \text{otherwise,} \end{cases} \tag{2.17}$$

using the inner product (2.14). Also

$$Q^s Q^{s*} = LI, \qquad Q^{s*} Q^s = LP^s, \tag{2.18}$$

where P^s is the projection onto configurations which are constant on surface bonds and zero on interior bonds. Furthermore, by (2.13),

$$QQ^{s*} = I. \tag{2.19}$$

Finally, we define an edge average Q^e associated with lattice plaquettes. The natural scalar product for functions on lattice plaquettes is

$$\langle f, g \rangle_a = \sum_p a^d f_p g_p, \tag{2.20}$$

where p is a plaquette on the a-lattice. Let p' denote an L-lattice plaquette. The set of edge plaquettes $B^e(p')$ are defined as those unit plaquettes p which (i) are parallel to p', (ii) whose bonds touch four distinct L-blocks, and (iii) such that these L-blocks contain the corners of p' (see Fig. 1). The edge average Q^e is defined by

$$(Q^e f)(p') = L^{-(d-2)} \sum_{p \in B(p')} f(p). \tag{2.21}$$

Then

$$(Q^{e*} f)(p) = \begin{cases} L^2 f(p'), & \text{if } p \in B^e(p'), \\ 0, & \text{otherwise.} \end{cases} \tag{2.22}$$

Also

$$Q^e Q^{e*} = L^2 I, \qquad Q^{e*} Q^e = L^2 P^e, \tag{2.23}$$

where P^e is the orthogonal projection onto configurations which are constant on edge plaquettes and zero elsewhere.

We also need the k-fold averaging operators $Q_k \equiv (Q)^k$, etc. Especially important are $Q_k^e \equiv (Q^e)^k$ and $Q_k^s = (Q^s)^k$ which satisfy

$$Q_k^e Q_k^{e*} = L^{2k} I = \eta^{-2} I, \qquad Q_k^s Q_k^{s*} = L^k I = \eta^{-1} I, \tag{2.24}$$

where $\eta = L^{-k}$.

Finally we consider a scale transformation $\mathscr{S}_{L^{-1}}$ which maps L-lattice fields to unit lattice fields. The canonical scaling for scalar fields

$$(\mathscr{S}_{L^{-1}} \phi)_x = L^{-(d-2)/2} \phi_{Lx}. \tag{2.25}$$

For gauge fields, the L-lattice action

$$\sum_p L^d \langle f_p, \sigma f_p \rangle$$

can be written in terms of a unit lattice action with

$$(\mathscr{S}_{L^{-1}} f)_p = L^{d/2} f_{Lp}, \tag{2.26}$$

so for a quadratic form σ,

$$\sum_{p \in T_L} L^d \langle f_p, \sigma f_p \rangle = \sum_{p \in T_1} \langle (\mathscr{S}_{L^{-1}} f)_p, \sigma(\mathscr{S}_{L^{-1}} f)_p \rangle. \tag{2.27}$$

Then the ε-lattice action

$$S_\varepsilon = \sum_{p \in T_\varepsilon} e^{-2} \varepsilon^{d-4}(1 - \operatorname{Re} u(p)) + \tfrac{1}{2} \sum_{b \in T_\varepsilon} \varepsilon^d |D_u \phi|^2 + \sum_{x \in T_\varepsilon} \varepsilon^d \mathscr{P}_\varepsilon(\phi_x) + E, \tag{2.28}$$

where $\mathscr{P}_\varepsilon(\xi) = \lambda |\xi|^4 - \tfrac{1}{4}(m^2 + \delta m^2)|\xi|^2$ is related to the unit lattice action (1.1) by

$$S_\varepsilon = S(\mathscr{S}_\varepsilon \phi, \mathscr{S}_\varepsilon u). \tag{2.29}$$

3. The First Renormalization Step

The renormalization transformation R defines a mapping

$$R: S \to S^{(1)}, \tag{3.1}$$

where S, $S^{(1)}$ are both unit lattice actions. This transformation involves two steps:

(i) Integrate high momentum degrees of freedom while fixing block spin averages.

(ii) Rescale the resulting L-lattice action to the unit lattice.

Thus we can write

$$R = \mathscr{S}_L \mathscr{T}, \tag{3.2}$$

where $\mathscr{T}$ is defined by

$$(\mathscr{T}e^{-S})(v, \psi) = \int e^{-S(u, \phi)} \delta_{Ax}(u) \delta(v/Qu) \delta_H(\psi - Q\phi) \mathscr{D}u \mathscr{D}\phi. \tag{3.3}$$

In (3.3) we specify the axial gauge with $\delta_{Ax}(u)$ which sets $u_b = 1$ on the set of bonds chosen as follows: Within each L-block $B(y)$ with corner y in the L-lattice, let $T(y)$ denote the tree composed of unit bonds in Γ_{yx}. Here x ranges over $B(y)$, and $T(y)$ is maximal – in the sense that adjoining any additional unit bond b in $B(y)$ to $T(y)$ makes $T(y)$ multiply connected. Then

$$\delta_{Ax}(u) = \prod_{y \in T_L} \prod_{b \in T(y)} \delta(u_b). \tag{3.4}$$

Let $\mathscr{G}_0$ denote the subgroup of gauge transformations which are specified by h's in (2.7) which are constant on each L-block $B(y)$. These gauge transformations preserve the axial gauge and affect only bonds on the unit lattice which connect different blocks. They generate the gauge group of the integral $\mathscr{T}(\exp - S)$.

The factor $\delta(v/Qu)$ is a delta function which specifies that the average gauge field in each block $B(y)$ is v. Thus for b' an L-lattice bond,

$$\delta(v/Qu) = \prod_{b' \in T_L} \delta(v_{b'}/(Qu)_{b'}). \tag{3.5}$$

The factor δ_H is an approximate delta function which specifies that the block averages of the scalar (Higgs) field equal ψ. In particular, if y denotes an L-lattice site,

$$\delta_H(\psi - Q\phi) = \prod_{y \in T_L} [(a/2\pi) \exp(-(1/2)a|\psi - Q(u)\phi|^2)]. \tag{3.6}$$

The integral (3.2) therefore has the normalization property

$$\int \mathscr{T}e^{-S} \mathscr{D}v \mathscr{D}\psi = \int e^{-S} \mathscr{D}u \mathscr{D}\phi, \tag{3.7}$$

where $\mathscr{D}v \mathscr{D}\psi$ is a measure corresponding to (3.3) but on the L-lattice.

Having defined the renormalization transformation, we now analyze its behavior for small fields. This will yield a Gaussian approximation to the integral (3.1) and a corresponding quadratic form for $S^{(1)}$. The quadratic form has the structure

$$S_Q^{(1)} = \tfrac{1}{2} \langle f^{(1)}, \sigma_1 f^{(1)} \rangle + \tfrac{1}{2} \langle \psi, \Delta_1(u_1)\psi \rangle, \tag{3.8}$$

which we now derive and analyze. In particular we will establish strict positivity of σ_1 and $\Delta_1(u_1)$ in (3.8).

We begin the study of (3.1) by rewriting the Wilson action contribution to $S(u, \phi)$. First define a unit lattice field u'_b with block averages equal to 1. In this way we can write the resulting u'_b in terms of a vector potential which fluctuates about zero. Let

$$(u')_b = \begin{cases} u_b, & \text{if} \quad b \notin B^s(b'), \\ u_b v_{b'}^{-1}, & \text{if} \quad b \in B^s(b') \end{cases} \tag{3.9}$$

be given in terms of the averaging operator Q of (2.10), where

$$v_{b'} = (Qu)_{b'}. \tag{3.10}$$

It then follows that

$$(Qu')_{b'} = 1 \tag{3.11}$$

for all L-lattice bonds b'. Now define A'_b by

$$(u')_b = \exp(ie(\varepsilon) A'_b). \tag{3.12}$$

Then the Wilson action can be written

$$\sum_{p \in T_1} e(\varepsilon)^{-2} (1 - \operatorname{Re} u(p)) = \tfrac{1}{2} \sum_{p \in T_1} |\partial A' + L^{-d/2} Q^{e*} F|^2 + O(e(\varepsilon)^2). \tag{3.13}$$

Here we define the L-lattice field $F(p')$ in terms of $v(p')$, where

$$v(p') \equiv \prod_{b' \in \partial p'} v_{b'} \equiv \exp[ie(L\varepsilon) F(p')]. \tag{3.14}$$

The factor $L^{-d/2}$ in (3.13) then arises from combining the factor L^{-2} from (2.22) with $L^{-(d-4)/2} = e(\varepsilon)/e(L\varepsilon)$.

The quadratic form in (3.13) can be reexpressed by translating to its minimum. By the Appendix, with $\alpha = \partial$ and $B = L^{-d/2} Q^{e*} F$, we have

$$A_{cl} = -L^{-d/2} C \partial^* Q^{e*} F. \tag{3.15}$$

Here $C = (\partial^* \partial)^{-1}$, restricted to the subspace of A satisfying both $QA = 0$ and the axial gauge condition. Then by Corollary A2, and

$$A' = A + A_{cl}, \tag{3.16}$$

$$\tfrac{1}{2} \sum_{p \in T_1} |\partial A' + L^{-d/2} Q^{e*} F|^2 = \tfrac{1}{2} \sum_{p \in T_1} |\partial A|^2 + \tfrac{1}{2} \langle F, \Sigma F \rangle, \tag{3.17}$$

where

$$\Sigma = L^{-d} Q^e (I - \partial C \partial^*) Q^{e*}. \tag{3.18}$$

The fluctuation field A lives on the unit lattice, and will be integrated. For small fluctuations we use this Gaussian approximation to yield a perturbative expression for the integral. Large fluctuations will lead to small probabilities.

The quadratic form $\langle F, \Sigma F \rangle$ provides the formula for the background field and it will come outside the integral, since the average fields v and hence F are fixed. This form will be scaled to the unit lattice. This scaling absorbs the factor L^{-d}, so

$$\mathscr{S}_L \langle F, \Sigma F \rangle = \langle f^{(1)}, \sigma_1 f^{(1)} \rangle, \tag{3.19}$$

where

$$\sigma_1 = Q^e(I - \partial C\partial^*)Q^{e*}, \qquad f^{(1)}(p) = \frac{1}{ie(L\varepsilon)} \ln v(p). \tag{3.20}$$

Here Q^{e*} maps the unit lattice to the L^{-1} lattice, and $\partial C\partial^*$ is an L^{-1} lattice operator.

Define the unit lattice field

$$u^{(1)} = u \exp(-ie(\varepsilon)A), \tag{3.21}$$

with A the fluctuation field defined by (3.16). Then after scaling back to the L^{-1} lattice,

$$u_1 \equiv \mathscr{S}_L^* u^{(1)}$$

is the gauge field on the L^{-1} lattice on which the unit lattice renormalized action depends.

Let us now direct attention to the boson part of the integral (3.1). We expand

$$S(u, \phi) + \tfrac{1}{2}a \|\psi - Q(u)\phi\|^2 \tag{3.24}$$

to identify the quadratic terms in ϕ. Using (3.23), we also expand u about its background value u_1, yielding additional contributions from A which are accompanied by (small) factors $O(e(\varepsilon))$. Thus the leading terms in the Higgs action are

$$\tfrac{1}{2}a \|Q(u^{(1)})\phi - \psi\|^2 + \tfrac{1}{2}\langle \phi, \Delta_{u^{(1)}}\phi \rangle, \tag{3.25}$$

with all corrections included in the potential $V_1(A, \phi, f_1)$ which have coefficients $O(\varepsilon)$.

Define

$$G = (-\Delta_{u^{(1)}} + aQ(u^{(1)})^*Q(u^{(1)}))^{-1}. \tag{3.26}$$

The quadratic form (3.25) is minimized by

$$\phi_{cl} = -aGQ(u')^*\psi. \tag{3.27}$$

See Appendix A. Writing $\phi = \phi_{cl} + \phi_{fl}$, expand (3.24) into background and fluctuation parts, yielding

$$\tfrac{1}{2}\langle \phi_{fl}, G^{-1}\phi_{fl} \rangle + \tfrac{1}{2}a\langle \psi, [I - aQGQ^*]\psi \rangle. \tag{3.28}$$

In (3.28) the field ϕ_{fl} is a unit lattice field but ψ is an L-lattice field. The scaling to the unit lattice will rewrite the second term in (3.28) in terms of a unit lattice field $\psi^{(1)} = \mathscr{S}_L^*\psi$. Let $G_1 = \mathscr{S}_L^* G \mathscr{S}_L = G_1(u_1)$ denote the scaled Green's function, which is expressed in terms of the L^{-1} lattice gauge field u_1.

4. Form of the kth Effective Action

In this section we state the quadratic or mean field approximation to the action which is produced after k renormalization transformations. We define the basic objects and give some detailed formulas for the quadratic action. Basically the

action is the sum of two parts: a gauge field action and a scalar field part which is quadratic in ϕ and depends on an average value of the gauge field. There are further interaction terms between the gauge field and the scalar field, but these interactions yield small corrections to our mean field action and are not considered here.

We begin with a discussion of the gauge field terms. The scalar field is more straightforward and is postponed to the final subsection. The formulas which we give here are justified in Sect. 6, when we show that renormalization of the kth effective action yields the quadratic form for the $(k+1)^{st}$ effective action. The action has the form

$$S_k(v, \psi) = \tfrac{1}{2}\langle f^{(k)}, \sigma_k f^{(k)}\rangle + \tfrac{1}{2}\langle \psi, \Delta_k(u_k)\psi\rangle + \text{interaction terms}, \qquad (4.1)$$

where the interaction terms will be considered in detail in a later paper.

4.1. The Axial Gauge Propagator and Minimizer

We require an axial gauge propagator to express the effective action. Define $G_{k,Ax}$ by the functional integral

$$\exp(\tfrac{1}{2}\langle J, G_{k,Ax}J\rangle) = Z_{k,Ax}^{-1}\int \mathscr{D}A\,\delta(Q_kA)\,\delta_{k,Ax}(A)\exp(-\tfrac{1}{2}\|\partial A\|^2 + \langle A, J\rangle).$$
$$(4.1.1)$$

Here the normalization factor $Z_{k,Ax}$ can be computed by setting $J=0$. The averaging operator Q_k is the k-fold composition of the 1-step averaging operators Q for bond variables, $Q_k = (Q)^k$. It follows that Q_k is given by the formula (2.13), where L is replaced by L^k. Finally, the axial gauge is fixed by the delta function

$$\delta_{k,Ax}(A) \equiv \prod_{j=0}^{k-1} \delta_{Ax}(Q_jA). \qquad (4.1.2)$$

The jth factor in the product acts on the $L^j\eta = L^{j-k}$ lattice. It sets bond field averages Q_jA to zero on contours Γ_{yx} defined in the $L^j\eta$ lattice in the same way that the axial gauge was fixed in Sect. 3 on contours in the unit lattice.

The reader may wonder whether ∂ has zero modes on the subspace of gauge fields satisfying $Q_kA = 0$ and satisfying the axial gauge condition. Such zero modes do not occur, and as a consequence the integral (4.1.1) is convergent also for noncompact gauge fields. A proof of this fact for $k=1$ follows by considering on each plaquette in a lattice block the condition $\partial A = 0$ and the axial gauge condition. This shows that A can be nonzero only on bonds connecting different blocks, and it must be constant on the bonds connecting two given blocks. The condition $QA = 0$ then ensures that A is everywhere zero. Similar, elementary reasoning yields an inductive proof for $k > 1$, but we leave out the details.

The propagator $G_{k,Ax}$ itself is the second moment of the measure $\exp(-\tfrac{1}{2}\|\partial A\|^2)$, restricted to the space of gauge fields which satisfy the kth axial gauge condition and condition on the kth average fields vanishing. We do not know a simple operator theoretic definition of $G_{k,Ax}$, and we generally establish the properties of the propagator by appealing to the definition (4.1.1).

Another aspect of the action $S_G(A) = \tfrac{1}{2}\|\partial A\|^2$ concerns what configurations minimize $S_G(A)$ subject to the constraints of a gauge condition and a condition that the average field Q_kA equals a given value B. We introduce a transformation

$H_{k, Ax}$ which maps B into such a minimizing configuration for axial gauge, and we call $H_{k, Ax}$ the *axial gauge minimizer*. Explicitly,

$$H_{k, Ax} B = Z_{k, Ax}(B)^{-1} \int \mathcal{D} A \delta(Q_k A - B) \delta_{k, Ax}(A) A \exp(-\tfrac{1}{2} \|\partial A\|^2), \quad (4.1.3)$$

where

$$Z_{k, Ax}(B) = \int \mathcal{D} A \delta(Q_k A - B) \delta_{k, Ax}(A) \exp(-\tfrac{1}{2} \|\partial A\|^2). \quad (4.1.4)$$

Note that by definition

$$Q_k H_{k, Ax} B = B. \quad (4.1.5)$$

4.2. The Quadratic Action and Plaquette Fields

We begin with the definition of the basic quadratic form σ_k. Again we use a functional integral definition

$$\exp(-\tfrac{1}{2} \langle f, \sigma_k f \rangle) = Z_{k, Ax}^{-1} \int \mathcal{D} A \delta(Q_k A) \delta_{k, Ax}(A) \exp(-\tfrac{1}{2} \|\partial A - Q_k^{e*} f\|^2). \quad (4.2.1)$$

Here f is any real (Lie algebra) valued field defined on unit lattice plaquettes.

Clearly σ_k is related to the propagator $G_{k, Ax}$ of Sect. 4.1. Comparing (4.2.1) with (4.1.1) we find

$$\sigma_k = Q_k^e (I - \partial G_{k, Ax} \partial^*) Q_k^{e*} = \eta^{-2} I - Q_k^e \partial G_{k, Ax} \partial^* Q_k^{e*}. \quad (4.2.2)$$

Here we have used $Q_k^e Q_k^{e*} = \eta^{-2}$, see (2.24). Note that if $k = 1$, then $G_{k, Ax} = C$ and (4.2.2) agrees with (3.21).

The right side of (4.2.2) involves the combination $\partial G_{k, Ax} \partial^*$ which is invariant under a change of gauge. Hence the gauge chosen to define σ_k is irrelevant, and the quadratic form σ_k is gauge invariant. We use this property in Sect. 7 when we derive an explicit momentum-space representation for σ_k. Using that representation we also establish a uniform, positive lower bound

$$0 < c \leq \sigma_k. \quad (4.2.3)$$

The quadratic part of the effective action is obtained by evaluating σ_k on a special field configuration $f^{(k)}$ related to the gauge field after k renormalization steps. Let v be defined for unit lattice bonds, with values in the group U(1). Define the plaquette field $f^{(k)}$ on the unit lattice by

$$f^{(k)} = (i e_k)^{-1} \ln v(\partial p), \quad p \in T_1^{(k)}, \quad \text{where} \quad e_k = e(L^k \varepsilon)^{(4-d)/2}. \quad (4.2.4)$$

The quadratic part of the action for the gauge field is then

$$\tfrac{1}{2} \langle f^{(k)}, \sigma_k f^{(k)} \rangle. \quad (4.2.5)$$

We also use a field f_k which lives on the $\eta = L^{-k}$ lattice and is defined by

$$f_k = (I - \partial G_{k, Ax} \partial^*) Q_k^{e*} f^{(k)}. \quad (4.2.6)$$

Thus $f_k(p)$ is a function with $p \in T_\eta$. The action (4.2.5) also can be written

$$\tfrac{1}{2} \langle f^{(k)}, \sigma_k f^{(k)} \rangle = \tfrac{1}{2} \|f_k\|^2. \quad (4.2.7)$$

This identity is a consequence of the fact that $\partial G_{k,Ax}\partial^*$ is a projection operator. Alternatively, we compute from (4.1.1) and (4.2.1) that

$$\exp(-\tfrac{1}{2}\langle f, \sigma_k f\rangle) = \exp(\tfrac{1}{2}\langle \partial^* Q_k^{e*} f, G_{k,Ax}\partial^* Q_k^{e*} f\rangle)\exp(-\tfrac{1}{2}\|Q_k^{e*} f\|^2),$$

from which (4.2.7) follows.

4.3. Quadratic Action for Curls

The quadratic form σ_k simplifies on curls, namely for fields f of the form $f = \partial B$. In particular

$$\langle \partial B, \sigma_k \partial B\rangle = \langle B, \Delta_k B\rangle,\tag{4.3.1}$$

which defines an action Δ_k. To give a functional integral representation for Δ_k, we note that

$$\begin{aligned}
\exp(-\tfrac{1}{2}\langle \partial B, \sigma_k \partial B\rangle) &= Z_{k,Ax}^{-1}\int \mathscr{D}A\delta(Q_k A)\delta_{k,Ax}(A)\exp(-\tfrac{1}{2}\|\partial A - Q_k^{e*}\partial B\|^2)\\
&= Z_{k,Ax}^{-1}\int \mathscr{D}A\delta(Q_k A - B)\delta_{k,Ax}(A)\exp(-\tfrac{1}{2}\|\partial A\|^2)\\
&= \exp(-\tfrac{1}{2}\langle B, \Delta_k B\rangle).
\end{aligned}\tag{4.3.2}$$

Here we have performed a translation of A on surface bonds $A \to A + Q_k^{s*}B$. We use the fact that the axial gauge condition is independent of the surface bonds, we use $\partial Q_k^{s*} = Q_k^{e*}\partial$, and we also use (2.19).

The action Δ_k yields the unit lattice propagator $C^{(k)}$, defined as follows:

$$\exp(\tfrac{1}{2}\langle J, C^{(k)}J\rangle) = (Z^{(k)})^{-1}\int \mathscr{D}B\delta(QB)\delta_{Ax}(B)\exp(-\tfrac{1}{2}\langle B, \Delta_k B\rangle + \langle B, J\rangle).\tag{4.3.3}$$

The actions Δ_k and propagators $C^{(k)}$ were studied by Balaban [6II] who established that $C^{(k)}$ is well defined and is bounded in norm

$$\|C^{(k)}\| \leqq c\tag{4.3.4}$$

uniformly in k. Furthermore $C^{(k)}$ has a kernel which decays exponentially, uniformly in k.

$$|C^{(k)}(x,y)| \leqq a\exp(-b|x-y|).\tag{4.3.5}$$

4.4. Landau Gauge Propagator and Minimizer

The axial gauge has many useful properties, especially in providing convenient definitions of Green's functions, and for establishing many algebraic identities in Sect. 5. However, the axial gauge does not provide either good regularity or good decay properties for the Green's functions. These are much better in Landau gauge.

Landau gauge is defined for our lattice theory in a complicated way, because of the restrictions on gauge transformations. We introduce a gauge fixing function $\mathscr{G}(\partial^* A)$. The exact form of $\mathscr{G}$ is important for the proof of estimates. An appropriate choice is

$$\mathscr{G}(\partial^* A) = \exp(-\tfrac{1}{2}\|\partial^* A\|^2)/(\int d\lambda\delta(Q'\lambda)\exp(-\tfrac{1}{2}\|\partial^* A - \Delta\lambda\|^2)),\tag{4.4.1}$$

where Q' denotes the ordinary average over k-blocks. Such a Faddeev-Popov type choice leads to the formula for the Landau gauge minimizer

$$H_k B = Z_k(B)^{-1} \int \mathscr{D}A \delta(Q_k A - B) \mathscr{G}(\partial^* A) A \exp(-\tfrac{1}{2}\|\partial A\|^2), \qquad (4.4.2)$$

where

$$Z_k(B) = \int \mathscr{D}A \, \delta(Q_k A - B) \mathscr{G}(\partial^* A) \exp(-\tfrac{1}{2}\|\partial A\|^2). \qquad (4.4.3)$$

We could define a well-behaved, Landau gauge propagator G_k by a formula similar to (4.4.1), but with $\mathscr{G}(\partial^* A)$ replacing $\delta_{k,Ax}(A)$. However, we do not use this propagator in our consideration of the Higgs effect. Rather, we define a propagator $\mathscr{D}_k$ in terms of minimizers H_j and the covariance operators $C^{(j)}$ of Sect. 4.3. Let

$$\mathscr{D}_k \equiv \sum_{j=0}^{k-1} H_j C^{(j)} H_j^*. \qquad (4.4.4)$$

Here the transformations H_j are now defined by the same formulas as (4.4.2); however, they act on the $\eta = L^{-k}$ lattice instead of the L^{-j} lattice and they involve j^{th} order averages Q_j. Similarly the propagators $C^{(j)}$ are defined by the formula (4.3.3) but on the $L^j \eta = L^{j-k}$ lattice, rather than on the unit lattice. They still involve one-step averages Q.

We remark that $\mathscr{D}_k$, $G_{k,Ax}$, and G_k are related by change of gauge formulas, some of which we establish in Sect. 5.

4.5. Bond Fields

The first renormalization step in Sect. 3 yielded an effective action $S^{(1)}(v, \psi)$ depending on a unit lattice field v, through the L^{-1} lattice field u_1. We need to generalize this picture to the k-step action $S^{(k)}$. After k renormalization steps we again have as independent variables the gauge field v defined on the unit lattice. We introduce a generalization of the configuration u_1 denoted u_k. To begin, we have to define a transformation Q_k^{s*} from unit-lattice, group-valued configurations v to η-lattice, group-valued configurations.

We can represent an arbitrary v as

$$v_b = \exp(ie_k B_b), \qquad (4.5.1)$$

where $B_b = (ie_k)^{-1} \ln v_b$, and the branch of the logarithm was chosen in (2.11). Then we extend the definition Q_s^* from the Lie algebra to group variables v by the definition

$$(Q_k^{s*} v)_b = \exp((ie_k \eta Q_k^{s*} B)_b), \qquad b \in T_\eta. \qquad (4.5.2)$$

This raises the question of what happens if we choose another branch of the logarithm. The independence of the resulting transformation on this choice follows from the equivalent definition:

$$(Q_k^{s*} v)_b = \begin{cases} 1 & \text{if } b \text{ is strictly contained in a } k\text{-block} \\ & \text{(both endpoints belong to the block)} \\ v_c & \text{if the } \eta\text{-lattice bond } b \text{ belongs to the corridor} \\ & \text{of bonds connecting the two blocks } B^k(c_-) \text{ and} \\ & B^k(c_+). \text{ Here } c \text{ is a unit lattice bond}. \end{cases} \qquad (4.5.3)$$

Now we define

$$(u_k)_b = (Q_k^{s*}v)_b \exp[-ie_k\eta(\mathscr{D}_k\partial^*Q_k^{e*}f^{(k)})_b].$$ (4.5.4)

This field configuration u_k is the minimal configuration for the approximate action, up to a gauge transformation. It serves as the background field (external gauge field) in the expressions for the scalar field action and propagators.

4.6. The Scalar Field Action

The fundamental propagator $G_k(u_k)$ for the scalar field in this theory can also be defined by a functional integral,

$$\exp(\tfrac{1}{2}\langle h, G_k(u_k)h\rangle) = Z_k(u_k)^{-1}\int\mathscr{D}\phi\exp[-\tfrac{1}{2}\|D_{u_k}\phi\|^2 - \tfrac{1}{2}a_k\|Q_k(u_k)\phi\|^2 + \langle\phi, h\rangle].$$ (4.6.1)

Since no restrictions on ϕ occur in the Gaussian integral (4.6.1), we can also write

$$G_k(u_k) = [-\Delta_{u_k} + a_kQ_k^*(u_k)Q_k(u_k)]^{-1},$$ (4.6.2)

where

$$-\Delta_{u_k} = D_{u_k}^*D_{u_k}.$$ (4.6.3)

The coefficients a_k are produced by iterating one-step renormalization transformations which use a constant a in the Gaussian. This yields $a_k = a(1-L^{-2})(1-L^{-2k})^{-1}$ in the k-step transformation [3].

The quadratic form which arises for the scalar field is $\langle\psi, \Delta_k(u_k)\psi\rangle$, where ψ is the unit lattice scalar field and where

$$\Delta_k(u_k) = a_kI - a_k^2Q_k(u_k)G_k(u_k)Q_k^*(u_k).$$ (4.6.4)

The scalar field action depends on the unit lattice field ψ through the η-lattice minimizer ψ_k, where

$$\psi_k = a_kG_k(u_k)Q_k^*(u_k)\psi.$$

The other propagators such as $C^{(k)}(u_k)$ are defined as in [3] with the only change that the background gauge field is u_k.

5. Relations Among Minimizers and Propagators

5.1. Change of Gauge for Minimizers

In this section we discuss relations among the operators H and G. We begin by relating $H_{k,Ax}$ and H_k. Recall that H_kB is the configuration which minimizes the action $\|\partial A\|^2$, subject to a gauge condition as well as the restriction $Q_kA = B$ on field averages. We claim that $H_{k,Ax}B$ and H_kB differ by a gauge transformation,

$$H_{k,Ax}B - H_kB = \partial\lambda.$$ (5.1.1)

We show that λ is an explicit, linear function of H_kB. In order to specify λ, we choose a sequence of points $x_0, \ldots, x_k$, where $x = x_0$, $y = x_k$, and where

$$x \in B^j(x_j), \qquad x_j \in T_{L^j\eta}^{(j)},$$ (5.1.2)

that is

$$x_j \in B(x_{j+1}). \tag{5.1.3}$$

The contour $\Gamma_{x_j, x}$ runs from x to x_1 in $B(x_1)$, from x_1 to x_2 in $B(x_2)$, etc.

Proposition 5.1.1. *The relation* (5.1.1) *holds with*

$$\lambda(x) = -\sum_{j=0}^{k-1} L^{j-k}\left[(Q_j H_k B)(\Gamma_{x_{j+1}, x_j}) - \sum_{x' \in B(x_{j+1})} (L^{-d} Q_j H_k B)(\Gamma_{x_{j+1}, x'})\right]. \tag{5.1.4}$$

Here we use $(Q_j H_k B)(\Gamma) = \sum_{b \in \Gamma}(Q_j H_k B)(b)$, *with no factor of the lattice spacing for* Γ.

Proof. The definition of $H_{k, Ax}B$ is

$$H_{k, Ax}B = Z_{k, Ax}(B)^{-1} \int dA\delta(Q_k A - B)\delta_{k, Ax}(A) \exp(-\tfrac{1}{2}\|\partial A\|^2)A, \tag{5.1.5}$$

where

$$\delta_{k, Ax}(A) = \prod_{j=0}^{k-1} \delta_{Ax}(Q_j A). \tag{5.1.6}$$

Furthermore,

$$H_k B = Z_k(B)^{-1} \int dA\delta(Q_k A - B)\mathcal{G}(\partial^* A) \exp(-\tfrac{1}{2}\|\partial A\|^2)A, \tag{5.1.7}$$

where $\mathcal{G}(\partial^* A)$ is a gauge fixing term, and where both integrals are normalized so that without the final factor A they equal 1. We use a variant of the Faddeev-Popov procedure and insert the factor 1 in the integrand (5.1.5), expressed as

$$1 = Z_k^{-1} \int d\lambda\delta(Q_k'\lambda)\mathcal{G}(\partial^* A - \Delta\lambda).$$

Therefore,

$$Z_{k, Ax}(B) = \int dA\delta(Q_k A - B)\delta_{k, Ax}(A) \exp(-\tfrac{1}{2}\|\partial A\|^2)$$
$$= Z_k^{-1} \int dA d\lambda\, \delta(Q_k'\lambda)\mathcal{G}(\partial^* A - \Delta\lambda)\delta(Q_k A - B)\delta_{k, Ax}(A) \exp(-\tfrac{1}{2}\|\partial A\|^2).$$

Making the gauge transformation $A \to A + \partial\lambda$, and using $Q_k A \to Q_k A + Q_k \partial\lambda = Q_k A + \partial Q_k'\lambda = Q_k A$, on account of the δ function yields

$$Z_{k, Ax}(B) = Z_k^{-1} \int dA\mathcal{G}(\partial^* A)\delta(Q_k A - B) \exp(-\tfrac{1}{2}\|\partial A\|^2)\int d\lambda\, \delta(Q_k'\lambda)\delta_{k, Ax}(A + \partial\lambda)$$
$$= Z_k^{-1} \int dA\mathcal{G}(\partial^* A)\delta(Q_k A - B) \exp(-\tfrac{1}{2}\|\partial A\|^2) = Z_k^{-1}Z_k(B). \tag{5.1.8}$$

If we perform the same steps leading to (5.1.8) in (5.1.5), we obtain by (5.1.8)

$$H_{k, Ax}B = Z_k(B)^{-1} \int dA\, \delta(Q_k A - B)\mathcal{G}(\partial^* A) \exp(-\tfrac{1}{2}\|\partial A\|^2)$$
$$\cdot \int d\lambda\delta(Q_k'\lambda)\delta_{k, Ax}(A + \partial\lambda)(A + \partial\lambda). \tag{5.1.9}$$

In order to evaluate (5.1.9), we solve the system of linear equations for λ which result from the delta functions $\delta(Q_k'\lambda)$ and $\delta_{k, Ax}(A + \partial\lambda)$. Using this value of $\lambda = \lambda(A)$, we have

$$H_{k, Ax}B = Z_k(B)^{-1} \int dA\, \delta(Q_k A - B)\mathcal{G}(\partial^* A) \exp(-\tfrac{1}{2}\|\partial A\|^2)(A + \partial\lambda(A)).$$

We now study the equations which determine $\lambda(A)$. The axial gauge fixing conditions in (5.1.9) entails

$$(Q_j A)(\Gamma_{z,z'}) = -(L^j\eta)^{-1}[(Q_j'\lambda)(z') - (Q_j'\lambda)(z)],$$

where z and z' are endpoints of a bond in $B(x_{j+1})$ which lies on an axial gauge fixing contour. Since every point in $B(x_{j+1})$ lies on some such contour, we can sum these identities to show that for all $x' \in B(x_{j+1})$,

$$(Q_j A)(\Gamma_{x_{j+1}, x'}) = -(L^j\eta)^{-1}[(Q_j'\lambda)(x') - (Q_j'\lambda)(x_{j+1})]. \qquad (5.1.10)$$

Summing over $x' \in B(x_{j+1})$ yields

$$(Q_j'\lambda)(x_{j+1}) = (Q_{j+1}'\lambda)(x_{j+1}) + L^j\eta Q'(Q_j A)(\Gamma_{x_{j+1}, \cdot}). \qquad (5.1.11)$$

We use the fact that $(Q_j'\lambda)(x_{j+1})$ is constant in this sum, and the final Q' acts on "$\cdot$". Substituting (5.1.11) in (5.1.10) with $x' = x_j$ then gives

$$(Q_j'\lambda)(x_j) = (Q_{j+1}'\lambda)(x_{j+1}) - L^j\eta[(Q_j A)(\Gamma_{x_{j+1}, x_j}) - Q'(Q_j A)(\Gamma_{x_{j+1}, \cdot})]. \qquad (5.1.12)$$

The identities (5.1.12) hold for $j = 0, 1, \ldots, k-1$. Setting $j = 0$ we have

$$\lambda(x) = (Q_0'\lambda)(x_0) = (Q'\lambda)(x_1) - \eta[A(\Gamma_{x_1, x}) - (Q'A)(\Gamma_{x_1}, \cdot)].$$

Furthermore, summing (5.1.12) from $j = 0$ to $j = k - 1$, the factors $(Q_j'\lambda)(x_j)$ cancel except for $j = 0$ and $j = k$. Thus

$$\lambda(x) = (Q_k'\lambda)(x_k) - \sum_{j=0}^{k-1} L^j\eta[(Q_j A)(\Gamma_{x_{j+1}, x_j}) - Q'(Q_j A)(\Gamma_{x_{j+1}, \cdot})].$$

Using the δ function for $Q_k'\lambda$, and $\eta = L^{-k}$ gives

$$\lambda(x) = \lambda(A, x) = -\sum_{j=0}^{k-1} L^{j-k}[(Q_j A)(\Gamma_{x_{j+1}, x_j}) - Q'(Q_j A)(\Gamma_{x_{j+1}, \cdot})]. \qquad (5.1.13)$$

We now notice that $\lambda = \lambda(A)$ is a linear function of A. Thus we can write $\lambda(A, x) = \lambda(H_k B, x) - \lambda(A - H_k B, x)$. The term $\lambda(H_k B, x)$ is independent of A and comes outside the A integral. Thus

$$H_{k, Ax}B = H_k B + \partial\lambda(H_k B) + Z_k(B)^{-1}\int dA\,\delta(Q_k A - B)\mathcal{G}(\partial^* A)$$
$$\cdot \exp(-\tfrac{1}{2}\|\partial A\|^2)\partial\lambda(A - H_k B). \qquad (5.1.14)$$

Since $H_k B$ minimizes the quadratic form $\|\partial A\|^2$ subject to the restrictions imposed by $\delta(Q_k A - B)$ and $\mathcal{G}$, the term in (5.1.14) involving $\partial\lambda(A - H_k B)$ is linear in $A - H_k B$ and hence its integral vanishes. Thus $H_{k, Ax}B = H_k B + \partial\lambda(H_k B)$, and the proof of Proposition 5.1.1 is complete.

5.2. Decomposition of Axial Gauge Green's Functions

Let $C^{(k)}$ denote the unit lattice propagator defined in (4.3.3). Here we give a simple relation between $C^{(k)}$ and the axial gauge propagator $G_{k, Ax}$ of (4.1.1). This relation is similar to the definition (4.4.4) of the Landau gauge propagator $\mathcal{D}_k$ in terms of the $C^{(j)}$. The algebraic identity for $G_{k, Ax}$ will be very useful, especially in computing the form of the action after renormalization, even though the axial gauge propagator has poor regularity and decay properties.

Proposition 6.2. *The* $G_{k,Ax}$ *satisfy the recursion relation*

$$G_{k,Ax} = H_{k-1,Ax} C^{(k-1)} H^*_{k-1,Ax} + G_{k-1,Ax}, \tag{5.2.1}$$

with the solution:

$$G_{k,Ax} = \sum_{j=0}^{k-1} H_{j,Ax} C^{(j)} H^*_{j,Ax}. \tag{5.2.2}$$

Proof. Clearly (5.2.1) and (5.2.2) are equivalent. We prove (5.2.1) starting from the definition of $G_{k,Ax}$, namely (4.1.1). Write $Q_k = QQ_{k-1}$, and expand the integrand of (4.1.1) using

$$\delta(Q_k A) \delta_{k,Ax}(A) = \int \mathscr{D}B\, \delta(QB) \delta(Q_{k-1} A - B) \delta_{Ax}(B) \delta_{k-1,Ax}(A).$$

Then translate the integrand with respect to A to the minimum of the quadratic form $\frac{1}{2}\|\partial A\|^2 - \langle A, J\rangle$, under the restriction $Q_{k-1} A = B$ and the appropriate axial gauge. The minimum is achieved at $H_{k-1,Ax} B$, so we write

$$A = A' + H_{k-1,Ax} B. \tag{5.2.3}$$

Inserting this and using (4.1.5) gives

$$\begin{aligned}
\exp(\tfrac{1}{2}\langle J, G_{k,Ax} J\rangle) = Z^{-1} \int &\mathscr{D}B\, \delta(QB) \delta_{Ax}(B) \\
&\cdot \exp(-\tfrac{1}{2}\|\partial H_{k-1,Ax} B\|^2 + \langle B, H^*_{k-1,Ax} J\rangle) \\
&\cdot \int \mathscr{D}A'\, \delta(Q_{k-1} A') \delta_{k-1,Ax}(A') \\
&\cdot \exp(-\tfrac{1}{2}\|\partial A'\|^2 + \langle A', J\rangle). \tag{5.2.4}
\end{aligned}$$

The A' integral yields $Z_{k-1,Ax} \exp(\tfrac{1}{2}\langle J, G_{k-1,Ax} J\rangle)$, while the B integral gives

$$ZZ^{-1}_{k-1,Ax} \exp(\tfrac{1}{2}\langle H^*_{k-1,Ax} J, C^{(k-1)} H^*_{k-1,Ax} J\rangle). \tag{5.2.5}$$

Thus (5.2.2) holds and the proposition follows.

Now that we have related H_k and $H_{k,Ax}$, we can find the relation between $\mathscr{D}_k$ and $G_{k,Ax}$. In fact we need only the compositions $\mathscr{D}_k \partial^*$ and $G_{k,Ax} \partial^*$, since only these products occur.

Proposition 5.2.2. *There is a gauge transformation* D *such that*

$$G_{k,Ax} \partial^* - \mathscr{D}_k \partial^* = \partial D. \tag{5.2.6}$$

Explicitly

$$D = \sum_{j=0}^{k-1} \lambda_j (H_j C^{(j)} H^*_j \partial^*), \tag{5.2.7}$$

where λ_j *is the function of Proposition 5.1.1 with k set equal to j. Here we write the operator identity, rather than the identity* $G_{k,Ax} \partial^* B = \mathscr{D}_k \partial^* B + \partial DB$ *for configurations.*

Proof. We use three facts: the formula (5.2.1), the fact that

$$H^*_{j,Ax} \partial^* = H^*_j \partial^*, \tag{5.2.8}$$

is gauge invariant, and formulas (5.1.1), (5.1.4) for the factor $H_{j,Ax}-H_j$ in the expression

$$G_{k,Ax}\partial^* - \mathscr{D}_k\partial^* = \sum_{j=0}^{k-1} (H_{j,Ax}-H_j)C^{(j)}H_j^*\partial^* . \tag{5.2.9}$$

Substitution then yields (5.2.7).

Remark 1. A consequence of the proposition is

$$\partial G_{k,Ax}\partial^* = \partial\mathscr{D}_k\partial^* . \tag{5.2.10}$$

Remark 2. The gauge field quadratic form $\langle f^{(k)}, \sigma_k f^{(k)}\rangle$ can now be written entirely as a function of the η-lattice gauge field u_k. In fact

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \sum_{p\in T_\eta} \eta^d |f_k(p)|^2 , \tag{5.2.11}$$

where

$$f_k(p) = (ie_k\eta^2)^{-1} \ln u_k(\partial p) . \tag{5.2.12}$$

This follows from the definition (4.5.4) of u_k, giving

$$u_k(\partial p) = \exp[ie_k\eta^2 Q_k^{e*}f^{(k)} - ie_k\eta^2\partial\mathscr{D}_k\partial^* Q_k^{e*}f^{(k)}] .$$

From this and (5.2.10) we get

$$(ie_k\eta^2)^{-1} \ln u_k(\partial p) = Q_k^{e*}f^{(k)} - \partial G_{k,Ax}\partial^* Q_k^{e*}f^{(k)} ,$$

coinciding with (4.2.6).

5.3. Minimizers and Green's Functions

To help define the fluctuation field in the $(k+1)^{st}$ step, we need one more identity, namely

$$H_{k,Ax}B = Q_k^{s*}B - G_{k,Ax}\partial^* Q_k^{e*}B . \tag{5.3.1}$$

This identity follows by inspecting the definition (4.1.3) after the translation $A = A' + Q_k^{s*}B$. Use the facts $Q_k Q_k^{s*} = I$ and $\partial Q_k^{s*}B = Q_k^{e*}\partial B$ to obtain

$$H_{k,Ax}B = Z_{k,Ax}(B)^{-1}\int \mathscr{D}A'\delta(Q_k A')\delta_{k,Ax}(A')(A'+Q_k^{s*}B)\exp(-\tfrac{1}{2}\|\partial A' + Q_k^{e*}\partial B\|^2) .$$

The second term yields $Q_k^{s*}B$. In the first term we translate to the minimum of the quadratic form which is $G_{k,Ax}\partial^* Q_k^{e*}\partial B$. We see this by integrating.

6. Renormalization Transformations Produce the Effective Action as Claimed

In this section we study the renormalization of the quadratic part of the kth action S_k in (4.1). This action depends on the plaquette field $f^{(k)}$, the (background) bond field u_k and the scalar field ψ. Under renormalization we prove that $S_k \to S_{k+1}$. We exhibit this transformation by writing S_k as a sum of two terms: the first is S_{k+1}, but scaled from the unit lattice to the L-lattice. The second term is the contribution of the fluctuation field on the unit lattice. The fluctuation field will be integrated in

performing the renormalization transformation, while the remaining terms scale back to give S_{k+1} on the unit lattice. In (4.1) we ignore the small corrections due to interactions and concentrate on the quadratic terms. Hence the result of this section is to justify our (otherwise ad hoc) choice of σ_k and $\varDelta_k(u_k)$ as the mean field quadratic forms for the kth effective action.

In accord with the notation of Sect. 3 we denote the variables in S_k as u and ϕ which take their values on the unit lattice. Thus we study

$$\int \mathscr{D}u\mathscr{D}\phi\, \delta(v/Qu)\delta_{Ax}(u)\exp[-\tfrac{1}{2}a\|\psi-Q(u_k)\phi\|^2 - S_k(u,\phi)]. \qquad (6.1)$$

In this integral we perform a preparatory translation (the first translation in Sect. 3) to isolate the average gauge field on the kth length scale. Let v denote the average gauge field on the L-lattice, and on the unit lattice let

$$u=u'Q^{s*}v, \qquad u'=\exp(ie_kB'). \qquad (6.2)$$

The plaquette field

$$f^{(k)}(p)=(ie_k)^{-1}\ln u(\partial p), \qquad p\in T_1, \qquad (6.3)$$

then decomposes as

$$f^{(k)}(p)=(ie_k)^{-1}\ln u'(\partial p)+(ie_k)^{-1}\ln(Q^{s*}v)(\partial p)=(\partial B')(p)+L^{-d/2}(Q^{e*}f^{(k+1)})(p), \qquad (6.4)$$

which is valid as long as $u'(p)$ and $(Q_k^{s*}v)(p)$ are close to 1. The new field $f^{(k+1)}$ is an L-lattice field, but Q_k^{e*} maps it back (to edge plaquettes) in the unit lattice.

The integral (6.1) will be studied in detail in later papers by dividing it into small and large field regions, defined by some restrictions on the range of ϕ and u. In this paper we are concerned with the small field regions which give the dominant contributions to the integral. In this region the identity (6.4) holds.

We will also see that the perturbative terms in the action are unaffected by the restriction to small fields, since these restrictions become trivial in the limit of zero coupling constants. In particular, this means that in the calculation of leading terms in the action S_{k+1} we can assume that the integrations over fields in (6.1) are restricted only by the delta functions which appear in the definition of the renormalization transformation.

6.1. $\sigma_k=\sigma_{k+1}+Fluctuation\ Form$

The goal here is to show that

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \mathscr{S}_L^{-1}\langle f^{(k+1)}, \sigma_{k+1}f^{(k+1)}\rangle + \langle B, \varDelta_k B\rangle. \qquad (6.1.1)$$

In other words, the quadratic form σ_k in S_k which occurs in the integrand (6.1) can be decomposed into the sum of two independent forms: the $(k+1)$-step quadratic form scaled to the L-lattice and the quadratic form for the fluctuation field B. Hence the fluctuation field can be integrated and, after rescaling to the unit lattice, we obtain the desired $(k+1)$-step gauge field quadratic form. Using (6.4),

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \langle \partial B', \sigma_k \partial B'\rangle + 2L^{-d/2}\langle \partial B', \sigma_k Q^{e*}f^{(k+1)}\rangle$$
$$+ L^{-d}\langle f^{(k+1)}, Q^e\sigma_k Q^{e*}f^{(k+1)}\rangle. \qquad (6.1.2)$$

Use definition (4.3.1) of Δ_k and the representation (4.2.2) for σ_k. In addition, note that $Q^e Q_k^e = Q_{k+1}^e$. Then

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \langle B', \Delta_k B'\rangle + 2L^{-d/2}\langle (I - \partial G_{k,Ax}\partial^*)Q_k^{e*}\partial B', Q_{k+1}^{e*} f^{(k+1)}\rangle$$
$$+ L^{-d}\langle Q_{k+1}^{e*} f^{(k+1)}, (I - \partial G_{k,Ax}\partial^*)Q_{k+1}^{e*} f^{(k+1)}\rangle. \qquad (6.1.3)$$

By (5.3.1), $H_{k,Ax} = Q_k^{s*} - G_{k,Ax}\partial^* Q_k^{e*}\partial$. Apply ∂ to this identity, and use the gauge invariance statement

$$\partial H_{k,Ax} = \partial H_k \qquad (6.1.4)$$

and the identity $\partial Q_k^{s*} = Q_k^{e*}\partial$. It follows that

$$\partial H_k = (I - \partial G_{k,Ax}\partial^*)Q_k^{e*}\partial. \qquad (6.1.5)$$

Substituting (6.1.5) in (6.1.3) gives

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \langle B', \Delta_k B'\rangle + 2L^{-d/2}\langle B', H_k^*\partial^* Q_{k+1}^{e*} f^{(k+1)}\rangle$$
$$+ L^{-d}\langle Q_{k+1}^{e*} f^{(k+1)}, (I - \partial G_{k,Ax}\partial^*)Q_{k+1}^{e*} f^{(k+1)}\rangle. (6.1.6)$$

Consider the quadratic form (6.1.6) as a function of B' and subject to the axial gauge and averaging restrictions of the integral (6.1). The minimum of the form is given by Proposition A3 of the Appendix, namely

$$B' = -L^{-d/2}C^{(k)}H_k^*\partial^* Q_{k+1}^{e*} f^{(k+1)}. \qquad (6.1.7)$$

The proposition is applicable with $\mathcal{H}_0$ the subspace of axial gauge configurations defined by the kth axial gauge delta function $\delta_{k,Ax}(B')$ and the delta function $\delta(QB')$. The covariance C of the proposition then agrees with $C^{(k)}$ defined in (4.3.3), and this is also the covariance in (6.1.7).

Furthermore, the translation

$$B' = B - L^{-d/2}C^{(k)}H_k^*\partial^* Q_{k+1}^{e*} f^{(k+1)} \qquad (6.1.8)$$

and the decomposition (A16) yield

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle$$
$$= \langle B, \Delta_k B\rangle + L^{-d}\langle Q_{k+1}^{e*} f^{(k+1)}, (I - \partial G_{k,Ax}\partial^* - \partial H_k C^{(k)}H_k^*\partial^*)Q_{k+1}^{e*} f^{(k+1)}\rangle. \qquad (6.1.9)$$

Using (6.1.4) and the representation (5.1.15), we can rewrite (6.1.9) as

$$\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \langle B, \Delta_k B\rangle + \mathcal{S}_L^{-1}\langle f^{(k+1)}, \sigma_{k+1} f^{(k+1)}\rangle,$$

where the scaling $\mathcal{S}_L$ absorbs the factor L^{-d} and where

$$\sigma_{k+1} = Q_{k+1}^e(I - \partial G_{k+1,Ax}\partial^*)Q_{k+1}^{e*}.$$

This verifies the inductive hypothesis.

6.2. $u_k = u_{k+1} \cdot$ Fluctuation (Modulo Gauge Transformation)

We now investigate the effect of the two translations defined in Sect. 6.1 on the field u_k. This allows us to separate a fluctuation field and a gauge transformation which

then leave u_{k+1}. Rewrite the definition of u_k (with u in place of v), so

$$u_k = Q_k^{s*} u \exp(-ie_k \eta \mathcal{D}_k \partial^* Q_k^{e*} f^{(k)}). \qquad (6.2.1)$$

By (6.2) we have

$$Q_k^{s*} u = (Q_{k+1}^{s*} v) e^{ie_k \eta Q_k^{s*} B'}, \qquad (6.2.2)$$

so substituting this and (6.3) into (6.2.1) we obtain

$$u_k = Q_{k+1}^{s*} v \exp[ie_k \eta Q_k^{s*} B' - ie_k \eta(\mathcal{D}_k \partial^* Q_k^{e*} \partial B') - ie_k \eta L^{-d/2}(\mathcal{D}_k \partial^* Q_{k+1}^{e*} f^{(k+1)})]. \qquad (6.2.3)$$

We have the identity $G_{k,Ax} \partial^* = \mathcal{D}_k \partial^* + \partial D$ to replace $\mathcal{D}_k \partial^*$ by $G_{k,Ax} \partial^*$. Then we apply again the identity (5.3.1) to replace the first two terms in the exponential by $ie_k \eta H_{k,Ax} B'$. This takes care of the first translation.

The second translation (6.1.5) then yields

$$u_k = Q_{k+1}^{s*} v \exp[ie_k \eta H_{k,Ax} B - ie_k \eta L^{-d/2} H_{k,Ax} C^{(k)} H_k^* \partial^* Q_{k+1}^{e*} f^{(k+1)}$$
$$- ie_k \eta L^{-d/2}(\mathcal{D}_k \partial^* Q_{k+1}^{e*} f^{(k+1)})] \cdot \text{(gauge trans)}. \qquad (6.2.4)$$

We make another gauge transformation to replace the axial gauge minimizers by Landau gauge minimizers, and we combine the second and third term into an operator $\mathcal{D}_{k+1}$. After rescaling, the $L^{-d/2}$ factor is absorbed and we have

$$u_k = e^{ie_k \eta H_k B} u_{k+1} \text{ (the gauge transformation)}, \qquad (6.2.5)$$

where the gauge transformation is generated by

$$\exp[ie_k D(Q_k^{e*} \partial B') + ie_k \lambda_k(H_k B) - ie_k L^{-d/2} \lambda_k(H_k C^{(k)} H_k \partial^* Q_{k+1}^{e*} f^{(k+1)})]. \qquad (6.2.6)$$

6.3. Scalar Field Renormalization

In Sect. 6.2 we established the general decomposition (6.2.5) for u_k,

$$u_k = u_{k+1} e^{ie_k \eta H_k B} e^{i\eta \partial \omega}, \qquad (6.3.1)$$

where ω denotes the gauge transformation (6.2.6). This gauge transformation does not influence the gauge field quadratic form σ_k, since this form is fully gauge invariant. More generally, expressions involving only the gauge field are gauge invariant, so we must discuss the scalar field part of the action.

In that case, the form $\Delta_k(u_k)$ does depend on the gauge transformation ω. Our procedure for constructing S_k has the general invariance

$$S_k(u_k e^{-i\eta \partial \lambda}, e^{i\lambda} \phi) = S_k'(u_k, \phi) \qquad (6.3.2)$$

for a gauge transformation λ. While we do not prove this invariance here, it is clear in the case of the quadratic forms for which we write explicit formulas.

The scalar field integral has the form

$$\int \mathcal{D}\phi \exp\{-[S_k(u_k, \phi) + \tfrac{1}{2} a \|\psi - Q(u_k)\phi\|^2]\}. \qquad (6.3.3)$$

If we apply the decomposition (6.3.1) for u_k, and the gauge invariance (6.3.2), then (6.3.3) becomes

$$\int \mathcal{D}\phi \exp(-[S_k(u_{k+1} e^{ie_k \eta H_k B}, \phi) + \tfrac{1}{2} a \|e^{-i\omega} \psi - Q(u_{k+1} e^{ie_k \eta H_k B})\phi\|^2]). \quad (6.3.4)$$

Here we use the gauge invariance of the measure $\mathcal{D}\phi$. The resulting integral still depends on ω. However, the integral of (6.3.4) over ψ is independent of ω, since $\mathcal{D}\psi$ is also gauge invariant.

We can either write the new action in terms of $\psi' = e^{-i\omega}\psi$, or use this gauge invariance and the fact that we are only interested in integrals of (6.3.4) over ψ, times other gauge invariant functions of ψ (as observables). We take the latter point of view and ignore ω.

Now we are in exactly the same situation as [3]. We expand the whole integrand with respect to $H_k B$ and perform the fluctuation field integral over boson fields as in that paper. It reproduces the $(k+1)$ quadratic form for the action.

7. Positivity and Localization of the Effective Action

Positivity bounds on σ_k and $\Delta_k(u_k)$ are a necessary step in proving that the renormalization transformation preserves stability of the effective action. The main difficulty centers on the quadratic form σ_k, which arises from the gauge field action. In particular, σ_k comes from the Wilson form of the action. Unlike the "Gaussian" action for the gauge field used in earlier studies of the U(1) Higgs model, the Wilson action has rather subtle positivity properties. We study this question first, yielding the main result of this section, $0 < c \leqq \sigma_k$. We then finish by stating minor generalizations of known estimates which apply to the other operators H_k, $\mathcal{D}_k$, and $\Delta_k(u_k)$ introduced in this paper.

7.1. Positivity of σ_k

Theorem 7.1.1. *There exists a constant $c > 0$, independent of k, such that*

$$c \leqq \sigma_k. \tag{7.1.1}$$

The strategy of the proof is to give an explicit formula for σ_k, which we then analyze in detail. Since we study periodic boundary conditions, σ_k is translation invariant. Thus it is natural to study σ_k as a multiplication operator $\sigma_k(p)$ in the Fourier transform representation. (Note that we now use p to denote a momentum variable, rather than a plaquette variable.) The momenta p have d components p_i, and $|p_i| \leqq \pi$.

We introduce a tensor notation with functions $f_{\mu\nu}$ on plaquettes given as antisymmetric functions on coordinate axes μ, ν which indicates plaquette orientation. They also depend on a lattice position. Then

$$\langle f^{(k)}, \sigma_k f^{(k)} \rangle = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle \tilde{f}^{(k)}(p), \sigma_k(p) \tilde{f}^{(k)}(p) \rangle \, dp, \tag{7.1.2}$$

where the momentum p inner product is defined by

$$\langle \tilde{f}(p), \sigma_k(p) \tilde{f}(p) \rangle = \sum_{\mu, \nu, \lambda, \kappa} \bar{\tilde{f}}_{\mu\nu}(p) \sigma_{k;\mu\nu\lambda\kappa}(p) \tilde{f}_{\lambda\kappa}(p). \tag{7.1.3}$$

In order to study $\sigma_k(p)$ we need standard momentum space operations such as derivatives ∂ or Laplacians Δ. Since we have two special lattice scales, the unit lattice leading to momenta $|p_i| \leqq \pi$ and the $\eta = L^{-k}$ lattice leading to momenta

$|p_i| \leqq \pi/\eta = \pi L^k$, we require operators on both scales. We use a superscript (1) to denote the unit lattice scale.

With these conventions, the derivative ∂ can be represented by a diagonal $d \times d$ matrix $\partial(p)$ with eigenvalues

$$\partial_\mu(p) = (\exp(i\eta p_\mu) - 1)/\eta. \tag{7.1.4}$$

The unit lattice derivative $\partial^{(1)}(p)$ is a similar matrix with eigenvalues

$$\partial_\mu^{(1)}(p) = \exp(ip_\mu) - 1. \tag{7.1.5}$$

The corresponding Laplacians are

$$\Delta(p) = \operatorname{Tr}\partial(p)^*\partial(p), \qquad \Delta^{(1)}(p) = \operatorname{Tr}\partial^{(1)}(p)^*\partial^{(1)}(p). \tag{7.1.6}$$

We require the operator $V(p)$ defined by

$$V(p) = \partial^{(1)}(p')\partial(p)^{-1}. \tag{7.1.7}$$

Here and below we use the notation $p_i' \in [-\pi, \pi]$ and

$$p' = p \bmod 2\pi, \tag{7.1.8}$$

so p' denotes a unit lattice momentum. The eigenvalues of $V(p)$ are $v_\mu(p) = \partial_\mu^{(1)}(p')/\partial_\mu(p)$. In terms of $V(p)$ we also define

$$u(p) = \det V(p) \tag{7.1.9}$$

and

$$\varphi_\mu(p') = \sum_{\substack{l \in 2\pi Z^d \\ |l_i| \leqq \pi/\eta}} |u(p'+l)v_\mu(p'+l)|^2 \Delta(p'+l)^{-1}. \tag{7.1.10}$$

In terms of these functions we can express the averaging operators Q_k^e, etc. For example

$$(Q_k^e f)_{\mu\nu}^{\sim}(p') = \sum_l u(p'+l)\tilde{f}_{\mu\nu}(p'+l)(v_\mu(p'+l)v_\nu(p'+l))^{-1}. \tag{7.1.11}$$

The basic object we wish to study is σ_k, defined in (4.2.2),

$$\sigma_k = Q_k^e(I - \partial G_{k,Ax}\partial^*)Q_k^{e*}. \tag{7.1.12}$$

The axial gauge Green's function $G_{k,Ax}$ has a very complicated momentum space structure. Thus we use gauge invariance of $\partial G_{k,Ax}\partial^*$ to replace it by the Landau gauge operator $\partial G_k\partial^*$. Here we emphasize that G_k is a slightly different operator from $\mathscr{D}_k$ introduced in Sect. 4. The transformation G_k is defined by replacing the gauge-fixing delta function $\delta_{k,Ax}(A)$ in (4.1.1) by the Landau-gauge fixing function $\mathscr{G}(\partial^*A)$ of (4.4.1). The operator G_k was given in the momentum representation in [6I, Eqs. (1.83) and (1.84)]. Starting from this expression, one can derive the following formulas for $\sigma_k(p)$ by straightforward, algebraic manipulation:

We express σ_k as a sum of two terms

$$\sigma_k = \tau_1 + \tau_2. \tag{7.1.13}$$

Here τ_1 vanishes on curls. Thus if $f = \partial B$, then $\tau_1 f = 0$. Explicitly

$$\tau_{1,\mu\nu\lambda\kappa}(p') = \tfrac{1}{2}\sum_l \left(\frac{|u|^2}{\bar{v}_\mu \bar{v}_\nu v_\lambda v_\kappa} \left[\delta_{\mu\lambda} - \frac{\partial_\mu \overline{\partial_\lambda}}{\Delta} \right] \left[\delta_{\nu\kappa} - \frac{\partial_\nu \overline{\partial_\kappa}}{\Delta} \right] \right)(p'+l), \tag{7.1.14}$$

and

$$\tau_{2,\mu\nu\lambda\kappa}(p') = \left(a_\mu \bar{a}_\lambda (\varphi_\nu \varphi_\kappa)^{-1/2} \left[\delta_{\nu\kappa} - \frac{\partial_\nu^{(1)} \overline{\partial_\kappa^{(1)}}}{\sqrt{\varphi_\nu \varphi_\kappa}} \left(\sum_\varrho |\partial_\varrho^{(1)}|^2 / \varphi_\varrho \right)^{-1} \right] \right)(p').$$
(7.1.15)

In the formula for $\tau_1(p')$, each term on the right side is evaluated at momentum $p'+l$. In the formula for τ_2, the averaging only occurs in

$$a_\mu(p') = \partial_\mu^{(1)}(p') \sum_l \left(\left| \frac{u}{v_\mu} \right|^2 \frac{1}{\Delta} \right)(p'+l).$$
(7.1.16)

In both τ_1 and τ_2 the expressions inside brackets [ ] are projection operators. Thus σ_k has the general form of a sum of projection operators, or tensor products of projection operators, sandwiched between averaging operators.

The fact that τ_1 vanishes on curls can be established as follows: The general form of τ_1 in configuration space is evident from (7.1.11), (7.1.14) namely

$$\tau_1 = Q_k^e (I - P_\partial) Q_k^{e*},$$
(7.1.17)

where P_∂ denotes the orthogonal projection onto curls. (If P_1 projects one-forms onto gradients, then $P_1 \otimes P_1$ projects two forms onto curls.) Furthermore $Q_k^{e*} \partial = \partial Q_k^{s*}$, so

$$\tau_1 \partial = Q_k^e (I - P_\partial) \partial Q_k^{s*} = 0.$$
(7.1.18)

Let $\mathcal{K}(p)$ denote the space of momentum p lattice one-forms B. Write the two-forms $\mathcal{H}(p)$ as a sum of curls and an orthogonal complement,

$$\mathcal{H}(p) = \partial \mathcal{K}(p) + (\partial \mathcal{K}(p))^\perp.$$
(7.1.19a)

Here we now introduce the scalar product,

$$(f,g) = \sum_{\mu,\nu} \overline{f_{\mu\nu}(p)} |v_\mu(p) v_\nu(p)|^{-2} g_{\mu\nu}(p) = \langle f, |V|^{-2} \otimes |V|^{-2} g \rangle,$$
(7.1.19b)

and the decomposition (7.1.19) is with respect to this inner product. This decomposition is natural in the study of $\sigma_k(p)$, since $\tau_1(p) \partial \mathcal{K}(p) = 0$.

Remark that there are strictly positive constants c_1, c_2 such that

$$c_1 \langle f, f \rangle \leqq (f,f) \leqq c_2 \langle f, f \rangle,$$

so the corresponding norms are equivalent. In fact

$$|v_\mu(p)| = \left[\frac{\sin(p_\mu/2)}{(p_\mu/2)} \right] \left[\frac{(p_\mu \eta/2)}{\sin(p_\mu \eta/2)} \right],$$

and since for $|x| \leqq \pi/2$,

$$\frac{2}{\pi} \leqq \frac{\sin x}{x} \leqq 1,$$

it follows that for $|p_i| \leqq \pi$,

$$\tfrac{2}{\pi} \leqq |v_\mu(p)| \leqq \tfrac{\pi}{2}.$$
(7.1.20)

Proposition 7.1.2. *Suppose there exists $\varepsilon > 0$, and $\tau_0(p) \leqq \tau_1(p)$ such that for $|p_j| \leqq \pi$,*

$$\varepsilon \leqq \tau_0(p) \restriction (\partial \mathcal{K}(p))^\perp \quad and \quad \varepsilon \leqq \tau_2(p) \restriction \partial \mathcal{K}(p). \tag{7.1.21}$$

Then Theorem 7.1.1 holds.

Proof. It is sufficient to show that there is a constant $c > 0$ such that

$$c \leqq \sigma_k(p) \tag{7.1.22}$$

for all $|p_j| \leqq \pi$. The inequality

$$|a_\mu(p)/\varphi_\nu(p)^{1/2}| \leqq c_2 \tag{7.1.23}$$

is an extension of (7.1.20). As a consequence,

$$\|\tau_2\| \leqq M, \tag{7.1.24}$$

where M is a constant independent of p and k.

Let us now decompose $\sigma_k(p)$ according to (7.1.19). For $f = \partial B + f^\perp \in \mathcal{H}(p)$, write

$$\begin{aligned}
\langle f, \sigma_k f \rangle &= \langle f, \tau_1 f \rangle + \langle f, \tau_2 f \rangle \\
&= \langle f^\perp, \tau_1 f^\perp \rangle + \langle f, \tau_2 f \rangle \geqq \langle f^\perp, \tau_0 f^\perp \rangle + \langle f, \tau_2 f \rangle \\
&\geqq \varepsilon \|f^\perp\|^2 + \langle f, \tau_2 f \rangle.
\end{aligned} \tag{7.1.25}$$

Here we use $\tau_1 \partial = 0$ and the hypothesis on τ_1. The idea is to bound $\langle f, \tau_2 f \rangle$ using the positivity of τ_2. In fact, $0 \leqq \tau_2$ is evident from (7.1.15). Then for any $\delta \in [0, 1]$,

$$\langle f, \sigma_k f \rangle \geqq \varepsilon \|f^\perp\|^2 + \delta \langle f, \tau_2 f \rangle. \tag{7.1.26}$$

Note that by (7.1.24) and the positivity of τ_2,

$$\begin{aligned}
\langle f, \tau_2 f \rangle &= \langle \partial B, \tau_2 \partial B \rangle + \langle \partial B, \tau_2 f^\perp \rangle + \langle f^\perp, \tau_2 \partial B \rangle + \langle f^\perp, \tau_2 f^\perp \rangle \\
&\geqq \langle \partial B, \tau_2 \partial B \rangle - 2M \|\partial B\| \, \|f^\perp\|.
\end{aligned}$$

For any $\lambda > 0$, we then have

$$\langle f, \tau_2 f \rangle \geqq \langle \partial B, \tau_2 \partial B \rangle - (M\lambda)^2 \|\partial B\|^2 - \lambda^{-2} \|f^\perp\|^2.$$

Use the lower bound (7.1.20) on $\tau_2 \restriction \partial \mathcal{K}$, and choose λ so $(M\lambda)^2 = \varepsilon/2$. Thus

$$\langle f, \tau_2 f \rangle \geqq \tfrac{1}{2}\varepsilon \|\partial B\|^2 - 2M^2 \varepsilon^{-1} \|f^\perp\|^2.$$

Inserting this in (7.1.26) yields

$$\langle f, \sigma_k f \rangle \geqq (\varepsilon - 2M^2 \delta \varepsilon^{-1}) \|f^\perp\|^2 + \tfrac{1}{2}\delta\varepsilon \|\partial B\|^2 \geqq \tfrac{1}{2}\varepsilon \|f^\perp\|^2 + \tfrac{1}{2}\delta\varepsilon \|\partial B\|^2, \tag{7.1.27}$$

where the last inequality follows for any $\delta < (\varepsilon/2M)^2 \leqq 1$. Let $\delta > 0$ and $c \equiv \tfrac{1}{2}\delta\varepsilon > 0$. Then

$$\langle f, \sigma_k f \rangle \geqq c(\|f^\perp\|^2 + \|\partial B\|^2) = c \|f\|^2,$$

as desired and completes the proof of the proposition.

Let us now prove the lower bound (7.1.21) on $\tau_0(p)$. Introduce the operator $\tau_0(p)$ obtained by restricting the sum in (7.1.14) to $l = 0$. Clearly each term in the

sum over l is a nonnegative operator. Hence

$$\tau_0(p) \leqq \tau_1(p). \tag{7.1.28}$$

Note that because $f^\perp$ is defined with respect to the inner product (7.1.19b), τ_0 simplifies on $f^\perp$. In particular the ∂ terms vanish and

$$\langle f^\perp, \tau_0 f^\perp \rangle = \tfrac{1}{2} |u(p)|^2 \sum_{\mu, \nu} |f_{\mu\nu}^\perp(p)/(v_\mu(p) v_\nu(p))|^2. \tag{7.1.29}$$

From (7.1.20) we then infer

$$\varepsilon \leqq \tau_0 \upharpoonright (\partial \mathscr{K})^\perp,$$

for some $\varepsilon > 0$, which with (7.1.28) yields the desired lower bound

$$\varepsilon \leqq \tau_1 \upharpoonright (\partial \mathscr{K})^\perp. \tag{7.1.30}$$

Next we consider the bound on $\tau_2 \upharpoonright \partial \mathscr{K}$. But since $\tau_1 \partial = 0$,

$$\langle \partial B, \tau_2 \partial B \rangle = \langle \partial B, \sigma_k \partial B \rangle = \langle B, \varDelta_k B \rangle, \tag{7.1.31}$$

using the definition (4.3.1) of $\varDelta_k$. An explicit formula for $\varDelta_k$ shows that

$$\langle \partial B, \sigma_k \partial B \rangle = \frac{1}{2} \left(\sum_{\varrho} |\partial_\varrho^{(1)}|^2 / \varphi_\varrho \right)^{-1} \sum_{\mu, \nu} |(\partial B)_{\mu\nu}|^2 / (\varphi_\mu \varphi_\nu).$$

which is bounded below by $\varepsilon \|\partial B\|^2$ (see also [6I, 8]). This completes the proof of Theorem 7.1.1.

7.2. Regularity and Decay of Propagators and Minimizers

We are interested in Landau gauge propagators and minimizers, because they have good regularity and decay. The key objects are H_k and $C^{(k)}$, while estimates on $\mathscr{D}_k$ follow by (4.4.4). The minimizer H_k can be expressed as an integral kernel. For $x \in T_\eta$,

$$(H_k B)_\mu(x) = \sum_{y \in T_1^{(k)}, \nu} H_{k, \mu\nu}(x; y) B_\nu(y). \tag{7.2.1}$$

The kernel $H_{k, \mu\nu}(x, y)$ and its gradient decay exponentially. In particular there exists $\delta > 0$ and for $0 \leqq \alpha < 1$ a constant $M = M(\alpha) < \infty$ such that for $|x - x'| \leqq 1$,

$$|H_{k, \mu\nu}(x, y)| + |\nabla H_{k, \mu\nu}(x, y)|$$
$$+ |x - x'|^{-\alpha} |\nabla H_{k, \mu\nu}(x, y) - \nabla H_{k, \mu\nu}(x', y)| \leqq M e^{-\delta|x-y|}. \tag{7.2.2}$$

This inequality is a consequence of Proposition 1.2 and the representation (1.103) of [6I].

The unit lattice propagator $C^{(k)}$ also has exponential decay,

$$|C_{\mu\nu}^{(k)}(x, y)| \leqq M e^{-\delta|x-y|}, \tag{7.2.3}$$

for $x, y \in T_1^{(k)}$. This inequality follows from the bound (2.157) in [6II] and from the general theorem on unit lattice operators in [7]. In fact this inequality also holds for propagators with Dirichlet boundary conditions outside a domain $\varLambda$, uniformly in $\varLambda$.

The gauge transformation λ in (5.1.1) is bounded and depends on B through an exponentially decaying kernel D_k:

$$\lambda(x) = (D_k B)(x), \; |D_k(x, b)| \leq M e^{-\delta \operatorname{dist}(x, b)}. \tag{7.2.4}$$

This estimate follows from (5.1.4) and (7.2.2).

The operators $\mathcal{D}_k$ have the same properties as the operators G_k in [6I], Proposition 1.2, with exponential decay but singularities on the diagonal. These properties follow from (4.4.4) and the above estimates on H_k, $C^{(k)}$.

7.3. Positivity of $\Delta_k(u_k)$

The scalar field quadratic form $\Delta_k(u_k)$ depends on the background field u_k. Hence we can only establish stability properties for $\Delta_k(u_k)$ with some restriction on u_k. In particular, let us assume that for the unit lattice field v,

$$|v(\partial p) - 1| \leq e_k \not p(e_k), \tag{7.3.1}$$

where $\not p(e_k) = (1 + \ln e_k^{-1})^{\not p}$. Then the stability estimate can be stated in two forms. For constants $\gamma > 0$, $\alpha > 0$, $M < \infty$,

$$\langle \phi, \Delta_k(u_k)\phi \rangle \geq \gamma \sum_{b \in T_1^{(k)}} |u_k(b)\phi(b_+) - \phi(b_-)|^2 - M e_k^{2-\alpha} \sum_{x \in T_1^{(k)}} |\phi(x)|^2. \tag{7.3.2}$$

The second form of the inequality substitutes v_b for $u_k(b)$ in the covariant derivative of ϕ. These inequalities can be proved by an extension of the proofs of [7]. The propagators arising from $\Delta_k(u_k)$, under the restriction (7.3.1) on the gauge field, also satisfy the regularity and decay estimates of [7]. In order to remain within the framework of this reference, we remark that by change of gauge u_k can be transformed in a local region Λ into a configuration of the form $\exp[ie_k\eta A]$, where A is smooth and small. In axial gauge for the configuration v in a domain $\Lambda' \supset \Lambda$ we can substitute $\mathcal{D}_k \partial^* = G_{k, Ax}\partial^* + \partial D$ in the formula for u_k, we use (5.3.1) to replace this by a minimizer in axial gauge. Then we use (5.1.1) to return the minimizers to Landau gauge. This is a local procedure since $f^{(k)}$ can locally be represented as a curl.

Appendix. Quadratic Forms

In the analysis of this paper we often wish to minimize a quadratic form on a Hilbert space $\mathcal{H}$, possibly subject to a constraint. We collect here a couple of elementary properties which we use, and which could serve as the start of a more general study of related problems.

Since we work with lattice fields, our space $\mathcal{H}$ is finite dimensional and all forms are bounded. Of course we may obtain limiting unbounded forms as $\dim \mathcal{H} \to \infty$ (e.g., as in infinite-volume or zero-lattice-spacing limit). Such forms are analyzed as limits of finite-dimensional approximations.

As a first example, let $0 \leq A = \alpha^*\alpha$ be a self-adjoint transformation on $\mathcal{H}$ and let $B \in \mathcal{H}$. Define the quadratic form

$$h(A) = \tfrac{1}{2}\|\alpha A + B\|^2. \tag{A1}$$

Let P denote the projection onto $\operatorname{Range}(\alpha)$.

Proposition A1. *Range* $\alpha^* = $ *Domain* Δ^{-1}, *so* $P = \alpha \Delta^{-1} \alpha^*$. *Furthermore*

$$h(A) \geqq h(A_{cl}) = \tfrac{1}{2} \langle B, (I-P)B \rangle, \tag{A2}$$

where

$$A_{cl} = -\Delta^{-1}\alpha^* B \tag{A3}$$

is the unique minimum of (A1).

Proof. Note that Range $\alpha^* = (\text{Kernel } \alpha)^\perp$ and Kernel $\alpha = $ Kernel Δ. By the spectral theorem $\mathcal{H} = \text{Kernel}(\Delta) \oplus \text{Range}(\Delta)$, so Range $\alpha^* = $ Range $\Delta = $ Domain Δ^{-1}. Therefore $P = \alpha \Delta^{-1} \alpha^*$ exists, $P^2 = P = P^*$, and $P\alpha = \alpha$. Thus P is the projection onto Range α. Writing

$$h(A) = \tfrac{1}{2} \| \Delta^{1/2} A + \Delta^{-1/2} \alpha^* B \|^2 + \tfrac{1}{2} \langle B, (I-P)B \rangle$$

yields (A2). Since $h(A)$ is convex, the minimum is unique, to complete the proof.

If we translate the quadratic form h, using the change of variables

$$A = A' + A_{cl},$$

then as a consequence of $(I-P)\alpha = 0$, and $h(A_{cl}) = \tfrac{1}{2} \langle B, (I-P)B \rangle$, $h(A)$ decomposes into a sum of independent quadratic forms. We state

Corollary A2. *With $h(A)$ given by* (A1) *and* $A = A' + A_{cl}$ *defined by* (A3),

$$h(A) = \tfrac{1}{2} \langle A', \Delta A' \rangle + h(A_{cl}). \tag{A4}$$

We now specialize to the case where $\mathcal{H} = L_2(\mathbb{R}^N, dx)$, with dx Lebesgue measure. We consider the Green's function C of Δ. Assuming $0 < \Delta$, then we have two convenient representations for C. First

$$C = \Delta^{-1}, \tag{A5}$$

by which we express C as the operator inverse of Δ. Secondly, we can express C as a moment of a Gaussian measure, namely

$$\exp(\tfrac{1}{2} \langle B, CB \rangle) = Z^{-1} \int_{\mathcal{H}} \exp(-\tfrac{1}{2} \langle x, \Delta x \rangle + \langle x, B \rangle) dx. \tag{A6}$$

We are often interested, however, in the case of operators Δ with zero modes. Thus Δ has an inverse only when restricted to a subspace $\mathcal{H}_0 \subset \mathcal{H}$ which does not contain zero modes. Likewise an integral such as (A6) converges only when restricted to such a subspace. In gauge theories it is often natural to choose various subspaces $\mathcal{H}_0$, and furthermore Δ is generally not diagonal under the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$. Thus we are led to study quadratic forms Δ constrained to act on $\mathcal{H}_0$.

Define the transformation C on $\mathcal{H}$ by

$$\exp(\tfrac{1}{2} \langle B, CB \rangle) = Z^{-1} \int_{\mathcal{H}_0} \exp(-\tfrac{1}{2} \langle x, \Delta x \rangle + \langle x, B \rangle) dx. \tag{A7}$$

Clearly C is self-adjoint in the inner product $\langle\ ,\ \rangle$. Furthermore, define

$$\Delta_0 = \Delta \upharpoonright \mathcal{H}_0. \tag{A8}$$

In general $\mathcal{H}_0 \neq \mathcal{H}$ and $C \upharpoonright \mathcal{H}_0 \neq \Delta_0^{-1}$. Thus we ask

(1) How are Δ_0 and C related?

(2) What is the minimizer of the quadratic form $\frac{1}{2}\langle x, \Delta_0 x\rangle - \langle x, B\rangle$ for $x \in \mathcal{H}_0$?

Let us make the following assumptions: Let V be given and

$$I \leqq \Delta_0 \quad \text{on} \quad \mathcal{H}_0, \tag{A9}$$

$$I \leqq \Delta + V \quad \text{on} \quad \mathcal{H}, \tag{A10}$$

where $\mathcal{H}_0 = \text{Null } V$. With these definitions, define

$$G = (\Delta + V)^{-1} \tag{A11}$$

and let $(,)$ be the inner product

$$(x, y) \equiv \langle x, (\Delta + V)y\rangle. \tag{A12}$$

Then let P denote the projection of $\mathcal{H}$ onto $\mathcal{H}_0$, where P is orthogonal in the inner product (A12).

Proposition A3. *Under the assumptions above,*

$$C = PG, \tag{A13}$$

so $\text{Range } C \subset \mathcal{H}_0$. *Furthermore*

and

$$C\Delta_0 C = C, \tag{A14}$$

$$HB = CB = Z^{-1} \int_{\mathcal{H}_0} \exp(-\tfrac{1}{2}\langle x, \Delta x\rangle + \langle x, B\rangle) x \, dx. \tag{A15}$$

is the minimum configuration of

$$\tfrac{1}{2}\langle x, \Delta x\rangle - \langle x, B\rangle, \quad x \in \mathcal{H}_0.$$

Thus with $x = y + CB$,

$$\tfrac{1}{2}\langle x, \Delta x\rangle - \langle x, B\rangle = \tfrac{1}{2}\langle y, \Delta y\rangle - \tfrac{1}{2}\langle B, CB\rangle, \quad x \in \mathcal{H}_0. \tag{A16}$$

Proof. For $x \in \mathcal{H}_0$, note that

$$\tfrac{1}{2}\langle x, \Delta_0 x\rangle - \langle x, B\rangle = \tfrac{1}{2}\langle x, (\Delta + V)x\rangle + \langle x, (\Delta + V)GB\rangle.$$

Thus (A7) can be written

$$\exp(\tfrac{1}{2}\langle B, CB\rangle) = Z^{-1} \int_{\mathcal{H}_0} \exp(-\tfrac{1}{2}\langle x - PGB, (\Delta + V)(x - PGB)\rangle$$

$$+ \tfrac{1}{2}\langle PGB, (\Delta + V)PGB\rangle) dx, \tag{A17}$$

where we have used

$$\langle PGB, (\Delta + V)x\rangle = (PGB, x) = (GB, Px) = (GB, x) = \langle GB, (\Delta + V)x\rangle. \tag{A18}$$

Thus (A17) equals

$$\exp(\tfrac{1}{2}\langle PGB, (\Delta + V)PGB\rangle) = \exp(\tfrac{1}{2}\langle GB, (\Delta + V)PGB\rangle) = \exp(\tfrac{1}{2}\langle B, PGB\rangle).$$

Thus $\langle B, CB\rangle = \langle B, PGB\rangle$, and (A13) holds as claimed. To verify (A14), note that

$$C\Delta_0 C = PG\Delta_0 PG = PG(\Delta + V)PG = P^2 G = PG = C.$$

Next note that (A17) displays CB as the minimum of $\frac{1}{2}\langle x, \Delta x\rangle - \langle x, B\rangle$, since linear terms in the fluctuation $x - PGB$ do not occur. Also from (A18) we have for $x \in \mathcal{H}_0$,

$$\langle CB, \Delta x\rangle = \langle B, x\rangle, \tag{A19}$$

so the integral in (A15) equals CB. Finally, for $x \in \mathcal{H}_0$, so by (A19), $\langle CB, \Delta y\rangle \equiv \langle B, y\rangle$. It follows that

$$\frac{1}{2}\langle x, \Delta x\rangle - \langle x, B\rangle = \frac{1}{2}\langle y, \Delta y\rangle + \langle \Delta y, CB\rangle + \frac{1}{2}\langle CB, \Delta CB\rangle - \langle y, B\rangle - \langle CB, B\rangle$$
$$= \frac{1}{2}\langle y, \Delta y\rangle - \frac{1}{2}\langle B, CB\rangle,$$

which is (A16).

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