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The Surfboard Schrödinger Equations

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Abstract. We study the large time behavior of solutions of time dependent Schrödinger equations $i\partial u/\partial t = -(\frac{1}{2})\Delta u + t^\alpha V(x/t)u$ with bounded potential $V(x)$. We show that (1) if $\alpha > -1$, all solutions are asymptotically free as $t \to \infty$, (2) if $\alpha \leqq -1$ a solution becomes asymptotically free if and only if it has the momentum support outside of supp V for large time, (3) if $-1 \leqq \alpha < 0$ all solutions are still asymptotically "modified free" as $t \to \infty$ and that (4) if $0 \leqq \alpha < 2$, for each local minimum x_0 of $V(x)$, there exist solutions which are asymptotically Gaussians centered at $x = tx_0$ and spreading slowly as $t \to \infty$.

1. Introduction

Several years ago Kuroda and Morita [6] proposed the study of the Schrödinger equations of the form

$$i\partial u/\partial t = -(\tfrac{1}{2})\Delta u + t^\alpha V(x/t)u, \quad t \geqq 1, \quad x \in \mathbb{R}^n, \tag{1.1}$$

in conjunction with their study of the equations

$$i\partial u/\partial t = -(\tfrac{1}{2})\Delta u + t^\alpha v(x/t^\beta)u, \quad t \geqq 1, \quad x \in \mathbb{R}^n, \tag{1.2}$$

with $\beta \neq 1$. Equation (1.1) is considered in the Hilbert space $L^2(\mathbb{R}^n) = \mathscr{H}$ of square integrable functions and was named the surfboard Schrödinger equation because of its obvious pictorial analogy with the motion of a surfboard: the potential spreads at the same rate as a free wave packet. In this paper we shall study the asymptotic behavior of the solution of (1.1) with $\alpha < 2$, and show the following results. We write the propagator for Eq. (1.1) as $U(t,s)$ and $H_0 = -(\frac{1}{2})\Delta$.

(I) If $\alpha < -1$, then for every $u \in \mathscr{H}$, the strong limit

$$\lim_{t \to \infty} U(1,t)\exp(-i(t-1)H_0)u = W_+ u \tag{1.3}$$

exists and the wave operator W_+ is unitary.

(II) If $\alpha \geqq -1$, the limit (1.3) exists if and only if $\mathfrak{F}u \in L^2((\text{supp } V)^c)$, the elements in $\mathscr{H}$ whose essential supports are in the complement of the support of V.

(III) If $0 > \alpha \geqq -1$, the modified wave operator still exists and is unitary.

(IV) If $0<\alpha<2$, corresponding to each of the local minima of $V(x)$ there exist solutions which are asymptotic to Gaussians traveling with velocity one and spreading at the rate $t^{1/2-\alpha/4}$ as $t\to\infty$. A similar result holds true for $\alpha=0$ if the eigenvalues of the Hessian matrix $\{\partial^2 V/\partial x_i\,\partial x_j(x_0)\}$ are larger than $\frac{2}{9}$.

We assume for simplicity the following:

Assumption (A). *$V(x)$ is a real-valued function on $\mathbb{R}^n$ and $V\in\mathscr{B}(\mathbb{R}^n)$, the set of infinitely differentiable functions which are bounded with all their derivatives.*

Under this assumption, it is well-known [4] that the operator $H(t)=(-\frac{1}{2})\Delta+t^\alpha V(x/t)$ with $D(H(t))=H^2(\mathbb{R}^n)$ is selfadjoint $\mathscr{H}$ and Eq. (1.1) generates a unique strongly continuous unitary propagator $\{U(t,s): t,s\geqq 1\}$ satisfying the following properties: (i) $U(t,s)$ is a unitary operator on $\mathscr{H}$ and is strongly continuous is (t,s); (ii) $U(t,s)U(s,r)=U(t,r)$ and $U(t,t)=\mathbb{1}$, the identity operator; (iii) $U(t,s)H^\alpha(\mathbb{R}^n)=H^\alpha(\mathbb{R}^n)$ for any $\sigma\in\mathbb{R}^1$, and if $f\in H^2(\mathbb{R}^n)$, then $i\partial/\partial t\, U(t,s)f=H(t)U(t,s)f$ and $-i\partial/\partial s\, U(t,s)f=U(t,s)H(s)f$. Here $H^\sigma(\mathbb{R}^n)$ is the Sobolev space of order σ and the derivatives are strong derivatives in $\mathscr{H}$.

In Sect. 2 we shall prove the statements (I) and (II). Since the third statement (III) is already proved in Kitada [5] we shall in Sect. 3 present a simpler proof of (III) only for $\frac{1}{2}<-\alpha\leqq 1$. The last statement (IV) will be proved in Sect. 4. For the proofs of (II) and (IV) we shall use the conjugate transform of Eq. (1.1) which will be explained at the beginning of Sect. 2.

The following notation and conventions are used in what follows. $\mathscr{S}(\mathbb{R}^n)=\mathscr{S}$ is the space of all rapidly decreasing functions and $\mathscr{S}'$ is its dual space. For $u\in\mathscr{S}'$, supp u is its support and $\mathfrak{F}u=\hat{u}$ is its Fourier transform:

$$(\mathfrak{F}u)(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx.$$

For $a\in\mathbb{C}$, $\bar{a}$ is its complex conjugate. For a nice function $p(x,\xi)$, $P(x,D)$ is the pseudo-differential operator with the symbol $p(x,\xi)$:

$$P(x,D)u(x)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi$$

(cf. Kumano-go [7]). Various constants are denoted by C when it is not necessary to trace them precisely. Thus the constants denoted by the same symbol C are different from context to context. $\|f\|_p$ is the L^p-norm of f and $\|f\|=\|f\|_2$. δ_{jk} is Kronecker's delta.

2. Conjugate Transform, the Proofs of (I) and (II)

The key technique for the proof of (II) and (IV) which we shall employ is the conjugate transform of Eq. (1.1) which we explain here. We suppose that $V(t,x)$ is a smooth real-valued function of $t>0$ and $x\in\mathbb{R}^n$ which is bounded with its derivatives. Then each of the equations

$$i\,\partial u/\partial t=-(\tfrac{1}{2})\Delta u+V(t,x)u,\quad t>0 \tag{2.1}$$

and

$$i\,\partial v/\partial t=-(\tfrac{1}{2})\Delta v+t^{-2}V(1/t,x/t)v,\quad t>0 \tag{2.2}$$

generates a strongly continuous unitary propagator on the Hilbert space $\mathscr{H}$. We denote it for (2.1) as $U(t,s)$ and for (2.2) as $\tilde{U}(t,s)$. We define for $t>0$,

$$T(t)v(x)=(1/it)^{n/2}\exp(ix^2/2t)\overline{v(x/t)}. \tag{2.3}$$

Lemma 2.1. *Let $U(t,s)$ and $\tilde{U}(t,s)$ be the propagators for Eq. (2.1) and (2.2), respectively, and let $T(t)$ be the anti-unitary operator defined by (2.3). Then for any $u\in\mathscr{H}$,*

$$\lim_{t\to\infty}\|\exp(-itH_0)u-T(t)\bar{\hat{u}}\|=0, \tag{2.4}$$

$$U(t,s)u=T(t)\tilde{U}(1/t,1/s)T(s)^{-1}u,\quad t,s>0. \tag{2.5}$$

Proof. The relation (2.4) is well-known and we omit its proof here (cf. for example Reed–Simon [9], p. 60). Since $T(t)$ is anti-unitary and $U(t,s)$ and $\tilde{U}(t,s)$ are unitary, it suffices to show (2.5) for $u\in C_0^\infty(\mathbb{R}^n)$. For $u\in C_0^\infty(\mathbb{R}^n)$, $u(t,\cdot)=T(t)\tilde{U}(1/t,1/s)T(s)^{-1}u$ is an $\mathscr{S}$-valued smooth function of t (Kato [4]). Clearly $u(s)=u$. On the other hand an elementary computation shows that $u(t,x)$ satisfies Eq. (2.1). It follows by the uniqueness of the propagator that $u(t,\cdot)=U(t,s)u$ and Eq. (2.5) is proved.

The transformation $T(t)$ is called the conjugate transform ([2], [10]).

We now prove the statements (I) and (II).

Theorem 2.2. *Suppose Assumption (A) is satisfied. Then Statements (I) and (II) of Sect. 1 hold.*

Proof. (I). By Dunhamel's identity, we have

$$U(t,1)u=e^{-i(t-1)H_0}u-i\int_1^t e^{-i(t-s)H_0}s^\alpha V(x/s)U(s,1)u\,ds, \tag{2.6}$$

$$U(t,1)u=e^{-i(t-1)H_0}u-i\int_1^t U(t,s)s^\alpha V(x/s)e^{-i(s-1)H_0}u\,ds. \tag{2.7}$$

Multiplying both sides of (2.7) (or (2.6)) by $U(t,1)^{-1}$ (or $\exp(i(t-1)H_0)$), we have

$$U(t,1)^{-1}\exp(-i(t-1)H_0)u=u+i\int_1^t U(1,s)s^\alpha V(x/s)e^{-i(s-1)H_0}u\,ds, \tag{2.8}$$

$$\exp(i(t-1)H_0)U(t,1)u=u-i\int_1^t e^{-i(1-s)H_0}s^\alpha V(x/s)U(s,1)u\,ds. \tag{2.9}$$

Since $\alpha<-1$, the integrands in the right-hand side of (2.8) and (2.9) are both integrable on $[1,\infty)$, and the limits

$$\lim_{t\to\infty}U(t,1)^{-1}\exp(-i(t-1)H_0)u=W_+u, \tag{2.10}$$

$$\lim_{t\to\infty}\exp(i(t-1)H_0)U(t,1)u=Z_+u \tag{2.11}$$

exist for every $u\in\mathscr{H}$. Moreover we clearly have

$$Z_+W_+u=W_+Z_+u=u. \tag{2.12}$$

Since Z_+ and W_+ are isometries, (2.12) implies that Z_+ and W_+ are unitary and $W_+=Z_+^{-1}$.

(II) For $u \in \mathscr{H}$, we write $u_1 = \overline{\mathfrak{F}(\exp(iH_0)u)}$. Assuming that one of the following three limits exists, we have by (2.4) and (2.5) that

$$\begin{aligned} W_+ u &= \lim_{t\to\infty} U(t,1)^{-1}\exp(-i(t-1)H_0)u = \lim_{t\to\infty} U(1,t)T(t)u_1 \\ &= \lim_{t\to 0} T(1)\tilde{U}(1,t)u_1 . \end{aligned} \tag{2.13}$$

Here $\tilde{U}(t,s)$ is the unitary propagator for the conjugate equation of (1.1):

$$i\,\partial v/\partial t = -(\tfrac{1}{2})\Delta v + t^{-2-\alpha}V(x)v. \tag{2.14}$$

Suppose now that $u \in \mathscr{S}(\mathbb{R}^n)$ and $\operatorname{supp}\mathfrak{F}u = \operatorname{supp} u_1 \subset (\operatorname{supp} V)^c$. Then by Taylor's expansion formula

$$\begin{aligned} \|V\exp(-itH_0)u_1\| &= \Bigg\| \sum_{k=0}^{N} V(-itH_0)^k u_1/k! \\ &\qquad + \int_0^1 \theta^N V e^{-it\theta H_0}\, d\theta\{(-itH_0)^{N+1}u_1/N!\}\Bigg\| \\ &\leqq \left(\int_0^1 \theta^N \|V\|_\infty\, d\theta\right) t^{N+1}\|H_0^{N+1}u_1\|/N! \\ &= C_N t^{N+1}\|y^{2(N+1)}u\|. \end{aligned} \tag{2.15}$$

Since $\tilde{U}(1,s)s^{-2-\alpha}V(x)\exp(-i(s-t)H_0)u_1$ converges strongly to $\tilde{U}(1,s)s^{-2-\alpha}\cdot V(x)\exp(-isH_0)u_1$ as $t\to 0$, and for $0<t<s$,

$$\begin{aligned} \|\tilde{U}(1,s)s^{-2-\alpha}V(x)\exp(-i(s-t)H_0)u_1\| &\leqq C_N s^{-2-\alpha}(s-t)^N\|y^{2N}u\|\,\|V\|_\infty \\ &\leqq C_N s^{N-2-\alpha}\|y^{2N}u\|\,\|V\|_\infty, \end{aligned}$$

with arbitrary large N by (2.15), we see that the limit as $t\to 0$ of

$$\tilde{U}(1,t)u_1 = \exp(-i(1-t)H_0)u_1 - i\int_t^1 \tilde{U}(1,s)s^{-2-\alpha}V(x)\exp(-i(s-t)H_0)u_1\, ds$$

exists by Lebesgue's dominated convergence theorem. Hence by (2.13) W_+u exists. Since the set of u's such that the limits in (2.13) exist forms a closed subspace of $\mathscr{H}$, W_+u exists for all $u\in\mathscr{H}$ with $\mathfrak{F}u \in L^2((\operatorname{supp} V)^c)$.

Suppose, on the contrary, that one (hence all) of the limits (2.13) exists and

$$\|Vu_1\| \neq 0, \qquad u_1 = \overline{\mathfrak{F}(\exp(iH_0)u)}. \tag{2.16}$$

We shall show that this will lead to a contradiction. We denote

$$u_2 = \lim_{t\downarrow 0} \tilde{U}(1,t)u_1 . \tag{2.17}$$

Then the following limits trivially exist:

$$\begin{aligned} u_1 &= \lim_{t\downarrow 0} \tilde{U}(t,1)u_2 = \lim_{t\downarrow 0} \exp(itH_0)\tilde{U}(t,1)u_2 \\ &= e^{iH_0}u_2 - i\lim_{t\downarrow 0}\int_1^t e^{isH_0}s^{-2-\alpha}V(x)\tilde{U}(s,1)u_2\, ds \end{aligned} \tag{2.18}$$

Therefore for any sequence $t_n \to 0$, $t_n > 0$,

$$\lim_{0<t_n<t_m\to 0} \left\| \int_{t_n}^{t_m} e^{isH_0} s^{-2-\alpha} V(x) \tilde{U}(s,1) u_2 \, ds \right\| = 0. \tag{2.19}$$

By (2.18), we see that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{0<t<\delta} \| \tilde{U}(t,1)u_2 - u_1 \| < \varepsilon, \tag{2.20}$$

$$\sup_{0<t<\delta} \| (e^{itH_0} - \mathbb{1}) V \tilde{U}(t,1) u_2 \| < \varepsilon. \tag{2.21}$$

It follows for $0 < t_n < t_m < \delta$,

$$\begin{aligned} \left\| \int_{t_n}^{t_m} e^{isH_0} s^{-2-\alpha} V \tilde{U}(s,1) u_2 \, ds \right\| &\geqq \left\| \int_{t_n}^{t_m} s^{-2-\alpha} V u_1 \, ds \right\| \\ &\quad - \int_{t_n}^{t_m} \| s^{-2-\alpha} V(\tilde{U}(s,1)u_2 - u_1) \| \, ds \\ &\quad - \int_{t_n}^{t_m} \| s^{-2-\alpha} (e^{isH_0} - \mathbb{1}) V \tilde{U}(s,1) u_2 \| \, ds \\ &\geqq \begin{cases} \left| \dfrac{t_m^{-1-\alpha} - t_n^{-1-\alpha}}{1+\alpha} \right| \{ \| V u_1 \| - \| V \|_\infty \varepsilon - \varepsilon \} \text{ if } \alpha > -1 \\ |\log(t_m/t_n)| \{ \| V u_1 \| - \| V \|_\infty \varepsilon - \varepsilon \} \text{ if } \alpha = -1. \end{cases} \end{aligned} \tag{2.22}$$

Since $\varepsilon > 0$ can be taken arbitrary small, (2.22) contradicts (2.19). This concludes the proof of Theorem 2.1.

3. The Case $-1 \leqq \alpha < 0$, the Modified Wave Operator

In this section we assume $-1 \leqq \alpha < 0$. In this case the existence and the unitarity of the modified wave operator are proved by Kitada [5] in a more general situation. Here, restricting ourselves to the case $-1 \leqq \alpha < -\frac{1}{2}$, we state the theorem and its proof in a simpler form. We refer the reader to [5] for the general case.

Theorem 3.1. *Suppose that Assumption (A) is satisfied and that $-1 \leqq \alpha < -\frac{1}{2}$. Then the limit*

$$W_+ u = \lim_{t\to\infty} U(t,1)^{-1} \exp(-itD^2/2 - it^{1+\alpha} V(D)/(1+\alpha)) u \tag{3.1}$$

exists for every $u \in \mathscr{H}$ and the modified wave operator W_+ is a unitary operator.

We write $H(t,D) = tD^2/2 + t^{1+\alpha} V(D)/(1+\alpha)$. For proving the theorem 3.1 we need the following two lemmas.

Lemma 3.2. *Let $f \in \mathscr{S}$. Then for each $j = 1, 2, \ldots, n$,*

$$\| ((x_j/t) - D_j) \exp(-iH(t,D)) f \| \leqq C t^\alpha, \quad t \geqq 1, \tag{3.2}$$

$$\| ((x_j/t) - D_j) U(t,1) f \| \leqq C t^\alpha, \quad t \geqq 1, \tag{3.3}$$

with a constant C independent of $t \geqq 1$.

Proof. By elementary computations of commutators we have

$$(d/dt)e^{iH(t,D)}(x_j - tD_j)e^{-iH(t,D)}f = e^{iH(t,D)}(t^\alpha(\partial V/\partial x_j)(D))e^{-iH(t,D)}f, \tag{3.4}$$

$$(d/dt)U(t,1)^{-1}(x_j - tD_j)U(t,1)f = U(t,1)^{-1}(t^\alpha(\partial V/\partial x_j)(x/t))U(t,1)f. \tag{3.5}$$

Integrating (3.4) and (3.5) from 1 to t by t and taking the norm in $\mathscr{H}$ in the resulting equation, we obtain

$$\|(x_j - tD_j)e^{-iH(t,D)}f\| \leqq \|(x_j - D_j)e^{-iH(1,D)}f\| + \frac{t^{\alpha+1}}{\alpha+1}\|\partial V/\partial x_j\|_\infty \|f\|, \tag{3.6}$$

$$\|(x_j - tD_j)U(t,1)f\| \leqq \|(x_j - D_j)f\| + \frac{t^{\alpha+1}}{\alpha+1}\|\partial V/\partial x_j\|_\infty \|f\|. \tag{3.7}$$

Dividing the both sides of (3.6) and (3.7) by t, we obviously have (3.2) and (3.3).

Lemma 3.3. *Let F be a C^∞-function on $\mathbb{R}^n$ which is bounded with its derivatives. Then*

$$F(x/t) - F(D) = \sum_{j=1}^n \left\{\int_0^1 (\partial F/\partial x_j)(\theta x/t + (1-\theta)D)d\theta\right\}\cdot(x_j/t - D_j)$$
$$+ (i/t)\int_0^1 (1-\theta)(\Delta F)(\theta x/t + (1-\theta)D)d\theta. \tag{3.8}$$

The identity (3.8) follows directly from the symbol calculus in pseudo-differential operators and its proof is omitted here (cf. also Enss [11]).

Proof of Theorem 3.1. Since V satisfies Assumption (A), $\int_0^1 \partial V/\partial x_j(\theta x/t + (1-\theta)D)d\theta$ and $\int_0^1 (1-\theta)\Delta V(\theta x/t + (1-\theta)D)d\theta$ are uniformly bounded operators on $\mathscr{H}$ for $t \geqq 1$ by Calderon–Vaillancourt's theorem [7]. Hence combining estimates (3.2) and (3.3) with (3.8) replacing V in place of F, we have

$$\|t^\alpha(V(x/t) - V(D))\exp(-iH(t,D))f\| \leqq Ct^{2\alpha}, \tag{3.9}$$

$$\|t^\alpha(V(x/t) - V(D))U(t,1)f\| \leqq Ct^{2\alpha}, \tag{3.10}$$

with the constant C independent of $t \geqq 1$. Thus the derivatives

$$\|(d/dt)U(t,1)^{-1}\exp(-iH(t,D))f\| = t^\alpha\|(V(x/t) - V(D))\exp(-iH(t,D))f\|,$$

and

$$\|(d/dt)\exp(+iH(t,D))U(t,1)f\| = t^\alpha\|(V(x/t) - V(D))U(t,1)f\|,$$

are both integrable on $[1,\infty)$ and this implies the existence of the limits

$$\lim_{t\to\infty} U(t,1)^{-1}\exp(-iH(t,D))f = W_+ f,$$

$$\lim_{t\to\infty} \exp(iH(t,D))U(t,1)f = Z_+ f.$$

As in the proof of Theorem 2.1, this implies the unitarity of W_+ and the proof of Theorem 3.1 is completed.

4. The Case $0 \leqq \alpha < 2$. The Existence of the Asymptotic Gaussians

In this section we shall show that when $0 < \alpha < 2$, corresponding to each local minimum $x_0 \in \mathbb{R}^n$ of $V(x)$ with positive definite Hessian matrix $\{\partial^2 V/\partial x_i \partial x_j(x_0)\}$, there exist solutions which are asymptotic to Gaussians centered about tx_0 with the width $Ct^{1/2-\alpha/4}$ times some oscillating factor. We also show almost the same statements are true for the case $\alpha = 0$, provided that the eigenvalues of the Hessian matrix $\{\partial^2 V/\partial x_i \partial x_j(x_0)\}$ are greater than $\frac{2}{9}$.

Postponing the precise statement of the theorem, we begin our discussion with the following lemma.

Lemma 4.1. *Let $l(t)$, $0 < t$, be a solution of the Riccati equation*

$$dl/dt + il^2(t) = i\lambda t^{-2-\alpha}, \quad \lambda \in \mathbb{C}, \tag{4.1}$$

and $\tilde{l}(t)$ be a solution of

$$i\, d\tilde{l}/dt = \tfrac{1}{2}\tilde{l}(t) l(t). \tag{4.2}$$

Then the function

$$u(t, x) = \tilde{l}(t) \exp(-l(t)x^2/2), \quad x \in \mathbb{R} \tag{4.3}$$

is a solution of the one dimensional Schrödinger equation

$$i\, \partial u/\partial t = -(\tfrac{1}{2})(\partial^2 u/\partial x^2) + (\tfrac{1}{2})\lambda t^{-2-\alpha} x^2 u. \tag{4.4}$$

This is a result of an elementary computation and we omit the proof here. We should remark that when $\lambda \in \mathbb{R}$ (4.1) and (4.2) imply

$$|\tilde{l}(t)|^4 = c \operatorname{Re} l(t), \tag{4.5}$$

with a constant c independent of $t > 0$. We assume $\lambda > 0$ hereafter.

When $\alpha \neq 0$, the solutions of the Riccati equation (4.1) can be obtained as fractions of Bessel functions (cf. [1]).

Lemma 4.2. *Suppose that $\alpha \neq 0$, and set*

$$\nu = -1/\alpha, \quad s = \lambda t^{-\alpha}/\alpha^2. \tag{4.6}$$

Then the general solution $l(t)$ of (4.1) *can be written by means of the Bessel functions $J_\nu(2\sqrt{s})$ and $Y_\nu(2\sqrt{s})$ as*

$$\begin{cases} l(t) = u(t)/t, \quad u(t) = i\alpha s w'(s)/w(s) \\ w(s) = s^{\nu/2}[cJ_\nu(2\sqrt{s}) + dY_\nu(2\sqrt{s})], \end{cases} \tag{4.7}$$

where c and d are arbitrary complex constants and $w' = dw/ds$.

When $\alpha = 0$, Eq. (4.1) has solutions

$$l_\pm(t) = (-i \pm \sqrt{4\lambda - 1})/2t, \tag{4.8}$$

and the general solution can be obtained by quadratures.

Lemma 4.3. *Let $\alpha = 0$ and $a \in \bar{\mathbb{C}}$ be an arbitrary constant. Then the general solution $l(t)$ of* (4.1) *can be written as follows:*

(i) *When* $\lambda > \frac{1}{4}$,

$$l(t) = -\frac{(i+\sqrt{4\lambda-1})(1-a\exp(i\sqrt{4\lambda-1}\log t + i\rho))}{2t(1-a\exp(i\sqrt{4\lambda-1}\log t))}, \tag{4.9}$$

with $\exp(i\rho) = (i-\sqrt{4\lambda-1})/(i+\sqrt{4\lambda-1})$.

(ii) *When* $\lambda = \frac{1}{4}$,

$$l(t) = -\frac{i}{2t} + \frac{1}{at + it\log t}. \tag{4.10}$$

(iii) *When* $\lambda < \frac{1}{4}$,

$$l(t) = -\frac{i((1+\sqrt{1-4\lambda})t^{\sqrt{1-4\lambda}} - a(1-\sqrt{1-4\lambda}))}{2t(t^{\sqrt{1-4\lambda}} - a)}. \tag{4.11}$$

Lemma 4.2 and Lemma 4.3 are well-known and we refer to Chapter 3 of Davis [1]. Using these expressions and the well-known asymptotic formulae for Bessel functions at real infinity $z \to +\infty$,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right) + O\left(\frac{1}{z}\right), \tag{4.12}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{(2\nu+1)\pi}{4}\right) + O\left(\frac{1}{z}\right), \tag{4.13}$$

and their derivatives

$$J'_\nu(z) = -\sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{(2\nu+1)\pi}{4}\right) + O\left(\frac{1}{z}\right), \tag{4.14}$$

$$Y'_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right) + O\left(\frac{1}{z}\right), \tag{4.15}$$

(cf. Magnus et al. [8]), we have the following lemma.

Lemma 4.4. *Suppose that* $\alpha > 0$. *Set*

$$\left.\begin{aligned} \nu = -1/\alpha, \quad s = \lambda t^{-\alpha}/\alpha^2, \\ p(s) = 2\sqrt{s} - (2\nu+1)\pi/4. \end{aligned}\right\} \tag{4.16}$$

Then with arbitrary real constants A and B, the real part $\operatorname{Re} l(t)$ *of the general solution of* (4.1) *has the following asymptotic behavior as* $t \downarrow 0$:

$$\operatorname{Re} l(t) = \frac{\alpha\sqrt{s}}{t}\left\{\frac{B}{(A\cos p(s) + \sin p(s))^2 + B^2\cos^2 p(s)} + O\left(\frac{1}{s^{1/4}}\right)\right\}. \tag{4.17}$$

Proof. By Lemma 4.2 and the relations (4.12)–(4.15), we have

$$l(t) = \frac{i\alpha s}{t}\left\{\frac{\nu}{2s} + \frac{-c\sin p(s) + d\cos p(s) + O(s^{-1/4})}{\sqrt{s}(c\cos p(s) + d\sin p(s) + O(s^{-1/4}))}\right\}.$$

Hence, writing $c/d = A + iB$, we see

$$\begin{aligned}\operatorname{Re} l(t) &= \operatorname{Im} \frac{\alpha\sqrt{s}}{t} \frac{c \sin p(s) - d \cos p(s) + O(s^{-1/4})}{c \cos p(s) + d \sin p(s) + O(s^{-1/4})} \\ &= \operatorname{Im} \frac{\alpha\sqrt{s}}{t} \frac{(A \sin p(s) - \cos p(s)) + iB \sin p(s) + O(s^{-1/4})}{A \cos p(s) + \sin p(s) + iB \cos p(s) + O(s^{-1/4})} \\ &= \frac{\alpha\sqrt{s}}{t} \frac{B}{(A \cos p(s) + \sin p(s))^2 + B^2 \cos^2 p(s) + O(s^{-1/4})}. \end{aligned} \tag{4.18}$$

Here in the final step we used the fact that $g(s) = (A \cos p(s) + \sin p(s))^2 + B^2 \cos^2 p(s) \geqq \gamma > 0$ for some $\gamma > 0$, since $g(s^2)$ is periodic in s and never vanishes. Equation (4.17) results from (4.18).

Taking the real parts in (4.9) $\sim$ (4.11), we have the following asymptotic formulae for $\operatorname{Re} l(t)$ for the case $\alpha = 0$.

Lemma 4.5. *Suppose $\alpha = 0$ and let A and B stand for arbitrary real constants. Then as $t \downarrow 0$, $\operatorname{Re} l(t)$ has the following asymptotic expression:*

(i) *When $\lambda > \frac{1}{4}$,*

$$\operatorname{Re} l(t) = \frac{(A^2 - 1)\sqrt{4\lambda - 1}}{2t(A^2 - 2A \cos(\sqrt{4\lambda - 1} \log t + B) + 1)}. \tag{4.19}$$

(ii) *When $\lambda = \frac{1}{4}$,*

$$\operatorname{Re} l(t) = \frac{A}{t(\log t)^2}\left(1 + O\left(\frac{1}{\log t}\right)\right). \tag{4.20}$$

(iii) *When $\lambda < \frac{1}{4}$,*

$$\operatorname{Re} l(t) = \frac{A\sqrt{1 - 4\lambda}}{t^{1 - \sqrt{1 - 4\lambda}}}(1 + O(t^{\sqrt{1 - 4\lambda}})), \quad -1 \leqq A \leqq 1. \tag{4.21}$$

Combining Lemma 4.1 $\sim$ Lemma 4.5 with the estimates similar to those used in semi-classical theory (cf. e.g. [3]), we shall obtain the solutions of the conjugate equation,

$$i\,\partial v/\partial t = -(\tfrac{1}{2})\Delta v + t^{-2-\alpha} V(x) v \tag{4.22}$$

of (1.1) which are asymptotic to Gaussian functions as $t \downarrow 0$. We let $x_0 \in \mathbb{R}^n$ be a local minimum of $V(x)$ and assume that the Hessian $\{\partial^2 V/\partial x_i \partial x_j(x_0)\}$ is positive definite there. After choosing suitable coordinates we may assume without losing generality that $\{\partial^2 V/\partial x_i \partial x_j(x_0)\}$ is diagonal with positive eigenvalues $0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots \leqq \lambda_n$. We denote by $l_j(t)$ a solution of the Riccati equation (4.1) with λ_j replacing λ which satisfies (4.17) with $B > 0$ when $\alpha > 0$; (4.19) with $A > 1$ when $\alpha = 0$ and $\lambda_j > \frac{1}{4}$; (4.20) with $A > 0$ when $\alpha = 0$ and $\lambda_j = \frac{1}{4}$; and (4.21) with $A > 0$ when $\alpha = 0$ and $0 < \lambda_j < \frac{1}{4}$. We also denote by $\tilde{l}_j(t)$ a solution of (4.2) with $l_j(t)$ replacing $l(t)$.

Lemma 4.6. *Let* $0 \leqq \alpha < 2$ *and let* $x_0 \in \mathbb{R}^n$, $0 < \lambda_1 \leqq \cdots \leqq \lambda_n$, $l_1(t), \ldots, l_n(t)$ *and* $\tilde{l}_1(t), \ldots, \tilde{l}_n(t)$ *be as above. Suppose that when* $\alpha = 0$, $\lambda_1 > \frac{2}{9}$. *Then the conjugate Schrödinger equation* (4.22) *has the solution* $v(t, x)$ *which satisfies*

$$\lim_{t \downarrow 0} \left\| v(t,x) - \left\{ \prod_{j=1}^{n} \tilde{l}_j(t) \exp(-l_j(t)(x_j - x_{0j})^2/2) \right\} \exp(it^{-1-\alpha} V(x_0)/1+\alpha) \right\| = 0. \tag{4.23}$$

Proof. We write

$$\tilde{v}(t,x) = \left\{ \prod_{j=1}^{n} \tilde{l}_j(t) \exp(-l_j(t)(x_j - x_{0j})^2/2 \right\} \exp(it^{-1-\alpha} V(x_0)/1+\alpha).$$

Since $\partial V/\partial x_j(x_0) = 0$ and $\partial^2 V/\partial x_j \partial x_k(x_0) = \lambda_j \delta_{jk}$, Lemma 4.1 implies that $\tilde{v}(t,x)$ satisfies the equation

$$\begin{aligned} i\,\partial\tilde{v}/\partial t = (-\tfrac{1}{2})\Delta\tilde{v} + t^{-2-\alpha}\Bigg\{ & V(x_0) + \sum_{j=1}^{n} \partial V/\partial x_j(x_0)(x_j - x_{j0}) \\ & + (\tfrac{1}{2}) \sum_{j,k} \partial^2 V/\partial x_j \partial x_k(x_0)(x_j - x_{0j})(x_k - x_{0k}) \Bigg\} \tilde{v}(t,x). \end{aligned} \tag{4.24}$$

By the choice of $l_j(t)$ and $\tilde{l}_j(t)$ as above, $\tilde{v}(t,x)$ also satisfies the estimate

$$t^{-2-\alpha} \| |x|^3 \tilde{v}(t,x) \|_{L^2(\mathbb{R}^n)} \leqq \rho(t), \tag{4.25}$$

where

$$\rho(t) = \begin{cases} Ct^{-2-\alpha}(t^{(1/2)+(\alpha/4)})^3 = Ct^{-(1/2)-(\alpha/4)}, & \text{when } \alpha > 0, \\ Ct^{-2}(t^{1/2})^3 = Ct^{-1/2}, & \text{when } \alpha = 0 \text{ and } \lambda_1 > \frac{1}{4}. \\ Ct^{-2}(t^{1/2}\log t)^3 = Ct^{-1/2}(\log t)^3, & \text{when } \alpha = 0 \text{ and } \lambda_1 = \frac{1}{4}. \\ Ct^{-2}(t^{(1/2)-1/2(\sqrt{1-4\lambda_1})})^3 = Ct^{-(1/2)-3/2\sqrt{1-4\lambda_1}}, & \text{when } \alpha = 0 \text{ and } \lambda_1 < \frac{1}{4}. \end{cases} \tag{4.26}$$

We note that $\rho(t)$ is integrable on $[0, 1]$ if $\lambda_1 > \frac{2}{9}$ when $\alpha = 0$. By (4.24)

$$\begin{aligned} f(t,x) &\equiv \{ i\partial/\partial t + (\tfrac{1}{2})\Delta - t^{-2-\alpha} V(x) \} \tilde{v}(t,x) \\ &= -t^{-2-\alpha} \left\{ V(x) - V(x_0) - (\tfrac{1}{2}) \sum_{j=1}^{n} \lambda_j x_j^2 \right\} \tilde{v}(t,x), \end{aligned} \tag{4.27}$$

and by Taylor's formula and Assumption (A),

$$\left| V(x) - V(x_0) - (\tfrac{1}{2}) \sum_{j=1}^{n} \lambda_j x_j^2 \right| \leqq C|x|^3. \tag{4.28}$$

It follows from (4.25)–(4.28) that

$$\int_0^1 \| f(t,\cdot) \|_{L^2(\mathbb{R}^n)}\, dt < \infty. \tag{4.29}$$

Writing the evolution operator for (4.27) as $\tilde{U}(t,s)$ and solving Eq. (4.27) for $\tilde{v}(t,x)$, we see

$$\tilde{v}(t,x)=\tilde{U}(t,1)\tilde{v}(1,\cdot)-i\int_1^t \tilde{U}(t,s)f(s,\cdot)ds$$

$$=\tilde{U}(t,1)\left[\tilde{v}(1,\cdot)-i\int_1^t \tilde{U}(1,s)f(s,\cdot)ds\right]. \tag{4.30}$$

When we set

$$v(t)=\tilde{U}(t,1)\left[\tilde{v}(1,\cdot)-i\int_1^0 \tilde{U}(1,s)f(s,\cdot)ds\right], \tag{4.31}$$

we obtain from (4.29) and (4.30) that

$$\|v(t)-\tilde{v}(t,x)\|\leqq\int_0^t \|f(s,\cdot)\|ds\to 0, \tag{4.32}$$

as $t\downarrow 0$. Since $v(t,x)$ is a solution of (4.22) by Definition (4.31), we have the statement of the lemma.

Combining Lemma 4.6 with Lemma 2.1 about the conjugate transform, we finally obtain the following theorem.

Theorem 4.7. *Let $V(x)$ satisfy Assumption (A) and let $0\leqq\alpha<2$. Suppose that $x_0\in\mathbb{R}^n$ is a local minimum of $V(x)$ with the positive definite $\{\partial^2V/\partial x_j\partial x_j(x_0)\}$. Suppose further that the smallest eigenvalue of $\{\partial^2V/\partial x_i\partial x_j(x_0)\}$ is larger than $\frac{2}{9}$ when $\alpha=0$, and the coordinates, the functions $l_j(t)$ and $\overline{l}_j(t)$, are taken as in Lemma 4.6. Then Eq. (1.1) has the solution $u(t,x)$ which satisfies the asymptotic formula*

$$\lim_{t\to\infty}\left\|u(t,x)-\left\{(1/it)^n\prod_{j=1}^n\overline{\overline{l}_j(1/t)}\right\}\exp\left[-it^{1+\alpha}V(x_0)/(1+\alpha)+ix\cdot x_0\right.\right.$$
$$\left.\left.-itx_0^2/2+\sum_{j=1}^n\{(i/2t)-\overline{(l_j(1/t)/t^2)}\}(x_j-tx_{0j})^2)\right]\right\|=0. \tag{4.33}$$

We note that modulo the oscillating factors which are bounded from above and below we have

$$\{\mathrm{Re}(l_j(1/t)/t^2)\}^{-1/2}\sim\begin{cases}Ct^{1/2-\alpha/4}, & \text{when } \alpha>0\\ Ct^{1/2}, & \text{when } \alpha=0 \text{ and } \lambda_j>\frac{1}{4},\\ Ct^{1/2}\log t, & \text{when } \alpha=0 \text{ and } \lambda_j=\frac{1}{4},\\ Ct^{(1+\sqrt{1-4\lambda})/2}, & \text{when } \alpha=0 \text{ and } \lambda_j<\frac{1}{4},\end{cases}$$

as $t\to\infty$ and the second function in the norm of (4.33) represents a slowly spreading Gaussian function with a linearly moving center tx_0.

Proof. Combine (4.23) with (2.3) and (2.5) to obtain (4.33).

References

1. Davis, H. T.: Introduction to non-linear differential and integral equations. New York: Dover, 1962
2. Ginibre, J., Velo, G.: Sur une équation de Schrödinger non linéaire avec interaction non locale, in "Nonlinear partial differential equations and their applications" Collège de France seminar II. Boston-London-Melborne: Pitman, 1981

3. Hagedorn, G. A.: Semi-classical quantum mechanics, I: The $\hbar \to 0$ limit for coherent states. Commun. Math. Phys. **71**, 77–93 (1980)
4. Kato, T.: Linear evolution equations of "hyperbolic type" I. J. Fac. Sci. Univ. of Tokyo, Sec. IA, **17**, 241–258 (1970)
5. Kitada, H.: Scattering theory for Schrödinger equations with time dependent potentials of long range type, preprint, Dept. Pure & Appl. Sci. University of Tokyo, 1981
6. Kuroda, S. T., Morita, H.: An estimate for solutions of Schrödinger equation with time-dependent potentials and associated scattering theory. J. Fac. Sci. Univ. of Tokyo Sec. IA, **24**, 459–475 (1977)
7. Kumano-go, H.: Theory of Pseudo-differential Operators. Tokyo: Iwanami Shoten, 1974
8. Magnus, W., Oberhettinger, F., Soni, R. P.: Formulas and theorems for the special functions of mathematical physics. Berlin. Heidelberg, New York: Springer, 1966
9. Reed, M., Simon, B.: Methods of modern mathematical physics, II, Fourier analysis, Selfadjointness. New York: Academic Press, 1975
10. Tsutsumi, Y., Yajima, K.: The asymptotic behavior of solutions of non-linear Schrödinger equations. Bull. Am. Math. Soc. (to appear)
11. Enss, V.: Scattering and spectral theory for three particle systems. In: Proc. International Conference on Differential Equations, Knowles, I., Lewis, R. (eds.). Amsterdam: North Holland

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