Commun. Math. Phys. 95, 345–391 (1984)

Communications in Mathematical Physics

Monopoles and Maps from S^2 to S^2; the Topology of the Configuration Space

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Abstract. The configuration space for the SU(2)-Yang-Mills-Higgs equations on $\mathbb{R}^3$ is shown to be homotopic to the space of smooth maps from S^2 to S^2. This configuration space indexes a family of twisted Dirac operators. The Dirac family is used to prove that the configuration space does not retract onto any subspace on which the SU(2)-Yang-Mills-Higgs functional is bounded.

A1. Introduction

In [1], the author announced a theorem which stated that the SU(2) Yang-Mills equations on $\mathbb{R}^3$ in the Prasad Sommerfield limit have an infinite number of non-minimal (and gauge inequivalent) solutions in each path component of the configuration space (monopole sector). It was also asserted that solutions exist in each path component with arbitrarily large action. These assertions are proved in a forthcoming article [2] with techniques from the calculus of variations.

The calculus of variations can be used to find solutions to a differential equation if that equation is the Euler-Lagrange variational equation for a functional f on a topological space M. If the pair (f, M) are "nice" in a suitable sense, then certain topological properties of M imply the existence of solutions to the differential equation. To make a concrete statement, one must study the functional f and the topology of the space M.

The purpose of this article is to explore those topological properties of the Yang-Mills-Higgs configuration space which are relevant for the proof of the existence theorem in [1].

This exploration leads, among other places, to the topology of the family of Dirac operators indexed by this configuration space; here the characteristic classes of the family of Dirac operators are of specific interest. As outlined in Sect. 2, these cohomology classes lead to a proof that there exist solutions in each path component of the configuration space with arbitrarily large action.

The work here is based upon the preliminary topological investigations in Sect. 3 of [3]. Most of the terminology and notation in the present article is the same as in [3].

For the uninitiated, the SU(2) Yang-Mills-Higgs equations are partial differential equations on $\mathbb{R}^3$ for an unknown, $c = (A, \Phi)$. Here A is a connection on

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the principle SU(2) bundle $P=\mathbb{R}^3\times \mathrm{SU}(2)$ over $\mathbb{R}^3$; and Φ is a section of the associated vector bundle $\mathrm{AdP}=\mathbb{R}^3\times \mathfrak{su}(2)$, the bundle with fibre the Lie algebra of SU(2), $\mathfrak{su}(2)$.

The Yang-Mills-Higgs equations are

$$D_A * F_A + [\Phi, *D_A\Phi]=0\,, \tag{A1.1a}$$

$$D_A * D_A\Phi=0\,, \tag{A1.1b}$$

with the boundary condition that $\lim_{|x|\to\infty} |\Phi|(x)=1$.

In Eq. (A1.1), F_A is the curvature of A; D_A is the exterior covariant derivative on $\Lambda T^*\otimes\mathrm{AdP}$; and $[\cdot,\cdot]$ is the natural graded bracket on $\Lambda T^*\otimes\mathfrak{su}(2)$. This is defined for AdP-valued p and q forms ω,η to be $[\omega,\eta]=\omega\wedge\eta-(-1)^{pq}\eta\wedge\omega$. The $*$ in Eq. (1.1) is the Hodge star on ΛT^* from the Euclidean metric on $\mathbb{R}^3$. Finally, the norm $|\cdot|$ on $\Lambda T^*\otimes\mathrm{AdP}$ is that induced from the Euclidean metric on T^* and the following Ad-invariant metric on $\mathfrak{su}(2)$: $|\sigma|^2=-2\,\mathrm{trace}_{\mathbb{C}^2}(\sigma^2)$.

Equation (A1.1) is the variational equation of the action functional

$$\mathfrak{A}(A,\Phi)=\tfrac{1}{2}\int_{\mathbb{R}^3}\{|F_A|^2+|D_A\Phi|^2\}d^3x\,. \tag{A1.2}$$

One is to consider $\mathfrak{A}$ as a functional on the set

$$\mathfrak{C}=\{\text{smooth } c=(A,\Phi):\mathfrak{A}(A,\Phi)<\infty \text{ and } (1-|\Phi|)\in L^6(\mathbb{R}^3)\}\,. \tag{A1.3}$$

The set $\mathfrak{C}$ is the configuration space, and it is topologized as follows: Let θ denote the flat product connection on $\mathbb{R}^3\times\mathrm{SU}(2)$. The topology on $\mathfrak{C}$ is the topology that is induced by the map of $\mathfrak{C}$ into $\times_{12}\, C^\infty(\mathbb{R}^3)\times[0,\infty)$, which sends (A,Φ) to $(A-\theta,\Phi,\mathfrak{A}(A,\Phi))$.

Acting on $\mathfrak{C}$ is the topological group $\mathfrak{G}=\mathrm{Aut}\,P\simeq C^\infty(\mathbb{R}^3;\mathrm{SU}(2))$. This group acts continuously on $\mathfrak{C}$ and it leaves $\mathfrak{A}$ invariant.

The subgroup $\mathfrak{G}_0=\{g\in\mathfrak{G}:g(x=0)=1\}$ acts freely on $\mathfrak{C}$. The functional $\mathfrak{A}$ descends to a continuous functional on $\mathfrak{B}=\mathfrak{C}/\mathfrak{G}_0$ if $\mathfrak{B}$ is given the quotient topology. [As a matter of notation, the orbit of $(A,\Phi)\in\mathfrak{C}$ under $\mathfrak{G}_0$ will be denoted by (A,Φ) also.]

There is an action of SU(2) on $\mathfrak{B}$ which leaves $\mathfrak{A}$ invariant. This SU(2) action can be partially eliminated by constructing a fibration of $\mathfrak{B}$ over S^2 with a fiber $\hat{\mathfrak{B}}$. The group SU(2) acts on S^2 via rotations and the fibration $\hat{n}:\mathfrak{B}\to S^2$, is equivariant. Therefore, no generality is lost by restricting $\mathfrak{A}$ to $\hat{\mathfrak{B}}=\hat{n}^{-1}$ (north pole).

This fibration is constructed in Sect. B5. Morally, $\hat{n}$ sends $(A,\Phi)\in\mathfrak{B}$ to the unit vector $\Phi/|\Phi|$ in the fibre of AdP at a suitably chosen point $x(A,\Phi)\in\mathbb{R}^3$. In practice, the actual construction is more complicated.

As suggested in [3], $\mathfrak{B}$ is intimately related to the space of smooth maps from S^2 to S^2, $\mathrm{Maps}(S^2;S^2)$. Evaluation fibers $\mathrm{Maps}(S^2;S^2)$ over S^2 with fiber $\Omega(S^2;S^2)$, the space of base point preserving maps. The construction of $\hat{\mathfrak{B}}$ suggests a relationship with $\Omega(S^2;S^2)$.

The first half of this article explores these relationships; there the following theorem is proved:

Theorem A1.1. *There exists an inclusion of maps* $\mathrm{Maps}(S^2; S^2)$ *into* $\mathfrak{B}$ *which induces the following commutative diagram where the vertical arrows are homotopy equivalences:*

$$\begin{array}{ccccc} \hat{\mathfrak{B}} & \longrightarrow & \mathfrak{B} & \searrow & \\ \uparrow & & \uparrow & & S^2 \\ \Omega(S^2;S^2) & \longrightarrow & \mathrm{Maps}(S^2;S^2) & \nearrow & \end{array} .$$

The path components of $\Omega(S^2; S^2)$ are the spaces $\{\Omega_n(S^2; S^2)\}_{n\in\mathbb{Z}}$, which are the spaces of maps of fixed degree. Correspondingly, $\hat{\mathfrak{B}} = \bigcup_{n\in\mathbb{Z}} \hat{\mathfrak{B}}_n$. (The space $\hat{\mathfrak{B}}_n$ is the n-monopole sector.)

It is known that if $c \in \hat{\mathfrak{B}}_n$, then

$$\mathfrak{A}(c) \geqq 4\pi|n| . \tag{A1.4}$$

There is equality for $c \in \mathfrak{C}_n$ if and only if $c=(A, \Phi)$ satisfies the Bogomol'nyi equation [4]

$$F_A = \mathrm{sign}(n) * D_A \Phi . \tag{A1.5}$$

Solutions to the Bogomol'nyi equations exist in each $\hat{\mathfrak{B}}_n$ [5, Chap. IV; 6].

Configurations which satisfy Eq. (1.5) are the minima of $\mathfrak{A}$ on $\hat{\mathfrak{B}}$. The newly discovered critical points of $\mathfrak{A}$, the solutions to Eq. (A1.1) that were announced in [1], are all non-minimal and unstable.

They are found with a convergent min-max theory. The min-max strategy is as follows [7]: Let M be a smooth manifold. A family $\mathfrak{F}$ of compact subsets of M is said to be *homotopy invariant* if it is true that for any continuous homotopy $\phi: [0,1] \times M \to M$ for which $\phi(0,\cdot)$ is the identity, the condition that $F \in \mathfrak{F}$ implies that $\phi(1,F) \in \mathfrak{F}$.

For example, let $H^l(M; \mathbb{Z})$ denote the l^{th} cohomology group of M with coefficients in the integers, $\mathbb{Z}$. Let $[z] \in H^l(M; \mathbb{Z})$, and let

$$\mathfrak{F} = \mathfrak{F}([z]) = \{F \subseteqq M : \text{The restriction map, } i_F^*: H^l(M;\mathbb{Z}) \to H^l(F;\mathbb{Z}) \text{ does not annihilate } [z]\} . \tag{A1.6}$$

The family $\mathfrak{F}$ above is homotopy invariant.

Let $\mathfrak{A}: \mathfrak{B} \to [0, \infty)$ be as in Eq. (A1.2). To a homotopy invariant family $\mathfrak{F}$, assign the number

$$\mathfrak{A}_{\mathfrak{F}} = \inf_{F\in\mathfrak{F}} \sup_{c\in F} \mathfrak{A}(c) . \tag{A1.7}$$

A main result in [1] is the theorem that any $\mathfrak{A}_{\mathfrak{F}}$ defined by Eq. (A1.7) is a critical value of $\mathfrak{A}$. To establish this strong result, it is necessary to understand compactness on $\mathfrak{C}$.

The topology of $\hat{\mathfrak{B}}_n$ influences $\mathfrak{A}$ on a more subtle level where no compactness conditions are required. It influences the a priori distribution in $[0, \infty)$ of the set

$$\mathrm{Crit}_n = \{\mathfrak{A}_{\mathfrak{F}} : \mathfrak{F} \text{ is a homotopy invariant family of compact subsets of } \hat{\mathfrak{B}}_n\} . \tag{A1.8}$$

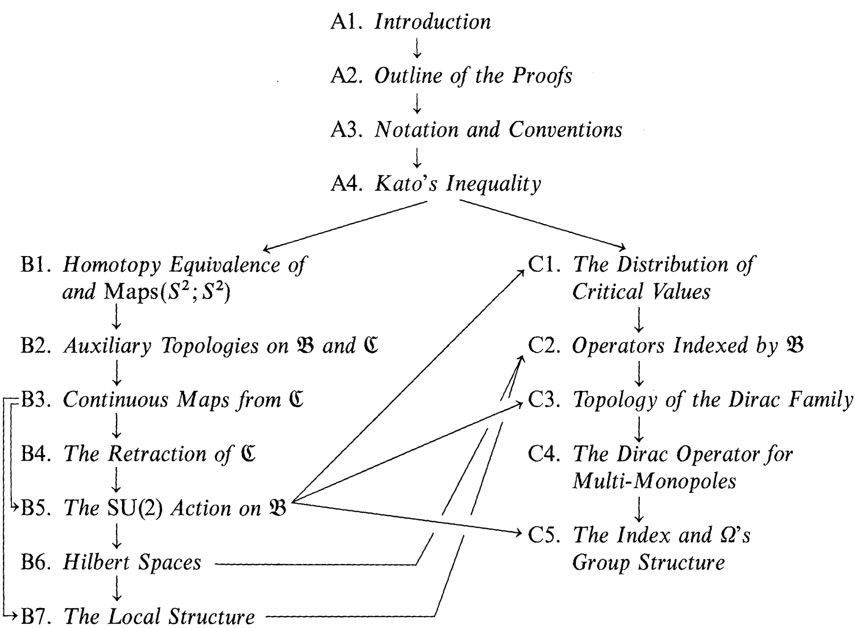
The second half of this article investigates $\mathrm{Crit}_n \subset [0, \infty)$ to prove

Theorem A1.2. *For each* $n \in \mathbb{Z}$, *define* $\mathrm{Crit}_n \subset [4\pi|n|, \infty)$ *for the functional* $\mathfrak{A}$ *on* $\hat{\mathfrak{B}}_n$ *by Eq.* (A1.8). *Then the set* Crit_n *is unbounded.*

Theorem A1.2 is proved by studying a family of Dirac operators, indexed by the configurations in $\hat{\mathfrak{B}}_n$. It is an application of the Atiyah-Singer Index Theorem for families [8]. A non-technical outline of the proof is given in the next section.

A2. Outline of the Proofs

The purpose of including this section in the article is to provide a non-technical outline of the proofs of Theorems A1.1 and A1.2 along with an outline of the contents of the article. The following flow chart should prove helpful.



Theorem A1.1 follows from Proposition B1.1 and B5.1 and its proof comprises most of Sects. B1–B5. The plan is to construct an embedding $I: \mathrm{Maps}(S^2; S^2) \to \mathfrak{B}$ with the property that $I(\mathrm{Maps}(S^2; S^2))$ is a deformation retract of $\mathfrak{B}$. The map I is defined in Sect. B1 and essentially it does the following: Identify the image S^2 with the unit sphere in $\mathfrak{su}(2) \simeq \mathbb{R}^3$. Then a map $e \in \mathrm{Maps}(S^2; S^2)$ defines an asymptotic (large $|x|$ on $\mathbb{R}^3$) model for $\Phi \in \Gamma(\mathrm{Ad}\,P)$. There is a way to extend e smoothly into $\mathbb{R}^3$ to define a global $\Phi(e) \in \Gamma(\mathrm{Ad}\,P)$; extend radially but bump to zero near the origin. A convenient $A(e)$ exists so that $I(e) = (A(e), \Phi(e)) \in \mathfrak{B}$.

The actual retraction of $\mathfrak{B}$ onto $I(\mathrm{Maps})$ is complicated by the fact that a given $(A, \Phi) \in \mathfrak{B}$ may not be sufficiently docile; i.e. Φ may not have a limit as $|x| \to \infty$ which defines a smooth map from the S^2 "at infinity" in $\mathbb{R}^3$ to the unit S^2 in $\mathfrak{su}(2)$.

This problem can be circumvented by solving Eq. (A1.1b) outside the unit ball for a new $\Phi_2(A, \Phi)$. Because $\Phi_2(A, \Phi)$ satisfies an elliptic differential equation, a priori estimates are available. These a priori estimates follow from Kato's inequality and the usual Sobolev inequalities on $\mathbb{R}^3$. Section A4 provides the inequalities in a convenient form. All of the a priori estimates in this paper are applications of the ideas in Sect. A4.

The assignment of (A, Φ) to $(A, \Phi_2(A, \Phi))$ is shown in Sect. B3 to define a continuous map from $\mathfrak{B}_n$ to $\mathfrak{B}_n$. This is the crucial result, and with it, the deformation retract can be readily constructed using the natural affine structure of $\mathfrak{C}$. The construction is presented in Sect. B4.

The fibration $\hat{\mathfrak{B}} \to \mathfrak{B} \xrightarrow{\hat{n}} S^2$ is constructed in Sect. B5. The map $\hat{n}$ is not precisely the evaluation of $\Phi/|\Phi|$; rather it is the evaluation of $\Phi_2(A, \Phi)/|\Phi_2(A, \Phi)|$ at a suitably chosen point $x(A, \Phi) \in \mathbb{R}^3$. The retraction of $\hat{\mathfrak{B}}$ onto $\Omega(S^2; S^2)$ is constructed in Sect. B5, where the proof of Theorem A1.1 is completed.

When studying the functional $\mathfrak{A}$ on $\mathfrak{B}$, one must deal with families of linear differential operators which are indexed by the configurations in $\mathfrak{B}$. Examples were studied in [9]. A convenient formalism is obtained by considering the assignment of the operator to a configuration in $\mathfrak{B}$ as defining a homomorphism between two Hilbert space vector bundles over $\mathfrak{B}$.

Hilbert space vector bundles over $\mathfrak{B}$ which are modeled after Sobolev spaces of sections over $\mathbb{R}^3$ of $\mathbb{R}^3 \times$ (finite dimensional vector space) are introduced in Sect. B6.

A part of this vector bundle construction requires a local embedding theorem (Proposition B6.1) which provides an injection of open neighborhoods of $\mathfrak{B}$ into a space of sections of $(T^* \otimes \mathrm{Ad}\,\mathrm{P}) \oplus \mathrm{Ad}\,\mathrm{P}$. This local embedding result is proved in Sect. B7.

Linear differential operators which are indexed by the configurations in $\mathfrak{B}$ arise in the proof of Theorem A1.2 in the following way: To establish, via min-max arguments, that the set of critical values of $\mathfrak{A}$ on $\hat{\mathfrak{B}}_n$ is unbounded, requires a priori knowledge of the relative homotopy or homology of the pair

$$(\hat{\mathfrak{B}}_n, \mathfrak{A}^{-1}([4\pi|n|, 4\pi|n| + \kappa))$$

for every $\kappa \in (0, \infty)$. In particular, for every such $\kappa, \hat{\mathfrak{B}}_n$ must not retract onto $(\mathfrak{A}^{-1}([4\pi|n|, 4\pi|n| + \kappa))$. Here it is sufficient that there should exist for every $\kappa > 0$, a homotopy invariant family of subsets of $\hat{\mathfrak{B}}_n$, $\mathfrak{F}(n, \kappa)$ such that $\mathfrak{A}_{\mathfrak{F}(n,\kappa)} > 4\pi|n| + \kappa$.

Such families $\mathfrak{F}(n, \kappa)$ are obtained via the natural stratification of $\hat{\mathfrak{B}}_n$ which comes from a family of twisted Dirac operators. The Dirac operator in question is presented in Sect. C1.

The assignment of $c \in \hat{\mathfrak{B}}_n$ to the Dirac operator defines a map δ from $\hat{\mathfrak{B}}_n$ into the space of Fredholm operators. This map is proved continuous in Sect. C2.

The space of Fredholm operators is homotopic to $\mathbb{Z} \times BU$, where BU is the classifying space for the infinite unitary group, $U = U(\infty)$. As described in the work of Koschorke [10] and also in [11], the map δ pulls back to $\hat{\mathfrak{B}}_n$ the universal Chern classes on BU. These pulled back classes are represented by elements in $H^*(\hat{\mathfrak{B}}_n; \mathbb{Z})$.

The strategy for proving Theorem A1.2 is to show that for any $n \in \mathbb{Z}$ and $\kappa < \infty$, there exists a class $\omega \in H^*(BU; \mathbb{Z})$ such that for $\delta^*\omega \in H^*(\hat{\mathfrak{B}}_n; \mathbb{Z})$, the homotopy invariant family $\mathfrak{F}(\delta^*\omega)$ of Eq. (A1.6) satisfies $\mathfrak{A}_{\mathfrak{F}(\delta^*\omega)} > 4\pi|n| + \kappa$.

This strategy is effected by adapting to the present situation the ideas of Koschorke as applied by Atiyah and Jones [12] to a family of twisted Dirac operators over S^4.

Koschorke defines for $p, q \geqq 0$, characteristic classes $\chi^{p,q} \in H^{2pq}(BU; \mathbb{Z})$ with the following property: $\delta^* \chi^{p,q} = \chi^{p,q}(\delta)$ is zero in $H^{2pq}(\hat{\mathfrak{B}}_n; \mathbb{Z})$ if for every $c \in \hat{\mathfrak{B}}_n$, the Dirac operator in question has kernel dimension less than p or cokernel dimension less than q. ($\chi^{p,q}$ is a specific determinant of Chern classes.)

By mimicking Atiyah and Jones, it is demonstrated in Sects. C2, C3 that there are arbitrarily large p, q with the property that $\chi^{p,q}(\delta) \neq 0$ in $H^*(\hat{\mathfrak{B}}_l; \mathbb{Z})$, provided that l is sufficiently large. This is accomplished by restricting the Dirac operator to a finite dimensional subspace in $\hat{\mathfrak{B}}_l$ which is homeomorphic to the configuration space of l unordered points in $\mathbb{R}^3, C_l$. This configuration space parametrizes l fundamental monopoles far apart. The restriction of $\chi^{p,q}(\delta)$ to $H^{2pq}(C_l; \mathbb{Z})$ is a characteristic class of a natural $\mathbb{C}^l$-vector bundle over C_l, one whose characteristic class were computed by Atiyah and Jones in [12].

The spaces $\hat{\mathfrak{B}}_n$, $n \in \mathbb{Z}$ are mutually homotopic; this corresponds to the well known fact that the mapping spaces $\Omega_n(S^2; S^2)$, $n \in \mathbb{Z}$ are mutually homotopic. In Sect. C5 it is demonstrated that the map which induces the homotopy between $\hat{\mathfrak{B}}_n$ and $\hat{\mathfrak{B}}_{n+1}$ has the property that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \hat{\mathfrak{B}}_{n+1} & & \\ \uparrow & \overset{\delta}{\searrow} & \\ & & BU\,. \\ & \underset{\delta}{\nearrow} & \\ \hat{\mathfrak{B}}_n & & \end{array}$$

This implies that for each $n \in \mathbb{Z}$, there are nonzero $\chi^{p,q}(\delta)$ in $H^{2pq}(\hat{\mathfrak{B}}_n; \mathbb{Z})$ for arbitrarily large p and q.

To prove Theorem A1.2, it remains to prove that $\mathfrak{U}_{\mathfrak{F}(\chi^{p,q}(\delta))}$ increases as p, q increase. The strategy here is to use Koschorke's assertion that a subset F of $\hat{\mathfrak{B}}_n$ is in $\mathfrak{F}(\chi^{p,q}(\delta))$ only if there exists $c \in F$ such that the Dirac operator indexed by c has kernel dimension p or greater. Now, the Weitzenbock formula for the Dirac operator (see Sect. C2) suggests that the curvatures $(F_A, D_A\Phi)$ must be large if p is large; the square of the Dirac operator is strictly positive save for an endomorphism which is linear in $(F_A, D_A\Phi)$. This suggestion is rigorously established in Sect. C2. The full proof of Theorem A1.2 is exhibited in Sects. C1–C5.

A3. Notation and Conventions

Because the topology of an infinite dimensional function space is the subject of this article, it is important at the outset to introduce the basic topologies on the spaces of smooth functions on $\mathbb{R}^3$. The starting point is $C^\infty(\mathbb{R}^3)$; the Fréchet space of infinitely differentiable functions on $\mathbb{R}^3$. A neighborhood $\mathfrak{N}(f)$ of f in $C^\infty(\mathbb{R}^3)$ is indexed by a compact set $K \subset \mathbb{R}^3$ and a sequence of positive numbers $\{\varepsilon_j > 0\}_{j=0}^{\infty}$,

$$\mathfrak{N}(f) = \{g \in C^\infty(\mathbb{R}^3) : (\|f - g\|_{C^j; K} < \varepsilon_j)_{j=0}^{\infty}\}\,.$$

Here $\|\cdot\|_{C^k; K}$ denotes the usual C^k-norm on K.

Denote by $C_0^\infty(\mathbb{R}^3) \subset C^\infty(\mathbb{R}^3)$ the subspace of compactly supported functions.

Sobolov spaces $L_k^p(\mathbb{R}^3)$ for $p \geqq 0$ and $k \geqq 1$ are defined in the usual way [13]; $f \in L_k^p(\mathbb{R}^3)$ if f is measurable and its derivatives through order k are in L^p. The space L^p is a Banach space with norm $\|f\|_p = [\int d^3x |f|^p]^{1/p}$ and the space L_k^p, $k > 0$ is a Banach space with norm

$$\|f\|_{p,k} = \left[\sum_{j=0}^{k} \sum_{(j)} \|\nabla^{(j)} f\|_p\right].$$

The inner sum, above, is over all multi-indices $(j) = (i_1, \ldots, i_j) \in \underset{j}{\times} \{1, 2, 3\}$.

If $\Omega \subset \mathbb{R}^3$ is a domain, the $L_k^p(\Omega)$ norm is denoted by $\|\cdot\|_{p,k;\Omega}$. For example,

$$\|f\|_{p;\Omega} = \left[\int_\Omega d^3x |f|^p\right]^{1/p}.$$

The space $L_{k;\mathrm{loc}}^p(\mathbb{R}^3)$ is defined for $p \geqq 2$ and $k \geqq 0$ to have as its underlying point set

$$\{\text{Measurable functions } f : f \in L_k^p(\Omega) \text{ for any bounded domain } \Omega \subset \mathbb{R}^3\}.$$

$L_{k;\mathrm{loc}}^p(\mathbb{R}^3)$ is topologized as a Fréchet space so that an open neighborhood $\mathfrak{N}(f)$ of $f \in L_{k;\mathrm{loc}}^p(\mathbb{R}^3)$ is indexed by a bounded domain $\Omega \subset \mathbb{R}^3$ and a number $\varepsilon > 0$:

$$\mathfrak{N}(f) = \{g \in L_{k;\mathrm{loc}}^p(\mathbb{R}^3) : \|f - g\|_{k,p;\Omega} < \varepsilon\}.$$

Finally, it is necessary to introduce the space $\bar{C}^{0,1/2}(\mathbb{R}^3)$. This is a Banach space which is the completion of the set of C^∞ functions on $\mathbb{R}^3$ with compact support in the norm

$$\|f\|_{0,1/2} = \sup_{x \neq y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|^{1/2}}.$$

Functions in $\bar{C}^{0,1/2}(\mathbb{R}^3)$ decay uniformly to zero as $|x| \to \infty$.

Now let V be a finite dimensional Hilbert space and let $E = \mathbb{R}^3 \times V$ be a vector bundle. The symbol $\Gamma(E)$ denotes the space of smooth sections of V; it is topologized via the projection $E \to V$ which identifies $\Gamma(E)$ with

$$C^\infty(\mathbb{R}^3; V) \simeq [C^\infty(\mathbb{R}^3)]^{\dim V}.$$

Define the spaces $\Gamma_0(E)$, $L_k^p(E)$, and $L_{k;\mathrm{loc}}^p(E)$ similarly; the L^p norms are defined with the given fibre metric $(\cdot,\cdot)$ on E (cf. [13]).

A useful example is $T^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$, the cotangent bundle of $\mathbb{R}^3$ with its Euclidean metric.

Let M be a smooth manifold with an isometric embedding $M \subseteqq \mathbb{R}^N$ for some $N \geqq 1$; for example, $\mathrm{SU}(2) \simeq S^3 \subset \mathbb{R}^4$. Denote by $C^\infty(\mathbb{R}^3; M)$ the space of smooth maps from $\mathbb{R}^3$ to M. It is topologized by its inclusion in $C^\infty(\mathbb{R}^3; \mathbb{R}^N) \simeq \underset{N}{\times} C^\infty(\mathbb{R}^3)$. See [13] for example.

Let $|\cdot|$ be a metric on E, and let $\alpha(E)$ denote the space of smooth metric compatible connections on E. The fixed projection $E \to V$ defines a fiducial product connection θ on E, and $\alpha(E) = \theta + \Gamma(\operatorname{End} E \otimes T^*)$. The connection spaces $\alpha^p_{k;\mathrm{loc}}(E)$ for $p \geqq 1$ and $k \geqq 0$ are defined similarly.

The summary of the basic topological spaces is now complete; the remainder of this section introduces some specific notation. To begin, it is convenient to fix an origin $0 \in \mathbb{R}^3$ and Cartesian coordinates $\{x^i\}_{i=1}^3$ that are centered at 0. With these coordinates, $|x| = \left(\sum_i x^i x^i\right)^{1/2}$.

The symbol d denotes the exterior derivative $d : \Gamma(\Lambda T^*) \to \Gamma(\Lambda T^*)$. The Laplacian, Δ, is always $\sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i}$.

If $E \to \mathbb{R}^3$ is a vector bundle with connection A, then $\nabla_A : \Gamma(E) \to \Gamma(E \otimes T^*)$ denotes the covariant derivative, while $D_A : \Gamma(E \otimes \Lambda T^*) \to \Gamma(E \otimes \Lambda T^*)$ denotes the covariant exterior derivative.

If $c = (A, \Phi) \in \mathfrak{C}$, then $A(c) = A$ and $\Phi(c) = \Phi$. Also $F(c) = F_A$ and $(D_A \Phi)(c) = D_A \Phi$.

It is convenient to introduce a fixed C^∞ bump function β. This function satisfies $\beta(x) = \beta(|x|)$ and

$$\begin{aligned} \beta(t) &= 1 \quad \text{if} \quad 0 \leqq t \leqq 1/2\,, \\ \beta(t) &= 0 \quad \text{if} \quad t \geqq 3/4\,, \\ 1 \geqq \beta(t) \geqq 0\,, &\quad \frac{\partial \beta}{\partial t} \leqq 8 \quad \text{for all} \quad t \in [0, \infty)\,. \end{aligned} \tag{A3.1}$$

If $r \in (0, \infty)$, then $\beta_r(x) = \beta(x/r)$. This definition is extended to $[0, \infty]$ by setting $\beta_0(x) \equiv 0$ and $\beta_\infty(x) \equiv 1$.

Finally, there is the convention for constants. In this article, the symbols ζ and z are numbers in $(0, \infty)$ whose precise value may change from line to line. This convention alleviates the necessity of indexing constants in long derivations.

A4. Kato's Inequality

The proofs in the future sections rely on function space norm inequalities which are covariant versions of standard Sobolev inequalities on $\mathbb{R}^3$. To set up the background, let $E \to \mathbb{R}^3$ be a vector bundle, and let $|\cdot|$ be a nondegenerate metric on E.

The idea here is to obtain L^p estimates for $\psi \in L^2_{1;\mathrm{loc}}(E)$ knowing only that $A \in \alpha^2_{1;\mathrm{loc}}(E)$ and $\nabla_A \psi \in L^2(E)$. These estimates all follow from

Kato's Inequality. *Let $E \to \mathbb{R}^3$ be a vector bundle with smooth fiber metric $\langle \cdot , \cdot \rangle$. Let $A \in \alpha^2_{1;\mathrm{loc}}(E)$. Let $\psi \in L^2_{1;\mathrm{loc}}(E)$. Then almost everywhere, $|\nabla_A \psi| \geqq |d|\psi||$.*

For a proof, cf. [5, Chap. IV].

The first observation stemming from Kato's inequality is

Lemma A4.1. *Let $A \in \alpha^2_{1;\mathrm{loc}}(E)$ and let $\delta \in (-1, \infty)$. Suppose that $\psi \in L^2_{1;\mathrm{loc}}(E)$ and $(1 + |x|)^\delta \nabla_A \psi \in L^2(E \otimes T^*)$. Then there exists a constant $c \in [0, \infty)$ such that for any*

$r\in[0,\infty)$,

$$\int_{|x|>r} d^3x(1+|x|)^{2\delta}|\nabla_A\psi|^2 \geqq z \int_{|x|>r} d^3x(1+|x|)^{2\delta-2}(c-|\psi|)^2\,, \tag{A4.1}$$

and

$$\|(1+|x|)^{\delta}|\nabla_A\psi|\,\|_2 > z\|(1+|x|)^{\delta}(c-|\psi|)\|_6\,. \tag{A4.2}$$

Further, the constant z, above, is positive and it is independent of $A\in\alpha^2_{1;\mathrm{loc}}(E)$, the metric on E and E itself. The constant z is only a function of δ.

Before proving Lemma A4.1, it is useful to introduce the Banach spaces K^δ, indexed by $\delta\in(-1,\infty)$. The space K^δ is the completion of $C_0^\infty(\mathbb{R}^3)$ in the norm $\|v\|=\|(1+|x|)^\delta dv\|_2$. The space K^δ, $\delta\in(-1,\infty)$ has the property that there exists $z(\delta)>0$ such that for any $v\in K^\delta$ and $r\in[0,\infty)$,

$$\int_{|x|>r} d^3x(1+|x|)^{2\delta}|dv|^2 \geqq z(\delta) \int_{|x|>r} d^3x(1+|x|)^{2\delta-2}v^2\,,$$

and

$$\|v\| \geqq z(\delta)\|(1+|x|)^\delta v\|_6\,. \tag{A4.3}$$

The first inequality above is proved by repeating the proof of Lemma 5.2 of [14] for $\delta\neq 0$. The second inequality in Eq. (A4.3) follows from a classical Sobolev inequality [15] which says that if $u\in K^0$ then $\|du\|_2\geqq z(0)\|u\|_6$. To apply this Sobolev inequality when $\delta\neq 0$, one must use the first inequality in Eq. (A4.3) with the following identity on $L^2_{1;\mathrm{loc}}$ functions: Almost everywhere,

$$d(1+|x|)^\delta f=(1+|x|)^\delta df+\delta(1+|x|)^{\delta-1}fd|x|\,.$$

Proof of Lemma A4.1. By using Kato's inequality, one sees that it is sufficient to prove the lemma with ψ replaced by a function, $f\in L^2_{1;\mathrm{loc}}(\mathbb{R}^3)$ for which $(1+|x|)^\delta df\in L^2$. Due to the remarks concerning K^δ, it is sufficient to prove that any function $f\in L^2_{1;\mathrm{loc}}$ with $(1+|x|)^\delta df\in L^2$ has the property that f-constant $\in K^\delta$. Such is the case, and the fact is proved by repeating Lemma 4.12 of [14] with $\delta\neq 0$. The details are left to the reader.

A second useful observation which stems ultimately from Kato's inequality is a "covariant" version of the Sobolev embedding of $L^6_1(\mathbb{R}^3)\to\bar{C}^{0,1/2}(\mathbb{R}^3)$ (cf. [15]).

Lemma A4.2. *Let $A\in\alpha^2_1(E)$. Let $\psi\in L^2_{1;\mathrm{loc}}(E)$ and suppose that*

$$\nabla_A\psi\in L^2(E\otimes T^*)\,;\qquad c-|\psi|\in L^2(\mathbb{R}^3)$$

for some $c\in[0,\infty)$; and $\nabla_A(\nabla_A\psi)$ $L^2(E\otimes T^*\otimes T^*)$. Then $\psi\in L^2_{2;\mathrm{loc}}(E)$, $c-|\psi|\in\bar{C}^{0,1/2}(\mathbb{R}^3)$ and the norm of $c-|\psi|\in\bar{C}^{0,1/2}(\mathbb{R}^3)$ depends on $A\in\alpha^2_{1;\mathrm{loc}}(E)$, the metric on E, and E only through the numbers $\|\nabla_A(\nabla_A\psi)\|_2$ and $\|\nabla_A\psi\|_2$.

Proof of Lemma A4.2. Due to Lemma A4.1, one may conclude that $\nabla_A\psi\in L^p$ for $p\in[2,6]$. In particular, $\nabla_A\psi\in L^4$. By writing $\nabla_A\psi=d\psi+\varrho(A)$, where $\varrho(A)$ is the matrix 1-form representing A, one can conclude with Hölder's inequality that $d\psi\in L^4_{\mathrm{loc}}$ and therefore, that $\psi\in C^0(E)$. (This is a Sobolev inequality, [15].) Then, returning to the expression $\nabla_A(\nabla_A\psi)$ and writing $\nabla_A=d+\varrho(A)$, one concludes that

$\psi \in L^2_{2;\mathrm{loc}}(E)$. Now, by Kato's inequality, $|d\psi| \in L^6$, and, therefore, $c - |\psi| \in L^6_1(\mathbb{R}^3)$. From this it follows that $c - |\psi| \in \bar{C}^{0,1/2}(\mathbb{R}^3)$ [15].

These Sobolev-type inequalities also have certain continuity aspects which are illustrated by the following lemma.

Lemma A4.3. *Let $\{A_i\} \in \alpha^2_{1;\mathrm{loc}}(E)$ converge to A in $\alpha^2_{1;\mathrm{loc}}(E)$. Let $\{\psi_i\} \in L^2_{1;\mathrm{loc}}(E)$. Suppose that $\{\psi_i\}$ converges strongly in L^2_{loc} to some $\psi \in L^2_{\mathrm{loc}}$. Assume that $\{\nabla_{A_i}\psi_i\} \in L^2$, where it converges weakly to $G \in L^2$; and that $\{\|\nabla_{A_i}\psi_i\|_2\}$ converges to $\|G\|_2$. Then $\psi \in L^2_{1;\mathrm{loc}}$, where it is the strong limit of $\{\psi_i\}$. Also, $\nabla_A\psi \in L^2$ and it is the strong L^2 limit of $\{\nabla_{A_i}\psi_i\}$. Further, there are constants $\{c_i\} \subset [0, \infty)$ with limit $c \in [0, \infty)$ such that $\{c_i - |\psi_i|\}$ converges in K^0 to $c - |\psi|$, therefore to $c - |\psi|$ in L^6 as well.*

Lemma A4.4. *Make the same assumptions as in Lemma A4.3 and assume additionally that $\{\psi_i\} \in L^2_{2;\mathrm{loc}}$, that $\{\nabla_{A_i}(\nabla_{A_i}\psi_i)\} \in L^2$, where it converges weakly to $G' \in L^2$, but that $\{\|\nabla_{A_i}(\nabla_{A_i}\psi_i)\|_2\}$ converges to $\|G'\|_2$. Let ψ and c be as in Lemma A4.3. Then $\{\psi_i\}$ converges strongly in $L^2_{2;\mathrm{loc}}$ to ψ; $\nabla_A(\nabla_A\psi) \in L^2$ and it is the strong limit, there, of $\{\nabla_A(\nabla_{A_i}\psi_i)\}$. Further, $\{c_i - |\psi_i|\}$ converges strongly in $L^6_1(\mathbb{R}^3)$ to $\{c - |\psi|\}$; hence in $\bar{C}^{0,1/2}(\mathbb{R}^3)$ also.*

The proof of Lemma A4.3 will be given shortly. The proof of Lemma A4.4 is along the same lines as the proof of Lemma A4.3 and so it is left to the reader. For the proof of Lemma A4.3 one requires the following facts:

Lemma A4.5. *Let $\{f_i\} \subset L^2$ be a sequence which converges weakly to $f \in L^2$ and is such that $\{\|f_i\|_2\}$ converges to $\|f\|$. Then $\{f_i\}$ converges strongly to f.*

Lemma A4.6. *Let $\{f_i\} \subset L^2$ be a sequence which converges strongly in L^2_{loc} and has the property that given $\varepsilon > 0$, there exists $r < \infty$ such that for all i, $\int_{|x|>r} |f_i|^2 < \varepsilon$. Then $\{f_i\}$ converges strongly in L^2.*

Proof of Lemmas A4.4 and A4.5. For Lemma A4.4, let f denote the weak limit of $\{f_i\}$. The assertion then follows from the identity

$$\|f - f_i\|_2 = -\|f\|_2 + \|f_i\|_2 + 2\langle f, f - f_i\rangle_2 .$$

For Lemma A4.5, the argument is a simple "$\varepsilon/3$" proof that is left to the reader.

Proof of Lemma A4.3. Lemma A4.5 insures that $\{\nabla_{A_i}\psi_i\}$ converges strongly to G. Using a test section, $\eta \in \Gamma_c(E \otimes T^*)$, one finds that $\{\langle \eta, \nabla_{A_i}\psi_i\rangle_2\}$ converges to $\langle \nabla_A\eta, \psi\rangle_2$ since $\{\psi_i\}$ converges strongly to ψ in L^2_{loc}, and $\{A_i\}$ converges to A in $L^2_{1;\mathrm{loc}}$. But this means that $\psi \in L^2_{1;\mathrm{loc}}$, $\nabla_A\psi \in L^2$, and $G = \nabla_A\psi$. In fact, since

$$\langle \eta, d(\psi - \psi_i)\rangle_2 = \langle \eta, \nabla_A\psi - \nabla_{A_i}\psi_i\rangle_2 - \langle \eta, (\varrho(A) - \varrho(A_i))\psi_i\rangle_2 - \langle \eta, \varrho(A)(\psi - \psi_i)\rangle_2 ,$$

$\{\psi_i\}$ converges weakly in $L^2_{1;\mathrm{loc}}$ to ψ; hence strongly to ψ in L^4_{loc}. Next, because

$$|d(\psi - \psi_i)| < |\nabla_A\psi - \nabla_{A_i}\psi_i| + |\varrho(A) - \varrho(A_i)|\,|\psi_i| + |\varrho(A)|\,|\psi - \psi_i| ,$$

one obtains strong $L^2_{1;\mathrm{loc}}$ convergence of $\{\psi_i\}$ to ψ. Lemmas A4.1 and A4.6 imply that there exist constants $\{c_i\} \in [0, \infty)$ such that $\{(1 + |x|)^{-1}(c_i - |\psi_i|)$ converges strongly in L^2. As $\{\psi_i\}$ converges strongly to ψ in L^2_{loc}, the sequence $\{c_i\}$ converges to some $c \in [0, \infty)$ and $(1+|x|)^{-1}(c-|\psi|)$ is the limit in L^2 of $\{c_i - |\psi_i|\}$. Then, $\{d(c_i - |\psi_i|)\}$ converges strongly in L^2 to $d(c-|\psi|)$ because of the $L^2_{1;\mathrm{loc}}$ convergence of $\{\psi_i\}$ to ψ and because of Lemmas A4.1 and A4.6.

B1. The Homotopy Equivalence of $\mathfrak{C}$ and Maps$(S^2; S^2)$

The exploration of $\mathfrak{B}$ begins here. Since $\mathfrak{B}$ is obtained as a quotient of $\mathfrak{C}$, the first consideration is to the topology of $\mathfrak{C}$. The global section of $P = \mathbb{R}^3 \times \mathrm{SU}(2)$ which sends $x \in \mathbb{R}^3$ to $(x, 1) \in \mathbb{R}^3 \times \mathrm{SU}(2)$ identifies $\mathfrak{C}$ with a subset of $\Gamma(\mathrm{Ad}\,P \otimes T^*) \times \Gamma(\mathrm{Ad}\,P)$. This identification will be implicity made throughout this article. In particular, the space $\mathfrak{C}$ is topologized via the map from $\mathfrak{C}$ into $\Gamma(\mathrm{Ad}\,P \otimes T^*) \times \Gamma(\mathrm{Ad}\,P) \times [0, \infty)$ which sends $c = (A, \Phi)$ to $(A, \Phi, \mathfrak{A}(c))$.

The fixed section of P, above, identifies the gauge group $\mathfrak{G} = \mathrm{Aut}\,P$ with $C^\infty(\mathbb{R}^3; \mathrm{SU}(2))$, and this is how $\mathfrak{G}$ is to be topologized. The group $\mathfrak{G}$ acts on $\mathfrak{C}$ by sending $(g, c = (A, \Phi))$ to $gc = (gAg^{-1} + gdg^{-1}, g\Phi g^{-1})$. This action is continuous (see Sect. 3 of [3]).

The subgroup, $\mathfrak{G}_0 = \{g \in \mathfrak{G} : g(0) = 1\}$ acts freely on $\mathfrak{C}$. Let $\mathfrak{B} = \mathfrak{C}/\mathfrak{G}_0$ be the quotient with the quotient topology. The projection, $\pi : \mathfrak{C} \to \mathfrak{B}$ defines a principal $\mathfrak{G}_0$-bundle which is isomorphic to $\mathfrak{B} \times \mathfrak{G}_0$, since $\mathfrak{G}_0$ is contractible [3, Sect. 3]. Thus $\mathfrak{B}$ and $\mathfrak{C}$ are homotopically the same. In fact, $\mathfrak{B}$ embeds in $\mathfrak{C}$ as [3, Sect. 3]

$$\mathfrak{B} = \left\{(A, \Phi) \in \mathfrak{C} : A(0) = 0 \text{ and } x^i \frac{\partial}{\partial x^i} \lrcorner A = 0\right\},$$

and $\mathfrak{C}$ retracts onto $\mathfrak{B}$ by contracting $\mathfrak{G}_0$ to 1.

The relationship between $\mathfrak{B}$ and Maps$(S^2; S^2)$ is due to the fact that the unit sphere in $\mathfrak{su}(2) \simeq \mathbb{R}^3$ is S^2. The explicit relationship is exhibited by considering Maps$(S^2; S^2)$ as a subset of Maps$(S^2; \mathfrak{su}(2))$ by choosing an identification of $\mathfrak{su}(2)$ with $\mathbb{R}^3$. Now introduce the map $I : \mathrm{Maps}(S^2; S^2) \to \mathfrak{B} \to \mathfrak{C}$ which sends $e \in \mathrm{Maps}(S^2; S^2)$ to

$$I(e) = (-(1 - \beta(|x|))\,[e(x/|x|), de(x/|x|)], (1 - \beta(|x|))e(x/|x|))\,. \tag{B1.1}$$

Here, Maps$(S^2; S^2)$ is identified with Maps$(\{x \in \mathbb{R}^3 : |x| = 1\}; \{\sigma \in \mathfrak{su}(2) : |\sigma| = 1\})$. The function $\beta(t) \in C^\infty(\mathbb{R})$ is a smooth, non-negative bump function which is identically one if $t \leqq \frac{1}{2}$ and identically zero if $t \geqq \frac{3}{4}$.

Proposition B1.1. *The map I of Eq.* (B1.1) *induces a homotopy equivalence between* $\mathfrak{B}$ *and* Maps$(S^2; S^2)$. *In fact, I*(Maps) *is a strong deformation retract of* $\mathfrak{C}$.

This proposition is a corollary to Proposition B2.1, still to come.

The path components of Maps$(S^2; S^2)$ are the spaces Maps$_n(S^2; S^2)$, labeled by $n \in \mathbb{Z}$. These are the maps of degree n. Correspondingly, $\mathfrak{C} = \bigcup_{n \in \mathbb{Z}} \mathfrak{C}_n$ and $\mathfrak{B} = \bigcup_{n \in \mathbb{Z}} \mathfrak{B}_n$, where $\mathfrak{B}_n = \mathfrak{C}_n/\mathfrak{G}_0$.

Groissier [16] showed that if $(A, \Phi) \in \mathfrak{C}_n$, then

$$\frac{1}{4\pi} \langle *F_A, D_A\Phi \rangle_2 = n\,. \tag{B1.2}$$

Proposition B1.1 shows that both $\mathfrak{B}$ and $\mathfrak{C}$ are globally nice. Because these spaces are defined implicitly, it is worthwhile to remark that they are not locally perverse either.

Proposition B1.2. *Both* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are paracompact.*

Proof of Proposition B1.2. The space $\Gamma(T^*) \times \mathfrak{su}(2) \times \Gamma(\mathrm{Ad}\,P)$ is paracompact as is $[0, \infty)$. Due to Lemma A4.1, both $\mathfrak{B}$ and $\mathfrak{C}$ are closed subsets of $\Gamma(T^*) \times \mathfrak{su}(2) \times \Gamma(\mathrm{Ad}\,P) \times [0, \infty)$. This implies that they also must be paracompact. For example, if $\{U_\alpha\}$ is an open cover of $\mathfrak{C}$, then there exist, by definition, open sets $\{U'_\alpha\}$ in $\Gamma(T^*) \times \mathfrak{su}(2) \times \Gamma(\mathrm{Ad}\,P) \times [0, \infty)$, such that for each α, $U_\alpha = U'_\alpha \cap \mathfrak{C}$. Now complete $\{U'_\alpha\}$ to an open cover of $\Gamma(T^*) \times \mathfrak{su}(2) \times \Gamma(\mathrm{Ad}\,P) \times [0, \infty)$ by adding the complement of $\mathfrak{C}$. Take a locally finite subcover. The restriction of this finite subcover to $\mathfrak{C}$ provides a locally finite refinement of the original cover, $\{U_\alpha\}$.

A final remark for this section is that there exists yet a residual SU(2) subgroup of $\mathfrak{G}$ which acts on $\mathfrak{C}$ via the embedding of SU(2) in $\mathfrak{G}$ as the constant maps from $\mathbb{R}^3$ to SU(2). The group SU(2) also acts on $\mathrm{Maps}(S^2; S^2)$ by rotating the image S^2. These two SU(2) actions are equivariant with respect to the sequence $\mathrm{Maps}(S^2; S^2) \xrightarrow{I} \mathfrak{B} \to \mathfrak{C}$.

The action of SU(2) on $\mathfrak{B}$ is not free, there are fixed points in the path component $\mathfrak{B}_0$ of $\mathfrak{B}$ which contains the image under I of the degree zero maps from S^2 to S^2. This SU(2) action is discussed further in Sect. B5.

B2. Auxiliary Topologies on $\mathfrak{C}$ and $\mathfrak{B}$

It is convenient while proving the convergence of min-max sequences for $\mathfrak{A}$ on $\mathfrak{C}$ to introduce additional topologies for $\mathfrak{C}$. Consider the topology on $\mathfrak{C}$ that is induced by the functional $\mathfrak{A}^\delta$, $\delta \in [0, \frac{1}{2})$ given by

$$\mathfrak{A}^\delta(A, \Phi) = \int_{\mathbb{R}^3} [(1 + |x|^2)^\delta |F_A|^2(x) + |\nabla_A \Phi|^2(x)].$$

The domain of $\mathfrak{A}^\delta$ is the set $\mathfrak{C}^\delta = \{c \in \mathfrak{C} : \mathfrak{A}^\delta(c) < \infty\}$. As $\mathfrak{A}^\delta$ is $\mathfrak{G}$-invariant, $\mathfrak{B}^\delta = \mathfrak{C}^\delta / \mathfrak{G}_0$ is well defined for each $\delta \in [0, \frac{1}{2})$. The set $\mathfrak{C}^\delta$ is given the induced topology from the map of $\mathfrak{C}^\delta$ into $\mathfrak{C} \times [0, \infty)$, which sends c to $(c, \mathfrak{A}^\delta(c))$. This also topologizes $\mathfrak{B}^\delta$.

The first observation is that for every $n \in \mathbb{Z}$, $\mathfrak{C}^\delta_n = \mathfrak{C}^\delta \cap \mathfrak{C}_n$ is non-empty. This is because I factors through $\mathfrak{C}^\delta$ for $\delta \in [0, \frac{1}{2})$.

The relationship between $\mathfrak{C}^\delta$ and $\mathfrak{C}$ is provided by the next two propositions. They say that $\mathfrak{C}^\delta$ lies in $\mathfrak{C}$ in a nice way.

Proposition B2.1. *There exist continuous maps* $c_2 : \mathfrak{C} \to \mathfrak{C}$, $\hat{e} : \mathfrak{C} \to \mathrm{Maps}(S^2; S^2)$ *and* $j : \mathfrak{C} \to \mathfrak{G}_0$ *with the following properties:*

(1) *On* $\mathrm{Maps}(S^2; S^2)$, $c_2 \circ I = I$, $\hat{e} \circ I = $ *identity and* $j \circ I = 1 \in \mathfrak{G}_0$.

(2) *The map which assigns* $(t, c) \in [0, 1) \times \mathfrak{C}^\delta$ *to*

$$\mathrm{r}(t, c) = \begin{cases} c - 2t[c - c_2(c)], & \text{for} \quad t \in [0, \frac{1}{2}] \\ c_2(c) - (2t - 1)[c_2(c) - j(c) I(\hat{e}(c))], & \text{for} \quad t \in [\frac{1}{2}, 1]; \end{cases}$$

defines a continuous map from $[0, 1] \times \mathfrak{C}^\delta$ *to* $\mathfrak{C}^\delta$ *for any* $\delta \in [0, \frac{1}{2})$.

Proposition B2.2. *There exists a continuous map $\tau:[0,1]\times[0,1]\times\mathfrak{C}\to\mathfrak{C}$ with the following properties:* (1) *τ maps $[0,1]\times[0,1]\times\mathfrak{C}^\delta$ continuously into $\mathfrak{C}^\delta$ for each $\delta\in[0,\frac{1}{2})$.* (2) *For any $\varepsilon\in(0,1]$ and $c\in\mathfrak{C}$, $\tau(\varepsilon,0,c)=c$ and $\tau(\varepsilon,1,c)\in\mathfrak{C}^\delta$ for every $\delta\in[0,\frac{1}{2})$. In fact, with $(A,\Phi)=\tau(\varepsilon,1,c)$,*

$$|F_A|(x)+|\nabla_A\Phi|(x)\leqq z(\varepsilon,c)(1+|x|)^{-2}.$$

(3) *For every $(\varepsilon,t,c)\in[0,1]\times[0,1]\times\mathfrak{C}$,*

$$|\mathfrak{A}(\tau(\varepsilon,t,c))-\mathfrak{A}(c)|<\varepsilon,$$

and for every $(t,c)\in[0,1]\times\mathfrak{C}$, $\tau(0,t,c)=c$.

The proofs of these two propositions are deferred to Sect. B4. There, the map τ is exhibited in terms of c_2, $\hat{e}$, and j from Proposition B2.1.

Proof of Proposition B1.1, given Proposition B2.1: Proposition B2.1 implies that for $\delta\in[0,\frac{1}{2})$, each $\mathfrak{C}^\delta$ deforms onto $I(\text{Maps})$. Indeed, if $c\in\mathfrak{C}$, then $\mathfrak{r}(1,c)$ is $j(c)I(e(c))$, with $j(c)\in\mathfrak{G}_0$. Now, $\mathfrak{G}_0$ retracts onto $1\in\mathfrak{G}_0$, and let $G:[0,1]\times\mathfrak{G}_0$ be such a retraction. The deformation retract of $\mathfrak{C}^\delta$ onto $I(\text{Maps})$ is provided by

$$\begin{cases}\mathfrak{r}(2t,c), & \text{for } t\in[0,\frac{1}{2}];\\ G(t,j(c))I(e(c)) & \text{for } t\in[\frac{1}{2},1].\end{cases}$$

The reader should note that Proposition B2.2 provides a deformation retract of $\mathfrak{C}$ onto $\mathfrak{C}^\delta$. In fact, it asserts that for $\delta\in[0,\frac{1}{2})$, $\mathfrak{C}^\delta$ is dense in $\mathfrak{C}$ and, as far as $\mathfrak{A}$ is concerned, $\mathfrak{C}^\delta$ approximates $\mathfrak{C}$ homotopically to any desired accuracy.

B3. Continuous Maps from $\mathfrak{C}$

The primary purpose of this section is to provide a priori estimates and continuity properties of certain maps from $\mathfrak{C}$. The first observation is

Proposition B3.1. *Let $\delta\in[0,\frac{1}{2})$. The map from $\mathfrak{C}^\delta$ to*

$$L^2\left(\operatorname{Ad}P\otimes\left(\bigwedge_2 T^2\oplus T^*\right)\right)\times L^6(\mathbb{R}^3)$$

which assigns to $c=(A,\Phi)$, the triple $((1+|x|)^\delta F_A,\nabla_A\Phi,1-|\Phi|)$ is continuous.

In later sections, it will be convenient to solve the "Φ" equation, to varying degrees. By doing this, one obtains maps from $\mathfrak{C}$ into $\mathfrak{C}$ whose properties are the next subject.

For each $c=(A,\Phi)\in\mathfrak{C}$, Proposition 4.8 of [3] establishes the existence and uniqueness of $c_0(c)=(A,\Phi_0(c))\in\mathfrak{C}$ with the property that

$$\nabla_A^2\Phi_0(c)=0, \tag{B3.1}$$

and $\Phi_0(c)-\Phi\in L^6(\operatorname{Ad}P)$. The proof of the convergence of min-max uses the map $c_0(\cdot)$.

By mimicking the proof of Proposition 4.8 of [3], one obtains a unique $c_1(c)=(A,\Phi_1(c))$ with the properties

$$\nabla_A^2\Phi_1(c)-[\Phi,[\Phi,\Phi_1(c)]]=0, \tag{B3.2}$$

and $\Phi_1(c)-\Phi\in L^6(\mathrm{Ad}\,P)$. This $\Phi_1(c)$ is obtained by minimizing the functional

$$\|\nabla_A(\Phi+(\cdot))\|_2^2+\|[\Phi,(\cdot)]\|_2^2$$

over $\{\eta\in\Gamma(\mathrm{Ad}\,P):\|\nabla_A\eta\|_2^2+\|[\Phi,\eta]\|_2^2<\infty$ and $|\eta|\in L^6(\mathbb{R}^3)\}$. The results in Sect. 7 require $c_1(c)$.

Similar arguments also establish the existence of a unique $\Phi_0'(c)\in L^2_{1;\mathrm{loc}}(\mathrm{Ad}\,P)$ which satisfies Eq. (B3.1) on $\{x\in\mathbb{R}^3:|x|>1\}$; is smooth on this set; satisfies $\Phi_0'(c)-\Phi\in L^6(\mathrm{ad}\,P)$ and which equals Φ on $\{x\in\mathbb{R}^3:|x|\leqq 1\}$. This $\Phi_0'(c)$ is obtained by minimizing $\|\nabla_A(\Phi+(\cdot))\|_2^2$ over the set

$$\{\eta\in\Gamma(\mathrm{Ad}\,P):\|\nabla_A\eta\|_2^2<\infty,\ |\eta|\in L^6(\mathbb{R}^3),\text{ and }\eta(x)=0\text{ if }|x|\leqq 1\}.$$

Let β be the cut-off function of Sect. A3 and set

$$\Phi_2(c)(x)=(1-\beta(x/2))\Phi_0'(c)(x)+\beta(x/2)\Phi(x).$$

Let $c_2(c)=(A,\Phi_2(c))$; this is also in $\mathfrak{C}$. This $c_2(\cdot)$ is used in Sect. B5.

Proposition B3.2. *Let $\delta\in[0,\frac{1}{2})$. For each $\lambda\in\{0,1,2\}$, the assignment of $c\in\mathfrak{C}^\delta$ to $c_\lambda(c)\in\mathfrak{C}^\delta$ defines a continuous map.*

Because $\Phi_\lambda(c)$ satisfies a differential equation, boot-strap arguments prove estimates for the second derivatives of $\Phi_\lambda(c)$:

Proposition B3.3. *Let $\delta\in[0,\frac{1}{2})$. For $c=(A,\Phi)\in\mathfrak{C}^\delta$, let $\lambda=0$ or 2. The map which sends $c\in\mathfrak{C}^\delta$ to $(1+|x|)^\delta\nabla_A\nabla_A\Phi_\lambda(c)\in\Gamma(\mathrm{Ad}\,P\otimes T^*\otimes T^*)$ factors continuously through L^2. In addition, the L^2 norm of $(1+|x|)^\delta\nabla_A\nabla_A\Phi_0(c)$ is bounded a priori knowing only $\mathfrak{A}^\delta(A,\Phi_0(c))$.*

Proposition B3.4. *Let $c=(A,\Phi)\in\mathfrak{C}$ and let $\Phi_1(c)$ be given by Eq.* (B3.2). *The map which sends $c\in\mathfrak{C}$ to $\nabla_A\nabla_A\Phi_1(c)$ factors continuously through L^2 with norm bounded a priori by $\mathfrak{A}(c)$.*

Corollary B3.5: *Let $c=(A,\Phi)\in\mathfrak{C}$. Let $\Phi_\lambda(c)$ satisfy Eq.* (B3.1) *with $\lambda\in\{0,1,2\}$. The assignment of c to $1-|\Phi_\lambda(c)|\in L^6(\mathbb{R}^3)$ factors continuously through $L_1^6(\mathbb{R}^3)$, and therefore $\bar{C}^{0,1/2}(\mathbb{R}^3)$.*

The proofs of these results occupy the remainder of this section.

Proof of Proposition B3.1. Lemma A4.5 establishes that the assignment of $c=(A,\Phi)\in\mathfrak{C}^\delta$ to $((1+x^2)^{\delta/2}F_A,\nabla_A\Phi)\in L^2$ is continuous. To obtain the continuity of the assignment of $(A,\Phi)\in\mathfrak{C}$ to $(1-|\Phi|)\in L^6(\mathbb{R}^3)$, use Lemma A4.3.

Proof of Proposition B3.2. The question here is independent of $\delta\in[0,\frac{1}{2})$. The proof begins by demonstrating that $\mathfrak{A}(c_\lambda(c))$ depends continuously on $c\in\mathfrak{C}$. The continuity of $\Phi_\lambda(\cdot)$ in $\Gamma(\mathrm{Ad}\,P)$ will follow by bootstrap arguments.

It is necessary to consider sequences $\{c_i=(A_i,\Phi_i)\}\subset\mathfrak{C}$. It is convenient to introduce the notation $\nabla_i=\nabla_{A(b)}$ for $b=c_i$ and for $\lambda\in\{0,1,2\}$, $\Phi^i_\lambda=\Phi_\lambda(c_i)$. The first step is to prove

Lemma B3.6. *The map $\mathfrak{A}(c_\lambda(\cdot)):\mathfrak{C}\to[0,\infty)$ is continuous.*

Proof of Lemma B3.6. For $c=(A, \Phi)\in\mathfrak{C}$, let $\Psi(c)=\Phi_0(c)$, $\Phi_1(c)$ or $\Phi_0'(c)$ as $\lambda=0, 1$ or 2, respectively. Let $\phi(c)=\Psi(c)-\Phi$. By construction (see Proposition 4.8 of [3]), $\phi(c)\in L^6(\mathrm{Ad}\,P)$ and $\nabla_A\phi(c)\in L^2$. For $\lambda=2$, ϕ vanishes on the unit ball. Let $\sigma=0$ if $\lambda=0$ or 2 and let $\sigma=1$ if $\lambda=1$.

Define $\mathfrak{s}(c)=\frac{1}{2}\|\nabla_A(\Phi+\phi(c))\|_2^2+\frac{1}{2}\sigma\|[\Phi, \phi(c)]\|_2^2$. By construction (again, Proposition 4.8 of [3]),

$$\mathfrak{s}(c)\leqq \inf_{\eta\in\hat{\Gamma}}\{\tfrac{1}{2}\|\nabla_A(\Phi+\eta)\|_2^2+\tfrac{1}{2}\sigma\|[\Phi, \eta]\|_2^2\},$$

where $\hat{\Gamma}=\Gamma_0(\mathrm{Ad}\,P)$ if $\lambda=0$ or 1 and $\hat{\Gamma}=\{\eta\in\Gamma_0(\mathrm{Ad}\,P): \eta(x)=0 \text{ if } |x|\leqq 1\}$ if $\lambda=2$.

For $r\in[1, \infty)$, let $\beta_r(x)=\beta(x/r)$, where $\beta(x)$ is the usual bump function of Sect. A3. Because $\phi(c)\in L^6$, there exists for each $\varepsilon>0$, an $r(\varepsilon)<\infty$ such that

$$\|\nabla_A(\Phi+\beta_r\phi(c))\|_2^2+\lambda\|[\Phi, \beta_r\phi(c)]\|_2^2<\mathfrak{s}(c)+\varepsilon.$$

As $\beta_r\phi(c)\in\hat{\Gamma}$, there exists a neighborhood $\mathfrak{N}(\varepsilon)\subset\mathfrak{C}$ of c such that for all $b\in\mathfrak{N}(\varepsilon)$,

$$\mathfrak{s}(b)\leqq\|\nabla_{A(b)}(\Phi(b)+\beta_r\phi(c))\|_2^2+\lambda\|[\Phi(b), \beta_r\phi(c)]\|_2^2<\mathfrak{s}(c)+\varepsilon.$$

Thus, $\mathfrak{s}(\cdot):\mathfrak{C}\to[0, \infty)$ is upper-semi-continuous.

Next, consider a sequence $\{c_i\}\subset\mathfrak{C}$ which converges to c. Let $\Psi^i=\Phi_0(c^i)$, $\Phi_1(c^i)$ or $\Phi_0'(c^i)$ as $\lambda=0, 1$ or 2. The sequence $\{\nabla_i\psi^i, \sigma[\Phi, \Psi^i]\}$ is bounded in L^2, so there is no loss of generality to assume that it converges weakly in L^2 to some pair $(G_1, G_2)\in L^2((T^*\oplus\mathbb{R})\otimes\mathrm{Ad}\,P)$.

As the L^2 norm is weakly lower semi-continuous,

$$\|G_1\|_2\leqq\lim_{i\to\infty}\|\nabla_i\Psi^i\|_2 \quad\text{and}\quad \|G_2\|_2\leqq\lim_{i\to\infty}\sigma\|[\Phi_i, \Psi_i]\|_2. \tag{B3.3}$$

If it can be shown that $G_1=\nabla_A\Psi(c)$ and that $G_2=\sigma[\Phi, \Psi(c)]$, then it follows from Eq. (B3.3) that $\mathfrak{s}:\mathfrak{C}\to[0, \infty)$ is continuous. But this implies (as the L^2 norms are weakly lower semi-continuous) that both of the maps $\|\nabla_{A(\cdot)}\Psi(\cdot)\|_2$ and $\sigma\|[\Phi(\cdot), \Psi(\cdot)]\|$ are continuous from $\mathfrak{C}$ into $[0, \infty)$. This would give Lemma B3.6 when $\lambda\in\{0, 1\}$. For $\lambda=2$, Lemma B3.6 would follow after using Lemma A4.1 to conclude that $\Phi_2(\cdot)-\Phi(\cdot)$ factors continuously through $L^6(\mathrm{Ad}\,P)$.

To summarize the previous paragraph, Lemma B3.6 follows from

Lemma B3.7. *If $\{c_i\}\subset\mathfrak{C}$ converges to $c\in\mathfrak{C}$, then $(\nabla_i\Psi^i, \sigma[\Phi_i, \Psi^i])$ converges weakly to $(\nabla_A\Psi, \sigma[\Phi, \Psi])$ in L^2.*

Proof of Lemma 3.7. Let $\phi^i=\Psi^i-\Phi_i$. As $\|\nabla_i\phi^i\|_2^2\leqq 2\mathfrak{A}(c_i)$, the sequence $\{\nabla_i\phi^i\}$ is uniformly bounded on L^2, and so $\{\phi^i\}$ is uniformly bounded in L^6 (cf. Lemma A4.1). Therefore, $\{\phi_i\}$ has a subsequent which converges weakly in L^6 to some η which is in $L^6\cap L^2_{1;\mathrm{loc}}$ (as A_i converges to A in C^∞ on any compact domain in $\mathbb{R}^3$). Hence, for each $\xi\in\Gamma_0(T^*\otimes\mathrm{Ad}\,P)$, $\{\langle\xi, \nabla_{A_i}\phi^i\rangle_2\}$ has a limit equal to $\langle\xi, \nabla_A\eta\rangle_2$. Therefore, $\nabla_A\eta$ is a weak L^2-limit of $\{\nabla_i\phi^i\}$. Similarly, one proves that $\sigma[\Phi, \eta]$ is a weak L^2-limit of $\sigma[\Phi_i, \phi^i]$.

The convergence to η of $\{\phi^i\}$ and the given fact that

$$0=\langle\nabla_i\xi, \nabla_i\Psi^i\rangle_2+\sigma\langle[\Phi_i, \xi], [\Phi_i, \Psi^i]\rangle_2$$

implies that

$$0=\langle \nabla_A\xi, \nabla_A(\Phi+\eta)\rangle_2+\sigma\langle[\Phi,\psi],[\Phi,\eta]\rangle_2 .$$

Since $\phi(c)$ is unique, this last equality implies that $\eta=\phi(c)$. This establishes Lemma B3.7.

The next step is to consider, for $\lambda\in\{0,1,2\}$, the continuity of $\Phi_\lambda(\cdot)$ in $\Gamma(\mathrm{Ad}\,P)$.

Lemma B3.8. *For $\lambda\in\{0,1,2\}$, the map $\Phi_\lambda(\cdot):\mathfrak{C}\to\Gamma(\mathrm{Ad}\,P)$ is continuous.*

Proof of Lemma B3.8. The continuity of $\mathfrak{A}(c_\lambda(\cdot)):\mathfrak{C}\to[0,\infty)$ and Lemmas A4.5 and B3.7 imply that $(\nabla_A\Phi_\lambda)(\cdot):\mathfrak{C}\to L^2$ is continuous. So $(\nabla_A\phi)(\cdot)$ is continuous in L^2. Due to Lemma A4.1, $(1+|x|)^{-1}|\phi(\cdot)|$ is uniformly bounded in L^2. Now, Lemmas A4.1 and A4.5 imply that $(1+|x|)^{-1}\phi(\cdot)$ is continuous in L^2; and so $\phi(\cdot)$ is continuous in L^2_{loc}. By Lemma A4.3, $\phi(\cdot)$ is continuous in $L^2_{1;\mathrm{loc}}$. Continuity of $\phi(\cdot)$ in $\Gamma(\mathrm{Ad}\,P)$ follows by the bootstrap arguments of Chap. 6 of [17]. This lemma completes of the proof of Proposition B3.2.

Proof of Proposition B3.3. The proofs for $\Phi_0(\cdot)$ and $\Phi_2(\cdot)$ are worked similarly, so only the Φ_0 case will be presented.

According to Proposition 4.8 of [3], $\|\Phi_0\|_\infty=1$, and so Theorem V8.1 of [5] is applicable. It states that $\nabla_A\nabla_A\Phi_0\in L^2(\mathrm{Ad}\,P\otimes T^*\otimes T^*)$ with *a priori* bound on the norm which is determined by $\mathfrak{A}((A,\Phi_0))$. This bound is obtained by differentiating once Eq. (B3.1) to obtain

$$\nabla_A^*\nabla_A(\nabla_A\Phi_0)=-*[*F_A,\nabla_A\Phi_0]+*D_A[*F_A,\Phi_0]. \tag{B3.4}$$

For $r<\infty$, let β_r be the cutoff function from Sect. A3. Let $\sigma=(1+\beta_r x^2)$. Let $\delta\in[0,\frac{1}{2})$. Suppose that $c_0=(A,\Phi_0)\in\mathfrak{C}^\delta$. Then from Eq. (B3.4) one obtains

$$\begin{aligned}\nabla_A^*\sigma^\delta\nabla_A(\nabla_A\Phi_0)=&-\sigma^\delta*[*F_A,\nabla_A\Phi_0]+*D_A[\sigma^\delta*F_A,\Phi_0]\\&+\delta\sigma^{\delta-1}(d\sigma,\nabla_A)\nabla_A\Phi_0-\delta\sigma^{\delta-1}*(d\sigma\wedge[*F_A,\Phi_0]).\end{aligned} \tag{B3.5}$$

Now, contract both sides of Eq. (B3.5) with $\nabla_A\Phi_0$ and integrate over $\mathbb{R}^3$. The resulting equation is

$$\begin{aligned}\|\sigma^{\delta/2}\nabla_A\nabla_A\Phi_0\|_2^2=&-\langle\sigma^\delta F_A,[\nabla_A\Phi_0,\nabla_A\Phi_0]\rangle_2+\|\sigma^{\delta/2}[F_A,\Phi_0]\|_2^2\\&+\delta\langle\nabla_A\Phi_0,\sigma^{\delta-1}(d\sigma,\nabla_A)\nabla_A\Phi_0\rangle_2-\delta\langle\sigma^{\delta-1}F_A,[\Phi_0,\nabla_A\Phi_0]\wedge d\sigma\rangle_2 .\end{aligned} \tag{B.3.6}$$

All but the first term on the right-hand side above are uniformly bounded as $r\to\infty$; indeed, $|d\sigma|\sigma^{\delta-1}<z$ which is independent of r as long as $\delta<\frac{1}{2}$.

To control the first term, observe first that

$$|\langle\sigma^\delta F_A[\nabla_A\Phi_0,\nabla_A\Phi_0]\rangle_2|\leqq\|\sigma^{\delta/2}F_A\|_2\|\nabla_A\Phi_0\|_2^{1/2}\|\sigma^{\delta/3}\nabla_A\Phi_0\|_6^{3/2},$$

by Holder's equality. Second,

$$\|\sigma^{\delta/3}\nabla_A\Phi_0\|_6^{3/2}\leqq z^{-1}\|\nabla_A\sigma^{\delta/3}\nabla_A\Phi_0\|_2^{3/2}$$

for some $z^{-1}(\delta)<\infty$ by Lemma 4.1. Third, since $|d\sigma^{\delta/3}|\leqq z(\delta)$ for some $z(\delta)$,

$$\|\nabla_A\sigma^{\delta/3}\nabla_A\Phi_0\|_2^{3/2}\leqq z'(\|\sigma^{\delta/3}\nabla_A\nabla_A\Phi_0\|_2^{3/2}+\|\nabla_A\Phi_0\|_2^{3/2}).$$

Finally, as $\sigma^{\delta/3} \leqq \sigma^{\delta/2}$, one obtains

$$|\langle \sigma^\delta F_A, [\nabla_A \Phi_0, \nabla_A \Phi_0] \rangle_2| \leqq \tfrac{1}{2} \|\sigma^{\delta/2} \nabla_A \nabla_A \Phi_0\|_2^2 + z(\delta)(\|\nabla_A \Phi_0\|_2^2 + \|\sigma^{\delta/2} F_A\|_2^4 \|\nabla_A \Phi_0\|_2^2). \quad \text{(B3.7)}$$

The conclusion from Eqs. (B3.6) and (B3.7) is that $\sigma^{\delta/2} \nabla_A \nabla_A \Phi_0$ is uniformly bounded in $L^2(\mathrm{Ad}\, P \otimes T^* \otimes T^*)$ independently of r and hence $(1+x^2)^{\delta/2} \nabla_A \nabla_A \Phi_0 \in L^2$.

To prove that the assignment of $c = (A, \Phi) \in \mathfrak{C}$ to $(1+x^2)^{\delta/2} \nabla_A \nabla_A \Phi_0(c) \in L^2$ is continuous, one makes use of Lemmas A4.5 and A4.6. With Lemma A4.6, it is a straightforward argument (using Lemma A4.1) to show that the $r = \infty$ limit of the right-hand side of Eq. (B3.6) is continuous. Then $\|(1+x^2)^{\delta/2} \nabla_A \nabla_A \Phi_0\|_2$ is continuous, and due to Lemma A4.5, $(1+x^2)^{\delta/2} \nabla_A \nabla_A \Phi_0$ varies continuously in L^2 as (A, Φ) varies in $\mathfrak{C}^\delta$.

Proof of Proposition B3.4. The argument here is essentially the same as the previous argument for the proof of Proposition B3.3. One additional fact is required. This is that the assignment of c to $\|[\Phi, [\Phi, \Phi_1(c)]]\|_2$ is continuous. To obtain this fact, one starts with the continuity of $\|[\Phi, \Phi_1(\cdot)]\|_2 : \mathfrak{C} \to [0, \infty)$. (This was established in the proof of Lemma B3.6.)

This start is used to prove that $\|[\Phi, \Phi_1(\cdot)]\|_6$ is also continuous. Here is the argument: Let $\Omega \subseteqq \mathbb{R}^3$ be a given domain. Then

$$\begin{aligned} \|[\Phi, \Phi_1]\|_{6;\Omega} &\leqq \||\Phi|^2 |[\Phi, \Phi_1]|\|_{2;\Omega}^{1/3} \\ &\leqq 2\|[\Phi, \Phi_1]\|_{2;\Omega}^{1/3} + 2\|(1-|\Phi|)^2 [\Phi, \Phi_1]\|_{2;\Omega}^{1/3} \\ &\leqq 2\|[\Phi, \Phi_1]\|_{2;\Omega}^{1/3} + 2\|(1-|\Phi|)\|_{6;\Omega}^{2/3} \|[\Phi, \Phi_1]\|_{6;\Omega}^{1/3}. \end{aligned} \quad \text{(B3.8)}$$

[In Eq. (B3.8), the fact that $\|\Phi_1\|_\infty = 1$ has been used.] The Minkowski inequality implies from Eq. (B3.8) the final inequality

$$\|[\Phi, \Phi_1]\|_{6;\Omega} \leqq 6\|[\Phi, \Phi_1]\|_{2;\Omega}^{1/3} + 4\|(1-|\Phi|)\|_{6;\Omega}. \quad \text{(B3.9)}$$

Thus, by taking Ω to be, consecutively, balls in $\mathbb{R}^3$ of radius $(1, 2, \ldots)$, one obtains a uniform bound for $\|[\Phi, \Phi_1]\|_6$ by $\mathfrak{A}(c)$. By taking Ω to be the exterior of a ball of radius $r < \infty$, one obtains with Lemma A4.6 that $\|[\Phi, \Phi_1(\cdot)]\|_6 : \mathfrak{C} \to [0, \infty)$ is continuous.

Next, observe that over any $\Omega \subseteqq \mathbb{R}^3$,

$$\begin{aligned} \|[\Phi, [\Phi, \Phi_1]]\|_{2;\Omega} &\leqq \|[\Phi, \Phi_1]\|_{2;\Omega} + \|(1-|\Phi|)|[\Phi, \Phi_1]|\|_{2;\Omega} \\ &\leqq \|[\Phi, \Phi_1]\|_{2;\Omega} + \|1-|\Phi|\|_{6;\Omega} \|[\Phi, \Phi_1]\|_{6;\Omega}^{1/2} \|[\Phi, \Phi_1]\|_{2;\Omega}^{1/2}. \end{aligned}$$

Therefore, the same argument as used for $\|[\Phi, \Phi_1]\|_6$ works for $\|[\Phi, [\Phi, \Phi_1]]\|_2$, and gives a uniform bound and continuity for $\|[\Phi, [\Phi, \Phi_1(\cdot)]]\|_2$ as a function on $\mathfrak{C}$.

Proof of Corollary B3.5. Propositions B3.3 and B3.4 state that for $\lambda \in \{0, 1\}$, the assignment of $c = (A, \Phi) \in \mathfrak{C}$ to $\nabla_A \nabla_A \Phi_\lambda(c) \in L^2$ is continuous. Therefore, one can use Proposition B3.2 and Lemma A4.4 to obtain the corollary.

B4. The Retraction of $\mathfrak{C}$

Propositions B2.1 and B2.2 are proved here by constructing the maps e, j, and ϱ from Sect. B2. The map e must be constructed first. In order to define e, introduce the map $c_2 : \mathfrak{C} \to \mathfrak{C}$ of Proposition B3.2. According to that proposition, c_2 maps $\mathfrak{C}^\delta$ continuously into the space

$$\mathfrak{L}^\delta = \{c \in \mathfrak{C}^\delta : (1+|x|)^\delta \nabla_A(\nabla_A \Phi) \in L^2(\operatorname{Ad} P \otimes T^* \otimes T^*)\}. \tag{B4.1}$$

This is provided that $\mathfrak{L}^\delta$ is topologized by the map which assigns $(A, \Phi) \in \mathfrak{L}^\delta$ to

$$(A, \Phi, (1+|x|)^\delta |\nabla_A(\nabla_A \Phi)|) \in \mathfrak{C}^\delta \times L^2(\mathbb{R}^3).$$

The space $\mathfrak{L}^\delta$ has a number of important properties, and it is timely to digress here to enumerate them. The first observation is

Lemma B4.1. *The assignment of $(A, \Phi) \in \mathfrak{L}^\delta$ to $(1-|\Phi|)$ factors continuously through $\bar{C}^{0,1/2}(\mathbb{R}^3)$, where its norm is bounded a priori given $\|\nabla_A(\nabla_A \Phi)\|_2$ and $\|\nabla_A \Phi\|_2$.*

Proof of Lemma B4.1. This is actually a corollary now to Lemma A4.4.

If $(A, \Phi) \in \mathfrak{L}^\delta$, then Lemma B4.1 implies that $1-|\Phi|(x)$ tends uniformly to zero as $|x| \to \infty$. Therefore, $R(A, \Phi) \in [1, \infty)$ exists with the property that $|\Phi(x)| > \frac{1}{2}$ if $|x| > R(A, \Phi)$. If $(A', \Phi') \in \mathfrak{L}^\delta$ is sufficiently close to (A, Φ), then also $|\Phi'(x)| > \frac{1}{2}$ if $|x| > R(A, \Phi)$. One would like to have $R(\cdot)$ depend continuously on $\mathfrak{L}^\delta$. The next lemma allows this.

Lemma B4.2. *There exists a continuous $\mathfrak{G}$-equivariant function $R(\cdot) : \mathfrak{L}^\delta \to [1, \infty)$ such that where $|x| > R(A, \Phi)$, then $|\Phi|(x) > \frac{1}{2}$.*

Proof of Lemma B4.2. Here one requires the following observation:

Lemma B4.3. *Let X be a topological space and suppose that $f : [0,1] \times X \to [0,1]$ is a continuous map with the property that for fixed $y \in X$, $f(\cdot, y)$ is not decreasing on $[0,1]$, and $f(0,y) = 0$. Then there is a continuous function $\varrho : [0,1] \times X \to [0,1]$ which satisfies for each $y \in X$, $\varrho(0,y) = 0$, $\varrho(\cdot, y)$ is increasing and $f(\varrho(\varepsilon, y), y) < \varepsilon$.*

Proof of Lemma B4.2 given Lemma B4.3. Use the previous lemma with the function $f : [0,1] \times \mathfrak{C} \to [0,1]$ given by

$$f(\varrho, (A, \Phi)) = 1 - \min\left(\inf_{|x| > 1/\varrho} |\Phi|(x), 1\right).$$

The continuity of f follows from Lemma B4.1.

Proof of Lemma B4.2. Let β denote the bump function from Sect. A3. Now set

$$\varrho(\varepsilon, y) = \int_0^\varepsilon dt \beta(\varepsilon^{-1} f(t, y)).$$

The function ϱ has the requisite properties for the following reasons: First, as $\varepsilon \to 0$, $\varrho(\varepsilon, y) \to 0$ and the limit is locally uniform in y. Second, $\beta(\varepsilon^{-1} f(t, y))$ has support only on $\{t \in [0,1] : f(t, y) < \varepsilon\}$ which is a connected open set containing $\{0\}$. Since $\beta \leqq 1$, $\varrho(\varepsilon, y)$ is in this set. Finally, $\varrho(\cdot, y) \not\equiv 0$ and $\left(\frac{\partial \varrho}{\partial \varepsilon}\right)(\varepsilon, y) \geqq 0$.

For $c=(A,\Phi)\in\mathfrak{L}^\delta$, $|\Phi|(x)$ goes uniformly to 1 as $|x|\to\infty$. Thus, each such Φ defines a splitting of $\mathrm{Ad}P$ over $\{x\in\mathbb{R}^3 : |x|>R(c)\}$ as $L\oplus\mathbb{R}$, where $\mathbb{R}$ embeds in $\mathrm{Ad}P_x$ as the span of $\Phi(x)$. It is useful to decompose the connection A according to this splitting. For this purpose, write $r=r(c)=2R(c)$ and $\hat{\Phi}(c)=\Phi/|\Phi|$. Then write

$$A=a\hat{\Phi}+a^T-(1-\beta)[\hat{\Phi},d\hat{\Phi}]+\beta_r A\,, \tag{B4.2}$$

where

$$a^T(c)=(1-\beta_r)[\hat{\Phi},[A,\hat{\Phi}]]+(1-\beta)[\hat{\Phi},d\hat{\Phi}] \text{ and } a(c)=(1-\beta_r)(-2\,\mathrm{trace}_{\mathbb{C}^2}(\hat{\Phi}A))\,. \tag{B4.3}$$

Due to Lemmas B4.1–2, the assignment of $c\in\mathfrak{L}^\delta$ to

$$(a(c),a^T(c))\in\Gamma(T^*)\otimes\Gamma(\mathrm{Ad}P\otimes T^*)$$

is continuous. The $\mathrm{Ad}P$ valued 1-form a^T has the properties that are listed below.

Lemma B4.4. *The assignment of (A,Φ) in $\mathfrak{L}^\delta$ to $(a^T,(1+|x|)^\delta\nabla_A a^T)$ as defined by Eqs.* (B4.2, 3) *factors continuously through*

$$[\Gamma(\mathrm{Ad}P\otimes T^*)\cap L^2(\mathrm{Ad}P\otimes T^*)]\times L^2(\mathrm{Ad}P\otimes T^*\otimes T^*)\,.$$

Proof of Lemma B4.4. First, observe that for $|x|>r$,

$$|\Phi|^{-2}[\Phi,\nabla_A\Phi]=a^T\,. \tag{B4.4}$$

Therefore, Lemmas A4.6 and B4.1, 2 imply that the assignment of $(A,\Phi)\in L^\delta$ to a^T factors continuously through $L^2(\mathrm{Ad}P\otimes T^*)$.

By differentiating Eq. (B4.4), one obtains

$$\begin{aligned}-2|\Phi|^{-4}(-2\,\mathrm{trace}_{\mathbb{C}^2}(\Phi\nabla_A\Phi))[\Phi,\nabla_A\Phi]+|\Phi|^{-2}[\nabla_A\Phi,\nabla_A\Phi]\\ +|\Phi|^{-2}[\Phi,\nabla_A(\nabla_A\Phi)]=\nabla_A a^T\,.\end{aligned} \tag{B4.5}$$

For a domain $\Omega\subseteqq\mathbb{R}^3$, Hölder's inequality implies that

$$\|(1+|x|)^\delta|\nabla_A\Phi|^2\|_{2;\Omega}^2\leqq\|\nabla_A\Phi\|_{2;\Omega}\|(1+|x|)^\delta\nabla_A\Phi\|_{6;\Omega}\,.$$

This last inequality, plus Lemmas A4.1, A4.4, and A4.6 imply that $(1+|x|)^\delta\nabla_A a^T$ factors continuously through $L^2(\mathrm{Ad}P\otimes T^*\otimes T^*)$.

Without an additional choice of gauge, $a(A,\Phi)$ of Eq. (B4.3) cannot be controlled. However, $\mathfrak{G}_0$ is contractible, so gauge fixing is possible. The process results in

Lemma B4.5. *There exists a continuous map $g:\mathfrak{L}^\delta\to\mathfrak{G}_0$ with the following properties:*

(1) *On I*(Maps), $g\equiv 1\in\mathfrak{G}_0$.

(2) *Let $c=(A,\Phi)\in\mathfrak{L}^\delta$. Then where $|x|>2R(c)$,*

$$\hat{\Phi}(g(c)\cdot c)(x)\equiv(g(c)\hat{\Phi}(c)g^{-1}(c))(x)=\hat{\Phi}(2R(c)\hat{x})\,.$$

(3) *The assignment of $c\in\mathfrak{L}^\delta$ to the real 1-form $a(g(c)\cdot c)$ of Eq.* (B4.3) *defines via*

$$c\mapsto(1+|x|)^\delta(a(g(c)\cdot c),\nabla a(g(c)\cdot c))$$

a continuous map from $\mathfrak{L}^\delta$ into $[\Gamma(T^*)\cap L^6(T^*)]\times L^2(T^*\otimes T^*)$.

Proof of Lemma B4.5. With the C^∞ topology on the participating function spaces, the map below defines a principal fibration with fiber $\mathrm{Maps}(S^2; S^1)$:

$$\begin{array}{c}\mathrm{Maps}(S^2;\mathrm{SU}(2)) \times \mathrm{Maps}(S^2;S^2) \\ \downarrow T \\ \mathrm{Maps}(S^2;S^2) \times \mathrm{Maps}(S^2;S^2)\end{array} \quad . \tag{B4.3}$$

Here T sends (h, σ) to $(h\sigma h^{-1}, \sigma)$. $\mathrm{Maps}(S^2;S^1)$ embeds as the fiber over (σ, σ) by assigning

$$\exp(\sqrt{-1} f) \in \mathrm{Maps}(S^2;S^1)$$

to $(\exp(f\sigma), \sigma)$ in $\mathrm{Maps}(S^2;\mathrm{SU}(2)) \times \mathrm{Maps}(S^2;S^2)$.

A continuous map (l, l') from $\mathfrak{L}^\delta$ into Eq. (B4.3) is obtained by assigning $c = (A, \Phi)$ as follows

$$\begin{array}{ccc} & \overset{l'}{\nearrow} & (1, \hat{\Phi}(2R(c)\hat{x})) \\ & & \downarrow T \\ \mathfrak{L}^\delta & \overset{l}{\longrightarrow} & (\hat{\Phi}(2R(c)\hat{x}), \hat{\Phi}(2R(c)\hat{x}))\end{array} \quad . \tag{B4.4}$$

Here $\hat{\Phi} = \Phi/|\Phi|$ and $\hat{x} = x/|x|$. Observe that $T \circ l' = l$.

A homotopy of l, a map $\phi : [1, \infty) \times \mathfrak{L}^\delta$ to $\mathrm{Maps}(S^2;S^2) \times \mathrm{Maps}(S^2;S^2)$ is defined by sending $(t, c = (A, \Phi))$ to

$$(\hat{\Phi}(tR(c)\hat{x}), \hat{\Phi}(R(c)\hat{x}))\,. \tag{B4.5}$$

The existence of a lifting ϕ', of ϕ, which commutes with T is guaranteed by the homotopy lifting property of a fibration [18]. There is no obstruction to choosing ϕ' to satisfy $\phi'(t, I(e)) = (1, e(\hat{x}))$ for $e \in \mathrm{Maps}(S^2;S^2)$ and $t \in [1, \infty)$.

Let $\pi_1 : \mathrm{Maps}(S^2;\mathrm{SU}(2)) \times \mathrm{Maps}(S^2;S^2)$ be the projection onto the first factor. Then $g' \equiv \pi_1 \circ \phi'$ maps $[1, \infty) \times \mathfrak{L}^\delta$ into $\mathrm{Maps}(S^2;\mathrm{SU}(2))$ continuously and it satisfies $g'(1, c) \equiv 1 \in \mathrm{SU}(2)$.

The homotopy lifting allows the extension of g' to define a map $g_1 : \mathfrak{L}^\delta \to \mathfrak{G}_0$ which satisfies for $|x| \geqq 2R(c)$,

$$g_1(c)(x) = g'\left(\frac{|x|}{2R(c)}, c\right)\left(\frac{x}{|x|}\right),$$

and which satisfies $g_1(c) \equiv 1 \in \mathfrak{G}_0$ when $c \in I(\mathrm{Maps})$.

The map g of the lemma will be of the form $g = h \cdot g_1$. To obtain h requires the following observation:

Lemma B4.6. *The assignment of $c \in \mathfrak{L}^\delta$ to $a(g_1(c) \cdot c) \in \Gamma(T^*)$ has the property that*

$$c \mapsto (1 + |x|)^\delta da(g_1(c) \cdot c)$$

defines a continuous map from $\mathfrak{L}^\delta$ into $L^2\left(\bigwedge_2 T^*\right)$.

Proof of Lemma B4.6. For $|x| \geqq 2R(c)$, $\hat{\Phi}(g_1(c) \cdot c)(x) \equiv \sigma(x)$ is only a function of $\hat{x} = x/|x|$. Thus, $|d\sigma| \leqq \text{const}|x|^{-1}$. Let $a_1^T = a^T(g_1(c) \cdot c)$. Let $g_1(c) \cdot c = (A, \Phi)$. Then when $|x| \geqq 2R(c)$,

$$da(g_1(c) \cdot c) = (\sigma, F_A) + 4\,\text{trace}_{\mathbb{C}^2}(d\sigma \wedge a_1^T) + (\sigma, d\sigma \wedge d\sigma) - (\sigma, a_1^T \wedge a_1^T).$$

Here, $(\alpha, \beta) \equiv -2\,\text{trace}_{\mathbb{C}^2}(\alpha, \beta)$ for $\alpha, \beta \in \mathfrak{su}(2)$.

The continuity of $(1+|x|)^\delta da(g_1(c) \cdot c)$ follows Lemmas A4.5 and B4.4.

To complete the construction of g for Lemma B4.5 one uses the previous lemma in the following way: Note that the Hodge theorem for $\mathbb{R}^3$ (cf. Proposition 7.6 of [4] or [19]) allows that there exists a unique $A^L \in \Gamma(T^*) \cap L^6$ with $\nabla A^L \in L^2(T^* \otimes T^*)$ which satisfies $d{*}A^L = 0$ and $dA^L = da(g_1(c) \cdot c)$. The existence proof implicity asserts that A^L varies continuously in $\Gamma(T^*) \cap L^6(T^*)$ and that ∇A^L varies continuously in $L^2(T^* \otimes T^*)$ as c varies in $\mathfrak{L}^\delta$. Lemma B4.6 with arguments as in the proof of Proposition B3.2 insure that the assignment of $c \in \mathfrak{L}^\delta$ to $(1+|x|)^\delta(A^L(c), \nabla A^L(c))$ defines a continuous map from c into $L^6(T^*) \cap L^2(T^* \otimes T^*)$.

The Poincaré lemma for $\mathbb{R}^3$ provides a function $\Lambda(c) \in C^\infty(\mathbb{R}^3)$ which satisfies $d\Lambda(c) = a(g_1(c) \cdot c) - A^L(c)$. By demanding that $\Lambda(c)(0) = 0$, the function $\Lambda(c)$ is uniquely determined by $c \in \mathfrak{L}^\delta$ and the assignment of $c \in \mathfrak{L}^\delta$ to $\Lambda(c) \in C^\infty(\mathbb{R}^3)$ is then continuous.

Define now

$$h(c) = \exp(\tfrac{1}{2}\Lambda(c)(1 - \beta_{R(c)})\hat{\Phi}(g_1(c) \cdot c)). \tag{B4.6}$$

Then, let $g(c) = h(c) \cdot g_1(c)$. This g defines a continuous map from $\mathfrak{L}^\delta$ into $\mathfrak{G}_0$, and it satisfies the requirements of Lemma B4.5. In fact, where $|x| \geqq 2R(c)$, $a(g(c) \cdot c) = A^L(c)$. Thus, conditions (2) and (3) of Lemma B4.5 are satisfied. For condition (1), remember that g_1 is identically 1 on $I(\text{Maps})$, while $a(\cdot)$ on $I(\text{Maps})$ is identically zero. Thus $\Lambda|_{I(\text{Maps})} \equiv 0$ and $g|_{I(\text{Maps})} \equiv 1$ as required.

Before proving Proposition B2.2, it is timely to define the maps e and j: define $\hat{e}: \mathfrak{C} \to \text{Maps}(S^2 : S^2)$ by requiring that

$$\hat{e}(c)(\hat{x}) = \hat{\Phi}(c_2(c))(2R(c_2(c))\hat{x}). \tag{B4.7}$$

Define $j: \mathfrak{C} \to \mathfrak{G}_0$ by requiring that

$$j(c) = g^{-1}(c_2(c)), \tag{B4.8}$$

with $g(\cdot): \mathfrak{L}^\delta \in \mathfrak{G}_0$ given by Lemma B4.5.

Proof of Proposition B2.1. For condition (1), observe that $c_2(I(e)) = I(e)$ by construction since $(\nabla_A \Phi)(I(e))$ vanishes on $\{x \in \mathbb{R}^3 : |x| \geqq 1\}$. For this reason,

$$\hat{e}(I(e))(\hat{x}) = \hat{\Phi}(I(e))(2R(I(e))\hat{x}) = e(\hat{x})$$

for all $\hat{x} \in S^2$. Also, $j(I(e)) = g^{-1}(I(e)) = 1 \in \mathfrak{G}_0$ due to Lemma B4.5. For condition (2), observe that

$$c - c_2(c) = (0, \Phi(c) - \Phi_2(c)), \tag{B4.9}$$

and so Propositions B3.1 and B3.2 imply that $\mathfrak{r}(\cdot)$ as a map from $[0, \frac{1}{2}] \times \mathfrak{C}^\delta$ to $\mathfrak{C}^\delta$ is continuous. To obtain the continuity of $\mathfrak{r}$ *on* $[\frac{1}{2}, 1] \times \mathfrak{C}^\delta$, observe that for $t \in [\frac{1}{2}, 1]$,

$$\mathfrak{r}(t, c) = j(c)[g \cdot c_2 - (2t-1)(g \cdot c_2 - I(\hat{e})],$$

where $c_2=c_2(c)$, $\hat{e}=\hat{e}(c)$ and $g=g(c_2(c))$. Keep in mind that $c_2(c_2(c))=c_2(c)$, and so $\hat{e}(c_2(c))=\hat{e}(c)$. For $|x|>2R(c_2)$,

$$g\cdot c_2-I(\hat{e})=(a(g\cdot c_2)\hat{e}+a^T(g\cdot c_2),(|\Phi(c_2)|-1)\hat{e})\,. \tag{B4.10}$$

Thus, continuity of $\mathfrak{r}$ as a map from $[\frac{1}{2},1]\times\mathfrak{C}^\delta$ to $\mathfrak{C}^\delta$ follows with Hölder's inequality and Lemmas B4.2, B4.4, and B4.5. The reader can supply the algebra.

Proof of Proposition B2.2. For $(\varrho,t,c)\in[0,1]\times[0,1]\times\mathfrak{C}$, consider

$$\hat{c}(\varrho,t,c)=\begin{cases}c-2t(1-\beta_{1/\varrho})\,[c-c_2] & \text{for} \quad t\in[0,\frac{1}{2}];\\ c_2-(2t-1)\,(1-\beta_{2/\varrho})\,[c_2-j(c)\,I(\hat{e})]+\beta_{1/\varrho}[c-c_2] & \text{for} \quad t\in[\frac{1}{2},1]\,.\end{cases} \tag{B4.11}$$

Because $(1-\beta_{1/\varrho})$ has support only where $|x|>1/\varrho$, $\hat{c}(\varrho,t,c)$ converges to c in C^∞ of any bounded domain in $\mathbb{R}^3$ as $\varrho\to 0$. This convergence is locally uniform in $(t,c)\in[0,1]\times\mathfrak{C}$. Independent of ϱ, $\hat{c}(\varrho,0,c)=c$.

On $\{x\in\mathbb{R}^3:|x|>2/\varrho\}$, $\hat{c}(\varrho,1,c)=j(c)I(\hat{e}(c))$, and therefore, for $\varrho>0$ and $(A,\Phi)=\hat{c}(\varrho,1,c)$,

$$|F_A|(x)+|\nabla_A\Phi|(x)\leqq\text{const}(c,\varrho)\,(1+|x|)^{-2}\,.$$

Thus, $\hat{c}(\varrho,1,c)\in\mathfrak{C}^\delta$ for any $\varrho\in(0,1]$, and every $c\in\mathfrak{C}$ and $\delta\in[0,\frac{1}{2})$.

As I maps $\text{Maps}(S^2;S^2)$ continuously into $\mathfrak{C}^\delta$, $\delta\in[0,\frac{1}{2})$, and $\hat{e}$ is continuous, $\hat{c}(\varrho,1,\cdot):\mathfrak{C}\to\mathfrak{C}^\delta$ continuously for any $\varrho\in(0,1]$ and $\delta\in[0,\frac{1}{2})$.

With Eq. (B4.9) and Propositions B3.1 and B3.2, one concludes that for any $\varrho\in(0,1]$ and $\delta\in[0,\frac{1}{2})$, $\hat{c}(\varrho,\cdot):[0,\frac{1}{2}]\times\mathfrak{C}^\delta\to\mathfrak{C}^\delta$ continuously. Lemma A4.1 and the fact that $|d\beta_{1/\varrho}|\in L^3(\mathbb{R}^3)$ with norm independent of ϱ implies that for any $c\in\mathfrak{C}^\delta$,

$$\lim_{\varrho\to 0}\sup_{t\in[0,1/2]}|\mathfrak{A}^\delta(\hat{c}(\varrho,t,c))-\mathfrak{A}^\delta(c)|=0\,. \tag{B4.12}$$

Lemma A4.3 with Proposition B3.2 implies that this limit is locally uniform on $\mathfrak{C}^\delta$. Therefore, $\hat{c}(\cdot)$ extends as a continuous function from $[0,1]\times[0,\frac{1}{2}]\times\mathfrak{C}^\delta$ to $\mathfrak{C}^\delta$ for every $\delta\in[0,\frac{1}{2})$.

Lemmas B4.2, B4.4, and B4.5 with Eq. (B4.10) imply that for any $\varrho\in(0,1]$ and $\delta\in[0,\frac{1}{2})$, $\hat{c}(\varrho,\cdot):[\frac{1}{2},1]\times\mathfrak{C}^\delta\to\mathfrak{C}^\delta$ continuously.

Proposition B3.1 and Lemmas B4.4 and B4.5 imply that for any $c\in\mathfrak{C}^\delta$,

$$\lim_{\varrho\to 0}\sup_{t\in[1/2,1]}|\mathfrak{A}^\delta(\hat{c}(\varrho,t,c))-\mathfrak{A}^\delta(c)|=0\,. \tag{B4.13}$$

This is proved by breaking the integrals into their contributions from the sets $\{x\in\mathbb{R}^3:|x|<\frac{1}{2}\varrho\}$, $\{x\in\mathbb{R}^3:\frac{1}{2}\varrho<|x|<1/\varrho\}$, and $\{x\in\mathbb{R}^3:|x|>1/\varrho\}$, and then using Hölder's inequality with the aforementioned lemmas. The details are straightforward and omitted. This proof also establishes that the limit in Eq. (B4.13) is locally uniform with respect to $c\in\mathfrak{C}^\delta$.

One concludes from this discussion that $\hat{c}$ extends as a continuous function from $[0,1]\times[0,1]\times\mathfrak{C}^\delta$ to $\mathfrak{C}^\delta$ for every $\delta\in[0,\frac{1}{2})$.

Using Eqs. (B4.12) and (B4.13), Lemma B4.3 provides a continuous function $\varrho(\cdot):[0,1]\times\mathfrak{C}\to[0,1]$ which (1) maps $\{0\}\times\mathfrak{C}$ to $\{0\}$ and for fixed c maps $(0,1)$ into $(0,1)$; (2) for every $c\in\mathfrak{C}$,

$$\sup_{t\in[0,1]}|\mathfrak{A}(\hat{c}(\varrho(\varepsilon,c),t,c))-\mathfrak{A}(c)|<\varepsilon\,.$$

The map τ of Proposition B2.1 is given by the assignment of

$$(\varepsilon, t, c) \in [0,1] \times [0,1] \times \mathfrak{C}$$

to $\hat{c}(\varrho(\varepsilon, c), t, c)$.

B5. The SU(2) Action on $\mathfrak{B}$

As pointed out in Sect. B1, SU(2) acts on $\mathfrak{B}$ via the action of the constant matrices in $\mathfrak{G}$. It acts on $\mathrm{Maps}(S^2; S^2)$ by rotating the image S^2 and I is an equivariant map. This SU(2) action can be reduced to an S^1-action by fixing Φ at a point. The reduction is constructed as follows: Let $n \in S^2$ be the north pole. Let $\Omega(S^2; S^2) \hookrightarrow \mathrm{Maps}(S^2; S^2)$ be the subspace of maps which take n to n. The S^1 subgroup of SU(2) which fixes n acts on $\Omega(S^2; S^2)$ while the orbit of $\Omega(S^2; S^2)$ in $\mathrm{Maps}(S^2; S^2)$ under SU(2) is all of $\mathrm{Maps}(S^2; S^2)$.

One has the fibration

$$\Omega(S^2; S^2) \to \mathrm{Maps}(S^2; S^2) \xrightarrow{\hat{n}} S^2,$$

with $\hat{n}(e) = e(n)$. The above mentioned S^1 acts equivariantly on this fibration, and $\Omega/S^1 \simeq \mathrm{Maps}/\mathrm{SU}(2)$.

There is a corresponding fibration for $\mathfrak{B}^\delta$ if $\delta \in [0, \frac{1}{2})$. This is defined with the map $c_2(\cdot): \mathfrak{C}^\delta \to \mathfrak{C}^\delta$ of Sect. B3 and the map $R: \mathfrak{C}^\delta \to [1, \infty)$ of Lemma B4.2. A continuous map, $\hat{n}: \mathfrak{B}^\delta \to S^2$ is defined as follows: Let $n \in S^2$ be the point $(1,0,0) \in \mathbb{R}^3$. Set $\hat{n}(c)$ to be the point $(\Phi_2(c)/|\Phi_2(c)|)\,(2R(c_2(c))n) \in S^2$. This map $\hat{n}$ induces a fibration,

$$\hat{\mathfrak{B}}^\delta \to \mathfrak{B}^\delta \xrightarrow{\hat{n}} S^2, \tag{B5.1}$$

where the fiber over $n \in S^2$ is

$$\hat{\mathfrak{B}}^\delta = \{c \in \mathfrak{B}^\delta : (\Phi_2(c)/|\Phi_2(c)|)\,(2R(c_2(c))n) = n\}. \tag{B5.2}$$

Proposition B3.2 and Lemma B4.2 insure that $\hat{\mathfrak{B}}^\delta$ is closed in $\mathfrak{B}^\delta$.

The orbit of $\hat{\mathfrak{B}}^\delta$ in $\mathfrak{B}^\delta$ under SU(2) is all of $\mathfrak{B}^\delta$. But, the S^1 subgroup of SU(2) which fixes $n \in S^2$ acts on $\mathfrak{B}^\delta$. This S^1 action is equivariant with respect to the commutative diagram below:

$$\begin{array}{ccccc} \hat{\mathfrak{B}}^\delta & \longrightarrow & \mathfrak{B}^\delta & \searrow & \\ \uparrow I & & \uparrow I & & S^2 \\ \Omega(S^2; S^2) & \longrightarrow & \mathrm{Maps}(S^2; S^2) & \nearrow & \end{array} \tag{B5.3}$$

It is pertinent to remark here that because $\mathfrak{A}^\delta$ is $\mathfrak{G}$-equivariant, it only "sees" the topology of $\hat{\mathfrak{B}}^\delta$, at least as far as min-max is concerned.

Proposition B5.1. *For each $\delta \in [0, \frac{1}{2})$, the map I embeds $\Omega(S^2; S^2)$ into $\hat{\mathfrak{B}}^\delta$ and the image under I of $\Omega(S^2; S^2)$ is a deformation retract of $\hat{\mathfrak{B}}^\delta$.*

The proof of this proposition will be given shortly. Observe that Propositions B1.1 and B5.1 yield Theorem A1.1 as a corollary.

Proposition B5.2. *There exists a continuous map* $q:[0,1]\times[0,1]\times\hat{\mathfrak{B}}\to\hat{\mathfrak{B}}$ *such that:*

(1) *For each* $\delta\in[0,\frac{1}{2})$, q *maps* $[0,1]\times[0,1]\times\hat{\mathfrak{B}}^\delta$ *into* $\hat{\mathfrak{B}}^\delta$ *continuously.*
(2) *For each* $\varepsilon\in(0,1]$, $q(\varepsilon,0,\cdot)$ *is the identity map on* $\hat{\mathfrak{B}}$.
(3) *For any* $\delta\in[0,\frac{1}{2})$ *and for any* $\varepsilon\in(0,1)$, $q(\varepsilon,1,\cdot)$ *maps* $\hat{\mathfrak{B}}$ *continuously into* $\hat{\mathfrak{B}}^\delta$.
(4) *For any* $t\in[0,1]$, $q(0,t,\cdot)$ *is the identity on* $\hat{\mathfrak{B}}$.
(5) *Finally, for any* $\varepsilon\in[0,1]$ *and for all* $c\in\hat{\mathfrak{B}}$,

$$\sup_{t\in[0,1]}|\mathfrak{A}(q(\varepsilon,t,c))-\mathfrak{A}(c)|<\varepsilon\,.$$

Proof of Proposition B5.1. Let $\mathfrak{r}(\cdot):[0,1]\times\mathfrak{C}\to\mathfrak{C}$ be the map of Proposition B2.1. Let $\Pi:\mathfrak{C}\to\mathfrak{B}$ denote the projection. By restricting $\mathfrak{r}$ to $[0,1]\times\mathfrak{B}$ and then composing with Π, one obtains a map $\Pi\circ\mathfrak{r}:[0,1]\times\mathfrak{B}\to\mathfrak{B}$. By associating each c in $\mathfrak{B}$ to the point $\hat{n}(c)\in S^2$, one obtains a continuous map $(l,\hat{l})$ of $\mathfrak{B}$ into the fibration

$$\begin{array}{ccc} & & S^1 \\ & & \downarrow \\ & \nearrow^{\hat{l}} & \mathrm{SU}(2)\times S^2 \\ & & \downarrow T \\ \mathfrak{B} & \xrightarrow{l} & S^2\times S^2 \end{array} \qquad \text{(B5.4)}$$

Here T sends (g,σ) to $(g\sigma g^{-1},\sigma)$ and $l(\cdot)=(\hat{n}(\cdot),\hat{n}(\cdot))$ while $\hat{l}(\cdot)=(1,\hat{n}(\cdot))$.

A homotopy of l is defined by $\phi:[0,1]\times\mathfrak{B}\to S^2\times S^2$ which sends (t,c) to $(\hat{n}(\Pi\circ\mathfrak{r}(t,c)),\hat{n}(c))$. The homotopy lifting property provides a continuous map, $k:[0,1]\times\mathfrak{B}\to\mathrm{SU}(2)$ with the property that $k(0,\cdot)=1$ and for all $(t,c)\in[0,1]\times\mathfrak{B}$,

$$k(t,c)\hat{n}(c)k^{-1}(t,c)=\hat{n}(\Pi\circ\mathfrak{r}(t,c))\,.$$

There is no loss of generality by assuming that $k(t,\cdot)|_{I(\mathrm{Maps})}=1$ for all $t\in[0,1]$. Indeed, if this is not the case, replace k above by

$$k'(t,c)=k(t,c)\cdot k^{-1}(t,I(\hat{e}(c)))\,,$$

with $\hat{e}:\mathfrak{C}\to\mathrm{Maps}(S^2;S^2)$ given by Proposition B2.1.

Define the retraction of $\hat{\mathfrak{B}}^\delta$ onto $I(\Omega(S^2;S^2)$ by sending $(t,c)\in[0,1]\times\hat{\mathfrak{B}}^\delta$ to

$$k^{-1}(t,c)\cdot(\Pi\circ\mathfrak{r})(t,c)\,.$$

Proof of Proposition B5.2. The proof here is identical in most respects to the proof of Proposition B5.1. One replaces $\mathfrak{r}$ by the map τ of Proposition B2.2 and then $(l,\hat{l})$ maps $[0,1]\times\mathfrak{B}$ into the fibration of Eq. (B5.4) by sending (ε,c) to $l(\varepsilon,c)=(\hat{n}(\tau(\varepsilon,0,c)),\ \hat{n}(\tau(\varepsilon,0,c)))=(\hat{n}(c),\hat{n}(c))$, while $\hat{l}(\varepsilon,c)=(1,\hat{n}(c))$. The homotopy ϕ sends $(t,\varepsilon,c)\in[0,1]\times[0,1]\times\mathfrak{B}$ to $(\hat{n}(\Pi\circ\tau(\varepsilon,t,c)),\hat{n}(c))$. The remaining aspects of the proof are the same.

B6. Hilbert Spaces

Hilbert space vector bundles over $\mathfrak{B}$ play a crucial role in this article, and one in particular, provides a useful local embedding theorem for $\mathfrak{B}$ (see Proposition B6.1).

To begin, let V be a finite dimensional Hilbert space on which SU(2) acts by an orthogonal representation, ϱ. Let $\varrho_* : \mathfrak{su}(2) \to \operatorname{End} V$ denote the induced representation. Let $E = P \times_\varrho V$.

Each pair $c = (A, \Phi)$ of $L^2_{1;\mathrm{loc}}$ connection on P and $L^2_{1;\mathrm{loc}}$ section of AdP defines a metric on $\Gamma_0(E)$ by

$$\langle \psi, \eta \rangle_c = \langle \nabla_A \psi, \nabla_A \eta \rangle_2 + \langle \varrho_*(\Phi)\psi, \varrho_*(\Phi)\eta \rangle_2 .$$

Lemma A4.1 insures that $\langle \cdot, \cdot \rangle_2$ is nondegenerate.

A Hilbert space, $H_c(E)$, is obtained by completing $\Gamma_0(E)$ in the norm $\| \cdot \|_c^2 = \langle \cdot, \cdot \rangle_c$. Notice that the assignment of $c = (L^2_{1;\mathrm{loc}}$ connection on P, $L^2_{1;loc}$ section of AdP) to $H_c(E) \subset L^2_{1;\mathrm{loc}}(E)$ is $\mathfrak{G}$ equivariant; when $g \in \mathfrak{G}$, then $H_{g \cdot c}$ and $g \cdot H_c$ are the same in $L^2_{1;\mathrm{loc}}(E)$.

Of particular importance is the case $E \equiv Q = \operatorname{Ad}P \otimes (T^* \oplus \mathbb{R})$. In this case, define for each $c \in \mathfrak{C}$, $\Gamma^c = H_c(Q) \cap \Gamma(Q)$, and topologize Γ^c by the inclusions in $H_c(Q)$ and $\Gamma(Q)$.

Proposition B6.1. *Let $n \in \mathbb{Z}$ and let $c \in \mathfrak{B}_n$. About each $b \in \mathfrak{B}_n$, there exists a neighborhood of b, $\mathfrak{N}(b)$ and a continuous map $h(b)(\cdot) : \mathfrak{N}(b) \to \mathfrak{G}_0$ with the following property: the map which sends $b' \in \mathfrak{N}(b)$ to $m_b(b') = h(b)(b') \cdot b' - c$ embeds $\mathfrak{N}(b)$ in Γ^c.*

Later in this section, Propositions 6.1 and 6.2, below, are used to define a C^0 vector bundle structure over $\mathfrak{B}$ for the set $\{H_b(E), b \in \mathfrak{B}\}$.

Proposition B6.2. *Let $c \in \mathfrak{C}$ and let $\psi \in H_c(Q)$. Let $E \to \mathbb{R}^3$ be an associated vector bundle to $P \to \mathbb{R}^3$. The identity map on $\Gamma_0(E)$ induces an isomorphism between $H_{c+\psi}(E)$ and $H_c(E)$ with the following property: for any $\eta \in H_{c+\psi}$,*

$$| \|\eta\|_{c+\psi} - \|\eta\|_c | \leqq z \cdot \|\eta\|_c \|\psi\|_c$$

with $z < \infty$ a constant that is independent of ψ, η and dependent only on $\mathfrak{A}(c)$.

Proposition B6.1 is proved in Sect. B7, and Proposition B6.2 will be proved here shortly.

Let $E \to \mathbb{R}^3$ be an associated vector bundle to P. A C^0 vector bundle structure for the set

$$H(E) = \bigcup_{b \in \mathfrak{B}} H_b(E)$$

is constructed as follows: Let $n \in \mathbb{Z}$, and choose $c \in \mathfrak{B}_n$. For each $b \in \mathfrak{B}_n$, there is a neighborhood, $\mathfrak{N}(b)$, and a Hilbert space isomorphism $\varrho(l_b(b')) : H_{b'}(E) \to H_c(E)$, which is defined for all $b' \in \mathfrak{N}(b)$ by the sequence

$$H_{b'}(E) \to \varrho(h(b)(b')) \cdot H_{b'}(E) \to H_{h(b)(b') \cdot b'}(E) \to H_c(E) .$$

Here, $h(b)(\cdot) : \mathfrak{N}(b) \to \mathfrak{G}_0$ is given by Proposition B6.1, and the last two maps are induced by the identity map of $L^2_{1;\mathrm{loc}}(E)$.

Let $p: H(E) \to \mathfrak{B}$ be the obvious projection, and topologize $H(E)$ by demanding that for each $b \in \mathfrak{B}$, the bundle map from $p^{-1}(\mathfrak{N}(b))$ to $\mathfrak{N}(b) \times H_c(E)$ which sends $(b' \in \mathfrak{N}(b), \psi \in H_{b'}(E))$ to

$$L_b(b', \psi) = (b', \varrho(l_b(b')) \cdot \psi). \tag{B6.1}$$

is a homeomorphism. Proposition B6.3, below, asserts that this defines consistently a topology on $H(E)$ such that for each $n \in \mathbb{Z}$, $p: H(E) \to \mathfrak{B}_n$ is a C^0-vector bundle over $\mathfrak{B}_n$(cf. [20]).

It is also necessary to provide a C^0-vector bundle structure for the set $L(E) = \bigcup_{b \in \mathfrak{B}} L^2(E)$ with its canonical projection $p: L(E) \to \mathfrak{B}$. Topologize $L(E)$ by demanding that for each $b \in \mathfrak{B}$, the map from $p^{-1}(\mathfrak{N}(b))$ to $\mathfrak{N}(b) \times L^2(E)$ which sends (b', ψ) to $L_b(b', \psi)$ of Eq. (B6.1) is a homeomorphism. Proposition B6.3, below, asserts that this has defined a conssitent topology on $L(E)$ such that for each $n \in \mathbb{Z}$, $p: L(E) \to \mathfrak{B}_n$ is a C^0-vector bundle over $\mathfrak{B}_n$.

Proposition B6.3. *For each $n \in \mathbb{Z}$, choose $c \in \mathfrak{B}_n$. The sets $H(E)$ and $L(E)$ with projection p to $\mathfrak{B}_n$ are consistently topologized as C^0-vector bundles over $\mathfrak{B}_n$ by demanding that the open cover $\{\mathfrak{N}(b): b \in \mathfrak{B}_n\}$ with the bundle maps $\{L_b: p^{-1}(\mathfrak{N}(b)) \to \mathfrak{N}(b) \times H_c(E)$ and $\mathfrak{N}(b) \times L^2(e)$, respectively$\}$ given in Eq.* (B6.1) *form a basis for the local trivializations. With this vector bundle structure, the inclusions of $\mathfrak{B}_n \times \Gamma_0(E)$ into $H(E)$ and $L(E)$ are continuous. Furthermore, the assignment of $b \in \mathfrak{B}_n$ to the fibre metric $\langle \cdot, \cdot \rangle_b$ defines a continuous section of* $\mathrm{Sym}_2(H(E)^*)$. *Finally, two different choices of $c \in \mathfrak{B}_n$ define isomorphic $H(E)$'s and $L(E)$'s.*

The remainder of this section contains the proofs of Proposition B6.2. Proposition B6.3 is proved at the end of Sect. B7.

Proof of Proposition B6.2. This proposition is a direct and easy consequence of

Lemma B6.4. *Let $c \in \mathfrak{B}$ and let $\psi \in H_c(\mathrm{Ad}\,P)$ and $\eta \in H_c(E)$. Then*

$$\|\varrho_*(\psi)\eta\|_2 \leqq \zeta(1 + \mathfrak{A}(c)) \|\psi\|_c \|\eta\|_c, \tag{B6.2}$$

where $\zeta < \infty$ is independent of c, ψ, and η.

Proof of Lemma B6.4. This is a generalization of Lemma 6.6 of [3]. Let $c = (A, \Phi)$ and let $\Omega(c) = \{x \in \mathbb{R}^3 : |\Phi|(x) < \frac{3}{4}\}$. Then Ω is open, and because of Kato's inequality and Lemma A4.1, one has

$$\mathfrak{A}(c) \geqq \tfrac{1}{2}\|d(1 - |\Phi|)\|_2^2 \geqq \tfrac{1}{2}\zeta \cdot \|(1 - |\Phi|)\|_6^2 \geqq \tfrac{3}{8}\zeta \cdot (\mathrm{vol}\,\Omega)^{1/3}. \tag{B6.3}$$

Here, $\zeta < \infty$ is independent of $c \in \mathfrak{C}$. Thus, Ω has finite volume. Now let $\beta \in C^\infty[0, \infty)$ be the bump function of Sect. A3 and let $b(x) = \beta(|\Phi|(x))$. Thus, $\mathrm{supp}\, b \subset \Omega$. Given $\eta \in H_c(E)$, decompose it as $\eta = \eta^L + \eta^T + \eta^\Omega$, where $\eta^\Omega = b\eta$, $\eta^T = -(1 - b)|\Phi|^{-2}\varrho_*(\Phi)\varrho_*(\Phi)\eta$, and $\eta^L = (1 - b)\eta - \eta^T$.

Now, due to Lemma A4.1,

$$\begin{aligned} \|\eta^\Omega\|_2 + \|\eta^\Omega\|_6 &\leqq \zeta^{-1}(1 + (\mathrm{vol}\,\Omega)^{1/3}) \|\eta\|_c, \\ \|\eta^L\|_6 &\leqq \zeta^{-1} \|\eta\|_c, \\ \|\eta^T\|_2 + \|\eta^T\|_6 &\leqq (2 + \zeta^{-1}) \|\eta\|_c, \end{aligned} \tag{B6.4}$$

where $\zeta < \infty$ is independent of $c \in \mathfrak{C}$ and $\eta \in H_c(E)$. To obtain the lemma, use the following two facts: First, if $\eta \in H_c(E)$ and $\psi \in H_c(\operatorname{Ad} P)$, then

$$\varrho_*(\psi^L)\eta^L = 0. \tag{B6.5}$$

Second, if $v \in L^6$ and $u \in L^2 \cap L^6$, then

$$\|vu\|_2 \leqq \|v\|_6 \|u\|_6^{1/2} \|u\|_2^{1/2}. \tag{B6.6}$$

Lemma B6.4 follows from Eqs. (B6.3)–(B6.6).

B7. The Local Structure of $\mathfrak{B}$

The purpose of this section is to establish Proposition B6.1, the local embedding theorem for domains in $\mathfrak{B}$ and Proposition B6.3. Before beginning the proof of Proposition B6.1, it is convenient to study the space $\mathfrak{L} = \mathfrak{L}^0$ of Eq. (B4.1) in greater detail. Observe first that there exists a continuous map $\sigma : \mathfrak{L} \to \operatorname{Maps}(S^2; S^2)$, given by the assignment of $b \in \mathfrak{L}$ and $\hat{x} \in S^2$ to $\hat{\Phi}(b)(2R(b)\hat{x})$, with $\hat{\Phi} = \Phi/|\Phi|$ and $R : \mathfrak{L} \to [1, \infty)$ given by Lemma B4.2. Let $\mathfrak{L}_n = \mathfrak{L} \cap \mathfrak{C}_n$, $n \in \mathbb{Z}$. Then σ maps $\mathfrak{L}_n$ onto $\operatorname{Maps}_n(S^2; S^2)$.

Second, observe that if $\sigma_1, \sigma_0 \in \operatorname{Maps}_n(S^2; S^2)$, then there exists an open neighborhood $\mathfrak{O}(\sigma_1)$ and a continuous map $h : \mathfrak{O} \to \operatorname{Maps}(S^2; \mathrm{SU}(2))$ with the property that for every $\sigma_2 \in \mathfrak{O}$,

$$h(\sigma_2)\sigma_2 h^{-1}(\sigma_2) = \sigma_0. \tag{B7.1}$$

This is just the statement that Eq. (B4.3) is a fibration.

These two observations imply

Lemma B7.1. *Let $n \in \mathbb{Z}$ and let $b, c \in \mathfrak{L}_n$. There exists a neighborhood $\mathfrak{Q}(b, c)$ of b in $\mathfrak{L}_n$ and a continuous map $q : \mathfrak{Q} \to \mathfrak{G}_0$ with the property that for every $b' \in \mathfrak{Q}$, $q(b')\sigma(b')q^{-1}(b') = \sigma(c)$.*

Proof of Lemma B7.1. Choose a contractible neighborhood $\mathfrak{O}$ of $\sigma(b)$ in $\operatorname{Maps}_n(S^2; S^2)$ for which Eq. (B7.1) is true with $\sigma_0 = \sigma(c)$. As $\mathfrak{O}$ is contractible, the homotopy lifting property for fibrations implies that the map h of Eq. (B7.1) has an extension, $h' : \mathfrak{O} \to \mathfrak{G}_0$ satisfying $h'(x) = h(x/|x|)$ for $|x| \geqq \frac{1}{2}$. Take $q(b')$ to be $h'(\sigma(b'))$.

Now, fix $\omega \in \operatorname{Maps}_n(S^2; S^2)$ and define $\mathfrak{L}[\omega] = \{b \in \mathfrak{L} : \sigma(b) = \omega\}$. Let $g : \mathfrak{L} \to \mathfrak{G}_0$ be given by Lemma B4.5.

Lemma B7.2. *The assignment of $b \in \mathfrak{L}[\omega]$ to $g(b) \cdot b - I(\omega) \in \Gamma(Q)$ defines a continuous map of $\mathfrak{L}[\omega]$ into $\Gamma^{I(\omega)}(Q)$.*

Proof of Lemma B7.2. Observe that on the set $\{x \in \mathbb{R}^3 : |x| \geqq 2R(b)\}$

$$g \cdot b - I(\omega) = (a(g \cdot b)\omega + a^T(g \cdot b), (|\Phi(b)| - 1)\omega),$$

where $g = g(b)$. Thus, Proposition B3.1 and Lemmas B4.4 and B4.5 imply that $g \cdot b - I(\omega) \in \Gamma^b$. Proposition B6.2 implies now that $g \cdot b - I(\omega) \in \Gamma^{I(\omega)}$. The continuity of this assignment $b \mapsto g(b) \cdot b - I(\omega) \in \Gamma^{I(\omega)}$ follows from Proposition B3.1 and Lemmas B4.4 and B4.5.

Before proving Proposition B6.1, it is necessary to point out that the map $c_1 : \mathfrak{C} \to \mathfrak{C}$ of Sect. B3 maps $\mathfrak{C}$ continuously into $\mathfrak{L}$ (Proposition B3.4) and for all $c \in \mathfrak{C}$,

$$c_1(c) - c \in \Gamma^{c_1(c)}, \Gamma^c . \tag{B7.2}$$

Proof of Proposition B6.1. Given $b, c \in \mathfrak{B}_n$, let $\mathfrak{Q}(c_1(b), c_1(c))$ be the neighborhood of $c_1(b)$ that is provided by Lemma B7.1. Let $\mathfrak{N}(b) = c_1^{-1}(\mathfrak{Q}) \subset \mathfrak{B}_n$. Define $h(b)(\cdot) : \mathfrak{N} \to \mathfrak{G}_0$ by assigning $b' \in \mathfrak{N}$ to

$$h(b)(b') = g^{-1}(c_1(c)) \cdot g(q(c_1(b')) \cdot c_1(b')) \cdot q(c_1(b')),$$

where $q : Q \to \mathfrak{G}_0$ is given by Lemma B7.1 and $g : \mathfrak{L} \to \mathfrak{G}_0$ is given by Lemma B4.5. The first claim is that $m_b(b') = h(b)(b')b' - c$ defines a continuous map from $\mathfrak{N}(b)$ into Γ^c. Indeed,

$$\begin{aligned} g(c_1(c))(h(b)(b') \cdot b' - c) &= g(q(c_1(b')) \cdot c_1(b'))q(c_1(b')) \cdot b' - g(c_1(c)) \cdot c \\ &= [g' \cdot q \cdot (b' - c_1(b'))] + [g' \cdot q \cdot c_1(b') - I(\sigma)] \\ &\quad - [g_c \cdot (c - c_1(c))] - [g_c \cdot c_1(c) - I(\sigma)] . \end{aligned} \tag{B7.3}$$

Here, $g' = g(q(c_1(b')) \cdot c_1(b'))$, $g_c = g(c_1(c))$, $q = q(c_1(b'))$, and $\sigma = \sigma(c_1(c)) = \sigma(q \cdot c_1(b'))$. Due to Lemma B7.2 and Eq. (B7.2), each bracket in Eq. (B7.3) is in $\Gamma^{I(\sigma)}$. Thus, Eq. (B7.2) implies that $h(b)(b') - c$ is in Γ^c. The continuity of the map follows readily also with Lemma B7.2 and Proposition B3.4.

The second claim is that the map $m_b(\cdot) : \mathfrak{N} \to \Gamma^c$ is a homeomorphism onto its image. For this, one requires

Lemma B7.3. *Let $c \in \mathfrak{C}$ and define Γ^c as in Sect. B6. The assignment of $\psi \in \Gamma^c$ to $c + \psi$ defines a continuous map from Γ^c into $\mathfrak{C}$.*

Proof of Lemma B7.3. This is a straightforward calculation using Lemma B6.4 and Hölder's inequality. The reader is referred to Proposition 5.1 of [3].

To complete the proof of Proposition B6.1, it is enough now to observe that the map $m_b(\cdot)$ has a continuous inverse, namely, the map $\hat{1}$ which sends $\psi \in \Gamma^c$ to $\Pi(c + \psi) \in \mathfrak{B}$, where $\Pi : \mathfrak{C} \to \mathfrak{B}$ is the projection. Indeed, let $b', b'' \in \mathfrak{N}(b)$. Suppose that $(\hat{1} \circ m_b)(b') = (\hat{1} \circ m_b)(b'')$. Then, $b' = h \cdot h_b(b')^{-1} \cdot h_b(b'') \cdot b''$ for some $h \in \mathfrak{G}_0$. But since $h_b(\cdot)$ maps into $\mathfrak{G}_0, b'$ must equal b''.

Proof of Proposition B6.3. As the arguments for $H(E)$ and $L(E)$ are similar, only the former case will be considered. To show that $H(E)$ has a well defined topology, it is sufficient to establish that for any pair b_1 and $b_2 \in \mathfrak{B}_n$ with intersecting $\mathfrak{N}(b_1)$ and $\mathfrak{N}(b_2)$, the transition function which sends $b \in \mathfrak{D} \equiv \mathfrak{N}(b_1) \cap \mathfrak{N}(b_2)$ to [cf. Eq. (B6.1)],

$$l(b) = l_{b_1}(b) l_{b_2}(b)^{-1} \in \mathfrak{G}_0 \tag{B7.4}$$

defines via $b \to \varrho(l(b))$ a continuous map from $\mathfrak{D}$ into the Banach space of bounded, linear endomorphisms of $H_c(E)$ (cf. [20, Chap. I]). This will imply immediately [20, Chap. III] that $p : H(E) \to \mathfrak{B}$ is a C^0-vector bundle with the asserted basis for its local trivializations. If one accepts for the moment this last assertion as a fact, then the remaining assertions follow in a straightforward manner from

Propositions B6.1 and B6.2; the details are left to the reader. Thus, Proposition B6.3 follows from the following:

Lemma B7.4. *Let* $n\in\mathbb{Z}$ *and let* $c\in\mathfrak{B}_n$. *If* $\mathfrak{D}\equiv\mathfrak{N}(b_1)\cap\mathfrak{N}(b_2)\neq\emptyset$ *for* $b_1,b_2\in\mathfrak{B}_n$, *define* $l:\mathfrak{D}\to\mathfrak{G}_0$ *by Eq.* (B7.4). *Then* $\varrho(l(b))$ *is a bounded, linear automorphism of* $H_c(E)$ *for each* $b\in\mathfrak{D}$; *and given* $b\in\mathfrak{D}$ *and* $\varepsilon>0$, *there exists an open neighborhood* $\mathfrak{D}'\subset\mathfrak{D}$ *of* b *such that for all* $b'\in\mathfrak{D}'$ *and* $\psi\in H_c(E)$,

$$\|\varrho(l(b))\psi-\varrho(l(b'))\psi\|_c\leqq\varepsilon\|\psi\|_c\,. \tag{B7.5}$$

Proof of Lemma B7.4. First, let $m:\mathfrak{N}(b_1)\to H_c(Q)$ and let $j:\mathfrak{N}(b_2)\to H_c(Q)$ be the two embeddings from Proposition B6.1. If $\psi\in\Gamma_0(E)$ and $b\in\mathfrak{D}$, then

$$\begin{aligned}\|\varrho(l(b))\psi\|_c&\leqq z_1(b)\,\|\varrho(l(b))\psi\|_{c+m(b)}\,,\\&\leqq z_1(b)\,\|\psi\|_{c+j(b)}\,,\\&\leqq z\|\psi\|_c\,.\end{aligned} \tag{B7.6}$$

Here Proposition B6.2 and the $\mathfrak{G}$-equivariance of the $\|\cdot\|_c$ norm have been used. The constant $z<\infty$ is independent of ψ, and uniform on a neighborhood of b. Thus, for each $b\in\mathfrak{D}$, $\varrho(l(b))$ extends to a bounded linear automorphism of $H_c(E)$ with inverse $\varrho(l(b)^{-1})$.

To establish Eq. (B7.5), the following lemma is needed:

Lemma B7.5. *Under the same assumptions as in Lemma B7.4, let* $b\in\mathfrak{D}$. *There is a neighborhood* $\mathfrak{D}'$ *of* b *in* $\mathfrak{D}$ *such that the assignment of* $b'\in\mathfrak{D}'$ *to the number* $d(b,b')\equiv\|l(b)-l(b')\|_{\bar{C}^0(\mathbb{R}^3)}$ *defines a continuous map of* $\mathfrak{D}'$ *into* $[0,2]$.

Proof of Lemma B7.4 assuming Lemma B7.5. For $c=(A,\Phi)\in\mathfrak{C}$ and $\psi\in\Gamma(E)$, let

$$\nabla_c'\psi=(\nabla_A\psi,[\Phi,\psi])\in\Gamma(E\otimes Q)\,.$$

Because of Proposition B6.2, it is sufficient to check that given $\varepsilon>0$, there is a neighborhood $\mathfrak{D}'\subseteqq\mathfrak{D}$ such that for all $b'\in\mathfrak{D}'$ and $\psi\in H_c(E)$,

$$\|\varrho(l(b))\psi-\varrho(l(b'))\psi\|_{c+m(b)}\leqq\varepsilon\|\psi\|_{c+m(b)}\,.$$

For notational convenience, let $(l,l')=(l(b),l(b'))$ and define similarly (m,m') and (j,j'). Observe that for any $\psi\in H_c(E)$, $\|\psi\|_c=\|\nabla_c'\psi\|_2$. Now one calculates using the $\mathfrak{G}$-invariance that

$$\nabla_{c+m}'\varrho(l)\psi=\varrho(l)\nabla_{c+j}'\psi\,, \tag{B7.7}$$

while

$$\begin{aligned}\nabla_{c+m}'\varrho(l')\psi&=\nabla_{c+m'}'\varrho(l')\psi+\varrho(m-m')\varrho(l')\psi\\&=\varrho(l')\nabla_{c+j'}'\psi+\varrho(m-m')\varrho(l')\psi\\&=\varrho(l')\nabla_{c+j}'\psi+\varrho(l')\varrho(j'-j)\psi+\varrho(m-m')\varrho(l')\psi\,.\end{aligned} \tag{B7.8}$$

Equations (B7.6–8), Proposition B6.2 and Lemma B6.4 imply that

$$\|\varrho(l(b))\psi-\varrho(l(b'))\psi\|_c\leqq z\{d(b,b')+\|j-j'\|_c+\|m-m'\|_c\}\,\|\psi\|_c\,,$$

where z is independent of b' in a sufficiently small neighborhood of b. This last equation with Proposition B6.1 and Lemma B7.5 implies Lemma B7.4.

Proof of Lemma B7.5. By construction, l maps $\mathfrak{D}$ continuously into $\mathfrak{G}_0$. Let $c_1 : \mathfrak{C} \to \mathfrak{C}$ be as in Sect. B3 and write $c_1(c) = (A, \Phi)$. Due to Lemma B4.2, there exists $R < \infty$ such that for all b' in a neighborhood $\mathfrak{D}'$ of b in $\mathfrak{D}$, $|\Phi(c_1(b'))|(x) > \frac{1}{2}$ if $|x| > R$, and also $|\Phi|(x) > \frac{1}{2}$ if $|x| > R$. For $b' \in \mathfrak{D}'$ and for $|x| > R$, set

$$\hat{\Phi}(b') = \Phi(c_1(b'))/|\Phi(c_1(b'))| .$$

By construction, there exists $r \in [R, \infty)$ such that for all b' in a possibly smaller neighborhood $\mathfrak{D}'$ of b,

$$(h(b_1)(b')\hat{\Phi}(b')h(b_1)(b')^{-1})(x) = \hat{\Phi}(x) \quad \text{for} \quad |x| > r ; \tag{B7.9}$$

and also Eq. (B7.9) is satisfied by all $b' \in \mathfrak{D}'$ with $h(b_2)(\cdot)$ replacing $h(b_1)(\cdot)$. Thus, for all $b' \in \mathfrak{D}'$ $(l(b')\hat{\Phi}l(b')^{-1})(x) = \hat{\Phi}(x)$ for $|x| > r$. This last equation implies that if $b' \in \mathfrak{D}'$ and if $|x| > r$, then

$$l(b')(x) = \exp[\tfrac{1}{2}\lambda(b')\hat{\Phi}(x)] . \tag{B7.10}$$

The proof of Lemma B4.5 [see Eq. (B4.6)] implies that λ maps a possibly smaller neighborhood $\mathfrak{D}'$ of b in $\mathfrak{D}$ continuously into $C^\infty(\{x \in \mathbb{R}^3 : |x| > r\})$. Further, there exists $r' \in [r, \infty)$ such that for all $b' \in \mathfrak{D}'$

$$\Delta\lambda(b')(x) = 0 \quad \text{for} \quad |x| > r' \tag{B7.11}$$

and

$$|\nabla\lambda(b')|(\cdot) \in L^6(\{x \in \mathbb{R}^3 : |x| > r'\}) . \tag{B7.12}$$

Together, Eqs. (B7.11) and (B7.12) imply that for each $b' \in \mathfrak{D}'$, there exists $\mu(b') \in \mathbb{R}$ such that

$$|\lambda(b')(x) - \mu(b')| < M(b)|x|^{-1} \quad \text{if} \quad |x| > r' , \tag{B7.13}$$

where $M(b) < \infty$ depends only on b. Equation (B7.13) implies that the assignment of $b' \in \mathfrak{D}'$ to $\mu(b') \in \mathbb{R}$ defines a continuous map. This fact, Eqs. (B7.10) and (B7.13), and the continuity of $l : \mathfrak{D} \to \mathfrak{G}_0$ imply Lemma B7.5.

C1. The Distribution of Critical Values

As the functional $\mathfrak{A}$ only sees the topology of $\hat{\mathfrak{B}}$, min-max for $\mathfrak{A}$ is concerned with a homotopy invariant family of compact subsets of $\hat{\mathfrak{B}}$. Let $n \in \mathbb{Z}$ and let $\mathfrak{F}$ be such a family in $\hat{\mathfrak{B}}_n$. As discussed in Sects. A1 and A2, one associates to $\mathfrak{F}$ the number $\mathfrak{A}_{\mathfrak{F}}$ of Eq. (A1.7). Then the set of such $\mathfrak{A}_{\mathfrak{F}}$'s defines $\text{Crit}_n \subset [4\pi|n|, \infty)$ [see Eq. (A1.8)], and Theorem A1.2 asserts that Crit_n is unbounded.

Theorem A1.2 is proved through a series of arguments, outlined in Sect. C2, which involve the topology of a family of Dirac operators on $S = (P \times_{SU(2)} \mathbb{C}^2) \otimes \mathbb{S}$. Here, $\mathbb{S} \to \mathbb{R}^3$ is the spin bundle over $\mathbb{R}^3$; that is, the frame bundle of $\mathbb{R}^3$ is canonically $\mathbb{R}^3 \times SO(3)$. It is double covered by $\mathbb{R}^3 \times SU(2)$ and $\mathbb{S}$ is

$$\mathbb{R}^3 \times SU(2) \times_{SU(2)} \mathbb{C}^2 \simeq \mathbb{R}^3 \times \mathbb{C}^2 .$$

The Dirac operator on $\Gamma_0(\mathbb{S})$ is the first order elliptic operator which sends $\psi \in \Gamma_0(\mathbb{S})$ to

$$\partial\psi \equiv \sum_{i=1}^{3} dx^i \cdot \frac{\partial}{\partial x^i}\psi, \tag{C1.1}$$

where dx^i is Clifford multiplication. (Since $\mathbb{S}$ is flat, there is no loss of generality in identifying $\{dx^i\}_{i=1}^3$ on $\mathbb{C}^2$ with the fixed set of matrices,

$$\left\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right\}.$$

To each pair $c=(A,\Phi)\in\mathfrak{C}$, assign the operator, δ_c, on $\Gamma_0(S)$ which sends $\psi\in\Gamma_0(S)$ to

$$\sum_{i=1}^{3} dx^i \cdot \nabla_{A_i}\psi + \Phi\psi. \tag{C1.2}$$

Observe that the assignment of $(c,\psi)\in\mathfrak{C}\times\Gamma_0(S)$ to $\delta_c\psi\in\Gamma_0(S)$ is $\mathfrak{G}$ equivariant, and thus one obtains a family of operators, $\{\delta_c : c\in\mathfrak{B}\}$, on $\Gamma_0(S)$.

The Atiyah-Singer Index theorem for families (cf. [21, 12 and 8, 10]) assigns topological significance to continuous families of elliptic operators. The proof of Theorem A1.2 is an application of these ideas. Specifically, the approach in [12] is adapted here. In order to effect this adaptation, it is necessary to introduce certain technical details. The next section contains these technicalities.

C2. Operators Indexed by $\mathfrak{B}$

Let V be a finite dimensional vector space and let ϱ be a representation of $SU(2)$ on V. Let V' be a second finite dimensional vector space with a faithful representation, $\{\tau^i\}_{i=1}^3 \subset \operatorname{End} V'$, of the imaginary quaterions, (thus $\tau^i\tau^j = -\delta^{ij} - \varepsilon^{ijk}\tau^k$). Let $E = V\otimes V'$. For each $c=(A,\Phi)\in\mathfrak{C}$, denote by δ_c, the operator on $\Gamma_0(E)$ which sends ψ to

$$\delta_c\psi = \tau^i \nabla_{A_i}\psi + \varrho_*(\Phi)\psi. \tag{C2.1}$$

As in Sect. B6, let $H_c(E)$ denote the (Hilbert space) completion of $\Gamma_0(E)$ in the norm

$$\|\psi\|_c^2 = \langle \nabla_A\psi, \nabla_A\psi\rangle_2 + \langle \varrho_*(\Phi)\psi, \varrho_*(\Phi)\psi\rangle_2. \tag{C2.2}$$

A summary of the results in Sects. 6 and 7 of [9] concerning such δ_c on H_c is provided by the first lemma.

Lemma C2.1. *Let $c\in\mathfrak{C}$ and let E and $\delta_c\in\operatorname{End}(\Gamma_0(E))$ be as defined above. Then δ_c extends as a bounded Fredholm operator from $H_c(E)$ to $L^2(E)$.*

Proof of Lemma C2.1. A specific example is treated in detail as Lemma 7.4 of [9]. The general δ_c is analyzed similarly. Use Lemma B6.4.

As defined in Sect. B6, $H(E), L(E)\to\mathfrak{B}$ denote the C^0 vector bundles with fibre $H_c(E), L^2(E)$ respectively at $c\in\mathfrak{B}$. Because the assignment of $c\in\mathfrak{C}$ to δ_c as a linear operator from $H_c(E)$ to $L^2(E)$ is $\mathfrak{G}$ equivariant, this assignment defines a section over $\mathfrak{B}$ of $\operatorname{Hom}(H(E), L(E))$.

Lemma C2.2. *The assignment of* $c \in \mathfrak{B}$ *to* $\delta_c \in \mathrm{Hom}(H_c(E); L^2(E))$ *defines a continuous section of* $\mathrm{Hom}(H(E), L(E))$; *in fact, a Fredholm morphism.*

Proof of Lemma C2.2. With the local trivializations of $H(E)$ and $L(E)$ provided in Sect. B6, it is enough to check that for fixed $c \in \mathfrak{C}$, the assignment of $\psi \in H_c(Q) \cap \Gamma(Q)$ to $\delta_{c+\psi} \in \mathrm{Hom}(H_c(E); L^2(E))$ is continuous. In fact, $\delta_{c+\psi} - \delta_c$ is a compact operator, linear in ψ. This is established in Lemma 7.4 of [9] for a specific representation of SU(2) and the quaternions and the general case is handled similarly.

Being a Fredholm operator, δ_c for $c \in \mathfrak{B}$ has a finite dimensional kernel. For Theorem A1.2, a crucial observation is that $\dim(\ker \delta_c)$ is bounded a priori knowing only $\mathfrak{A}(c)$.

Proposition C2.3. *Given specific representations of* SU(2) *and the quaternions on vector spaces* V *and* V', *respectively, define* $E(V, V') \to \mathbb{R}^3$ *as above. There is a continuous function,* $z: [0, \infty) \to [1, \infty)$ *with the following significance. Let* $n \in \mathbb{Z}$. *Let* $c = (A, \Phi) \in \mathfrak{C}_n$. *Define* $\delta_c: H_c(E) \to L^2(E)$ *as above. Then*

$$\dim(\ker \delta_c) \leqq z(\|*F_A - \nabla_A \Phi\|_2) \cdot \|*F_A - \nabla_A \Phi\|_2 .$$

This proposition follows as a corollary to a more general result concerning bilinear forms on $H_c(E)$. The general result is Proposition C2.5, below.

The bilinear forms under consideration are of the following kind. Let $E \to \mathbb{R}^3$ be as above, let $c \in \mathfrak{C}$ and let J be a bounded, symmetric, bilinear form on $H_c(E)$. For Proposition C2.5, below, J will be restricted to be a form which sends $\psi, \eta \in H_c(E)$ to

$$J(\psi, \eta) = \langle \psi, \eta \rangle_c + \langle \psi, [\varrho_*(\mathfrak{R}_0) + \tau^i \varrho_*(\mathfrak{R}_i)]\eta \rangle_2 , \tag{C2.3}$$

where $\{\mathfrak{R}_0, \mathfrak{R}_i\}_{i=1}^3 \subset \Gamma(\mathrm{Ad} P)$ are such that $\mathfrak{R} = \mathfrak{R}_0 + \tau^i \mathfrak{R}_i$ is in $L^2(\mathrm{Ad} P \otimes \mathrm{End}\, V')$.

A form of this type satisfies Property $*$ with respect to the metric $\langle \cdot, \cdot \rangle_c$ on $H_c(E)$; this as defined in Sect. 6 of [9]. This fact is proved for three examples in Lemma 6.7 of [9], and the more general case is handled in a similar way; see Lemma C2.10, here. As a result of satisfying Property $*$, such a form J is bounded from below.

As in [9], an eigenvector of J with eigenvalue $\lambda \in \mathbb{R}$ is by definition a nonzero $\psi \in H_c(E)$ such that for all $\eta \in H_c(E)$,

$$J(\psi, \eta) = \lambda \langle \psi, \eta \rangle_c . \tag{C2.4}$$

Concerning bilinear forms on $H_c(E)$, the results in Sect. 6 of [9] imply

Lemma C2.4. *Let* J *be a symmetric bilinear form on* $H_c(E)$ *which satisfies* Property $*$ *with respect to* $\langle \cdot, \cdot \rangle_c$. *Let* $\lambda < 1$. *There exists* $N(\lambda, J) < \infty$ *eigenvectors,* $\{\psi_i\}_{i=1}^N \in H_c(E)$ *of* J *such that* J *is bounded from below by* λ *on the orthogonal complement in* $H_c(E)$ *of* $\{\psi_i\}_{i=1}^N$.

Proof of Lemma C2.4. Use Lemma 6.6 of [9] with the bilinear form $J(\cdot, \cdot) - \lambda \langle \cdot, \cdot \rangle_c$ on $H_c(E)$.

For those forms J as described in Eq. (C2.3), it is possible to obtain an upper bound on the number $N(\lambda, J)$ of the previous lemma.

Proposition C2.5. *Let $E\to\mathbb{R}^3$ be a vector bundle as described in the opening paragraph of this section. Let $c\in\mathfrak{C}$ and let $J\in\mathrm{Sym}_2 H_c(E)^*$ be a form as described in Eq.* (C2.3). *Let*

$$\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_i\tau^i\in L^2(\mathrm{Ad}\,P\otimes\mathrm{End}\,V')$$

be as defined in Eq. (C2.3). *For $\lambda<1$, let $N(\lambda, J)$ be the number of eigenvectors of J with eigenvalues smaller than λ. Then $N(\lambda, J)$ is bounded above by $z(\lambda, \mathfrak{A}(c), \|\mathfrak{R}\|_2)\cdot\|\mathfrak{R}\|_2$, where $z(\cdot)\in[1,\infty)$ depends on c, J, and λ only as indicated; and $z(\cdot)$ is continuous on $(-\infty, 1)\underset{2}{\times}[0,\infty)$.*

Proof of Proposition C2.3 given Proposition C2.5. The result follows from the Weitzenbock formula for δ_c which shows that the bilinear form $\langle\delta_c\cdot, \delta_c\cdot\rangle_2$ on $H_c(E)$ is of the kind under consideration in Proposition C2.5. In this Weitzenbock formula, the endomorphism $\mathfrak{R}$ of Eqs. (C2.2) and (C2.3) sends $\psi\in\Gamma(E)$ to

$$\sum_{i=1}^{3}\tau^i(\varrho_*(\nabla_A\Phi-*F_A)_i)\psi\,. \tag{C2.5}$$

Thus, $\|\mathfrak{R}\|_2\leqq\zeta\cdot\|*F_A-\nabla_A\Phi\|_2$, where $\zeta<\infty$ is independent of c. Furthermore, Groisser showed [16] (see also [4]) that when $c\in\mathfrak{C}_n$, then $\mathfrak{A}(c)=\frac{1}{2}\|F_A-*\nabla_A\Phi\|_2^2+4\pi n$, and so one obtains from Proposition C2.5 the required bound for $\dim(\ker\delta_c)$.

Proof of Proposition C2.5. The proof is motivated by the proof of Theorem 3.2 in [22]. The strategy in [22] is to consider $v\in H_c(E)$ which is a linear combination of orthonormal eigenvectors, $\{\psi_i\}$, of J, each with eigenvalue less than λ (<1). Assume that $\|v\|_c=1$, so

$$v=\sum_i\alpha_i\psi_i \quad\text{and}\quad \sum_i\alpha_i^2=1\,. \tag{C2.6}$$

Observe that

$$J(v,v)=\sum_i\lambda_i\alpha_i^2<\lambda\,.$$

But, in addition, Eq. (C2.3) implies that

$$1-\lambda\leqq|\langle v, \varrho_*(\mathfrak{R})v\rangle_2|\,. \tag{C2.7}$$

The goal is to obtain a contradiction from Eq. (C2.7) by assuming that $N(\lambda, J)$ is too large.

To begin, let $\psi\in H_c(E)$ be an eigenvector of J with eigenvalue $\lambda<1$. Then Eq. (C2.4) is an elliptic differential condition on ψ. As a consequence of some standard regularity theorems (cf. [17, Chap. 6]), $\psi\in\Gamma(E)$ and ψ satisfies the equation

$$\nabla_A^2\psi+\varrho_*(\Phi)^2\psi+\mathfrak{R}'\cdot\psi=0\,, \tag{C2.8}$$

where $\mathfrak{R}'=(1-\lambda)^{-1}\varrho_*(\mathfrak{R})$.

First observe that $\|\psi\|_\infty$ has an a priori bound.

Lemma C2.6. *Let $c\in\mathfrak{C}$ and let $\psi\in H_c(E)$ satisfy Eq.* (C2.8) *with $\|\psi\|_c=1$. Assume that $\mathfrak{R}'\in L^2(\mathrm{End}\,E)\cap\Gamma(\mathrm{End}\,E)$. Then $\|\psi\|_\infty\leqq z(1+\|\mathfrak{R}'\|_2)$, where $z<\infty$ is independent of c, $\mathfrak{R}'$, and ψ.*

Proof of Lemma C2.6. If ψ satisfies Eq. (C2.8), then almost everywhere on $\mathbb{R}^3$,

$$-\Delta|\psi| \leqq |\mathfrak{R}'||\psi|. \tag{C2.9}$$

The Green's function for $-\Delta$ is the kernel $(4\pi|x-y|)^{-1} \in (\Gamma_0(\mathbb{R}^3)\times\Gamma_0(\mathbb{R}^3))^*$. Let $\beta^x(y)=\beta(|x-y|)$, where $\beta(t)\in C^\infty([0,\infty))$ is the usual bump function (cf. Sect. A3). Multiply both sides of Eq. (C2.9) by $\beta^x(y)\,(4\pi|x-y|)^{-1}$ and integrate over $\mathbb{R}^3$. Two integration by parts yield the inequality

$$|\psi|(x) \leqq \frac{1}{4\pi}\int \frac{\beta^x}{|x-(\cdot)|}|\mathfrak{R}'||\psi| + z \int_{|x-(\cdot)|<1} |\psi|.$$

Here $z<\infty$ is a fixed constant. Now with Hölder's inequality and Lemma A4.1, one obtains

$$|\psi|(x) \leqq z\left(\|\mathfrak{R}'\|_2 \left\|\frac{\psi}{x-(\cdot)}\right\|_2 + \|\psi\|_6\right) \leqq z(1+\|\mathfrak{R}'\|),$$

which is what the lemma claimed was true.

The second step in the proof of Proposition C2.5 is the procurement of a uniform, a priori bound on the $\bar{C}^{0,1/2}$ norm of ψ, $\|\psi\|_{0,1/2}$.

Lemma C2.7. *Under the same assumptions as in Lemma C2.6,*

$$\|\psi\|_{0,1/2} \leqq z(1+\|\mathfrak{R}'\|_2)(1+\mathfrak{A}(c)+\|\mathfrak{R}'\|_2),$$

and

$$\|\nabla_A\psi\|_c \leqq z(1+\|\mathfrak{R}'\|_2)(1+\mathfrak{A}(c)+\|\mathfrak{R}'\|_2), \tag{C2.10}$$

where $z<\infty$ is a constant which is independent of c, ψ, and $\mathfrak{R}'$.

Proof of Lemma C2.7. Equation (C2.8) and the previous lemma imply that

$$\|\nabla_A^2\psi\|_2 \leqq \|\varrho_*(\Phi)^2\psi\|_2 + z(1+\|\mathfrak{R}'\|_2)\|\mathfrak{R}'\|_2.$$

By integrating by parts, one finds as in [5, Chap. 5] that

$$\|\nabla_A(\nabla_A\psi)\|_2 \leqq z[\|\nabla_A^2\psi\|_2 + (\|\psi\|_\infty + \|\nabla_A\psi\|_2)\|F_A\|_2],$$

where $z<\infty$ is independent of c and ψ. Thus using Lemma C2.6 again, one concludes that

$$\|\nabla_A(\nabla_A\psi)\|_2 \leqq z[\|\varrho_*(\Phi)^2\psi\|_2 + (1+\|\mathfrak{R}'\|_2)(\mathfrak{A}(c)^{1/2}+\|\mathfrak{R}'\|_2)]. \tag{C2.11}$$

Now, as in the proof of Lemma B6.4, let $\Omega=\{x\in\mathbb{R}^3 : |\Phi|(x)<\frac{3}{4}\}$. As in the proof of Lemma B6.4, let $b(x)=\beta(|\Phi|(x))$. On $\mathbb{R}^3\backslash\Omega$, set $\hat{\Phi}=\Phi/|\Phi|$, and observe that when $x\in\mathfrak{R}^3\backslash\Omega$,

$$\varrho_*(\Phi)^2\psi = \varrho_*(\hat{\Phi})\varrho_*(\Phi)\psi + (|\Phi|-1)\varrho_*(\hat{\Phi})^2\psi + (|\Phi|-1)^2\varrho_*(\hat{\Phi})^2\psi. \tag{C2.12}$$

Now, $|\varrho_*(\hat{\Phi})\psi| \leqq |\psi|$ and also since $x\notin\Omega$, $|\varrho_*(\hat{\Phi})\psi| \leqq 2|\varrho_*(\Phi)\psi|$. With these two facts, Eq. (C2.12), Hölder's inequality, Eq. (B6.3), and Lemma A4.1, one obtains the a priori bound

$$\begin{aligned}\|\varrho_*(\Phi)^2\psi\|_2 &\leqq \|b\psi\|_2 + \|\varrho_*(\Phi)\psi\|_2 + \|(1-b)(1-|\Phi|)\varrho_*(\hat{\Phi})\psi\|_2 \\ &\quad + \|(1-b)(1-|\Phi|)^2\psi\|_2 \\ &\leqq z[(\mathrm{Vol}\,\Omega)^{1/3} + 1 + \|1-|\Phi|\|_6^2] \\ &\leqq z[\mathfrak{A}(c)+1]\,.\end{aligned}$$

The last equation and Eq. (C2.12) yield the a priori bound

$$\|\nabla_A(\nabla_A\psi)\|_2 \leqq z(1+\|\mathfrak{R}'\|_2)(1+\mathfrak{A}(c)+\|\mathfrak{R}'\|_2)\,. \tag{C2.13}$$

The first assertion of Lemma C2.7 now follows from Lemma A4.12. The second assertion of Lemma C2.7 follows from the additional observation that

$$\begin{aligned}\|\varrho_*(\Phi)\nabla_A\psi\|_2 &\leqq \|b\nabla_A\psi\|_2 + \|(1-b)\varrho_*(\hat{\Phi})\nabla_A\psi\|_2 + \|(1-b)(1-|\Phi|)\varrho_*(\hat{\Phi})\nabla_A\psi\|_2 \\ &\leqq z[\|\nabla_A\psi\|_2 + \|\nabla_A\psi\|_2^{1/2}\|\nabla\ \psi\|_6^{1/2}\|1-|\Phi|\|_6] \\ &\leqq z(1+\|\mathfrak{R}'\|_2)(1+\mathfrak{A}(c)+\|\mathfrak{R}'\|_2)\,.\end{aligned}$$

The last inequality above uses Eq. (C2.13) and Lemma A4.1.

The application of Lemma C2.7 to the proof of Proposition C2.5 requires the introduction of the set $U \subseteqq \mathbb{R}^3$, where the L^2-norm of $\mathfrak{R}$ is large.

Definition C2.8. *Let $\beta^x(\cdot) \in C^\infty(\mathbb{R}^3)$ denote $\beta(|x-(\cdot)|)$, where β is the bump function of Sect. A3. Let $c=(A,\Phi)\in\mathfrak{C}$, and define for each $\kappa>0$,*

$$\hat{U}(\kappa) = \left\{x\in\mathbb{R}^3 : \int_{\mathbb{R}^3} \beta^x|\mathfrak{R}|^2 \geqq \kappa^2\right\},$$

and

$$U(\kappa) = \bigcup_{x\in\hat{U}(\kappa)} \{y\in\mathbb{R}^3 : |x-y|<1\}\,. \tag{C2.14}$$

The following lemma summarizes the first relevant properties of $U(\kappa)$:

Lemma C2.9. *Let $\mathfrak{R}\in L^2$ and define for $\kappa>0$ the set $U(\kappa)$ as in Definition C2.8. Then*

1) *$U(\kappa)$ is a bounded domain.*
2) *The number of path components of $U(\kappa)$ is less than $\kappa^{-2}\|\mathfrak{R}\|_2^2$.*
3) *If $x, y\in\mathbb{R}^3$ are in the same path component of $U(\kappa)$, then $|x-y|<4\kappa^{-2}\|\mathfrak{R}\|_2^2$.*
4) *$\mathrm{Vol}\,U(\kappa)\leqq 4^4\pi/3\kappa^{-6}\|\mathfrak{R}\|_2^6$.*

Proof of Lemma C2.9. Statement (1) is immediate since $\mathfrak{R}\in L^2$. Statement (2) follows since for each path component $q\subseteqq U(\kappa)$,

$$\kappa^2 \leqq \int_q |\mathfrak{R}|^2 \leqq \|\mathfrak{R}\|_2^2\,.$$

Statement (3) is true because if x, y are in a path component q of $U(\kappa)$, then there are at least $\frac{1}{4}|x-y|$ disjoint balls of radius 1 inside q, and the integral of $|\mathfrak{R}|^2$ over each one is larger than κ^2. On the other hand, the integral of $|\mathfrak{R}|^2$ over q is less than $\|\mathfrak{R}\|$. Statement (4) follows from Statement (3).

To complete the proof of Proposition C2.5, let $v \in H_c(E)$ be as specified in Eq. (C2.6). Given $\kappa > 0$, Eq. (C2.7) implies that

$$1 - \lambda \leqq \int_{\mathbb{R}^3 \setminus U(\kappa)} |(v, \varrho(\mathfrak{R})v)| + \zeta \cdot \int_{U(\kappa)} |\mathfrak{R}||v|^2, \tag{C2.15}$$

with $\zeta < \infty$ depending only on the SU(2) representation ϱ.

Lemma C2.10. *There exists a continuous function* $v : [0, \infty) \to (0, \infty)$ *such that for any* $c \in \mathfrak{C}$, $\kappa \in (0, \infty)$ *and for all* $\psi \in H_c(E)$,

$$\int_{\mathbb{R}^3 \setminus U(\kappa)} |(\psi, \varrho(\mathfrak{R})\psi)| \leqq v(\mathfrak{A}(c))\, \kappa^{1/3} \|\mathfrak{R}\|_2^{2/3} \|\psi\|_c^2 .$$

The proof of this lemma is postponed until the end of this section.

Proof of Proposition C2.5, assuming Lemma C2.10. Suppose that for a given $\kappa > 0$, there exists $\varepsilon > 0$ such that $|v|(x) < \varepsilon$ when $x \in U(\kappa)$. Then Lemma C2.10 and Eq. (C2.15) imply that

$$1 - \lambda \leqq \zeta[(1 + \mathfrak{A}(c))\kappa^{1/3} \|\mathfrak{R}\|_2^{2/3} + \varepsilon^2 \|\mathfrak{R}\|_2 (\operatorname{Vol} U(\kappa))^{1/2}] .$$

With Lemma C2.9, one could conclude that

$$1 - \lambda \leqq \zeta[(1 + \mathfrak{A}(c))\kappa^{1/3} \|\mathfrak{R}\|_2^{2/3} + \|\mathfrak{R}\|_2^4 \varepsilon^2 \kappa^{-3}] . \tag{C2.16}$$

Let $N = N(J, \lambda)$ and let $d = \dim V \cdot \dim V'$. Given N/d points in $\mathbb{R}^3$, there is a linear combination of eigenvectors, $v = \sum_i \alpha_i \psi_i$, as in Eq. (C2.6), such that v vanishes at the given points. Due to Eq. (C2.10),

$$\|v\|_{0, 1/2} \leqq \zeta \cdot (1 - \lambda)^{-2} (1 + \|\mathfrak{R}\|_2)(1 + \mathfrak{A}(c) + \|\mathfrak{R}\|_2) . \tag{C2.17}$$

Thus, if v vanishes at x, then $|v(y)| < \varepsilon$ for $y \in \mathbb{R}^3$ with

$$|x - y| < \|v\|_{0,1/2}^{-2} \varepsilon^2 .$$

Therefore, one can find $N' = [\operatorname{Vol} \cdot U(\kappa)] \cdot 3/4\pi \cdot d \cdot \|v\|_{0,1/2}^6 \cdot \varepsilon^{-6}$ points in $U(\kappa)$ such that *if* v vanished at each, *then* $|v(\cdot)|$ would be less than ε on $U(\kappa)$.

Now choose $\kappa^{1/3} \leqq \frac{1}{4}(1 - \lambda)\zeta^{-1}(1 + \mathfrak{A}(c))^{-1}(1 + \|\mathfrak{R}\|_2^{2/3})^{-1}$. And choose $\varepsilon^2 = \frac{1}{4}\zeta^{-1}\kappa^3 \|\mathfrak{R}\|_2^{-4}(1 - \lambda)$. Then, because Eq. (C2.16) is true, one concludes that

$$N < d \cdot [\operatorname{Vol} U(\kappa)] \cdot \tfrac{3}{4}\pi \cdot \|v\|_{0,1/2}^6 \varepsilon^{-6} . \tag{C2.18}$$

Proposition C2.5 follows directly from Eqs. (C2.17) and (C2.18) and Lemma C2.9.

Proof of Lemma C2.10. It is sufficient to prove the lemma with c replaced by $c_1(c)$ as defined in Sect. B3. To see this, use Eq. (B7.2) and Proposition B6.2. Use also that $\|c_1(c) - c\|_c^2 \leqq 4\mathfrak{A}(c)$ [to derive, contract Eq. (B3.2) with $\Phi_1(c) - \Phi(c)$ and then integrate over $\mathbb{R}^3$ and integrate by parts]. Write $c_1(c) = (A, \Phi)$. The proof works by adapting a trick due to Morrey (see [17, Lemma 5.2.1]). Decompose $\psi \in H_{c_1}(E)$ as $\psi = \psi^L + \psi^T + \psi^\Omega$, this as in the proof of Lemma B6.4.

Observe that if $\eta, \psi \in H_{c_1}(E)$, and if $\sigma \in \Gamma(\operatorname{Ad} P)$, then

$$(\eta^L, \varrho_*(\sigma)\psi^L) = 0 . \tag{C2.19}$$

This is because

$$\varrho_*(\sigma)\psi^L = -\varrho_*([\hat{\Phi}[\hat{\Phi}, \sigma]])\psi^L = -\varrho_*(\hat{\Phi})^2 \varrho_*(\sigma)\psi^L .$$

Therefore,

$$|(\psi, \varrho_*(\mathfrak{R})\psi)| \leqq \zeta |\mathfrak{R}| |\psi| |\psi'|, \tag{C2.20}$$

where $|\psi'| = |\psi^T| + |\psi^\Omega|$. Here, $\zeta < \infty$ is a fixed constant. Observe that Eqs. (B6.3) and (B6.4) imply that

$$\|\psi\|_6 \leqq \zeta \|\psi\|_{c_1} \quad \text{and} \quad \|\psi^T\|_2 + \|\psi^\Omega\|_2 \leqq \zeta(1 + \mathfrak{A}(c)) \|\psi\|_{c_1}. \tag{C2.21}$$

Now, to apply Morrey's trick, cover the $\mathbb{R}^3$ by a uniform countable set of unit balls, $\{B_\alpha\}$. Let $\{\omega_\alpha^2\}$ be a subordinate partition of unity, for which $|d\omega_\alpha|$ is bounded, independently of α. Be aware that

$$\int_{B_\alpha \cap (\mathbb{R}^3 \setminus U(\kappa))} |\mathfrak{R}|^2 < \kappa. \tag{C2.22}$$

Holder's inequality with Eq. (C2.20) yields

$$\int_{\mathbb{R}^3 \setminus U(\kappa)} (\psi, \varrho_*(\mathfrak{R})\psi) \leqq \zeta \cdot \|\psi\|_6 \cdot \|\mathfrak{R}\|_2^{2/3} \cdot \left[\int_{\mathbb{R}^3 \setminus U(\kappa)} |\mathfrak{R}|^{2/3} |\psi'|^2\right]^{1/2}. \tag{C2.23}$$

Now evaluate

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus U(\kappa)} |\mathfrak{R}|^{2/3} |\psi'|^2 &\leqq \sum_\alpha \int_{B_\alpha \cap (\mathbb{R}^3 \setminus U(\kappa))} \omega_\alpha^2 |\mathfrak{R}|^{2/3} |\psi'|^2, \\ &\leqq \zeta \cdot \kappa^{2/3} \sum_\alpha \|\omega_\alpha |\psi'|\|_6^\alpha \\ &\leqq \zeta \cdot \kappa^{2/3} \sum_\alpha \int [\omega_\alpha^2 |d|\psi'||^2 + |d\omega_\alpha|^2 |\psi'|^2] \\ &\leqq \kappa^{2/3} v'(\mathfrak{A}(c)) \|\psi\|_{c_1}. \end{aligned} \tag{C2.24}$$

The second inequality above is Hölder's inequality with Eq. (C2.22). The third inequality is a Sobolev inequality (cf. Lemma A4.1) and the fourth inequality uses Lemma A4.1, Proposition B3.4, and Eq. (C2.21). The constant, ζ, changes from line to line, but it is always independent of c and ψ. Here, $v'(\cdot) \in C^0([0, \infty), (0, \infty))$. The lemma follows directly now from Eqs. (C2.23) and (C2.24).

C3. Topology of the Dirac Family

Both $H(S)$ and $L(S)$ as defined in Sects. C1 and B6 are vector bundles over the paracompact space $\hat{\mathfrak{B}}$. According to Lemma C2.2, the family of Dirac operators of Eq. (C1.2) indexed by $\hat{\mathfrak{B}}$ defines a Fredholm morphism, $\delta: H(S) \to L(S)$. As discussed in [8] and [10], δ defines in a natural way, characteristic classes,

$$\chi^{p,q}(\delta) \in H^{2pq}(\hat{\mathfrak{B}}; \mathbb{Z}) \quad \text{for} \quad p, q \in (0, 1, \ldots).$$

Here $H^l(\cdot\,; \mathbb{Z})$ is the l^{th} (compactly supported) cohomology group with coefficients in $\mathbb{Z}$. The class $\chi^{p,q}(\delta)$ is a determinant of Chern classes of the "virtual" vector bundle $\operatorname{Index}\delta = [\ker\delta] - [\operatorname{coker}\delta]$ in the K theory of $\hat{\mathfrak{B}}$.

Proposition C3.1. *There exists an unbounded set $\Lambda \subsetneq \times_2 [0, 1, \ldots)$ such that for every $n \in \mathbb{Z}$ and $p, q \in \Lambda$, $0 \neq \chi^{p,q}(\delta) \in H^{2pq}(\hat{\mathfrak{B}}_n; \mathbb{Z})$.*

With Propositions C3.1 and C2.3 one obtains immediately a proof of Theorem A2.1.

Proof of Theorem A2.1. For fixed $n\in\mathbb{Z}$, let $(p,q)\in\Lambda$ be such that $\chi^{p,q}(\delta)\neq 0$. Consider the family, $\mathfrak{F}$, of compact subsets $F\subseteq\hat{\mathfrak{B}}_n$ such that the restriction map from $H^{2pq}(\hat{\mathfrak{B}}_n;\mathbb{Z})$ to $H^{2pq}(F;\mathbb{Z})$ does not annihilate $\chi^{p,q}(\delta)$. As discussed in Sect. A1, this family is homotopy invariant. Define $\mathfrak{A}^n_{p,q}\equiv\mathfrak{A}_{\mathfrak{F}}$ by Eq. (A1.7). According to Proposition 5.3 of Koschorke [10], $\chi^{p,q}(\delta)=0$ in $H^{2pq}(F;\mathbb{Z})$ if for all $c\in F$, $\dim(\ker\delta_c)<p$ and $\dim(\operatorname{coker}\delta_c)<q$. Therefore, each $F\in\mathfrak{F}$ must contain a configuration c for which $\dim(\ker\delta_c)\geqq p$. In fact, since the index of δ_c is $-n$ ([9] and also [23]; but use Lemma C2.2), each $F\in\mathfrak{F}$ must also contain a configuration c for which $\dim(\ker\delta_c)\geqq q-n$. Proposition C2.3 now provides a lower bound for $\mathfrak{A}^n_{p,q}$. As Λ is unbounded, one concludes from Proposition C2.3 that the set $\{\mathfrak{A}^n_{p,q}:(p,q)\in\Lambda\}\subset\operatorname{Crit}_n\subset[0,\infty)$ is also unbounded. This proves Theorem A2.1.

The proof of Proposition C3.1 follows closely the discussion in [12] by Atiyah and Jones and especially their proof of Theorem 4.6 of [12].

The proof begins with a digression. For $t\in[0,\infty)$ and $0<n\in\mathbb{Z}$, let $C_{n,t}$ denote the space of unordered n-tuples of points in $\mathbb{R}^3$ which are mutually separated by a distance larger than t. Thus $C_{n,t}$ is the quotient by the symmetric group, Σ_n, of

$$\tilde{C}_{n,t}=\{(x_1,\dots,x_n)\in(\mathbb{R}^3)^n:\inf_{\alpha\neq\beta}|x_\alpha-x_\beta|>t\}.$$

The spaces $C_{n,t}$ for $t>0$ are strong deformation retracts of C_n, which is the space of unordered sets of n distinct points in $\mathbb{R}^3$.

A construction in [5, Chap. 4] provides a set of approximate solutions in $\hat{\mathfrak{B}}_n$, $n>0$, of the Bogomol'nyi equation, Eq. (A1.5). This set, $V_n\subset\hat{\mathfrak{B}}_n$ is described next (see Definition C4.2). It parametrizes the configurations of n widely spaced, Prasad-Sommerfield solutions to Eq. (A1.5). (These are exhibited in Eq. (C4.8), and discovered in [24].)

Lemma C3.2. *There exists a family of configurations, $V_n=\{c(n,t;\{x_\alpha\}_{\alpha=1}^n)\}\subset\hat{\mathfrak{B}}_n$ which is parametrized by $t\in[1,\infty)$ and $\{x_\alpha\}_{\alpha=1}^n\in C_{n,t}$ with the properties below:*

(1) *There exists for each n, $t_n\in(1,\infty)$ such that for each $t\in[t_n,\infty)$, the assignment of $c(n,t,\{x_\alpha\})\in\hat{\mathfrak{B}}_n$ to $\{x_\alpha\}\in C_{n,t}$ defines an embedding $J:C_{n,t}\to\hat{\mathfrak{B}}_n$.*

(2) *For fixed $t\in[t_n,\infty)$ and for $y=\{x_\alpha\}\in C_{n,t}$, there exists $g_\alpha\in\mathfrak{G}$ such that $g_\alpha c(n,t;y)(x)=c^1(x-x_\alpha)$ if $x\in\mathbb{R}^3$ satisfies $|x-x_\alpha|<\ln t$. Here, $c^1=(A^1,\Phi^1)\in\mathfrak{C}_1$ is the unique (up to $\mathfrak{G}$) Prasad Sommerfield solution to Eq.* (A1.5) *with $\Phi^1(x=0)=0$.*

(3) *There exists ζ which is independent of $n\in\mathbb{Z}$, $t\in[t_n,\infty)$ and $y\in C_{n,t}$ with the following property: Let $c(n,t;y)=(A,\Phi)$. Then*

$$||\Phi|-1|\leqq\zeta\sum_\alpha(1+|x-x_\alpha|)^{-1}\quad\textit{and}\quad|*F_A+\nabla_A\Phi|\leqq\zeta\sum_\alpha(1+|x-x_\alpha|)^{-2}.$$

(4) *If $x\in\mathbb{R}^3$ satisfies* $\inf|x-x_\alpha|<\ln t$, *then $*F_A-\nabla_A\Phi=0$. If* $\inf_\alpha|x-x_\alpha|\geqq\ln t$, *then $|*F_A-\nabla_A\Phi|\leqq\zeta t^{-1/2}\sum_\alpha|x-x_\alpha|^{-2}$.*

The proof of this lemma and also the proof of Lemma C3.4 is provided in Sect. C4.

The first step in proving Proposition C3.1 is to examine the kernel and cokernel of δ_c as c varies in V_n; the purpose is to calculate the restriction of $\chi^{p,q}(\delta)$ to $H^*(V_n;\mathbb{Z})$.

Lemma C3.3. *For each $0<n\in\mathbb{Z}$, there exists $t(n)\in[t_n,\infty)$ such that for each $t>t(n)$ and $c\in J(C_{n,t})$,* $\ker\delta_c=\emptyset$ *and* $\dim(\operatorname{coker}\delta_c)=n$.

Proof of Lemma C3.3. This follows from Proposition C2.3 and property (4) of the set V_n.

Now it is a straightforward argument using Lemma C2.2 and property (1) of V_n to establish that for $t>t(n)$, the assignment to $c(n,t;y)\in J(C_{n,t})$ of the vector space $\operatorname{coker}\delta_c\simeq\mathbb{C}^n$ defines a C^0 vector bundle over $J(C_{n,t})$. This bundle is denoted by N_n.

Consider the bundle $N_n\to J(C_{n,t})$:

Lemma C3.4. *Given $0<n\in\mathbb{Z}$ and $t>t(n)$, let $N_n\to J(C_{n,t})$ be as defined above. There exists $t'(n)<\infty$ such that for $t>t'(n)$, the pull-back of N_n to $C_{n,t}$ is isomorphic to the vector bundle associated to the standard representation σ_n of $\pi_1(C_{n,t})\simeq\Sigma_n$.*

The proof of Lemma C3.4 is deferred to Sect. C4. Suffice it to say that the construction in Sect. C4 assigns to each $c(n,t,\{x_\alpha\})$ n approximate "zero modes" of δ_c^*, one concentrated near each x_α. Then Proposition C2.3 provides a tool to project isomorphically the $\mathbb{C}^n\subset H_c(E)$ of this approximate cokernel onto the cokernel of δ_c.

For $0<n\in\mathbb{Z}$, let $t'(n)$ be as in Lemma C3.4. Assume that $t>t'(n)$. The Chern classes of $J^{-1}(N_n)\to C_{n,t}$ are computed by Atiyah and Jones in Proposition 4.5 of [12].

Because $J^{-1}(N_n)\to C_{n,t}$ is the pull-back of the virtual bundle, $\operatorname{Index}\delta\in K(\hat{\mathfrak{B}}_n)$, the computation in [12] asserts that certain specific Chern classes of $\operatorname{Index}\delta$ are therefore nonvanishing in $H^*(\hat{\mathfrak{B}}_n:\mathbb{Z})$. In particular, one obtains from [12]:

Lemma C3.5. *Given any $k\geqq 0$, pick a prime $p>k+1$. Then in $H^*(\hat{\mathfrak{B}}_{p(p-1-k)};\mathbb{Z})$, the class*

$$\chi^{p-1-k,p-1}(\delta)\not\equiv 0 \bmod p\,.$$

The second step of the proof of Proposition C2.5 is to establish a relationship between $\chi^{p,q}(\delta)$ in $H^*(\hat{\mathfrak{B}}_n;\mathbb{Z})$ and $\chi^{p,q}(\delta)$ in $H^*(\hat{\mathfrak{B}}_l;\mathbb{Z})$ for $l\neq n$. Atiyah and Jones faced an analogous problem in [12] and the solution here is adapted from their solution.

Let $\Omega_n=\Omega_n(S^2;S^2)$. Observe that I induces an isomorphism by pull-back of $H^*(\hat{\mathfrak{B}};\mathbb{Z})$ with $H^*(\Omega_n;\mathbb{Z})$ (see Lemma B5.1). To avoid confusion, denote by $\chi_n^{p,q}$ the class $I^*\chi^{p,q}(\delta)$ in $H^{2pq}(\Omega_n;\mathbb{Z})$.

The characteristic classes $\{\chi_n^{p,q}\}$ can be viewed as follows. First, a straightforward argument which is similar to the proof of Proposition B6.3 shows that the inclusions of $H(S)$ in $\hat{\mathfrak{B}}\times L^2_{1;\mathrm{loc}}(S)$ and of $L(S)$ in $\hat{\mathfrak{B}}\times L^2(S)$ induce vector bundle isomorphisms

$$I^*H(S)\simeq\Omega(S^2;S^2)\times L^2_1(S)\,,$$

and

$$^*L(S)\simeq\Omega(S^2;S^2)\times L^2(S)$$

(cf. Sects. 6 and 7 of [9] and also [23]). Therefore, $I^*\delta$ defines a continuous map,

$$\begin{aligned} I^*\delta:\Omega(S^2;S^2)\longrightarrow \operatorname{Fred}&(L^2_1(S);L^2(S))\\ &\wr\\ &\mathbb{Z}\times BU\,.\end{aligned}$$

The cohomology ring of BU is a polynomial ring on the universal Chern classes, and each $\chi_n^{p,q}(\delta)$ is the pull-back of a specific such polynomial by $I^*\delta$.

As is well known, the Ω_n's are mutually homotopic. The homotopy of Ω_k with Ω_{k+1} is induced by the commutative addition operation on $\pi_2(S^2; n)$. The effect of addition on $\chi_n^{p,q}$ is summarized next.

Lemma C3.6. *There exists a map, $t: \Omega_n \to \Omega_{n+1}$ which induces a homotopy equivalence and is such that for every pair (p, q) of non-negative integers, $t^*\chi_{n+1}^{p,q}$ and $\chi_n^{p,q}$ are cohomologous in $H^{2pq}(\Omega_n; \mathbb{Z})$.*

The proof of Lemma C3.6 is provided in Sect. C5.

Proof of Proposition C3.1 given Lemmas C3.2, C3.3, and C3.6. Lemma C3.5 establishes that there exists arbitrarily large, non-negative integers (p, q) and an integer $n>0$ such that $\chi^{p,q}(\delta) \neq 0$ in $H^{2pq}(\hat{\mathfrak{B}}_n; \mathbb{Z})$. But then $\chi_n^{p,q} \neq 0$ in $H^*(\Omega_n; \mathbb{Z})$ and Lemma C3.6 establishes that $\chi_n^{p,q} \neq 0$ in $H^{2pq}(\hat{\mathfrak{B}}_l; \mathbb{Z})$ for every $l \in \mathbb{Z}$.

As a parenthetical remark, the Dirac operator defines, via its total symbol, a second map from $\Omega(S^2; S^2)$ into $\mathbb{Z} \times BU$. Indeed, the assignment to $e \in \Omega(S^2; S^2)$ of the total symbol $\sigma(\delta_{I(e)})$ defines upon restriction to the unit ball S^5 in T^* (the set $\{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x|^2 + |\xi|^2 = 1\}$), the following continuous map from S^5 into $U(\mathbb{C}^2 \otimes \mathbb{C}^2) = U(4)$ [25];

$$\hat{\sigma}(e)(x, \xi) = i\xi. \otimes 1 + |x| 1 \otimes e .$$

Here $\xi.$ is Clifford multiplication by ξ.

As e varies in $\Omega(S^2; S^2)$, one obtains a continuous map, $\hat{\sigma} : \Omega(S^2; S^2) \to \Omega^5(\mathrm{U}(4))$. Now, $\mathrm{U}(4)$ includes in $\mathrm{U}(n)$ for $n>4$ as

$$\left(\begin{array}{c|c} \mathrm{U}(4) & 0 \\ \hline 0 & 1_{n-4} \end{array}\right),$$

and so $\hat{\sigma}$ defines by direct limit, a continuous map $\hat{\sigma} : \Omega(S^2; S^2) \to \Omega^5(U)$, where U is the group of unitary automorphisms of a separable complex Hilbert space [$L^2(S)$, for example].

Following Atiyah-Jones [12] one uses the Bott periodicity theorem to identify up to homotopy $\Omega^5(U) \sim \mathbb{Z} \times BU$. Thus, $\hat{\sigma}$ maps $\Omega(S^2; S^2)$ continuously into $\mathbb{Z} \times BU$.

The Atiyah-Singer index theorem for families asserts that the two maps, $I^*\delta$ and $\hat{\sigma}$ are homotopic as maps from $\Omega(S^2; S^2)$ into $\mathbb{Z} \times BU$.

C4. The Dirac Operator for Multi-Monopoles

This section contains the proof of Lemmas C3.2 and C3.4. These lemmas with Lemma C3.6 are the superstructure which holds up the somewhat formal relationship between the Dirac operator and the cohomology of $\hat{\mathfrak{B}}$.

Proof of Lemma C3.2. The configurations of V_n are constructed in [5, Chap. 4.7], though the language there is oriented more to physicists than it is here. For this reason, the construction will be reviewed briefly. Let $y = (n, t, \{x_\alpha\})$. The configuration $c(y) \in V_n$ is specified by the following cohomological data: a finite open cover of $\mathbb{R}^3$, $\{U_j(y)\}$; transition functions $\{g_{ij}(y): U_i \cap U_j \to SU(2)\}$; and fields $\{c_j(y) = (A_j, \Phi_j)(y)\}$ on $U_j(y)$ satisfying on $U_i \cap U_j$ the cocycle relation $c_i(y) = g_{ij}(y) \cdot c_j(y)$. The fields $\{c_j\}$ and the transition functions $\{g_{ij}\}$ are C^∞.

The data $\{U_j(y), g_{ij}(y)\}$ for fixed y defines a principal SU(2) bundle, $P(y) \to \mathbb{R}^3$. The data $\{c_j(y)\}$ defines a smooth pair, $c'(y)$, of a smooth connection on $P(y)$ and section of $\mathrm{Ad}\,P(y)$.

An isomorphism $\eta: P \to P(y)$ defines by pull-back $\eta^* c'(y) \in \mathfrak{C}_n$. As any two such isomorphisms differ by an element in $\mathfrak{G} = \mathrm{Aut}\,P$, the assignment to $y = (n, t, \{x_\alpha\})$ of the data $(P(y), c'(y))$ defines a unique element, $[c'(y)] \in \mathfrak{C}_n/\mathfrak{G} = \mathfrak{B}_n/\mathrm{SU}(2) = \hat{\mathfrak{B}}_n/S^1$.

For the reader's convenience, the cohomological data $\{U_j(y), g_{ij}(y), c_j(y)\}$ for $(P(y), c'(y))$ is presented explicitly in Definition C4.2 at the end of this section. The reader should refer there during the discussion that follows.

Statements (2–4) of Lemma C3.2 follow by direct calculation with $(P(y), c'(y))$. The reader is referred to [5, Chap. 4] for the details.

Give the quotient topology to $\hat{\mathfrak{B}}_n/S^1$. For fixed $t \in [t_n, \infty)$, the continuity of $[c'(\cdot)] \in \hat{\mathfrak{B}}_n/S^1$ as a map of $C_{n,t}$ is straightforward to check. This can be done by taking a subordinate cover, $\{U'_j\}$ to $\{U_j(y)\}$ such that for $y' \in C_{n,t}$, in a neighborhood of y, $\{U'_j\}$ also defines a subordinate cover to $\{U_j(y')\}$. This allows one to directly compare the data for y and y' that Definition C4.2 provides. The details are left to the reader.

Observe that $[c'(\cdot)]$ is an embedding for fixed $t \in [t_n, \infty)$ of $C_{n,t}$ into $\hat{\mathfrak{B}}_n/S^1$. This is because the assignment to $[c = (A, \Phi)] \in \hat{\mathfrak{B}}_n/S^1$ of

$$Z([c]) = \{x \in \mathbb{R}^3 : \Phi(x) = 0\}$$

defines, upon restriction to $\mathrm{Im}[c'(\cdot)]$, a continuous inverse to $[c'(\cdot)]$.

Finally, it remains to show that for fixed $t \in [t_n, \infty)$, $[c'(\cdot)]$ lifts to an embedding, $J: C_{n,t} \to \hat{\mathfrak{B}}_n$. Let $y = (n, t, \{x_\alpha\})$ with $t \in [t_n, \infty)$. Identify $P(y)$ over $U_{(1)} \subset \mathbb{R}^3$ with $U_{(1)} \times \mathrm{SU}(2)$. Here, $(1) = \underset{n}{(1, 1, \ldots) \times \{-1, 1\}}$ as defined in Definition C4.2. Let $x(y) = \left(4 \sum_\alpha |x_\alpha|, 0, 0\right) \in U_{(1)}$. A point in $P(y)|_{x(y)}$ is $p(y) = (x(y), 1)$.

Let $q(y) \in P(y)|_0$ be the parallel transport of $p(y)$ to $0 \in \mathbb{R}^3$, by the connection in $c'(y)$, along the line segment between $x(y)$ and 0.

As for $[c'(\cdot)]$, the assignment for fixed $t \in [t_n, \infty)$ of $y = (n, t, \{x_\alpha\})$ to $(P(y), c'(y), q(y))$ defines a continuous embedding of $C_{n,t}$ into $(\mathfrak{C}_n \times P_0)/\mathfrak{G}$ which will be denoted $[c'(\cdot), q(\cdot)]$.

Let $(c, q) \in \mathfrak{C}_n \times P_0$. The parallel transport of q by the connection $A(c)$ along the rays from $0 \in \mathbb{R}^3$ defines a continuous section of $(\mathfrak{C}_n \times P_0) \underset{\mathfrak{G}}{\times} \Gamma(P)$ over $(\mathfrak{C}_n \times P_0)/\mathfrak{G}$. Denote this section by $\mathfrak{s}$.

The assignment of $(c, q) \in \mathfrak{C}_n \times P_0$ to $\mathfrak{s}^*(c, q) \cdot c \in \mathfrak{B}_n$ defines a continuous map from $(\mathfrak{C}_n \times P_0)/\mathfrak{G}$ onto $\mathfrak{B}_n$. Let $j: C_{n,t} \to \mathfrak{B}_n$ denote the image of $[c'(\cdot), q(\cdot)]$ in $\mathfrak{B}_n$. As $j(\cdot)$ covers $[c'(\cdot)]$ with respect to the projection $\mathfrak{B}_n \to \mathfrak{B}_n/\mathrm{SU}(2)$, one concludes that $j(\cdot)$ is an embedding. For $z = \{x_\alpha\} \in C_{n,t}$, let $j(z) = (A(z), \Phi(z))$, and let $x(z) = \left(4 \sum_\alpha |x_\alpha|, 0, 0\right)$. By construction $\Phi(z)(x(z)) = |\Phi(z)|(x(z)) \cdot \sigma^3$.

With Definition C4.2, one can check that $j(z) \in \mathfrak{L}^0$ as defined by Eq. (B4.1). Let $R(\cdot)$ be the map of Lemma B4.2. Notice that $|\Phi(z)|$ is nonvanishing on the line segment between $x(z)$ and $(2R(j(z)), 0, 0)$. Therefore, the homotopy lifting property of the fibration $S^1 \to S^3 \to S^2$ provides a continuous map, $g(\cdot): C_{n,t} \to \mathrm{SU}(2)$ such that for all $y \in C_{n,t}$,

$$g_1(z)\,(\Phi(z)/|\Phi|(z))\,(R(j(z))n)g_1^{-1}(z) = \sigma^3 .$$

Finally, $j(\cdot)$ and $c_2(j(\cdot))$ are homotopic as maps from $C_{n,t}$ into $\mathfrak{Q}^0$. Thus, the homotopy lifting property provides a continuous $g(\cdot): C_{n,t} \to \mathrm{SU}(2)$ such that for all $y \in C_{n,t}$,

$$g(z)\,(\Phi_2(j(z))/|\Phi_2(j(z))|)\,(R(c_2(j(z)))n)g^{-1}(z) = \sigma^3 .$$

Therefore, $J(\cdot) = g(\cdot)j(\cdot)$ maps $C_{n,t}$ continuously into $\mathfrak{B}_n$ and since $J(\cdot)$ covers $[c'(\cdot)]$, $J(\cdot)$ is an embedding.

Proof of Lemma C3.4. The Prasad-Sommerfield solution, c_1, of Eq. (C4.8) has the property that $\dim(\ker \delta_{c_1}) = 1$. Let $\psi \in \Gamma(S)$ be an $L^2(S)$ normalized generator for $\operatorname{coker} \delta_{c_1}$. The explicit form of ψ from [16] shows that for all $x \in \mathbb{R}^3$

$$|\psi(x)| \leqq \zeta e^{-|x|} . \tag{C4.1}$$

For fixed $t \in [t_n, \infty)$ and $z = \{x_\alpha\} \in C_{n,t}$, define for each $\alpha \in \{1, \ldots, n\}$,

$$\psi_\alpha(z)\,(x) = \beta_{\ln t}(\alpha)\,(x) g_\alpha^{-1}(x)\psi(x - x_\alpha) g_\alpha(x) . \tag{C4.2}$$

Here, $g_\alpha(\cdot) \in \mathfrak{G}$ is as specified in statement (2) of Lemma C3.2. Also, $\beta_\varrho(\alpha)\,(x) \equiv \beta(|x - x_\alpha|/\varrho)$.

Using Eq. (C4.1) and statement (2) of Lemma C3.2, one finds that there exists $\zeta \in [0, \infty)$, which is independent of $t \in [t_n, \infty)$ and $z \in C_{n,t}$ such that for each $\alpha, \beta \in \{1, \ldots, n\}$,

$$|\langle \psi_\alpha(z), \psi_\beta(z) \rangle_2 - \delta_{\alpha\beta}| \leqq \zeta t^{-1/2} . \tag{C4.3}$$

Thus for $t > t(n) \gg \zeta^2$, $\operatorname{Span}\{\psi_1(z), \ldots, \psi_n(z)\}$ in $L^2(S)$ is n-dimensional.

With Definition C4.2, it is straightforward to check that for $t > t(n)$, the assignment of

$$\operatorname{Span}\{\psi_1(z), \ldots, \psi_n(z)\} \subset L^2(S)$$

to $z \in C_{n,t}$ defines a continuous map of $C_{n,t}$ into the Grassmanian of n-dimensional subspaces of $L^2(S)$. The graph of this map is a $\mathbb{C}^n$-vector bundle, $N'_n \to C_{n,t}$. Since interchanging points x_γ, x_β in $(x_1, \ldots, x_n) \in \tilde{C}_{n,t}$ induces the interchange of ψ_γ with ψ_β, the bundle N'_n is $\tilde{C}_{n,t} \underset{\Sigma_n}{\times} \mathbb{C}^n$.

The vector bundle $N_n \to C_{n,t}$ of Lemma C3.4 is also defined by a map from $C_{n,t}$ into the Grassmanian of n-dimensional subspaces of $L^2(S)$; in this case z is sent to $\operatorname{coker} \delta_{J(z)}$. A continuous section over $C_{n,t}$ of $\operatorname{Hom}(N'_n, N_n)$ is defined by the $L^2(S)$-orthogonal projection of $\operatorname{Span}\{\psi_\alpha(z)\}_{\alpha=1}^n$ onto $\operatorname{coker} \delta_{J(z)}$. Denote this section by Π. Then Lemma C3.4 follows from

Lemma C4.1. *There exists $t'(n) \in [t_n, \infty)$ such that for $t > t'(n)$, $L^2(S)$ orthogonal projection $\Pi: N'_n \to N_n$ is a bundle isomorphism.*

Proof of Lemma C4.1. For $t \in [t(n), \infty)$ and $z \in C_{n,t}$, let $\delta^*_{J(z)}: L^2(S) \to H_{J(z)}(S)$ denote the adjoint of $\delta_{J(z)}$. Lemma C2.2 implies that $\delta^*_{J(z)}$ defines a continuous Fredholm section of $\mathrm{Hom}(L^2(S); J^*H(S))$ over $C_{n,t}$. With Eq. (C4.3) and Statement (2) of Lemma C3.2 and Lemma A4.1, one finds that

$$\langle \delta^*_{J(z)}\psi_\alpha(z), \delta^*_{J(z)}\psi_\alpha(z)\rangle_2 \leqq \zeta \cdot t^{-1/2}, \tag{C4.4}$$

for each $\alpha \in \{1, \ldots, n\}$. Again, $\zeta \in [0, \infty)$ is independent of $t, z \in C_{n,t}$ and α.

Consider now $\psi \in \mathrm{Span}\{\psi_\alpha(z)\}$ with $\Pi(z)\psi = 0$. Then the Fredholm alternative implies that $\psi = \delta_{J(z)}u$ for some $u \in H_{J(z)}(S)$.

Equation (C2.21) and statements (3) and (4) of Lemma C3.2 imply that $H_{J(z)}(S)$ embeds in $L^2(S)$ and this embedding constant v is independent of $t \in [t(n), \infty)$ and $z \in C_{n,t}$.

Therefore, since

$$\delta^*_{J(z)}\delta_{J(z)}u = \delta^*_{J(z)}\psi,$$

Eq. (C4.4) implies that

$$\|\delta_{J(z)}u\|_2^2 \leqq \zeta \cdot \|u\|_2 t^{-1/4}\|\psi\|_2. \tag{C4.5}$$

But the Weitzenbock formula for $\delta_{J(z)}$ [Eqs. (C2.3) and (C2.5)] with Lemma A4.1 and Statement (4) of Lemma C3.4 imply that there exists $t''(n) \in [t(n), \infty)$ such that for $t \geqq t''(n)$ and $z \in C_{n,t}$,

$$\|\delta_{J(z)}u\|_2 \geqq \frac{1}{2}\|u\|_{J(z)} \geqq \frac{v}{2}\|u\|_2. \tag{C4.6}$$

Now Eqs. (C4.5) and (C4.6) imply that $\|\psi\|_2^2 \leqq 2\zeta v^{-1}t^{-1/4}\|\psi\|_2^2$. Hence, when $t > t'(n) = \max(16\zeta^4 v^{-4}, t''(n))$, then $\ker \Pi(z) = \emptyset$ for any $z \in C_{n,t}$.

This section ends with the definition of the space V_n. To present the definition, introduce for each $\alpha \in \{1, \ldots, n\}$, polar coordinates $(r_\alpha, \theta_\alpha, \chi_\alpha)$ on $\mathbb{R}^3$, where $r_\alpha = |x - x_\alpha|$; the polar angle θ_α is such that $\theta_\alpha = 0$ is the ray in $\mathbb{R}^3$ with base x_α and direction parallel to $n = (1, 0, 0) \in \mathbb{R}^3$; the angle χ_α is such that $(\theta_\alpha = \pi/2, \chi_\alpha = 0)$ is the ray on $\mathbb{R}^3$ with base point x_α and direction parallel to $(0, 1, 0) \in \mathbb{R}^3$.

For $\varrho > 0$ and $\alpha \in \{1, \ldots, n\}$, let $B_\varrho(x)$ denote the open ball in $\mathbb{R}^3$ with center x_α and radius ϱ. Let $\beta_\varrho(\alpha)(x) = \beta(|x - x_\alpha|/\varrho)$, and let

$$w_\varrho(\alpha)(x) = \prod_{B \neq \alpha} (1 - \beta_\varrho(\alpha)(x)).$$

Let $\{\sigma^i\}_{i=1}^3$ be a fixed basis for $\mathfrak{su}(2)$ satisfying $[\sigma^i, \sigma^j] = -2\varepsilon^{ijk}\sigma^k$.

Definition C4.2. The cohomological data for the configuration $c(n, t, \{x_\alpha\})$: Let $t_n = 8n$.

(A) The cover $\{U_j\}$:

For $\alpha \in \{1, \ldots, n\}$, set $U_\alpha = B_{2\ln t}(\alpha)$.

For $\alpha \in \{1, \ldots, n\}$ and $\varepsilon \in \{-1, 1\}$, set

$$U_{\alpha,\varepsilon} = (B_{3\ln t}(\alpha) \backslash B_{\ln t}(\alpha)) \cap \{x \in \mathbb{R}^3 : \varepsilon(\theta_\alpha(x) - \pi/2) < \pi/4\}.$$

For each of the 2^n points $(\varepsilon) = (\varepsilon_1, \ldots, \varepsilon_n) \in \times_n \{-1, 1\}$,

$$U_{(\varepsilon)} = \{x \in \mathbb{R}^3 : \text{for each } \alpha \in \{1, \ldots, n\}, \varepsilon_\alpha(\theta_\alpha(x) - \pi/2) < \pi/4\} \backslash \bigcup_\alpha \bar{B}_{3\ln t}(\alpha). \tag{C4.7}$$

(B) The transition functions $\{g_{ij}\}$:
On $U_\alpha \cap U_{\alpha,1}$, set

$$g_{(\alpha,1)\alpha} = \cos\tfrac{1}{2}\theta_\alpha + \sin\tfrac{1}{2}\theta_\alpha(-\sin\chi_\alpha\sigma^1 + \cos\chi_\alpha\sigma^2).$$

On $U_\alpha \cap U_{\alpha,-1}$, set

$$g_{(\alpha,-1)\alpha} = \sin\tfrac{1}{2}\theta_\alpha\sigma^2 + \cos\tfrac{1}{2}\theta(\cos\chi_\alpha + \sin\chi_\alpha\sigma^3).$$

On $U_{\alpha,1} \cap U_{\alpha,-1}$, set

$$g_{(\alpha,-1)(\alpha,1)} = \exp\chi_\alpha\sigma^3.$$

On $U_{(\alpha,\varepsilon')} \cap U_{(\varepsilon)}$, set

$$g_{(\alpha,\varepsilon')(\varepsilon)} = \exp(\tfrac{1}{2}(\varepsilon_\alpha - \varepsilon')\chi_\alpha\sigma^3).$$

On $U_{(\varepsilon')} \cap U_{(\varepsilon)}$, set

$$g_{(\varepsilon')(\varepsilon)} = \prod_{\alpha=1}^{n} \exp(\tfrac{1}{2}(\varepsilon_\alpha - \varepsilon'_\alpha)\chi_\alpha\sigma^3). \tag{C4.8}$$

(C) The configuration $\{c_j\}$:
For $\alpha \in \{1, \ldots, n\}$, set $c_\alpha = (A_\alpha, \Phi_\alpha)$ on U_α to be

$$\begin{aligned}\Phi_\alpha &= \tfrac{1}{2}(\coth r_\alpha - 1/r_\alpha)(\cos\theta_\alpha\sigma^3 + \sin\theta_\alpha(\cos\chi_\alpha\sigma^1 + \sin\chi_\alpha\sigma^2)),\\ A_\alpha &= \tfrac{1}{2}(1 - r_\alpha/\sinh r_\alpha)(\sin^2\theta_\alpha d\chi_\alpha\sigma^3 + (\cos\chi_\alpha\sigma^2 - \sin\chi_\alpha\sigma^1)d\theta_\alpha\\ &\quad - (\cos\chi_\alpha\sigma^1 + \sin\chi_\alpha\sigma^2)\sin\theta_\alpha\cos\theta_\alpha d\chi_\alpha).\end{aligned} \tag{C4.9}$$

For $\alpha \in \{1, \ldots, n\}$ and $\varepsilon \in \{-1, 1\}$, set $c_{\alpha,\varepsilon} = (A_{\alpha,\varepsilon}, \Phi_{\alpha,\varepsilon})$ on $U_{\alpha,\varepsilon}$ to be

$$\begin{aligned}\Phi_{\alpha,\varepsilon} &= (1 - 1/r_\alpha + \beta_{2\,\mathrm{Int}}(\alpha)(\coth r_\alpha - 1))\tfrac{1}{2}\sigma^3,\\ A_{\alpha,\varepsilon} &= \tfrac{1}{2}\{(\varepsilon - \cos\theta_\alpha)d\chi_\alpha\sigma^3 + \beta_{2\,\mathrm{Int}}(\alpha)\exp(\tfrac{1}{2}(1-\varepsilon)\chi_\alpha\sigma^3)\\ &\quad \cdot [(\sin\chi_\alpha\sigma^1 - \cos\chi_\alpha\sigma^2)d\theta_\alpha + (\cos\chi_\alpha\sigma^1 + \sin\chi_\alpha\sigma^2)\sin\theta_\alpha d\chi_\alpha]\exp(-\tfrac{1}{2}(1-\varepsilon)\chi_\alpha\sigma^3)\}.\end{aligned}$$

For $(\varepsilon) \in \times_n \{-1, 1\}$, set $c_{(\varepsilon)} = (A_{(\varepsilon)}, \Phi_{(\varepsilon)})$ on $U_{(\varepsilon)}$ to be

$$\begin{aligned}\Phi_{(\varepsilon)} &= \left(1 - \sum_{\alpha=1}^{n} w_{4\,\mathrm{Int}}(\alpha)\frac{1}{r_\alpha}\right)\frac{1}{2}\sigma^3,\\ A(\varepsilon) &= \sum_{\alpha=1}^{n} w_{4\,\mathrm{Int}}(\alpha)(\varepsilon_\alpha - \cos\theta_\alpha)d\chi_\alpha\frac{1}{2}\sigma^3.\end{aligned} \tag{C4.10}$$

C5. The Index and Ω's Group Structure

The map $I^*\delta$ from $\Omega(S^2; S^2)$ into $\mathrm{Fred}(L_1^2(S); L^2(S))$ sends Ω_n into $F_{-n} \subset \mathrm{Fred}(L_1^2(S), L^2(S))$, the space of Fredholm operators from $L_1^2(S)$ to $L^2(S)$ with index $-n$.

Denote the homotopy addition operation on $\Omega(S^2; S^2)$ by # [see Eq. (C5.3) below]. The homotopy equivalence of $\Omega_n(S^2; S^2)$ with $\Omega_{n+1}(S^2; S^2)$ is induced by the map $t:\Omega_n\to\Omega_{n+1}$ which sends $e\in\Omega_n$ to $t(e)\equiv e_1 \# e\in\Omega_{n+1}$, where $e_1\in\Omega$, is the identity map of S^2. With t one obtains the continuous map

$$t^*I^*\delta:\Omega_n\to F_{-n-1}\,. \tag{C5.1}$$

Each F_n is homotopic to every other [10]. The composition with a fixed operator of index -1, $l\in\mathrm{Fred}(L_1^2(S);\,L_1^2(S))$ [see Eq. (C5) below] induces the homotopy equivalence between F_{-n} and F_{-n-1}. Thus one obtains the continuous map

$$I^*\delta(\cdot)\cdot l:\Omega_n\to F_{-n-1}\,. \tag{C5.2}$$

Lemma C3.6 is proved by showing that for each n, the two maps $t^*I^*\delta$ and $I^*\delta\cdot l$ are homotopic as maps from Ω_n into F_{-n-1}. Thus, Lemma C3.6 follows from Lemma C5.1, below, which makes this assertion.

To be more explicit, let $\{\tau^i\}_{i=1}^3$ be an orthonormal basis for $\mathfrak{su}(2)\in\mathrm{End}\,\mathbb{C}^2$ with $(\tau^i)^2=-\frac{1}{4}$. Let (θ,χ) be coordinates on $S^2\subset\mathfrak{su}(2)$ with $\theta\in[0,\pi]$ the azimuthal angle and $\chi\in[0,2\pi]$ the equatorial angle. Require that $\theta=0$ is the point $\frac{1}{2}\tau^3$.

Choose C^∞ maps, $\sigma_\pm\in\Omega_0$, which have the following properties: $\sigma_+^*(\theta,\chi)=(\theta^+,\chi)$, where $d\theta^+/d\theta\geqq 0$; $\theta^+(\theta)=2\theta$ when $\theta\in[0,\pi/4]$ and $\theta^+=\pi$ when $\theta\in[3\pi/8,\pi]$. The map σ_- has $\sigma_-^*(\theta,\chi)=(\theta^-,\chi)=(\pi-\theta^+(\pi-\theta),\chi)$.

Let $e=\Omega_n$ be arbitrary and set $(\theta_e,\chi_e)=e^*(\theta,\chi)$. Also set $(\theta_e^-,\chi_e^-)=e^*(\theta^-,\chi)$. Let $e_1\in\Omega_1$ be the identity map.

Now, define $t(e)$ by setting

$$t(e)=\begin{cases}\cos\theta^+\tau^3+\sin\theta^+(\cos\chi\tau^1+\sin\chi\tau^2) & \text{for}\quad \theta\in[0,\pi/2]\,,\\ -\cos\theta_e^-\tau^3+\sin\theta_e^-(\cos\chi_e^-\tau^1-\sin\chi_e^-\tau^2) & \text{for}\quad \theta\in[\pi/2,\pi]\,.\end{cases} \tag{C5.3}$$

The reader can check that $t(e)\in\Omega_{n+1}$ if $e\in\Omega_n$.

To define the homotopy equivalence from F_{-n} to F_{-n-1}, introduce the pseudo-differential operator $l:\Gamma_0(s)\to\Gamma(s)$ by setting

$$l\psi=(\partial\!\!\!/-\tau^3)\cdot\delta_{I(e_1)}\cdot[(-\Delta+\tfrac{1}{4})^{-1}\psi]\,. \tag{C5.4}$$

Here $\partial\!\!\!/$ is defined by Eq. (C1.1).

The operator l is an elliptic, Fredholm operator from $L^2(S)$ to $L^2(S)$ for the following reasons: First, $(-\Delta+\frac{1}{4})^{-1}$ is, up to a constant, an isomorphism from $L_1^2(S)$ to $L_3^2(S)$. Then, $\delta_{I(e_1)}$ is Fredholm of degree -1 from $L_3^2(S)$ to $L_2^2(S)$ (cf. the arguments in Sects. 6, 7 of [9] or [23] plus [25]). Similarly, the operator $\partial\!\!\!/-\tau^3$ is an isomorphism from $L_2^2(S)$ to $L_1^2(S)$ since

$$(\partial\!\!\!/-\tau^3)^*(\partial\!\!\!/-\tau^3)=-\Delta+\tfrac{1}{4}\,. \tag{C5.5}$$

Lemma C5.1. *For each $n\in\mathbb{Z}$, the map $t^*I^*\delta:\Omega_n\to F_{-n-1}$ as defined by Eqs.* (C1.2) *and* (C5.3) *and the map $I^*\delta\cdot l:\Omega_n\to F_{-n-1}$ as defined by Eqs.* (C1.2) *and* (C5.4) *are homotopic.*

Proof of Lemma C5.1. If $e\in\Omega(S^2;S^2)$, then for all $k\in(0,1,2,\ldots)$,

$$|\partial^k A(I(e))|\leqq z(e,k)\,(1+|x|)^{-k-1}\,.$$

This implies that multiplication by $A(I(e))$ defines an operator from $L^2_{k+1}(S)$ to $L^2_k(S)$ which is compact. As a consequence, $t^*I^*\delta$ is homotopic to the map which sends $e\in\Omega(S^2;S^2)$ to the Fredholm operator

$$\hat{t}(e)=\not{\partial}+(1-\beta)t(e)\,. \tag{C5.6}$$

Also, $I^*\delta\cdot l$ is homotopic to the map which sends $\varepsilon\in\Omega(S^2;S^2)$ to

$$\hat{l}(e)=(\not{\partial}+(1-\beta)e)(\not{\partial}-\tau^3)(\not{\partial}+(1-\beta)e_1)(-\Delta+\tfrac{1}{4})^{-1}\,. \tag{C5.7}$$

The identity map of S^2 is homotopic to the maps $\sigma_\pm:S^2\to S^2$ and it is also homotopic to the reflection $v:S^2\to S^2$ which sends (θ,χ) to $(\pi-\theta,-\chi)$. By composing with these homotopies, $\hat{l}(\cdot)$ is homotoped to $l_1:\Omega_n\to F_{-n-1}$, which sends e to $l_2(e)(-\Delta+\frac{1}{4})^{-1}$, where

$$l_2(e)=(\not{\partial}+(1-\beta)v\cdot e\cdot\sigma_-)(\not{\partial}-\tau_3)(\not{\partial}+(1-\beta)e_1\cdot\sigma_+)\,. \tag{C5.8}$$

To examine $l_2(e)$, let (r,θ,χ) be spherical coordinates on $\mathbb{R}^3$. For $\theta\in[0,5\pi/8]$,

$$l_2(e)=[-\beta\tau^3(\not{\partial}-\tau^3)+(-\Delta+\tfrac{1}{4})](\not{\partial}+(1-\beta)\,t(e))\,. \tag{C5.9}$$

For $\theta\in[3\pi/8,\pi]$,

$$l_2(e)=(\not{\partial}+(1-\beta)\,t(e))(-\tau^3(\not{\partial}-\tau^3)(\beta\,\cdot)+(-\Delta+\tfrac{1}{4}))\,. \tag{C5.10}$$

One can readily check that Eqs. (C5.9) and (C5.10) imply that

$$l_2(e)=(\not{\partial}+(1-\beta)t(e))(-\Delta+\tfrac{1}{4})+\mathfrak{R}(e)\,, \tag{C5.11}$$

where $\mathfrak{R}(e):L^2_3(S)\to L^2(S)$ is compact. This implies that $l_1(\cdot)$ is homotopic to $\hat{t}(\cdot)$.

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Communicated by A. Jaffe

Received April 9, 1984