Topological Charges in Monopole Theories

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Abstract. Let us consider a monopole theory with a compact, simply connected gauge group and the Higgs field in the adjoint representation. Using root theory we show that.

(i) The homotopy class of the Higgs field is a p-tuple of integers where p is the dimension of the centre of the residual symmetry group. These "Higgs charges" can be expressed as surface integrals of differential forms.

(ii) To any invariant polynomial on the Lie algebra is associated a topological invariant which turns out to be a combination of the Higgs charges.

(iii) Electric charge is quantized. The monopole's magnetic charge is a combination – with the Higgs charges as coefficients – of p basic magnetic charges which satisfy generalized Dirac conditions.

The example of $G = SU(N)$ is worked out in detail.

1. Introduction

The fundamental role played by topological invariants in monopole theory has been recognized since the very beginning [1, 2, 3, 18]. Essentially three types of such "charges" have been considered so far:

(i) Assume that the full gauge group G is broken spontaneously to H by the vacuum expectation value of the Higgs field Φ. The usual requirements concerning the asymptotic behaviour of Φ imply the existence of a map from S^2, the 2-sphere at infinity, into an orbit G/H of G. We have thus a homotopy class

$$[\Phi] \in \pi_2(G/H). \tag{1.1}$$

This first topological invariant shall be referred to as the Higgs charge.

(ii) Let F denote the field strength tensor. If the Higgs field is covariantly constant, the quantity

$$I = \int_{S^2} \mathrm{tr}(F \cdot \Phi) \tag{1.2}$$

is also invariant under smooth deformations. It appears for instance in the formula of Bogomolny for the lower bound of the energy [4]. Remarkably, the integrand of this "trace invariant" (1.2) is just the topological charge density of the 4-dimensional pure Yang-Mills theory to which the static Yang-Mills system is equivalent if the Higgs potential vanishes [5].

(iii) To each generator of the unbroken symmetry group H corresponds a massless gauge particle one of which is identified with the photon of electromagnetic interactions [2, 3]. The possible electric charges are eigenvalues of the charge operator

$$Q_{em} = e_0 \Phi/|\Phi| . \tag{1.3}$$

The electromagnetic field itself is the projection of the gauge-field onto the Higgs-direction [7, 8, 9]. The monopole's magnetic charge is defined to be

$$g = (1/4\pi e_0) \int_{S^2} \mathrm{Tr}(F \cdot \Phi/|\Phi|) . \tag{1.4}$$

These charges are of course intimately related. The aim of this paper is to give a systematic study of all the aforementioned topological invariants in a gauge theory with a compact, simply connected gauge group, broken spontaneously by a Higgs field in the adjoint representation.

We show first that the Higgs charge (1.1) is an p-tuple of integers

$$\Phi = (m_1, ..., m_p) , \tag{1.5}$$

where p is the dimension of $Z(\mathfrak{h})$, the centre of the Lie algebra of H [6, 9]. The m_i's are also expressed as surface integrals

$$m_i = (1/2\pi\sqrt{-1}) \int_{S^2} \Phi^* \omega^{(i)} . \tag{1.6}$$

The closed (but not exact) 2-forms $\omega^{(i)}$ on G/H correspond with properly scaled vectors of a basis of $Z(\mathfrak{h})$.

Next we show that, for any invariant polynomial f on the Lie algebra of G which is homogeneous of order $k+1$ the quantity

$$I^{(f)} = \int_{S^2} f(F, \Phi, ..., \Phi) \tag{1.7}$$

is again a topological invariant, supposing that $D\Phi = 0$. This is proved by showing that $I^{(f)}$ can be written, just like the Higgs charges in (1.6), as the integral of a closed 2-form. Comparing the two formulae we see that $I^{(f)}$ is a linear combination of the Higgs charges:

$$I(f) = 2\pi\sqrt{-1} \sum_{k=1}^{p} \lambda_k m_k . \tag{1.8}$$

The coefficients λ_k are explicitly obtained. They depend only on the modulus of the Higgs field and the group structure (see Sect. 3).

In particular, we give formulae for computing the "higher order invariants" [6] of the form

$$f(F, \Phi, ..., \Phi) = \mathrm{tr}(F \cdot \Phi^n), \quad n = 1, 2, ... \tag{1.9}$$

Equation (1.8) generalizes the corresponding expression for SU(3) [10, 11].

We study then the electromagnetic properties. The electric charge operator is defined by the Higgs field [8, 13, 14]. In order to have quantized electric charge Φ must define a U(1) (rather then merely an R) subgroup. In this case there exists an (unique up to sign) minimal U(1) generator ζ_0 pointing in the electromagnetic direction. This minimal generator plays an important role in the sequel. First:

All electric charges in the theory are integer multiples of

$$q_{\min} = e_0/|\zeta_0|. \tag{1.10}$$

ζ_0 here is the minimal U(1) generator determined by the Higgs field's direction (see Sect. 5).

The magnetic charge g of the monopole is in turn a combination – with ζ_0-dependent coefficients – of p partial magnetic charges $g_1, ..., g_p$ each of which is the multiple by the corresponding Higgs charge m_k of a basic magnetic charge $g_k^{(0)}$:

$$g_k = m_k g_k^{(0)}. \tag{1.11}$$

$q_{\min}$ and g satisfy a generalized Dirac condition [13, 14].

The basic magnetic charges correspond to the "fundamental monopoles" in [27, 28].

The general theory is illustrated on the example of $G = \mathrm{SU}(N)$ [8, 15, 18].

We conclude that root theory [16, 17], combined with that of differential forms [19, 23] provides us with a *via regia* to solve any problem concerning topological invariants in monopole theory.

2. Roots, Weights and Characters

The basic facts and definitions of root theory we summarize here without proof will be used through the whole paper. For more details the reader should consult refs. [16, 17].

Let G denote a compact, simply connected (and hence semisimple) Lie group having Lie algebra $\mathscr{G}$. Let us choose a point ξ_0 from $\mathscr{G}$, and denote by H its stablizer under the adjoint action of G. The orbit of ξ_0 is identified with G/H.

Let us choose a maximal torus $T \subset G$ in such a way that $\mathscr{T}$, its Lie algebra, contains ξ_0. T is a maximal torus also for H, and $\mathscr{T}^C$, the complexification of $\mathscr{T}$, is a Cartan subalgebra of $\mathscr{G}^C$.

Denote by Δ the set of roots. $\Delta \subset \sqrt{-1}\, \mathscr{T}^*$. Let Δ^+ be a positive root system and consider the simple roots

$$\alpha_1, ..., \alpha_r \ (r = \dim T = \text{rank of } G). \tag{2.1}$$

Define the subset of indices $I = \{i_1, ..., i_p\}$ by

$$i_k \in I \quad \text{if} \quad \alpha_{i_k}(\xi_0) \neq 0. \tag{2.2}$$

I is empty if and only if ξ_0 vanishes. Hence I contains p elements, $0 < p < r$.

Let us choose the vectors $\xi_1, ..., \xi_r$ from $\mathscr{T}$ so that

$$\alpha_i(\xi_j) = \sqrt{-1}\, \delta_{ij}, \quad i, j = 1, ..., r. \tag{2.3}$$

The ξ_j's form a basis of $\mathscr{T}$, dual to the α_i's. Let $Z(\mathfrak{h})$ denote the centre of $\mathfrak{h}$, the Lie algebra of H.

Theorem 2.1.

$$\{\xi_{i_1}, ..., \xi_{i_p}\} \text{ is a basis for } Z(\mathfrak{h}).$$

Consequently p is the dimension of $Z(\mathfrak{h})$. Since ξ_0 is in the centre, it is a linear combination of the ξ_{i_k}'s, $i_k \in I$:

$$\xi_0 = \sum_{i_k \in I} a_k \xi_{i_k}, \tag{2.4}$$

where

$$a_k = \alpha_{i_k}(\xi_0)/\sqrt{-1}.$$

G is semisimple and admits thus a negative definite Killing form $B(\cdot, \cdot)$. To each root $\alpha \in \Delta$ we may associate a unique $\eta_\alpha \in \mathscr{T}^C$ defined by

$$B(\eta_\alpha, \zeta) = \alpha(\zeta), \quad \zeta \in \mathscr{T}^C. \tag{2.5}$$

If μ is a linear function on $\mathscr{T}^C$ with values in C, we may define tentatively a map χ from T to $U(1)$ by

$$\chi(\exp \zeta) = \exp(\mu(\zeta)), \quad \zeta \in \mathscr{T}. \tag{2.6}$$

χ is clearly a character of T as soon as it is well-defined. This happens if for any root $\alpha \in \Delta$,

$$2\mu(\eta_\alpha)/\alpha(\eta_\alpha) \in Z. \tag{2.7}$$

Any root being an integral combination of simple roots, it is enough to require (2.7) to hold for the simple roots. Let us define $\mu_1, ..., \mu_r$ by

$$2\mu_i(\eta_{\alpha_j})/\alpha_j(\eta_{\alpha_j}) = \delta_{ij}. \tag{2.8}$$

Theorem 2.2. *The μ_i's form a Z-basis for those linear functions μ on $\mathscr{T}^C$ which exponentiate to characters.*

These μ_i's are the *fundamental weights*. Their associated characters are denoted by $\chi_1, ..., \chi_r$.

Every weight of any representation of G is an integer combination of these fundamental weights and so the eigenvalues in any representation of an element ξ of have the form

$$\sum_{i=1}^{r} s_i \mu_i(\xi). \tag{2.9}$$

Theorem 2.3. *The character χ_i of T extends to a character of H if and only if $i = i_k \in I$. The extension is obtained at the Lie algebra level by setting $\mu_i = 0$ on $\mathfrak{h} \cap \mathscr{T}^\perp$.*

The characters are useful in describing the homotopy of H. Indeed, let $\gamma: S^1 \to H$ be a loop and consider a character χ of H. The composed map $\chi \circ \gamma$ goes from S^1 to $U(1)$ and has thus a degree (or winding number) $m_\chi(\gamma) \in Z$. If γ_1 and γ_2 are homotopic loops in H, the degrees are the same since any smooth function with integer values is constant. So we get a homomorphism

$$m_\chi: \pi_1(H) \to Z.$$

The fundamental theorem of Bott [17] tells us now that we have enough characters to determine $\pi_1(H)$ completely:

Theorem 2.4. *Consider those characters associated to the indices in* I, *and denote*

$$m_k(\gamma) = m_{\chi_{i_k}}(\gamma). \tag{2.10}$$

The map

$$\pi_1(H) \ni [\gamma] \to (m_1, \ldots, m_p) \in Z^p \tag{2.11}$$

is an isomorphism.

If $(m_1, \ldots, m_p)$ is a p-tuple of integers, it is easy to construct a loop γ having this p-tuple as its homotopy class. Indeed,

$$\gamma(t) = \exp\left(2\pi\sqrt{-1}\, t \sum_{i_k \in I} m_k \eta_{i_k}\right), \quad t \in [0, 1] \tag{2.12}$$

has this property. To see this let us apply the character χ_{i_j}. We get

$$\chi_{i_j}(\gamma(t)) = \exp(2\pi\sqrt{-1}\, t \sum m_k \mu_{i_j}(\eta_{i_k})) = \exp(2\pi\sqrt{-1}\, t m_{i_j}).$$

This map has clearly degree m_{i_j}.

We can switch from the μ_i's to the α_i's by the Cartan matrix (C_{ij}):

$$\alpha_i = \sum_j^r C_{ij}\mu_j. \tag{2.13}$$

C_{ij} has integer entries. Set

$$\eta_i = 2\eta_{\alpha_i}/\alpha_i(\eta_{\alpha_i}), \quad i = 1, \ldots, r. \tag{2.14}$$

The η_i's are dual to the μ_j's:

$$\mu_i(\eta_j) = \delta_{ij}. \tag{2.15}$$

Equation (2.13) implies

$$\eta_i = (1/\sqrt{-1}) \sum_j^r C_{ji}\xi_j. \tag{2.16}$$

The μ_j's and ξ_j's are related in turn by

$$\mu_j = (\alpha_j(\eta_{\alpha_j})/2\sqrt{-1}) \cdot B(\xi_j, \ldots). \tag{2.17}$$

It follows that

$$B(\xi_i, \eta_j) = \delta_{ij} \cdot (2\sqrt{-1}/\alpha_i(\eta_{\alpha_i})). \tag{2.18}$$

Observe that, according to (2.17), the μ_{i_k}'s ($i_k \in I$) are represented by correctly scaled elements of the centre of $\mathfrak{h}$. The corresponding η_{i_k}'s however do not belong necessarily to $Z(\mathfrak{h})$.

Γ, the unit lattice of $\mathcal{G}$, consists of those vectors $\zeta \in \mathcal{T}$ which satisfy

$$\exp(2\pi\zeta) = 1. \tag{2.19}$$

Theorem 2.5. *Any vector ζ of the unit lattice Γ is a combination of $(\sqrt{-1}$ times) the η_i's defined in (2.14) with integer coefficients:*

$$\zeta = \sqrt{-1} \sum_{i=1}^{r} n_i \eta_i . \tag{2.20}$$

The coefficients here are computed by

$$n_i = \mu_i(\zeta)/\sqrt{-1} . \tag{2.21}$$

$\zeta_0 \in \Gamma$ is said to be a minimal generator if $\exp(2\pi\zeta_0 t) = 1$ the first time for $t = 1$. This happens iff the integers n_i in its expansion (2.20) have no common diviser. The set of minimal generators will be denoted by Γ_0.

If $\zeta \in \Gamma$, there exists a (unique up to sign) minimal generator parallel to it.

We illustrate the general theory on the example $G = SU(N)$. We choose our maximal torus T to consist of diagonal matrices in the standard representation.

$$T = \left\{ g = \mathrm{diag}(u_1, ..., u_N), |u_i| = 1, \prod_{i}^{N} u_i = 1 \right\} .$$

The rank of $SU(N)$ is $(N - 1)$.

For any diagonal matrix $a = \mathrm{diag}(a_1, ..., a_N)$, let ε_i be the map giving its i's entry, $\varepsilon_i(a) = a_i$. Then $\{\varepsilon_i - \varepsilon_j | i \neq j\}$ gives a root system and $\{\varepsilon_i - \varepsilon_j | i < j\}$ a positive system whose simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, ..., \alpha_{N-1} = \varepsilon_{N-1} - \varepsilon_N . \tag{2.22}$$

The dual basis $\xi_1, ..., \xi_{N-1}$ reads

$$\xi_i = (\sqrt{-1}/N) \begin{bmatrix} N-i & & & & \\ & \ddots & N-i & & \\ & & & -i & \\ & & & & \ddots \\ & & & & & -i \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\} i \\ \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\} N-i \end{matrix} . \tag{2.23}$$

If we choose the trace form Tr (rather then the Killing form $B = -\mathrm{Tr}/N$) we get

$$\eta_{\alpha_i} = \eta_i = \mathrm{diag} \Big(0, ..., \overset{i}{1}, \overset{i+1}{-1}, 0, ..., 0 \Big), \tag{2.24}$$

$\alpha_i(\eta_{\alpha_i}) = 2$ in this case.

The fundamental weights are now

$$\mu_1 = \varepsilon_1, \mu_2 = \varepsilon_1 + \varepsilon_2, ..., \mu_{N-1} = \varepsilon_1 + ... + \varepsilon_{N-1} . \tag{2.25}$$

The associated fundamental characters are

$$\chi_j(u) = \prod_{i=1}^{j} u_i, u = \mathrm{diag}(u_i) \in T, \quad j = 1, ..., N-1 . \tag{2.26}$$

Consider the base point

$$\xi_0 = \sqrt{-1}\,\mathrm{diag}(\lambda_1, ..., \lambda_N), \lambda_j \in R, \sum \lambda_j = 0 . \tag{2.27}$$

$\lambda_1 \geq \lambda_2 \geq \ldots$ can be assumed without loss of generality. (Such a situation can namely always be achieved by applying a Weyl transformation which corresponds physically to a constant gauge transformation.) ξ_0 is decomposed to blocks

$$\xi_0 = \sqrt{-1}\begin{bmatrix} \lambda_{i_1} & & & & & & \\ & \ddots & \lambda_{i_1} & & & & \\ \hline & & & \lambda_{i_2} & & & \\ & & & & \ddots & \lambda_{i_p} & \\ \hline & & & & & \lambda_{i_{p+1}} & \\ & & & & & & \ddots & \lambda_{i_{p+1}} \end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}i_1 \\ \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}i_p \\ \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}N\end{matrix} \qquad (2.28)$$

Observe that there are $(p+1)$ blocks. p is hence, as anticipated by the notation, the dimension of the centre of $H \cdot Z(\mathfrak{h})$ is generated in fact by those ξ_{i_k}'s, $i_i \in I$.

The little group is

$$H = S\{U(i_1) \times U(i_2 - i_1) \times \ldots \times U(N - i_p)\}. \qquad (2.29)$$

Those characters belonging to the i_k's ($i_k \in I$) extend to H. Denote $\det_k$ the determinant of the block between i_k and i_{k-1}. Then

$$\chi_{i_k} = \det_1(h) \ldots \det_k(h), \qquad h \in H, k \leq p. \qquad (2.30)$$

Let us fix $0 < i_1 < \ldots < i_p < i_{p+1} = N$. The dimension of the corresponding orbit $\mathcal{O}_{i_1, \ldots, i_p}$ is calculated by noting that the orbit is the same as it was for $U(N)$. But for $U(N)$ the stability group would be

$$U(i_1) \times U(i_2 - i_1) \times \ldots U(N - i_p),$$

so

$$\dim \mathcal{O}_{i_1, \ldots, i_p} = 2\{i_1(i_2 - i_1) + \ldots + i_p(N - i_p)\}. \qquad (2.31)$$

3. The Higgs Charge

The first of the topological charges we would like to study is the one associated to the way the gauge symmetry is broken by the vacuum expectation value of the Higgs field.

Let us consider in fact a Yang-Mills-Higgs theory in Minkowski space with a compact and simply connected gauge group G. The Higgs field Φ is assumed to be in the adjoint representation. We choose the standard Lagrangian

$$-L = (1/4)F_{\mu\nu}F^{\mu\nu} + (1/2)D_\mu\Phi D^\mu\Phi + V(\Phi), \qquad (3.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi]$ (the fields are rescaled so that the coupling constant is 1).

Let us fix a gauge where A_0 is zero and suppose all the fields are static. We are interested in everywhere regular, finite energy (monopole) solutions. We assume, as usual, that the Higgs potential $V(\Phi)$ takes its minimum on an orbit of G. If $\xi_0 = \Phi(x_0)$ is a reference point having H as stability group, the orbit in question can be identified with the homogeneous space G/H. Assuming that the Higgs field takes its values asymptotically in G/H; we get a map (we denote also by Φ) from S^2, the two-sphere at infinity, into the orbit G/H. Φ defines a class $[\Phi]$ in the second

homotopy group of the orbit, and it is well-known that two situations are gauge-related and are hence physically equivalent if the homotopy classes are the same [3]. In the sequel we shall call this class

$$[\Phi] \in \pi_2(G/H) \tag{3.2}$$

the Higgs charge of Φ.

The aim of this section is to describe $\pi_2(G/H)$. $\pi_1(G) = 0$ by hypothesis and so $\pi_2(G)$ is also trivial. The long exact homotopy sequence implies that

$$\delta : \pi_2(G/H) \to \pi_1(H) \tag{3.3}$$

is an isomorphism. We describe it explicitly as follows: Denote by U_1 (respectively U_2) $S^2 \backslash \{$south pole$\}$ (respectively $S^2 \backslash \{$north pole$\}$). Let us choose a Φ representing a homotopy class in $\pi_2(G/H)$. Since the U_i's are contractible, $\Phi | U_i$, the restriction of Φ to U_i, lifts to a smooth map

$$g_i : U_i \to G, \tag{3.4}$$

such that

$$\Phi(x) = \mathrm{Ad}_{g_i} \xi_0, \qquad x \in U_i. \tag{3.5}$$

$g_i(x_0) = e$ can be assumed without loss of generality.

In $U_1 \cap U_2$, g_1 and g_2 are related by

$$g_2 = g_1 \cdot \gamma_{12}. \tag{3.6}$$

γ_{12} is smooth in $U_1 \cap U_2$ and takes its values in H. Denote by γ its restriction to S^1, the equatorial circle. γ can be considered also as a loop in H having a homotopy class $[\gamma] \in \pi_1(H)$. Furthermore, this homotopy class is the same for homotopic (in $\pi_2(G/H)$) Φ's. We define

$$\delta([\Phi]) = [\gamma] \in \pi_1(H). \tag{3.7}$$

Using Bott's theorem (Theorem 2.4) we get

Theorem 3.1. *The Higgs charge of Φ is a p-tuple of integers,*

$$[\Phi] = (m_1, \ldots, m_p) \in Z^p, \tag{3.8}$$

where m_k is the degree of the map

$$\chi_{i_k}(\delta(\Phi)) : S^1 \to U(1). \tag{3.9}$$

The degree of a map $f : S^1 \to U(1)$ can also be calculated by a contour integral viewing $U(1) \subset C$ and f as a contour:

$$\deg f = (1/2\pi \sqrt{-1}) \int_f d\chi/\chi.$$

Applied to $f = \chi(\gamma)$

$$m_\chi(\gamma) = (1/2\pi \sqrt{-1}) \int_\gamma d\chi/\chi. \tag{3.10}$$

The Higgs charges can be expressed also as integrals on the two-sphere at infinity of suitable closed (but not exact) 2-forms [6, 9]. To see this observe that

each $\varrho \in Z(\mathfrak{h})$ determines an invariant 2-form ω^{ϱ} on the orbit G/H. By invariance it is enough to define it at ξ_0:

$$\omega_{\xi_0}\varrho(\eta_{\xi_0}, \zeta_{\xi_0}) = B(\varrho, [\zeta, \eta]), \qquad \zeta, \eta \in \mathscr{G}. \tag{3.11}$$

ω^{ϱ} is a closed 2-form on G/H. Its pullback by Φ is thus a closed 2-form on S^2.

Theorem 3.2. *Let us consider the two-forms*

$$\omega^{(k)} = \{\alpha_{i_k}(\eta_{\alpha_{i_k}})/2\sqrt{-1}\} \cdot \omega^{\xi_i}k, \qquad i_k \in I. \tag{3.12}$$

The Higgs charges can be calculated as

$$m_i(\Phi) = (1/2\pi\sqrt{-1}) \int_{S^2} \Phi^* \omega^{(k)}, \qquad i_k \in I. \tag{3.13}$$

Proof. Let us define U_1^{ε} and U_2^{ε} respectively as the upper (respectively lower) part of S^2 between $0 < \theta < \pi/2 + \varepsilon$ (respectively $\pi/2 - \varepsilon < \theta < \pi$). Their intersection is a narrow "collar" of width 2ε containing the equator. Let $\varrho \in Z(\mathfrak{h})$ be arbitrary. Then

$$\int_{S^2} \Phi^* \omega^{\varrho} = \lim_{\varepsilon \to 0} \int_{U_1^{\varepsilon}} \Phi^* \omega^{\varrho} + \int_{U_2^{\varepsilon}} \Phi^* \omega^{\varrho}.$$

Let us choose lifts g_i in U_i^{ε} as in (3.5). Then we have the commuting diagram

$$\begin{array}{ccc} & & G \\ & \overset{g_i}{\nearrow} & \downarrow \pi \\ U_i & \overset{\phi}{\longrightarrow} & \mathcal{O} \simeq G/H \end{array} \tag{3.14}$$

so

$$\Phi^* \omega^{\varrho} = (\pi \circ g_i)^* \cdot \omega^{\varrho} = g_i^* \pi^* \omega^{\varrho}.$$

$\pi^* \omega^{\varrho}$ is the (unique) extension to G as a left-invariant 2-form whose value at $e \in G$ is $B(\varrho, [\zeta, \eta])$. This 2-form is exact, since $H^2(\mathscr{G}; R) = 0$. It is in fact the exterior derivative of the left-invariant 1-form θ^{ϱ} whose value at $e \in G$ is given by

$$\theta_e^{\varrho}(\eta) = B(\varrho, \eta), \qquad \eta \in \mathscr{G}. \tag{3.15}$$

It is easy to see that

$$\theta^{\varrho} = B(\varrho, \theta), \tag{3.16}$$

where $\theta = g^{-1}dg$ is the Maurer-Cartan 1-form on G [19].
$\pi^* \omega^{\varrho} = d\theta^{\varrho}$ and thus

$$\Phi^* \omega^{\varrho} = g_i^* d\theta^{\varrho} = d(g_i^* \theta^{\varrho}) = d(B(\varrho, g_i^{-1}dg_i)).$$

But on the equator $g_2(x) = g_1(x) \cdot \gamma(x)$ and so, taking the respective orientations into account, we get

$$\int_{S^2} \Phi^* \omega^{\varrho} = \int_{S^1} B(\varrho, \gamma^{-1}d\gamma) = \int_{\gamma} \theta^{\varrho}. \tag{3.17}$$

In particular, for

$$\varrho_k = \{\alpha_{i_k}(\eta_{\alpha_{i_k}})/2\sqrt{-1}\} \cdot \xi_{i_k} \qquad (i_k \in I), \tag{3.18}$$

$$B(\varrho_k, \cdot) = \mu_{i_k} = d\chi_{i_k}. \tag{3.19}$$

So

$$\int_{S^2} \Phi^* \omega^{(k)} = \int_\gamma \theta^{(k)} = \int_\gamma d\chi_{i_k}/\chi_{i_k} = 2\pi\sqrt{-1}\, m_k(\Phi). \qquad (3.20)$$

Q.E.D.

Note for further reference that for any $\varrho \in Z(\mathfrak{h})$

$$\Phi^* \omega^\varrho = 2\pi\sqrt{-1} \sum_{i_k \in I} m_k(\Phi) B(\eta_k, \varrho). \qquad (3.21)$$

Remark 3.3. The 2-forms ω^ϱ on $\mathcal{O}_{\xi_0}$ we associated to the elements $\varrho \in Z(\mathfrak{h})$ are closely related to the coadjoint orbit construction of Kirillov, Kostant and Souriau [21–23]. Observe in fact that $\mathcal{O}_{\xi_0}$ projects onto $\mathcal{O}_\varrho$, the orbit of ϱ. Indeed, $\eta = \mathrm{Ad}_g \xi_0 \in \mathcal{O}_\xi$ is mapped to $\pi_\varrho(\eta) = \mathrm{Ad}_g \varrho$. It follows from $\varrho \in Z(\mathfrak{h})$ that π_ϱ is well-defined.

$\mathcal{G}$ can be identified with $\mathcal{G}^*$, its algebraic dual, through the Killing form, so $\mathcal{O}_\varrho$ can alternatively be considered as a coadjoint orbit, carrying a natural symplectic form. This is defined as the invariant extension to ϱ of

$$\Omega_\varrho^{(\varrho)}(\eta_\varrho, \zeta_\varrho) = B(\varrho, [\eta, \zeta]), \qquad \eta, \zeta \in \mathcal{G}. \qquad (3.22)$$

Plainly,

$$\omega^\varrho = \pi_\varrho^* \Omega^\varrho. \qquad (3.23)$$

Observe that, according to (3.13), the 2-form Ω^ϱ must be integral which means that, for any 2-sphere lying in $\mathcal{O}_\varrho$,

$$(1/2\pi\sqrt{-1}) \int_{S^2} \Omega^\varrho \in Z. \qquad (3.24)$$

In other words the orbit $\mathcal{O}_\varrho$ endowed with its canonical symplectic structure is prequantizable.

Coadjoint orbits have been studied extensively in the literature [21–24]. Their knowledge makes it easy to describe the Higgs charges (see Sect. 7).

Remark 3.4. The construction of the isomorphism $\delta : \pi_2(G/H) \to \pi_1(H)$ we gave here differs slightly from the usual procedure [8, 14, 18]. It is not difficult to show however that our loop γ and the standard one are homotopic.

4. Generalized Invariants

At the "Prasad-Sommerfield limit" of vanishing Higgs potential [25] the static Yang-Mills Higgs system becomes equivalent to a pure Yang-Mills system [5]. To the 4-dimensional topological charge corresponds the quantity

$$I = \int_{S^2} B(F, \Phi). \qquad (4.1)$$

To the self-duality equations in 4 dimensions correspond the equations of Bogomolny [4] whose solutions have, for a fixed value of I, minimal energy. Furthermore, this energy is proportional to I.

Recently [6] (4.1) has been generalized to

$$I^{(n)} = \int_{S^2} \mathrm{tr}(F\Phi^n), \qquad (4.2)$$

where Φ is in some matrix representation so it is meaningful to speak of its nth power.

In this section we

(i) further generalize (4.1) and (4.2)

and

(ii) relate them to the Higgs charges and to other characteristics of the Higgs field.

In fact we associate a topological invariant to any invariant polynomial on the Lie algebra of G.

A polynomial is a sum of homogenous terms so it is enough to restrict our attention to homogenous invariant polynomials. Remember that an invariant polynomial of order n is a function $f: \mathscr{G} \to R$ such that for any $t \in R$ and $\zeta \in \mathscr{G}$,

$$f(t\zeta) = t^n f(\zeta) \tag{4.3}$$

and

$$f(\mathrm{Ad}_g \zeta) = f(\zeta). \tag{4.4}$$

Alternatively, we can view f as an invariant n-linear function on the Lie algebra obtained as

$$f(\zeta_1, ..., \zeta_n) = \frac{1}{n!} \frac{\partial^n f(t_1 \zeta_1 + ... t_n \zeta_n)}{\partial t_1 ... \partial t_n}. \tag{4.5}$$

This f satisfies

$$f([\zeta, \zeta_1], ..., \zeta_n) + ... + f(\zeta_1, ..., [\zeta, \zeta_n]) = 0. \tag{4.6}$$

Let us now consider a static Yang-Mills system (A_μ, Φ) and assume (to have finite energy) that the Higgs potential is asymptotically covariantly constant:

$$D_\mu \Phi = 0. \tag{4.7}$$

We intend to show

Theorem 4.1. *Let f be an invariant $(n+1)$-linear function on the Lie algebra of G;*

$$I^{(f)} = \int_{S^2} f(F, \Phi, ..., \Phi) \tag{4.8}$$

is a topological invariant, actually independent of A as long as $D_\mu \Phi = 0$.

Theorem 4.1 is proved by relating (4.8) to integrals of 2-forms on the two-sphere at infinity. Let us define in fact

$$\omega_\xi^{(f)}(\eta_\xi, \zeta_\xi) = f([\zeta, \eta], \xi, ..., \xi), \quad \xi \in G/H. \tag{4.9}$$

That $\omega^{(f)}$ is a smooth, G-invariant, closed 2-form on G/H is verified in the standard way.

Similarly, let us define the 1-form $\theta^{(f, \xi_0)}$ on G as the left-invariant extension of

$$\eta \to f(\eta, \xi_0, ..., \xi_0). \tag{4.10}$$

Clearly,

$$\theta^{(f, \xi_0)} = f(g^{-1}dg, \xi_0, ..., \xi_0). \tag{4.11}$$

If π is the projection $G \to G/H$ defined by $\pi(g) = \mathrm{Ad}_g \xi_0$, then

$$\pi^* \omega^{(f)} = d\theta^{(f, \xi_0)}.$$

Let us now calculate $\Phi^* \omega^{(f)}$. Φ lifts locally to a map $g : U \to G$ so that $\Phi(x) = \pi(g(x))$.

$$\Phi^* \omega^{(f)} = g^* \pi^* \omega^{(f)} = g^* d\theta^{(f, \xi_0)} = d(g^* \theta^{(f, \xi_0)}). \tag{4.12}$$

Consider

$$a = \mathrm{Ad}_{g^{-1}} A + g^{-1} dg.$$

It follows from $D_\mu \Phi = 0$ that a takes its values in the Lie algebra of H. Indeed,

$$[a, \xi_0] = [\mathrm{Ad}_{g^{-1}} A, \xi_0] + [g^{-1} dg, \xi_0] = g^{-1}([A, \Phi] + d(\mathrm{Ad}_g \xi_0))g = g^{-1}(D\Phi)g = 0.$$

On the other hand,

$$da + (1/2)[a \wedge a] = \mathrm{Ad}_{g^{-1}} F \tag{4.13}$$

so using the invariance of f,

$$g^* \theta^{(f, \xi_0)} = f(a, \xi_0, \ldots, \xi_0) - f(A, \Phi, \ldots, \Phi).$$

Hence by (4.13)

$$\Phi^* \omega^{(f)} = f(da, \xi_0, \ldots, \xi_0) - d(f(A, \Phi, \ldots, \Phi)). \tag{4.14}$$

Lemma 4.2.

$$f([\mathfrak{h}, \mathscr{G}], \xi_0, \ldots, \xi_0) = 0. \tag{4.15}$$

Indeed, it follows from (4.6) that

$$f([\mathfrak{h}, \mathscr{G}], \xi_0, \ldots, \xi_0) + n \cdot f(\mathscr{G}, [\mathfrak{h}, \xi_0], \ldots, \xi_0) = 0,$$

but $[\mathfrak{h}, \xi_0] = 0$ by the definition of $\mathfrak{h}$.

It follows now from (4.13), (4.15) and the invariance of f that the first term in (4.14) is just $f(da, \xi_0, \ldots, \xi_0) = f(F, \Phi, \ldots, \Phi)$. We conclude that

$$\Phi^* \omega^{(f)} = f(F, \Phi, \ldots, \Phi) - d\{f(A, \Phi, \ldots, \Phi)\}. \tag{4.16}$$

What we have proved is the following:

Theorem 4.3. *Equation* (4.8) *is expressed as a surface integral at infinity:*

$$I^{(f)} = \int_{S^2} \Phi^* \omega^{(f)}. \tag{4.17}$$

Theorem 4.1 is proved now by noting that the integral (4.17) depends only on the homotopy class of Φ in $\pi_2(G/H)$. It is also independent of A.

Having expressed $I^{(f)}$ as the integral of a 2-form, it is easy to relate it to the Higgs charges. The point is that $\omega^{(f)}$ is just ω^ϱ (Definition (3.11)) for a suitable $\varrho \in Z(\mathfrak{h})$. To see this observe that $\eta \to f(\eta, \xi_0, \ldots, \xi_0)$ is a linear function on $\mathscr{G}$ and thus, by the non-degeneracy of the Killing-form B, there exists a unique $\varrho \in \mathscr{G}$ such that

$$f(\eta, \xi_0, \ldots, \xi_0) = B(\varrho, \eta), \qquad \eta \in \mathscr{G}. \tag{4.18}$$

From (4.15) we get on the other hand

$$0 = B(\varrho, [\mathcal{G}, \mathfrak{h}]) = B([\mathfrak{h}, \varrho], \mathcal{G}),$$

so $\varrho \in Z(\mathfrak{h})$. Comparing (4.9) and (3.21) we conclude

Proposition 4.4. $\omega^{(f)} = \omega^{\varrho}$.

To find ϱ explicitly expand it as

$$\varrho = \sum_{i_k \in I} b_k \xi_{i_k}$$

(ϱ is in the centre!). Using (4.18) and (2.14) we get

$$\varrho = \sum_{i_k} \{[\alpha_{i_k}(\eta_{\alpha_{i_k}})/2\sqrt{-1}] \cdot f(\eta_{i_k}, \xi_0, ..., \xi_0)\} \cdot \xi_{i_k}. \tag{4.19}$$

Using (3.21) this implies

$$\Phi^* \omega^{\varrho} = 2\pi \sqrt{-1} \sum_{i_k} f(\eta_{i_k}, \xi_0, ..., \xi_0) \Phi^* \omega^{(k)}. \tag{4.20}$$

Comparing to (3.12) we conclude

Theorem 4.5. *The topological invariant $I^{(f)}$ is a linear combination of the Higgs charges $m_k(i_k \in I)$,*

$$I^{(f)} = 2\pi \sqrt{-1} \sum_{i_k \in I} \lambda_k^{(f)} m_k, \tag{4.21}$$

where the coefficients are given as

$$\lambda_k^{(f)} = f(\eta_{i_k}, \xi_0, ..., \xi_0). \tag{4.22}$$

This works in particular for $f(F, \Phi) = B(F, \Phi)$:

Corollary 4.6. *The trace-(or rather Killing) invariant (4.1) is expressed as*

$$I = 2\pi \sqrt{-1} \sum_{i_k} \lambda_k^{(B)} m_k, \tag{4.23}$$

where

$$\lambda_k^{(B)} = B(\eta_{i_k}, \xi_0) = \{2\sqrt{-1}/\alpha_{i_k}(\eta_{\alpha_{i_k}})\} \cdot \alpha_{i_k}(\xi_0). \tag{4.24}$$

Equations (4.23) · (4.24) generalize the formula known for SU(3) [10, 11].

Even better, it applies to the higher-order invariants (4.2). Indeed, choosing a matrix-representation for the Lie algebra Φ^n, the nth power of Φ, becomes meaningful and we can consider

$$f(F, \Phi, ..., \Phi) = B(F, \Phi^n). \tag{4.25}$$

Φ appears n times in the argument so f is an $(n+1)$-linear function. Equation (4.25) is also invariant, so Theorem 4.4 yields

Corollary 4.7. $I^{(n)} = B(F, \Phi^n)$ *is a topological invariant. It is a linear combination of the Higgs charges,*

$$I^{(n)} = 2\pi \sqrt{-1} \sum_{i_k \in I} \lambda_k \cdot m_k, \tag{4.26}$$

where the coefficients are given by

$$\lambda_k = B(\eta_{i_k}, \xi_0^n).$$

(4.27)

5. Electromagnetic Charges

The last charges we want to study are those associated to the electromagnetic properties. Indeed, it is a fundamental requirement that the residual symmetry group contains the electromagnetic gauge group $U(1)$ as a subgroup. The electromagnetic direction must be defined at every point. This can be done in a gauge-invariant way by a Higgs field.

Indeed, let us define the electromagnetic field as the projection onto the Higgs field's direction of the gauge field:

$$\mathcal{F} = (1/e_0) B(F, \Phi/|\Phi|).$$

(5.1)

($D\Phi = 0$ on S^2 implies that the definition of the electromagnetic field is unambigous [7–9, 18].)

According to general principles [26], the elementary particles appearing in the theory are multiplets belonging to some irreducible representation of the gauge group. The possible electric charges are the eigenvalues of the electric charge operator

$$Q_{em} = e_0 \Phi/|\Phi|.$$

(5.2)

The electric charges observed in nature are quantized so Q_{em} must generate a $U(1)$ rather then merely an R-subgroup in H [12, 13]. This happens if a suitable real multiple of ξ_0 belongs to Γ, the unit lattice (see Sect. 2.) If so, there exists an (up to sign unique) minimal generator ζ_0 parallel to ξ_0. Hence by (2.20)

$$\zeta_0 = \sqrt{-1} \sum_i^r n_i \eta_i, \qquad n_i \in Z,$$

(5.3)

where the coefficients have no common divisor.

It is easy to see what condition must ξ_0 satisfy for this. ξ_0 is in the centre so

$$\xi_0 = \sum_{i_k \in 1} a_k \xi_{i_k}.$$

(5.4)

Inverting (2.16) we express ξ_{i_k} in terms of the η_i's and the Cartan matrix C_{ij}:

$$\xi_{i_k} = \sqrt{-1} \sum_j^r (C^{-1})_{i_k j} \eta_j.$$

But [16]

$$(C^{-1})_{ij} = (\text{adj}\,C)_{ij}/\det C = (\text{adj}\,C)_{ij}/|Z(G)|,$$

since $\det C = |Z(G)|$, the number of elements in the centre of G. Thus

$$\xi_0 = (\sqrt{-1}/|Z(G)|) \sum_j^r \left\{ \sum_{i_k \in I} a_k (\text{adj}\,C)_{i_k j} \right\} \eta_j.$$

(5.5)

$(\text{adj}\,C)_{ij}$ and $|Z(G)|$ are integers so

Proposition 5.1. ζ_0 *generates a* $U(1)$ *if and only if its coefficients* a_k *in the expansion* (5.4) *are rationally related.*

In what follows we shall assume ξ_0 has this property. Let us choose a minimal generator ζ_0 parallel to ξ_0 and consider the new Higgs field $\Psi(x)$ obtained from $\Phi(x)$ by rescaling:

$$\Psi(x) = (|\zeta_0|/|\xi_0|)\,\Phi(x)\,. \tag{5.6}$$

$|\Psi(x)| = |\zeta_0|$, so Q_{em} reads also

$$Q_{\text{em}} = e_0 \cdot \Psi(x)/|\zeta_0|\,. \tag{5.7}$$

(This apparently complicated procedure is needed since the only coherent way to define the electric charge operator is through a Higgs field.)

Representation theory [16, 17] – see Sect. 2 – tells us that in any irreducible representation the eigenvalues of Q_{em} belong to the weight lattice: there exist integer coefficients k_i such that

$$q = \sum k_i \mu_i(Q_{\text{em}})\,. \tag{5.8}$$

Using the expansion (5.3) and the duality property (2.15) it is easy to evaluate q:

Theorem 5.2.

(i) *Electric charge is quantized:*

$$q = \left(\sum_{i=1}^{r} k_i n_i\right) q_{\min}\,. \tag{5.9}$$

(ii) *The minimal charge* $q_{\min}$ *is determined by the length of the minimal generator* ζ_0

$$q_{\min} = e_0/|\zeta_0|\,. \tag{5.10}$$

The minimal charge $q_{\min}$ is attained in some representation: the k_i's can be chosen to satisfy $\sum k_i n_i = 1$ since the n_i's have no common factor.

ζ_0 belongs also to the centre and so

$$\zeta_0 = \sum b_k \cdot \xi_{i_k}\,. \tag{5.11}$$

Thus

$$|\zeta_0|^2 = B(\zeta_0, \zeta_0) = \sum\sum \sqrt{-1}\, n_j b_k B(\eta_j \cdot \xi_{i_k}) = \sum \{2/\alpha_{i_k}(\eta_{\alpha_{i_k}})\} \cdot b_k n_{i_k}\,. \tag{5.12}$$

So the minimal charge reads

$$q_{\min} = e_0 \left| \sum_{i_k \in I} \{2/\alpha_{i_k}(\eta_{\alpha_{i_k}})\} \cdot n_{i_k} b_k \right|^{-1/2}\,. \tag{5.13}$$

On the other hand, monopoles carry also a magnetic charge defined as

$$g = (1/4\pi) \int_{S^2} \mathcal{F}\,. \tag{5.14}$$

Using the definition (5.1) of the electromagnetic field and the fact that on $S^2 |\Phi| = |\xi_0|$, (5.14) becomes

$$g = (1/4\pi e_0|\xi_0|) \int_{S^2} B(F, \Phi)\,. \tag{5.15}$$

Here we recognize the trace invariant – topological charge   of Sect. 4. Consequently:

Corollary 5.3. *The magnetic charge (5.14) is proportional to the topological charge (or, more exactly, to the trace invariant) I:*

$$g = (1/4\pi e_0|\xi_0|) I(\Phi) = (1/4\pi e_0|\zeta_0|) I(\Psi). \tag{5.16}$$

Equation (5.16) can be further developed using the results of Sect. 4.

$$g = (1/2e_0) \sum_{i_k \in I} m_k \sqrt{-1} B(\eta_{i_k}, \xi_0/|\xi_0|)$$

$$= (1/2e_0) \sum_{i_k \in I} m_k \sqrt{-1} B(\eta_{i_k}, \zeta_0/|\zeta_0|)$$

$$= (1/2e_0) \sum_{i_k \in I} m_k \{2/\alpha_{i_k}(\eta_{\alpha_{i_k}})\} \cdot b_k/|\zeta_0|.$$

Notice that $b_k/|\zeta_0| = a_k/|\xi_0|$. Finally we get:

Theorem 5.4. *The magnetic charge of the monopole is a combination of p "partial magnetic charges" $g_1, ..., g_p$:*

$$g = \sum a_k(|\xi_{i_k}|/|\xi_0|) g_k = \sum b_k(|\xi_{i_k}|/|\zeta_0|) g_k, \tag{5.17}$$

where each of the g_k's is the multiple by m_k of a fundamental charge $g_k^{(0)}$:

$$g_k = m_k \cdot g_k^{(0)}, \tag{5.18}$$

$$g_k^{(0)} = \frac{1}{2e_0|\xi_{i_k}|} \left[\frac{2}{\alpha_{i_k}(\eta_{\alpha_{i_k}})} \right], \quad (k = 1, ..., p). \tag{5.19}$$

How are the electric – and magnetic charges related? (5.9) and (5.19) imply that

$$2 g_{\min} g_k = [2/\alpha_{i_k}(\eta_{\alpha_{i_k}})]/|\zeta_0| |\xi_{i_k}|, \tag{5.20}$$

so we conclude:

Theorem 5.5. *The total magnetic charge g and the minimal electric charge $q_{\min}$ satisfy a generalized Dirac condition:*

$$2 q_{\min} g = \sum_{i_k \in I} \left| \frac{2 m_k a_k}{\alpha_{i_k}(\eta_{\alpha_{i_k}})} \right| \Big/ \sum_{i_j \in I} \left| \frac{2 n_{i_j} a_j}{\alpha_{i_j}(\eta_{\alpha_{i_j}})} \right|. \tag{5.21}$$

It is interesting to notice that the magnetic charge admits also a cohomological expression, i.e. can be written as the surface integral of a 2-form. Indeed, from (5.15) we get, using (3.21), that

$$g = \sum_{i_k \in I} (a_k |\xi_{i_k}|/|\xi_0|) \frac{1}{2e_0|\xi_{i_k}|} \int_{S^2} \Phi^* \omega^{\xi_{i_k}}. \tag{5.22}$$

comparing to (5.19):

$$g_k = \frac{1}{2e_0|\xi_{i_k}|} \int_{S^2} \Phi^* \omega^{\xi_{i_k}}. \tag{5.23}$$

We see that the "basic charges" $g_k^{(0)}$ are associated with the "basic 2-forms" $\omega^{(\xi_{i_k})}$ of Sect. 3 [9].

The situation is particularly simple if ξ_0 is parallel to one of the fundamental ξ_i's – say to ξ_k. Then the centre of $\mathfrak{h}$ is 1-dimensional [13]. The Lie algebra of H is decomposed now as

$$\mathfrak{h} = u(1) + \mathcal{K} ,\tag{5.24}$$

where $\mathcal{K} = [\mathfrak{h}, \mathfrak{h}]$ is semisimple. Physically, this corresponds to the situation [13, 14] when the residual symmetry group is (locally) the product of an electromagnetic $U(1)$ and a color group K.

$I = \{k\}$, so we have just one Higgs charge $m = m_k$. The minimal electric charge reads now

$$q_{\min} = e_0 \{\alpha_k(\eta_{\alpha_k})/2n_k b_k\}^{1/2} .\tag{5.25}$$

There exists one basic magnetic charge, $g_k^{(0)} = g_{\min}$ and any magnetic charge is the multiple of it by the Higgs charge:

$$g = m \cdot g_{\min}\tag{5.26}$$

with

$$g_{\min} = (1/2e_0) \{2b_k/\alpha_k(\eta_{\alpha_k})n_k\}^{1/2} .\tag{5.27}$$

(This follows also from (5.22) and (3.12).)

The generalized Dirac condition becomes

$$2q_{\min} \cdot g_{\min} = 1/n_k .\tag{5.28}$$

The integer n_k here counts the number of times the $U(1)$ – and the semisimple subgroups intersect. Indeed, H has the form [13]

$$H = [U(1) \times K]/Z ,\tag{5.29}$$

where $Z = U(1) \cap K$. $Z \subset Z(K)$ so it has a finite number of elements. Equation (5.29) shows also that n_k is the number of times $U(1) \times K$ covers H [15, 18]. Or this number is recovered by applying the weight μ_{i_k} to the generator of the $U(1)$ subgroup of H, which is ζ_0.

Notice that the minimal magnetic charge is – up to normalisation – the reciprocal of the length of the fundamental ξ_k to which ξ_0 is parallel [14]:

$$g_{\min} = \{e_0 \alpha_k(\eta_{\alpha_k}) |\xi_k|\}^{-1} .\tag{5.30}$$

This follows from (5.19) noting that $\zeta_0 = b_k \xi_k$ now.

6. Application: $G = SU(N)$

The aim of this last section is to illustrate the general theory outlined above on the example of $G = SU(U)$. This is also of practical interest, since most of the gauge groups considered in physics belong to this class.

$G = SU(2)$. Let us start, in order to get a feeling what is going on, with $G = SU(2)$. The base point now has the form

$$\xi_0 = \sqrt{-1} \begin{vmatrix} \lambda & \\ & -\lambda \end{vmatrix} = \sqrt{-1}\,\lambda\sigma_3 \in SU(2).\tag{6.1}$$

Identifying $SU(2)$ with R^3, the orbit of ξ_0 will be a 2-sphere of radius $|2\lambda|$. The little group consists of the rotations around the third axis:

$$H = \{\exp(2\pi\sqrt{-1}\,\sigma_3 t),\ t \in [0,1]\}\,. \tag{6.2}$$

$I = \{1\}$ so $Z(\mathfrak{h}) \simeq \mathfrak{h} \simeq u(1)$ is generated by

$$\xi_1 = \sqrt{-1}\,\sigma_3/2\,. \tag{6.3}$$

There is just one Higgs charge $m = m_1$. It is computed, according to Theorem 3.2, by

$$m(\Phi) = (1/2\pi\sqrt{-1})\int_{S^2}\Phi^*\omega\,, \tag{6.4}$$

where

$$\omega = \omega^{(1)} = \omega^{\xi_1}/\sqrt{-1}\,. \tag{6.5}$$

Observe that the orbit of ξ_0 – the 2-sphere of radius $|2\lambda|$ – projects to the orbit of ξ_1 – the sphere of radius $1/\sqrt{2}$,

$$\omega = \pi^*\Omega, \quad \text{where} \quad \pi : \mathcal{O}_{\xi_0} \to \mathcal{O}_{\xi_1}\,. \tag{6.6}$$

Ω here is the canonical symplectic (surface) form of the 2-sphere of radius $1/\sqrt{2}$. Expressed by the group elements, Ω reads

$$\Omega_\zeta = -d(\mathrm{Tr}\,\sigma_3 \cdot g^{-1}dg)\,, \quad (\zeta = \mathrm{Ad}_g\xi_0)\,. \tag{6.7}$$

In 3-dimensional notation Ω is expressed cf. [3, 7, 9].

$$(1/r^3)\varepsilon_{ijk}dx^j \otimes dx^k\ (r = |x|)\,. \tag{6.8}$$

In these terms the Higgs charge (6.4) becomes

$$m(\Phi) = (1/4\pi)\int_{S^2}\mathrm{Tr}(\Phi[\partial_\mu\Phi, \partial_\nu\Phi])d\sigma_{\mu\nu}\,. \tag{6.9}$$

The orbit of ξ_1 is prequantizable: the integral of Ω on it is 2π.

The trace invariant or topological charge (Sect. 4) reads now

$$I = -4\pi\lambda\,m\,. \tag{6.10}$$

The higher order invariants are also easily computed:

$$I^{(n)} = \begin{cases} 4\pi(-1)^{n+1/2}\lambda^n m & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even}. \end{cases} \tag{6.11}$$

ξ_0 is automatically a $U(1)$ generator. The minimal generator parallel to it is

$$\zeta_0 = 2\xi_1 = \sqrt{-1}\,\sigma_3 = \sqrt{-1}\,\eta_1\,. \tag{6.12}$$

The particles having an $SU(2)$ internal structure belong to irreducible representations characterized by an integer k (twice the isospin). Their electric charge reads hence, according to (5.25),

$$q = k \cdot q_{\min} = k \cdot (e_0/\sqrt{2})\,. \tag{6.13}$$

Similarly, the magnetic charge is obtained by (5.26) as

$$g = m \cdot g_{\min} = m/\sqrt{2}e_0,\qquad(6.14)$$

$2qg = km \in Z$, so Dirac's condition is satisfied.

$G = \mathrm{SU}(3)$. Let us consider now what happens for $G = \mathrm{SU}(3)$. The base point ξ_0 has now the form

$$\xi_0 = \sqrt{-1}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix},\ \lambda_i \in R,\ \sum_{i=1}^{N}\lambda_i = 0.\qquad(6.15)$$

There are two possibilities.

(i) Two of the λ_i's are equal. One can arrange, by applying a Weyl transformation, that $\lambda_1 = \lambda_2$. By (2.29),

$$H = S\{\mathrm{U}(2) \times \mathrm{U}(1)\} = \mathrm{U}(2).\qquad(6.16)$$

The orbit is the so-called minimal one. It has dimension 4 and has the topology of $P_2(C)$. $I = \{2\}$ so $Z(\mathfrak{h})$ is 1-dimensional and is generated by

$$\xi_2 = (\sqrt{-1}/3)\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}\qquad(6.17)$$

(or by ξ_0 which is proportional to ξ_2). The orbit of ξ_2 is also a minimal one and $\mathcal{O}_{\xi_0}$ projects obviously onto $\mathcal{O}_{\xi_2}$. The canonical symplectic structure of the latter is calculated the simplest way by representing the elements of $\mathrm{SU}(3)$ by elements of C^3: $\mathrm{SU}(3) \ni g = (Z_1, Z_2, Z_3)$, where $Z_i \in C^3$, $Z_i Z_j = \delta_{ij}$, $\det g = 1$.

The pullback to $\mathrm{SU}(3)$ of the canonical symplectic structure of $\mathcal{O}_{\xi_2}$ reads then

$$\Omega^{(2)} = \sqrt{-1}\,(dZ_1 \wedge dZ_1 + dZ_2 \wedge dZ_2).\qquad(6.18)$$

There is again one Higgs charge, $m = m_2$, calculated as:

$$m = (1/2\pi\sqrt{-1})\int_{S^2}\Phi^*\omega^{(2)} = (1/2\pi\sqrt{-1})\int_{S^2}g_i^*\Omega^{(2)}.\qquad(6.19)$$

(g_i here is the lift of Φ on U_i. The result does not depend on the choice of the lift.)

The Trace invariant becomes

$$I = -6\pi\lambda m.\qquad(6.20)$$

Similarly, the higher order invariants are expressed as

$$I^{(n)} = 2\pi\lambda^n \cdot (-1)^{(n+1)/2}[1 - (-2)^n] \cdot m.\qquad(6.21)$$

ξ_0 is parallel to $\xi_2 = (1/3\lambda)\xi_0$, so it generates a U(1). In fact,

$$\mathfrak{h} = \mathfrak{u}(1) + \mathfrak{su}(2).\qquad(6.22)$$

The minimal generator ζ_0 reads

$$\zeta_0 = 3\xi_2 = \sqrt{-1}\,(\eta_1 + 2\eta_2),\qquad(6.23)$$

so $b_2 = 3$ and $n_2 = 2$. The electric charge of a particle belonging to an irreducible representation of SU(3) is a multiple of

$$q_{min} = e_0/|\zeta_0| = e_0/\sqrt{6}. \tag{6.24}$$

There is one basic magnetic charge, $g_2^{(0)} = g_{min}$, and

$$g_2 = m \cdot g_{min} = \sqrt{6}\, m/4e_0. \tag{6.25}$$

The generalized Dirac condition becomes hence

$$2q_{min} \cdot g_{min} = 1/2 \tag{6.26}$$

(the same result is obtained directly from (5.28) observing that $n_2 = 2$).

But there is also another possibility, namely when
(ii) all the λ_i's are different. $I = \{1, 2\}$, yielding the residual symmetry group

$$H = U(1) \times U(1). \tag{6.27}$$

The orbit of ξ_0 is 6-dimensional and has the topology of $PTP_2(C)$. This is the maximal orbit.

$$Z(\mathfrak{h}) = \mathfrak{h} = \mathfrak{u}(1) + \mathfrak{u}(1). \tag{6.28}$$

It is generated by ξ_2 and

$$\xi_1 = (\sqrt{-1}/3)\begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix} \tag{6.29}$$

The orbit of ξ_1 is again a minimal one. Its canonical symplectic structure is given by

$$\Omega^{(1)} = \sqrt{-1}\, dZ_1 \wedge d\bar{Z}_1. \tag{6.30}$$

$\mathcal{O}_{\xi_0}$ projects now to both $\mathcal{O}_{\xi_1}$ and $\mathcal{O}_{\xi_2}$. There are two Higgs charges:

$$m_j = (1/2\pi\sqrt{-1})\int_{S^2} \Phi^* \omega^{(j)} = (1/2\pi\sqrt{-1})\int_{S^2} g_i^* \Omega^{(j)}, \quad j = 1, 2. \tag{6.31}$$

(g_i denotes again the lift of Φ on U_i.) The $\omega^{(j)}$'s are of course the pullbacks to $\mathcal{O}_{\xi_0}$ of the $\Omega^{(j)}$'s. Both orbits $\mathcal{O}_{\xi_j}$ ($j = 1, 2$) are prequantizable.

The trace invariant becomes [11]

$$I = -2\pi\{(\lambda_1 - \lambda_2)m_1 + (2\lambda_2 + \lambda_1)m_2\}. \tag{6.32}$$

For the higher order invariants our formulae yield

$$I^{(n)} = 2\pi(-1)^{(n+1)/2}[(\lambda_1^n - \lambda_2^n)m_1 + (\lambda_2^n - \{-\lambda_1 - \lambda_2\}^n)m_2]. \tag{6.33}$$

ξ_0 generates a U(1) only if λ_1 and λ_2 are rationally related. In this case $n_1 = r\lambda_1$ and $n_2 - n_1 = r\lambda_2$ are integers for a suitable real number r. Let us choose the least such positive r. Then

$$r\xi_0 - \zeta_0 = \sqrt{-1}\,(n_1\eta_1 + n_2\eta_2). \tag{6.34}$$

In terms of the ξ_i's ζ_0 is expanded as

$$\zeta_0 = (2n_1 - n_2)\xi_1 + (2n_2 - n_1)\xi_2. \tag{6.36}$$

For the minimal electric charge we get from (5.10) (or (5.13)

$$q_{\min} = e_0/|\zeta_0| = e_0/\sqrt{2}\,(n_1^2 - n_1 n_2 + n_2^2)\,. \tag{6.36}$$

On the other hand there are now two basic magnetic charges,

$$g_1^{(0)} = g_2^{(0)} = \sqrt{6}/4e_0\,. \tag{6.37}$$

The monopole's magnetic charge is hence

$$g = \frac{m_1(2n_1 - n_2) + m_2(2n_2 - n_1)}{2e_0|\zeta_0|}\,. \tag{6.38}$$

So the generalized Dirac conditions become

$$2q_{\min}g = \frac{n_1(2m_1 - m_2) + n_2(2m_2 - m_1)}{2(n_1^2 - n_1 n_2 + n_2^2)}\,. \tag{6.39}$$

$G = SU(N)$. The generalization to $G = SU(N)$ is now straightforward. Indeed, an element of $SU(N)$ can be represented by N-tuples,

$$g = \{(Z_1, \ldots, Z_N) | Z_i \in C^N,\, Z_j Z_k = \delta_{jk},\, \det(Z_1, \ldots, Z_N) = 1\}\,. \tag{6.40}$$

The base point is given, after rearrangement, by (2.28) (see Sect. 2). The residual symmetry group is hence

$$H = S\{U(i_1) \times \ldots \times U(N - i_p)\}\,. \tag{6.41}$$

The orbit is described as [24]

$$\xi_0 = \sqrt{-1}\sum_{i=1}^{N-1} \Lambda_i(Z_i Z_i - 1/N)\,, \tag{6.42}$$

where

$$\Lambda_i = \lambda_1 + \ldots + 2\lambda_i + \ldots + \lambda_{N-1}\,. \tag{6.43}$$

H has a p-dimensional centre whose Lie algebra is generated by those ξ_{i_k}'s, $i_k \in I$ (cf. (2.23)). The corresponding orbits are prequantizable. Their canonical symplectic forms pull back to G as

$$\Omega^{(i_k)} = \sqrt{-1}\sum_{i=1}^{i_k} dZ_j \wedge dZ_j\,, \qquad i_k \in I\,. \tag{6.44}$$

The trace and respectively higher order invariants are obtained by (4.23) and (4.24),

$$\mathrm{Tr}(\eta_{i_k}, \xi_0^n) = (-1)^n(\lambda_{i_k}^n - \lambda_{i_{k+1}}^n)\,, \qquad i_k \in I\,.$$

Consequently,

$$I^{(n)} = 2\pi(-1)^{(n+1)/2}\sum_{i_k \in I} (\lambda_{i_k}^n - \lambda_{i_{k+1}}^n)\cdot m_k\,. \tag{6.45}$$

Let us study finally the electromagnetic properties. First, ξ_0 generates a $U(1)$ if the λ_i's in (2.28) are rationally related. If so, let us consider the minimal generator ζ_0

parallel to it. Let b_k $(k=1,...,p)$ and $n_i \in Z$, $i=1,...,r$ its coefficients in its expansions in terms of the ξ_{i_k}'s and the η_i's respectively. The formulae of Sect. 5 work beautifully: the minimal charge is expressed as

$$q_{\min} = e_0 \{\sum n_{i_k} b_k\}^{-1/2}. \tag{6.46}$$

There are p basic magnetic charges,

$$g_k^{(0)} = 1/2e_0 |\xi_{i_k}| = (1/2e_0)\sqrt{N/(N-i_k)}. \tag{6.47}$$

The generalized Dirac conditions are those given in (5.20) and (5.21) respectively.

In particular, if we choose [15],

$$\xi_0 = \sqrt{-1}\left.\begin{bmatrix} 1/N-M & & & & & \\ & \ddots & & & & \\ & & 1/N-M & | & & \\ \hline & & & | & -1/M & \\ & & & | & & \ddots \\ & & & | & & -1/M \end{bmatrix}\right\}\begin{matrix} N-M \\ \\ M \end{matrix}, \tag{6.48}$$

then there are just two blocks, $I = \{N-M\}$ and the 1-dimensional centre is generated by

$$\xi_{N-M} = \{(N-M) \cdot M/N\} \xi_0.$$

Being parallel to one of the fundamental ξ_i's, (6.48) generates a U(1). The minimal generator in its direction is

$$\zeta_0 = n\xi_0,$$

where n is the least common multiple of $(N-M)$ and M. This is obvious from (6.48). Alternatively, we can see this from the decomposition

$$\xi_0 = \sqrt{-1}\left[\sum_{j=1}^{N-M} (j/N-M) \cdot \eta_j + \sum_{j=1}^{M-1} (M-j)/M) \cdot \eta_{N-M+j}\right].$$

This shows also that

$$n_{N-M} = \mu_{N-M}(\zeta_0) = n.$$

The minimal electric charge is hence

$$q_{\min} = (1/n)\sqrt{(N-M)M/N},$$

while the basic magnetic charge is

$$g_{\min} = g_{N-M} = (1/2)\sqrt{N/(N-M)M}.$$

Thus the generalized Dirac condition becomes

$$2g_{\min}q_{\min} = 1/n.$$

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