Existence, Uniqueness, and Nondegeneracy of Positive Solutions of Semilinear Elliptic Equations

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Abstract. We study positive solutions of the Dirichlet problem: $\Delta u(x) + f(u(x)) = 0$, $x \in D^n$, $u(x) = 0$, $x \in \partial D^n$, where D^n is an n-ball. We find necessary and sufficient conditions for solutions to be nondegenerate. We also give some new existence and uniqueness theorems.

In this paper we study positive solutions of the Dirichlet problem

$$\Delta u(x) + f(u(x)) = 0, \quad x \in \Omega, \tag{1}$$

$$u(x) = 0, \quad x \in \partial \Omega, \tag{2}$$

where Ω is an n-ball D_R^n of radius R. Our original interest was with the degeneracy problem for solutions of (1), (2). That is, we wanted to find conditions under which 0 is not in the spectrum of the linearized equations; in symbols,

$$\text{if } \begin{cases} \Delta v(x) + f'(u(x))v(x) = 0, & x \in \Omega \\ v(x) = 0, & x \in \partial \Omega \end{cases}, \quad \text{then} \quad v \equiv 0.$$

When this holds, we say that the solution u of (1), (2) is *non-degenerate*; otherwise u is called *degenerate*. The interest in this notion comes from the fact that the non-degeneracy of a solution allows application of certain topological techniques to it, whereby its stability properties can be investigated [8, Chap. 24, Sect. D]. In pursuing this problem, we were led quite naturally to existence and uniqueness questions for (1), (2), and we also obtain some new results in these directions.

From a result of Gidas et al. [4], all positive solutions of (1), (2) on $\Omega = D_R^n$ are (monotone decreasing) functions of the radius, and must therefore satisfy a non-autonomous ordinary differential equation. Our uniqueness results follow from a general theorem concerning non-bifurcation of solutions of equations of the form

$$u'' + g(u, u', t) = 0, \tag{3}$$

satisfying linear boundary conditions. Again using the fact that positive solutions
of (1), (2) on balls satisfy an ordinary differential equation, we prove some new
existence theorems. Our hypotheses are only concerned with the behavior of f at
infinity; in particular, we do not require any sign conditions on $f(0)$ (compare with
[6]). Thus for example, if $f(u) = O(u^k)$ as $u \to +\infty$, we give a general condition, [Eq.
(36)], for solutions to exist. This enables us to prove, for example, that if f is
superlinear, and $f'' \leqq 0$, then for each n, positive solutions exist for some range of
R's.

In order to study the non-degeneracy of solutions of (1), (2), we use the
decomposition of a solution of the linearized equations in terms of the
eigenfunctions of the Laplacian on the $(n-1)$-sphere S^{n-1}. We prove that as a
consequence of the monotonicity of the positive solution, all modes of higher order
than the second must vanish. The fact that the lowest mode is zero follows from a
general non-degeneracy theorem which we give for positive solutions of (3).
Indeed, we find a necessary and sufficient condition for a positive solution of (1), (2)
to be non-degenerate. This condition is stated in terms of the associated "time-
map," $T(p)$, $p = u(0)$, (see [9, 10]), and the statement is that a positive solution is
non-degenerate if and only if both $T'(p) \neq 0$, and $u'(R) \neq 0$. We remark that our
existence and uniqueness theorems are also obtained from studying properties of
the time-map.

We illustrate all of our theorems by considering the special cases where $f(u)/u$
is monotone. For example if $f(u)/u$ is a decreasing function, then (1), (2) has at most
one positive solution. If in addition f is positive, then solutions exist on all balls D_R^n,
$R \geqq \delta > 0$ provided that $f(u)/u \to 0$, while if $f(u)/u \geqq \lambda^2 > 0$, in $u > 0$, solutions exist
on D_R^n only for a bounded range of R's, $0 < R < R(\lambda, n)$. These positive solutions are
always non-degenerate. In fact, even more is true; namely, whenever $(f(u)/u)' < 0$,
then for any bounded domain Ω (not necessarily an n-ball), the entire spectrum
of the linearized operator lies in the open subset of $\mathbb{R}$, $x < -\eta$, for some
$\eta = \eta(f, n) > 0$. This means that the positive solution is a stable stationary
solution of the associated time-dependent problem

$$u_t = \Delta u + f(u), \quad (x, t) \in D_R^n \times \mathbb{R}_+,$$
$$u = 0, \quad (x, t) \in D_R^n \times \mathbb{R}_+.$$

Since the stationary solution depends only on the radius and is monotone, our
stability result can be interpreted as showing that asymptotically, at least, this
symmetry cannot be broken under small perturbations.

The case where both $(f(u)/u)' > 0$ and $f'' \leqq 0$ in $u > 0$ is also interesting. Namely,
we show that if $f(0) < 0$, the positive solution of (1), (2) exists only on balls D_R^n,
where $R_2 < R \leqq R_1$; that is on balls which are neither too small nor too large. These
solutions are the unique positive ones, and are non-degenerate if and only if
$R \neq R_1$. The degenerate solution on $D_{R_1}^n$ is the only one which also satisfies
homogeneous Neumann boundary conditions. Among other things, this observa-
tion shows that for solutions of (1), (2), the condition $u'(r) < 0$ in $0 < r < R$, (see [4]),
cannot be improved. In addition we show that in this case, the positive solution is
unique and is an unstable solution of the above time dependent problem. This
implies, using a well-known result, [8, p. 100], that the existence of a positive

solution cannot be obtained via the method of upper and lower solutions; nor can positive solutions be gotten as minima of (unconstrained) functionals.

Some of our results are extensions to non-autonomous systems of our earlier work, where we considered related questions for autonomous second-order equations; the proof of non-degeneracy given here is actually simpler and more intuitive than our earlier less general result in [9]. The uniqueness question, where $\Omega = \mathbb{R}^n$, has recently been treated by Peletier and Serrin [7]. See also the survey paper by Lions [6], which is concerned mainly with existence questions. In both of these articles entirely different methods than ours are used.

2. Non-Autonomous Ordinary Differential Equations

In this section we prove a general result which will be used in many of our applications. Thus we consider non-negative solutions of the equation

$$u'' + g(u, u', t) = 0, \quad 0 < t < L,\tag{3}$$

where g is a C^2-function together with the boundary conditions

$$u'(0) = u(L) = 0.\tag{4}$$

The theorem which we prove below is valid for any linear boundary conditions; we take (4) only for ease in notation, and for the application to (1), (2). Let $u(0) = p > 0$, $u'(0) = 0$ and let $u(t, p)$ denote the solution of (3), having $u(0) = p$. We set

$$A = \{p \in \mathbb{R}_+ : u(t, p) = 0 \text{ for some } t > 0\}.$$

Define a mapping $T : A \to \mathbb{R}_+$ by

$$T(p) = \min\{t > 0 : u(t, p) = 0\}.$$

Observe that $u \geqq 0$ is a solution to (3), (4) if and only if $u(0) = p \in A$ and $T(p) = L$. [$T(p)$ is differentiable; see the appendix.]

Now in order to prove the uniqueness of solutions, it is sufficient to prove that $T'(p) \neq 0$ for $p \in A^{\text{int}}$, (assuming that A is connected, as it is in the applications; otherwise we only prove local uniqueness, in other words non-bifurcation in the sense that the solutions are isolated). For, if $u(\cdot, p_1)$ and $u(\cdot, p_2)$ are solutions of (3), (4) and $T(p_1) = L = T(p_2)$, then $T'(p) = 0$ for some $p \in A^{\text{int}}$ by Rolle's theorem.

The analytical expression for T is fairly complicated and we shall avoid working with it here; instead, we shall proceed indirectly. To this end denote by $\sigma_t(q)$, the flow on $\mathbb{R}^3$ generated by (3) where $q = (u, v, t) \in \mathbb{R}^3$, with $u' = v$, $v' = -g(u, v, t)$, $t' = 1$. Thus, if $X = (v, -g, 1)$, then $\sigma_t'(q) = X_{\sigma_t(q)}$, $\sigma_0(q) = q$, and $\sigma_t(\sigma_s(q)) = \sigma_{t+s}(q)$. Let π be the projection defined by $\pi(u, v, t) = (u, v, 0)$. We begin with an easy lemma.

Lemma 1. *Assume that along an orbit $\{\sigma_t(q)\}$ of (3) that*

$$v^2 + ug(u, v, t) > 0,\tag{5}$$

where we are denoting u' by v. Then the vectors πq, X, and $\partial/\partial t$ form a basis at each point on $\{\sigma_t(q)\}$. (Here $q = q(t) = (u(t), v(t), t)$, $X = (v(t), -g(u(t), v(t), t)), 1)$, and $\partial/\partial t = (0, 0, 1)$.)

Proof. The matrix

$$(\pi q, X, \partial/\partial t) = \begin{pmatrix} u & v & 0 \\ v & -g & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

is non-singular since (5) holds.   □

For $p \in R$, we let $\bar{p} = (p, 0, 0) \in \mathbb{R}^3$, and $\sigma_t(\bar{p}) = q$. Now assuming (5), we can write

$$d\sigma_t(\bar{p})[\bar{p}] = a\pi q + bX + c\partial/\partial t, \tag{6}$$

where $a = a(t, p)$ and $b = b(t, p)$. But from (4)

$$\partial\sigma_{T(p)}(\bar{p})/\partial p = \partial(0, v(T(p)), T(p))/\partial p = (0, v'(T(p))T'(p), T'(p)),$$

and also from the chain rule,

$$\partial\sigma_{T(p)}(\bar{p})/\partial p = d\sigma_{T(p)}[\bar{p}/\|\bar{p}\|] + T'(p)X = a\pi q + (b + T'(p))X + c\partial/\partial t.$$

Thus if we equate the first components of both expressions for $\partial\sigma_{T(p)}(\bar{p})/\partial p$, and recall that $\pi q(T(p)) = (0, v, 0)$, we find $b + T'(p) = 0$, if $v(T(p)) \neq 0$, where $b = b(T(p), p)$. If (5) holds, then for local uniqueness it suffices to prove that $b \neq 0$. Note too that if we equate third components, we get $T'(p) = b + T'(p) + c$; i.e., $c = -b$.

Now we differentiate (6) with respect to t and use the well-known relation [5],

$$\frac{d}{dt} d\sigma_t(v) = dX d\sigma_t(v),$$

to obtain (where "dot" denotes differentiation with respect to t),

$$adX(\pi q) + bdX(X) + cdX(\partial/\partial t) = \dot{a}\pi q + a\dot{\pi}q + \dot{b}X + b\dot{X} + \dot{c}\partial/\partial t$$
$$= \dot{a}\pi q + a(X - \partial/\partial t) + \dot{b}X + bdX(X) - \dot{b}\partial/\partial t,$$

or

$$adX(\pi q) - bdX(\partial/\partial t) = \dot{a}\pi q + a(X - \partial/\partial t) + \dot{b}(X - \partial/\partial t). \tag{7}$$

If $\widetilde{\pi q} = (-v, u, 0)$, and $\widetilde{\pi X} = (g, v, 0)$, then taking inner products with (7) by these quantities gives successively the following two equations:

$$\widetilde{\pi q} \cdot (X - \partial/\partial t)\dot{b} = \widetilde{\pi q} \cdot [dX(\pi q) - (X - \partial/\partial t)]a - \widetilde{\pi q} \cdot (dX(\partial/\partial t))b$$
$$\widetilde{\pi X} \cdot (\pi q)\dot{a} = \widetilde{\pi X} \cdot [dX(\pi q) - (X - \partial/\partial t)]a - \widetilde{\pi X} \cdot (dX(\partial/\partial t))b. \tag{8}$$

Since

$$X = \begin{pmatrix} v \\ -g \\ 1 \end{pmatrix}, \quad \text{and} \quad dX = \begin{pmatrix} 0 & 1 & 0 \\ -g_2 & -g_2 & -g_3 \\ 0 & 0 & 0 \end{pmatrix},$$

the equations (8) become

$$-(v^2 + ug)\dot{b} = u(g - ug_u - vg_v)a + ug_t b$$
$$(v^2 + ug)\dot{a} = v(g - ug_u - vg_v)a + vg_t b. \tag{9}$$

We define the quantity ϕ by

$$\phi = g - ug_u - vg_v, \qquad (10)$$

and we then have the following theorem.

Theorem 2. *Suppose that along an orbit $\sigma_t(\bar{p})$ of (3), (4), condition (5) holds, and $uv\phi$ $g_t < 0$ and both $\phi > 0$, $g_t > 0$ in $0 < t < L$. Then Eq. (3) together with boundary conditions (4) has isolated (i.e., locally unique) solutions. If the domain of T is connected, then global uniqueness holds in the sense that this problem has at most one solution.*

Proof. Note $v(t) < 0$ on $0 < t < L$ since $u(t) > 0$ on this range. We show now that $b(t) < 0$ on $0 < t < \varepsilon$, for some $\varepsilon > 0$. To see this, consider the first equation in (9), which we write in the form, $b' + hb = ka$, where $h = (ug_t)(v^2 + ug)^{-1}$, and $k = -u\phi(v^2 + ug)^{-1}$. Note that $k < 0$, and $a > 0$ for small $t > 0$. If

$$H(t) = \int_\delta^t h(s)ds, \qquad \delta > 0,$$

then multiplying our equation by e^H and integrating, we get,

$$b(t) = e^{H(\varepsilon) - H(t)}b(\varepsilon) + e^{-H(t)} \int_\varepsilon^t e^{H(s)}k(s)a(s)ds.$$

Note that the above integral is negative, and that $\operatorname{sgn} h = \operatorname{sgn}(ug_t) > 0$ so that $H(\varepsilon) < 0$ if $\varepsilon < \delta$. Thus if we let $\varepsilon \to 0$ in the above equation, we find that $b(t) < 0$ for small $t > 0$ since $b(0) = 0$.

Now at $t = 0$, $a = 1$, and $b = 0$. Also, when $a > 0$ and $b = 0$, $\operatorname{sgn}\dot{b} = \operatorname{sgn}(-u\phi)$, and when $a = 0$, and $\operatorname{sgn}b = \operatorname{sgn}(-u\phi)$, $\operatorname{sgn}\dot{a} = \operatorname{sgn}(-uv\phi g_t) > 0$ if $t > 0$. Thus the "orbit" $(a(t), b(t)) \equiv [a(u(t), v(t)), b(u(t), v(t))]$ is trapped in the quadrant $a > 0$, $\operatorname{sgn}b = \operatorname{sgn}(-u\phi) < 0$ (see Fig. 1). Noting that $a \neq 0$ when $t = L$ (for otherwise $b(L) = 0$ and hence $c(L) = 0$, so that $d\sigma_L(\bar{p})[\bar{p}] = 0$; this is impossible since σ_L is a diffeomorphism and $\bar{p} \neq 0$), we see that $b(L) < 0$. This completes the proof.

3. Application to Uniqueness Problems

In this section we shall apply our theorem to solutions[1] of the Dirichlet problem (1), (2). Here is our first theorem.

Theorem 3. *Suppose that $f: \mathbb{R}_+ \to \mathbb{R}$ is C^2 and satisfies*

$$(f(u)/u)' < 0, \quad for \quad u > 0. \qquad (11)$$

Then there is at most one non-negative solution to the problem (1), (2). In addition $\operatorname{dom}(T)$ is connected.

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1   By a solution, we always mean a positive solution u, $u \not\equiv 0$
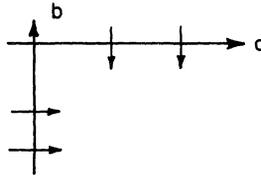
Fig. 1

Proof. First note that (11) implies $f(0) \geq 0$. From the results in [4], the non-negative solutions of (1), (2) are radially symmetric, and thus satisfy the ordinary differential equation

$$u'' + \frac{n-1}{r} u' + f(u) = 0, \quad 0 < r < R, \tag{12}$$

together with the boundary conditions

$$u'(0) = u(R) = 0. \tag{13}$$

In the notation of the last section, we have

$$g(u, u', r) = \frac{n-1}{r} u' + f(u);$$

thus $\phi = f(u) - uf'(u) > 0$, and $g_r = -(n-1)r^{-2}u' > 0$, since $u' < 0$ in $0 < r < R$ (see [4]). It follows that $uv\phi g_r < 0$ in $r > 0$, where we have set u' equal to v.

In order to apply Theorem 2, we must also show that

$$h(r) \equiv v(r)^2 + u(r)g(u(r), v(r), r) > 0 \tag{14}$$

along any solution $(u(r), v(r))$ of (12), (13), where, as above, $v(r) = u'(r)$. To do this it suffices to show two things; namely i) $h(0) > 0$, and ii) $h'(r) > 0$ when $h(r) = 0$.

For i), note that $u(0) > 0$, $v(0) = 0$, and $v(r) < 0$ if $r > 0$. Thus, since $\lim_{r \to 0} vr^{-1} = u''(0)$ and $nu''(0) + f(u(0)) = 0$, we see that $f(u(0)) \geq 0$. But $f(u(0)) \neq 0$ since otherwise we would be at a "rest" point. Thus

$$h(0) = u(0)\left[-(n-1)\frac{f(u(0))}{n} + f(u(0)) \right]$$

$$= \frac{1}{n} u(0)f(u(0)) > 0.$$

For ii), we have, when $v^2 = -ug$,

$$h'(r) = -2vg + vg + u(g_u v - g_v g + g_r)$$

$$= -vg + uvg_u - ugg_v + ug_r$$

$$= -vg + uvg_u + v^2 g_v + ug_r$$

$$= -v\phi - \frac{n-1}{r^2} uv$$

$$> 0,$$

since $v < 0$ in $0 < r \leq R^2$. Thus (14) holds.

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2   Thus if $v(R) = 0$, then $v'(R) = -f(0) < 0$. Hence $v(R - \varepsilon) > 0$ for some $\varepsilon > 0$; this is a contradiction

Finally, in order to complete the proof of the theorem we must prove that the domain of T is connected. To this end, first note that $f(0) \geqq 0$, and that f can be zero at most once in $u > 0$.

Now we shall show that $D = \mathrm{dom}(T)$ is open. Let $p \in D$; then $f(p) > 0$, and since f satisfies (11), $f(u) > 0$ for $0 < u \leqq p$. It follows that $v(r, p) < 0$ for $0 < r \leqq R = T(p)$. Thus $u(T(p) + \varepsilon, p) < 0$ for small $\varepsilon > 0$, so $u(T(p) + \varepsilon, q) < p$ for q near p; hence $q \in D$ and D is open.

Next, let $\bar{p} = \inf D$, and choose $p_n \in D$ with $p_n \searrow p$. Since $u(T(p_n), p_n) = 0$, and $T' > 0$ on D (from Theorem 2), we see that $\lim u(T(p_n), p_n) = u(\lim T(p_n), \bar{p}) = 0$; thus $\bar{p} \in D$. But as we have observed above, if $\bar{p} > 0$, then $u'(T(\bar{p}), \bar{p}) < 0$, and this would violate the definition of $\bar{p}$. Thus $\bar{p} = 0$ and $D = \{u > 0\}$. This completes the proof.

Remarks. It is not very hard to show that $D = \mathrm{dom}(T)$ is in fact, non-void provided that f satisfies (11). Thus, if $f(u)/u \searrow A > 0$ as $u \to \infty$, then $f(u) = O(u)$ as $u \to \infty$, and by Corollary 15 (below), there are solutions $u(\cdot, p)$ for all sufficiently large p, and hence for all p. If on the other hand, $f(u)/u \searrow 0$ as $u \to \infty$, [or $f(\beta) = 0$ for some $\beta > 0$], then since $\bar{u} > u$ implies $f(u)/u > f(\bar{u})/\bar{u}$, we see that $f'(0) = k > 0$. Therefore, defining $g(u) = k^{-1} f(u)$, we see there is a one-to-one correspondence between solutions of the Dirichlet problems for (1) and $\Delta u + g(u) = 0$, (for different R's, of course). Thus we may assume that $f'(0) = 1$, and this enables us to apply Theorem 1.4 of [6] (see also [1]) in order to obtain a solution of (1), (2). Thus again $D \neq \emptyset$.

Our next result considers the case where ϕ is negative; here we find it necessary to further restrict f [3].

Theorem 4. *Suppose that $f: \mathbb{R}_+ \to \mathbb{R}$ satisfies the conditions*

$$(f(u)/u)' > 0 \quad \text{and} \quad f''(u) \leqq 0 \tag{15}$$

for all $u > 0$. Then $\mathrm{dom}(T)$ is connected, and there is at most one non-negative solution of the problem (1), (2).

Proof. Note first that $f(0) \leqq 0$. As in Theorem 3, we first show that $T'(p) = -b(T(p)) < 0$ for orbits with $v(T(p), p) \neq 0$, and then show that the domain of T is an interval.

From (7), we have

$$a \begin{pmatrix} 0 \\ \phi \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ g_r \\ 0 \end{pmatrix} = a' \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + b' \begin{pmatrix} v \\ -g \\ 0 \end{pmatrix},$$

and this gives the equations

$$\begin{aligned} a'u + b'v &= 0 \\ a'v - b'g &= a\phi + bg_r. \end{aligned} \tag{16}$$

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3   If we do not have some further restriction on f, then solutions may not exist. For example, if $f(u) = u^k$, then as is well-known (see, e.g. [6]), solutions do not exist if $k > \dfrac{n+2}{n-2}$

Since $g_r = -(n-1)vr^{-2} > 0$, $\phi < 0$, $u > 0$ and $v < 0$, we have $uvg_r\phi > 0$, and the previous argument does not apply. However, since $a(0) = 1$, $b(0) = 0$, $u(0) > 0$, $v(0) = 0$, and since we have shown in the last theorem that $g > 0$ when $r = 0$, we conclude that $b(r) > 0$ for small r. We wish to show that $b(R) > 0$.

Now from (16), $b' > 0$ when $b = 0$, $a > 0$, so it suffices to show that $a(r) > 0$, or equivalently that $0 < \theta < \pi/2$, where $\tan\theta = b/a$. If we differentiate this equation with respect to r we get

$$\theta' = \frac{ab' - ba'}{a^2 + b^2} = \frac{-\dfrac{u}{v}aa' - ba'}{a^2 + b^2} = \frac{-(ua + vb)}{v(a^2 + b^2)}a'$$

$$= \frac{-(ua + vb)}{(a^2 + b^2)} \cdot \frac{(a\phi + bg_r)}{(v^2 + ug)}.$$

Next define β by $\tan\beta = -\phi/g_r$; then $0 \leq \beta < \pi/2$, and $\theta' = 0$ when $\theta = \beta$. Thus we shall show

$$(\beta - \theta)'|_{\beta = \theta} = \beta'|_{\beta = \theta} > 0,$$

from which it will follow that $\theta \leq \beta < \pi/2$. But this is easy since

$$\beta' = \frac{-g_r\phi + \phi g_{rr}}{g_r^2 + \phi^2} = (n-1)\frac{v}{r^2}\left[-vuf''(u) + \frac{2}{r}\phi\right] > 0,$$

in view of our hypotheses.

As above, we must show that (5) holds along each orbit for which $v(R) < 0$. To this end let $h(r)$ be defined as in (14); then $h(0) > 0$, and $h(R) = v(R)^2 > 0$. We claim that it suffices to show that the function

$$H(r) = \frac{h'(r) + \dfrac{n-1}{r}h(r)}{-v(r)} = f(u(r)) - u(r)f'(u(r)) \tag{17}$$

is non-increasing. For, if this were so and r_1 was the first zero of h, then $h'(r_1) \leq 0$, so $H(r_1) \leq 0$. But since $h(R) > 0$, we let r_2 be the largest value in (t_1, R) for which $h(r_2) = 0$ and then let $r_3 \in (t_2, R)$ be such that $h(r_3) > 0$ and $h'(r_3) > 0$. Then

$$H(r_3) = \frac{h'(r_3) + \dfrac{n-1}{r_3}h(r_3)}{-v(r_3)} > 0 \geq \frac{h'(r_1) + \dfrac{n-1}{r_1}h(r_1)}{-v(r_1)} = H(r_1);$$

this contradiction shows that no such r, exists; i.e., $h(r) > 0$, $0 \leq r \leq R$. To show that H is non-increasing, note that (17) gives $H'(r) = -uf''(u)v \leq 0$.

Observe now that if $\phi(u) = f(u) - uf'(u)$, then $f(0) = \phi(0) \leq 0$, and if $f(u) = 0$ for some $u > 0$, then $0 > \phi(u) = -uf'(u)$, so $f'(u) > 0$. Thus f can have at most one positive root; call it p_0. Furthermore, if $f(\bar{u}) > 0$ for some $\bar{u}$, then $f'(\bar{u}) > 0$, so $f'(u) > 0$ for all $u > \bar{u}$.

We shall now show that the domain of T is connected. In order to do this, we assume that $p_1 \in \text{domain}(T)$, $p_1 > 0$. We define $B = \{p > p_1 : p \notin \text{domain}(T)\}$, and then show that B is empty. If B is not empty, let $p_2 = \inf B$. Then $p_2 \geq p_1$, and we have

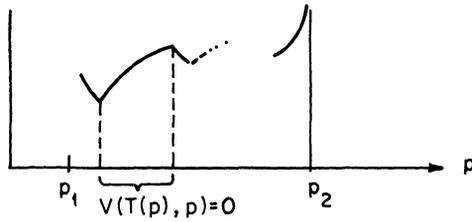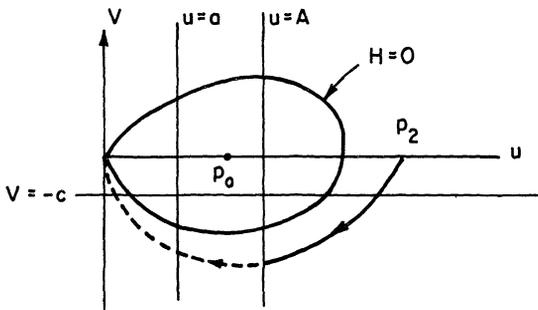**Fig. 2**

Fig. 3

Lemma A. $p_2 \in \text{dom}(T)$.

Proof. Let $p_n \in \text{dom}(T)$, $p_n \nearrow p$. If there is a subsequence $\{p_n'\}$ with $\lim_n T(p_n') = t_2$ $< \infty$, then since $u(T(p_n'), p_n') = 0$ for each n, we have $u(t_2, p_2) = 0$, and $p_2 \in \text{dom}(T)$. For example, if $f(0) = 0$, then no orbit has $v(T(p), p) = 0$. Thus $T'(p) < 0$ for $p < p_2$ and $\lim_{p \to p_2^-} T(p) = T(p_2)$ is finite, so $p_2 \in \text{dom}(T)$. We may thus assume that $f(0) < 0$.

Suppose now that $T(p_n) \to +\infty$. This can only occur if $v(T(p), p) = 0$ for a sequence of intervals approaching p_2 since $T'(p) < 0$ if $v(T(p), p) \neq 0$; see Fig. 2.

We let $H(u, v) = v^2/2 + F(u)$, where $F' = f$, and $F(0) = 0$. Then $H' \leq 0$, $H(p_0, 0) = F(p_0) < 0$, so $H < 0$ in a neighborhood of $(p_0, 0)$. Note that $H(0, 0) = 0$, and that $H(u, v) = 0$ is a bounded simple closed curve in the $u - v$ plane, which contains $(p_0, 0)$.

Let $r > 0$ be given, and let $p_1 < p < p_2$; then $H(u(T(p), v(T(p), p)) \geq 0$, and hence $H' \leq 0$ implies $H \geq 0$ below the curve $r = T(p)$ (see Fig. 2). Since $T(p) \to \infty$ as $p \to p_2^-$, we have $H(u(r, p_2), v(r, p_2)) > 0$ for all $r \geq 0$.

We now claim that the orbit through $(p_2, 0)$ meets $u = 0$ at a point with $v \leq 0$; this will imply that $p_2 \in \text{dom}(T)$, and will complete the proof of the lemma. To prove our claim, let $A > p_0$; then the orbit through $(p_2, 0)$ meets $u = A$ (see Theorem 8, below), at a point where $H > 0$. Since $H(p_0, 0) < 0$, there is a $c > 0$ and an interval (a, A) such that $H(u, c) = F(u) + c^2/2 < 0$ for $u \in (a, A)$. Thus the orbit through $(p_2, 0)$ meets $u = A$ at a point with $v < c$, and meets $u = a$ with $v < c$, for some $r < (A - a)/c$. That is, this orbit meets the line $u = a$ for finite r (see Fig. 3).

This implies that the orbit cannot stay in the region $u>0$ for all $r>0$. To see this, suppose the contrary. Since $f<0$ on $0\leq u\leq a$, we can find $D>0$ such that $f(u)\leq -D$ on this interval. Since $-(r^{n-1}v)'=f(u)r^{n-1}\leq -r^{n-1}D$, on this interval, we obtain by integrating from T to $T+t$ $[u(T)=a]$,

$$-(T+t)^{n-1}v(T+t)\leq D\left[\frac{-(T+t)^n}{n}+\frac{T^n}{n}\right]-T^{n-1}v(T).$$

This shows that $v(t+T)>0$ for large $t>0$. Thus the orbit through $(p_2,0)$ crosses $H=0$. This is a contradiction, so the claim holds, and the lemma is proved.

If $r_2=T(p_2)$, and $v(r_2,p_2)<0$, then $u(r,p_2)<0$ for $r>r_2$, and hence by the continuity of the flow, p_2 is in the interior of the domain of T; i.e. $p_2\neq \inf B$. This is impossible, so we must have $v(r_2,p_2)=0$. We now show

Lemma B. $B=\phi$.

Proof. Choose $\varepsilon>0$ so that $\varepsilon<p_0$ and $\phi(u)+\varepsilon f'(u)<0$ for $0\leq u\leq p_2+1$. (This can be achieved since $v(r_2,p_2)=0$ implies $f(0)<0$ so that $\phi(u)=f(u)-uf'(u)<0$ if $u\geq 0$.) Now choose δ, $0<\delta<1$ such that $u(r_2,p)<\varepsilon$ for $p_2\leq p\leq p_2+\delta$ (we show in the appendix that u depends smoothly on p).

Now we shall prove that there is a $\delta'\leq \delta$ such that $[p_2,p_2+\delta']\cap B=\phi$; this will be the desired contradiction since it will violate the definition of p_2. We begin with the following

Claim. There is a δ', $0<\delta'\leq \delta$, such that if $p\in [p_2,p_2+\delta']\cap B$, then $v(r_2,p)\leq 0$.

If the claim were false, then we could find $p_n\in B$, with $p_n\searrow p_2$ and $v(r_2,p_n)>0$. Then set $z_n(r)=u(r,p_n)-u(r,p_2)$, and note that $z_n''+(n-1)r^{-1}z_n'+f'(\xi_n(r))z_n=0$, where $\xi_n(r)$ is intermediate to $u(r,p_n)$ and $u(r,p_2)$. Also, $z_n(0)>0$, $z_n'(0)=0$, $z_n(r_2)>0$, and $z_n'(r_2)>0$. Since z_n satisfies a linear equation, $z_n(r)^2+z_n'(r)^2>0$ for $0\leq r\leq r_2$. Thus we may define $\theta_n(r)=\arctan z_n'(r)/z_n(r)$ [or $\theta_n(r)=\arccot z_n(r)/z_n'(r)$ near $z_n(r)=0$], and observe that $\theta_n(0)=0$ and $\theta_n(r_2)<-3\pi/2$. Then for any r, $0\leq r\leq r_2$, we have

$$\theta_n'(r)=\frac{-(n-1)}{r}\sin\theta_n\cos\theta_n-f'(\xi_n(r))\cos^2\theta_n-\sin^2\theta_n.$$

Let $z(r)=u'(r,p_2)$; then z satisfies the equation

$$z''+(n-1)r^{-1}z+f'(u(r,p_2))z=0,$$

and the boundary conditions $z(0)=z(r_2)=0$. Note that $z'(0)<0$ and $z'(r_2)>0$. Again, $z(r)^2+z'(r)^2>0$ and we may define $\theta(r)=\arctan z'(r)/z(r)$ (or $\theta(r)=\arctan z(r)/z'(r)$), and observe that $\theta(0)=-\pi/2$, $\theta(r_2)=-3\pi/2$, $-\pi/2\geq \theta(r)\geq -3\pi/2$, and for any r, $0\leq r\leq r_2$,

$$\theta'(r)=\frac{-(n-1)}{r}\sin\theta\cos\theta-f'(u(r,p_2))\cos^2\theta-\sin^2\theta.$$

Moreover, for n large, $f'(\xi_n(r))$ can be made uniformly close to $f'(u(r,p_2))$ on $0\leq r\leq r_2$. Also since $\theta(0)<\theta_n(0)$, and $\theta(r_2)>\theta_n(r_2)$, there is an $r_1\in (0,r_2)$ with $\theta(r_1)=\theta_n(r_1)$. Now there is an $r_3=r_3(p_n)\in (0,r_2)$ with $\theta_n(r_3)=-\pi/4$; thus $\theta_n(r_3)-\theta(r_3)$
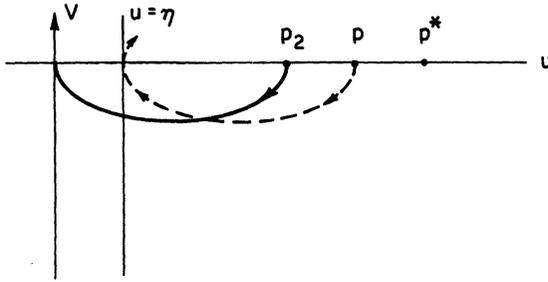
Fig. 4

$\geq -\pi/4 + \pi/2 = \pi/4$. Since

$$-\theta_n'(r) \leq f'(u(r, p_2)) \cos^2 \theta_n(r) + \sin^2 \theta_n(r),$$

and the right side is uniformly bounded on $[0, r_2]$, we see that $r_3(p_n) \geq \eta > 0$, where η is independent of n. Also $r_3(p_n) < r_2$. Thus for n large, θ_n and θ satisfy differential equations which are "close", $\theta_n(r_1) = \theta(r_1)$, but $\theta_n(r) - \theta(r)$ is *not* uniformly small. This violates the standard continuous dependence theorems. (Note that on $\eta \leq r \leq r_2$ the equation has continuous coefficients.) The proof of the claim is complete.

Now assume that there is some $\tilde{p} \in B \cap I$, where $I = [p_2, p_2 + \delta']$. Then $\varepsilon > u(r_2, \tilde{p}) > 0$, and $v(r_2, \tilde{p}) \leq 0$. For $p \in I$, let $g(p) = \min\{u(r, p) : r \in [0, r_2]\}$, and notice that if $g(p) \leq 0$, then there is an $r < r_2$ with $u(r, p) \leq 0$ so $p \in \text{dom}(T)$. Now $g(\tilde{p}) > 0$, and we shall show that this gives a contradiction. Let $\eta = \max\{g(p) : p \in I\}$; then $\eta > 0$. If $p^* = \inf g^{-1}(\eta)$; then for $p_2 \leq p < p^*$, $u(\bar{T}(p), p) = \eta$ for some $\bar{T}(p) < r_2$ [for such p we have $g(p) < \eta$ since $g(p) = \eta$ would imply $p = p^*$; thus $u(t, p) < \eta$ for some $t < r_2$]. Now $\max\{\bar{T}(p) : p \in I\} = r_2 = \bar{T}(p^*)$, and $\bar{T}(p^*) - \bar{T}(p_2) > 0$. Also, $v(\bar{T}(p), p) < 0$ for $p_2 < p < p^*$. To see this, suppose that $v(\bar{T}(p), p) \geq 0$. Thus referring to the above figure, $v(\bar{T}(p), p) > 0$ is clearly impossible, while if $v(\bar{T}(p), p) = 0$, then $v(r_2, p) > 0$, and this violates our earlier claim since $p \in B$. Thus $v(\bar{T}(p), p) < 0$ for $p_2 < p < p^*$.

Now we shift coordinates by writing $\bar{u} = u - \eta$. Since $\eta < \varepsilon$, $\phi(u) - \eta f'(u) < 0$ for $u \in [0, p_2 + \delta']$. The idea is that in this "shifted" frame, the "new" form $\bar{\phi}$ is still negative. Thus, consider the equation $\bar{u}'' + (n-1)r^{-1}\bar{u}' + f(\bar{u} + \eta) = 0$, together with the boundary conditions $\bar{u}'(0) = \bar{u}(R) = 0$. We have $\bar{u}(r_2, p) < 0$ if $p \in [p_2, p_2 + \delta']$, so that this interval is in $\text{dom}(\bar{T})$. Furthermore, $\bar{\phi}(\bar{u}) = f(\bar{u} + \eta) - (\bar{u} + \eta)f'(\bar{u} + \eta) = \phi(u) + \eta f'(u) < 0$, on this interval, and since $f''(\bar{u} + \eta) \leq 0$, we may conclude, as above, that $\bar{T}'(p) < 0$ on $[p_2, p_2 + \delta']$. But since $\bar{T}(p^*) = r_2$, and $\bar{T}(p_2) < r_2$, we have $0 < \bar{T}(p^*) - \bar{T}(p_2) = (p^* - p_2)\bar{T}'(\xi)$, for some $\xi \in (p^*, p_2)$. This implies $\bar{T}'(\xi) > 0$, which is the desired contradiction. The proof of the lemma is complete.

We have thus shown that the domain of T is connected. We must still investigate the behavior of T on the set

$$D = \{p \in \text{dom}(T) : u(T(p), p) = v(T(p), p) = 0\},$$

since this set may consist of intervals on which $T' > 0$, and this would violate uniqueness of solutions.

Lemma C. *D is a single point; namely* $D = \{\bar{p}\}$, *where* $\bar{p} = \inf \mathrm{dom}(T)$.

Proof. In Sect. 4, Corollary 16, we shall show that the domain of T is non-void if (15) holds, (and f is not everywhere negative!); thus $\bar{p}$ exists.

Suppose that $\tilde{p}_1$ and $\tilde{p}_2$ are in D. If there is a $p \notin D$, p between $\tilde{p}_1$ and $\tilde{p}_2$, then there is an $\bar{r} > 0$ with $u(\bar{r}, p) < 0$, $v(\bar{r}, p) = 0$. If $\sigma = u(\bar{r}, p)$, then if we shift coordinates, and write $\bar{u} = u - \sigma$, then Lemma B is violated in this frame; namely, $\tilde{p}_2 > p$ and $\tilde{p}_2 \notin \mathrm{dom}(T)$. Thus D must be an interval. Curiously, we can easily eliminate the case $D = \mathrm{dom}(T)$ by referring to an existence theorem in the next section, but the case where D is a finite interval is much more difficult. Namely, if D were an unbounded interval of the form $p \geq \bar{p}$, we again shift coordinates, $\bar{u} = u + \varepsilon$, where $\varepsilon > 0$. Note that the hypotheses (15) are valid in this new frame, yet there are no solutions of the Dirichlet problem; this violates Corollary 16 below. Thus D can only be a compact interval, $[\tilde{p}_1, \tilde{p}_2]$. (The argument we now give rules out both possibilities.) First observe that T must be monotone on this interval; otherwise there would be two distinct points, p_1', p_2' on this interval for which $T(p_1') = T(p_2')$, and $u(T(p_i'), p_i') = v(T(p_i'), p_i') = 0$, $i = 1, 2$. This violates the standard uniqueness theorem for ordinary differential equations.

We next show that $T' \leq 0$ on $D = [\tilde{p}_1, \tilde{p}_2]$[4]. Thus, suppose that $T' > 0$ on D. We may assume $\tilde{p}_2 > \tilde{p}_1$; if equality holds, then there is nothing to prove. Let $T_i = T(\tilde{p}_i)$, $i = 1, 2$; then $T_2 > T_1$. Choose $\varepsilon > 0$ so small that both $T_2 - \varepsilon > T_1$ and $F(A_\varepsilon) < 0$, where $A_\varepsilon = u(T_2 - \varepsilon, \tilde{p}_2)$; the latter can be achieved since $f(0) < 0$. Note that $v(T_2 - \varepsilon, \tilde{p}_2) \neq 0$ since $H(u(T_2 - \varepsilon, \tilde{p}_2), v(T_2 - \varepsilon, \tilde{p}_2)) > 0$; see Fig. 3. Consider now the equation $u(\tau, p) = A_\varepsilon$, $\tilde{p}_1 \leq p \leq \tilde{p}_2$. Since $u'(\tau, p) < 0$, we see that this equation defines a function, $\tau = \tau(p)$, $\tilde{p}_1 \leq p \leq \tilde{p}_2$. If ε is small, we have $\tau(\tilde{p}_2) = T(p_2) - \varepsilon > T(p_1) > \tau(\tilde{p}_1)$; i.e., $\tau(\tilde{p}_2) > \tau(\tilde{p}_1)$. On the other hand if we shift coordinates by writing $\bar{u} = u - A_\varepsilon$, then in this frame, if ε is small, A_ε is near zero, so (15) holds in this frame, and $v(\tau(p), p) \neq 0$ for $\tilde{p}_1 \leq p \leq \tilde{p}_2$. Thus $\tau'(p) = \bar{T}'(p) < 0$ so $\tau(\tilde{p}_1) > \tau(\tilde{p}_2)$. This is a contradiction, and so $T' \leq 0$ on D. (*The proof of Theorem 4 is now complete.*)

To finish the proof of Lemma C, we proceed in a manner similar to the proof of the above claim. Thus, suppose that D is a non-trivial bounded interval, and choose $\tilde{p}$ in D^{int}. Then $w(r) = u'(r, \tilde{p})$ satisfies $w'' + (n-1)r^{-1}w' + f'(u(r, \tilde{p})) = 0$, with $w(0) = w(T(\tilde{p})) = 0$, and $w'(0) < 0$, $w'(T(\tilde{p})) > 0$. We may define $\theta(r) = \arctan w'(r)/w(r)$, and note that $\theta(0) = -\pi/2$, $\theta(T(\tilde{p})) = -3\pi/2$, $-\pi/2 \geq \theta(r) \geq -3\pi/2$. Also for $0 \leq r \leq T(\tilde{p})$, $\theta'(r) = -(n-1)r^{-1}\sin\theta\cos\theta - f'(u(r, \tilde{p}))\cos^2\theta - \sin^2\theta$. If now $p > \tilde{p}$, $p \in D$, we set $z(r) = u(r, p) - u(r, \tilde{p})$, and so $z'' + (n-1)r^{-1}z' + f'(\xi(r))z = 0$, $\xi(r)$ between $u(r, p)$ and $u(r, \tilde{p})$. Also $z(0) > 0$, $z'(0) = 0$, $z(T(\tilde{p})) > 0$, $z'(T(\tilde{p})) > 0$, since $u(T(\tilde{p}), \tilde{p}) = 0 = u'(T(\tilde{p}), \tilde{p})$, and $T(\tilde{p}) > T(p)$ (equality would violate uniqueness), so $u'(T(\tilde{p}), p) > 0$ and $u(T(\tilde{p}), p) > 0$. Now we define $\tilde{\theta}(r) = \arctan z'(r)/z(r)$. Then $\tilde{\theta}$ satisfies an equation which is "close" to the z equation uniformly on $[0, T(\tilde{p})]$ if $\tilde{p}$ is close to p. Since $\theta(0) < \tilde{\theta}(0)$, and $\tilde{\theta}(T(\tilde{p})) < \theta(T(\tilde{p}))$ we can find $t_1 \in (0, T(\tilde{p}))$ with $\theta(t_1) = \tilde{\theta}(t_1)$. Also, there is $t_2 = t_2(p) \in [0, T(\tilde{p})]$ with

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4   Note that this will prove that T is a monotone function, and hence will complete the proof of Theorem 4

$\tilde{\theta}(t_2) = -\pi/4$, so $\tilde{\theta}(t_2) - \theta(t_2) \geq \pi/4$. Since $-\theta'$ is bounded from above, uniformly on $[0, T(\tilde{p})]$, we see that $t_2(p) \geq \eta > 0$, η independent of p. Thus $\theta(t_1) = \tilde{\theta}(t_1)$, but $\theta - \tilde{\theta}$ is not uniformly small; this is a contradiction, and the proof of Lemma C is complete. $\square$

As we have noted earlier, the proof of Theorem 4 is also complete. $\square$

We shall next show that if f satisfies (15), then[5]
  i) if $f(0) < 0$, dom$(T) = [\bar{p}, \infty)$ for some $\bar{p} > 0$, and the orbit starting at $(\bar{p}, 0)$ is the unique one going through the origin; i.e., $u(T(\bar{p}), \bar{p}) = 0 = v(T(\bar{p}), \bar{p})$,
  ii) if $f(0) = 0$, dom$(T) = [0, \infty)$ and $v(T(p), p) < 0$ for $p > 0$.
  Thus, suppose first that $f(0) < 0$, and let p_0 be the positive root[6] of f. If $p > p_0$, p near p_0, then if $H(u, v) = v^2/2 + F(u)$, $(F' = f)$, we have $H(0, p) < 0$, and since H decreases on orbits, such p cannot be in dom(T). Thus $\bar{p} = \inf \text{dom}(T) > 0$, and as we have seen earlier, the orbit through $(\bar{p}, 0)$ is the unique one which goes through the origin. If $f(0) = 0$, then in order for solutions to exist, we must have $f'(0) > 0$. It follows that for $u > 0$, u near 0, there is a $c > 0$ such that $f(u) \geq cu$. Now define $\tan \theta = v/u$, and for $u > 0$, u near 0, we have

$$\theta' = \frac{u\dot{v} - v^2}{u^2 + v^2} = \frac{-[v^2 + (n-1)r^{-1}uv + uf(u)]}{u^2 + v^2}$$

$$\leq \frac{-[v^2 + (n-1)r^{-1}uv + cu^2]}{u^2 + v^2}$$

$$= \frac{-[\tan^2\theta + (n-1)r^{-1}\tan\theta + c]}{\sec^2\theta}$$

$$= -[\sin^2\theta + (\sin\theta\cos\theta)(n-1)r^{-1} + c\cos^2\theta].$$

As $r \to \infty$, $\theta \to \pi/2$, and for large r, we see that θ' is uniformly negative; i.e., $\theta' \leq -\eta$ for some $\eta > 0$, for all sufficiently large r. It follows that $\theta = -\pi/2$ for finite r. That is, $p \in \text{dom}(T)$ for all sufficiently small $p > 0$; this proves (ii).
  We close this section with a few remarks. First note that if (15) holds, and $f(0) < 0$, (so for example if $f(u) = u - e^{-u}$), then positive solutions to the Dirichlet problem exist, and there *must* be exactly one positive solution for which $u(R) = u'(R) = 0$. In particular, this shows that the conclusion $u'(r) < 0$ for $0 < r < R$ in [3], cannot be improved. Finally, note that for this class of functions f, there is a real number R_1 such that if $R > R_1$, the problem (1), (2) has no positive solution; (this follows since $T' < 0$ and T is bounded. The proof of the latter fact is similar to what we have shown in the proof of Lemma A. Namely, $H = 0$ is a bounded closed curve containing $(p_0, 0)$. If $p_n \searrow \bar{p}$ and $T(p_n) \to +\infty$, then since $H(u(r, \bar{p}), v(r, \bar{p})) > 0$, the argument in the proof of Lemma A applies, and gives a contradiction.) But even more can be said about the domain of T. This will be discussed in the next section; *Theorem* 17.

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5   Assuming solutions exist; they do by Corollary 16, below
6   If f has no positive root then no positive solution exists

4. Existence of Radial Solutions

In this section we shall use techniques from the theory of ordinary differential equations in order to prove some existence theorems for positive solutions of (1), (2), or equivalently, of (12), (13). Our hypotheses are only concerned with the behavior of f at infinity; in particular, we do not require that $f(0) \geq 0$. Thus, for example, if $f(u) = O(u^k)$ as $u \to \infty$, we give a general condition for positive solutions to exist (Theorem 14). This enables us to prove, for example, that if $k < 2$ and $n \leq 3$, or if $k = 1$ and $n \geq 1$, positive radial solutions must exist. These results are applied to the case where f satisfies (15), and they enable us to prove existence of solutions for all n.

We begin by writing the Eq. (12) as a first-order system:

$$u' = v, \qquad v' = -(n-1)vr^{-1} - f(u). \tag{18}$$

In order to obtain solutions of this system, we find it convenient to consider the following two associated systems of equations:

$$\tilde{u}' = \tilde{v}, \qquad \tilde{v}' = -(n-1)\tilde{v}r^{-1} - B;$$
$$\tilde{u}(T) = A, \qquad \tilde{v}(T) = q \tag{19}$$

and

$$z' = w, \qquad w' = -(n-1)wT^{-1} - B,$$
$$z(T) = A, \qquad w(T) = q. \tag{20}$$

Here A, B, q and T will be suitably chosen constants. Our first result is a comparison theorem relating solutions of the above systems.

Theorem 5. *Suppose that $f(u) > B$ for $0 \leq u \leq A$, $u(T) = A > 0$, and $v(T) = q < 0$, for some $T > 0$. Then $\tilde{u}(r) \geq u(r)$ for $r \geq T$ on $0 \leq u \leq A$, and if $\tilde{v}(r) < 0$, $T \leq r \leq T_1$, then $z(r) \geq \tilde{u}(r)$ on this range.*

Proof. Let $h(r) = \tilde{u}(r) - u(r)$. Then $h(T) = 0$, $h'(r) = \tilde{v}(r) - v(r)$, and $h'(T) = 0$. Next, $h''(r) = \tilde{v}'(r) - v'(r) = -(n-1)r^{-1}h'(r) - B + f(u)$, so $h''(T) > 0$. Now if $h'(r_1) = 0$ for some $r_1 > T$, $0 \leq u \leq A$, then $h''(r_1) > 0$. It follows that $h'(r) > 0$ for all $r > T$ on $0 \leq u \leq A$, so that $\tilde{u}(r) > u(r)$ if $r > T$, $0 \leq u \leq A$. For the second part, let $g(r) = z(r) - \tilde{u}(r)$. Then $g(T) = 0$, $g'(r) = w(r) - \tilde{v}(r)$, $g'(T) = 0$, $g''(r) = w'(r) - \tilde{v}'(r)$, $g''(T) = 0$, $g'''(r) = w''(r) - \tilde{v}''(r) = -(n-1)T^{-1}w'(r) + (n-1)r^{-1}\tilde{v}'(r) - (n-1)r^{-2}\tilde{v}(r)$, and $g'''(T) = -(n-1)T^{-2}\tilde{v}(T) > 0$. Thus $g(r) > 0$ if $T < r < T + \varepsilon$, for some $\varepsilon > 0$. On the other hand, if $g'(r_1) = 0$ for some r_1, $T_1 \geq r_1 > T$, then $g''(r_1) = -(n-1)w(r_1)T^{-1} + (n-1)\tilde{v}(r_1)r_1^{-1} = -(n-1)\tilde{v}(r_1)(T^{-1} - r_1^{-1}) > 0$. Hence $g'(r) > 0$ for $T_1 \geq r > T$ and so $g(r) > 0$ on this range. $\square$

We shall apply this comparison theorem in order to prove that the Dirichlet problem (1), (2) has a solution for some $R > 0$, provided that (15) holds. We shall then discuss the range of R's. The existence theorem will follow from a general theorem, which will be applied in other contexts as well. We begin with a lemma.

Lemma 6. *Suppose that*

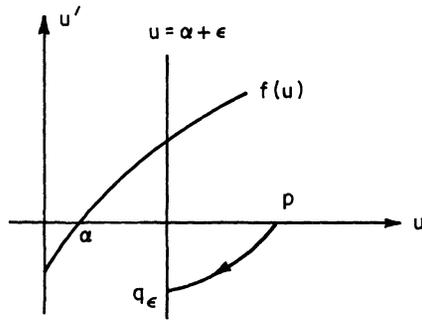$$m_p \leq f(u) \leq M_p, \qquad A \leq u \leq p.$$

Fig. 5

Then for any solution of (12), *there is an* $r_1 > 0$ *with* $u(r_1, p) = A$. *Moreover,*

$$\left[\frac{2n}{M_p}(p - A)\right]^{1/2} \leq r_1 \leq \left[\frac{2n}{m_p}(p - A)\right]^{1/2}. \tag{21}$$

Proof. We have, on $A \leq u \leq p$,

$$r^{n-1} m_p \leq (-r^{n-1} u')' = r^{n-1} f(u) \leq r^{n-1} M_p.$$

Thus integrating from 0 to r, gives

$$\frac{r m_p}{n} \leq -u' \leq \frac{r M_p}{n},$$

so that

$$\frac{r^2 m_p}{2n} \leq p - u(r) \leq \frac{r^2 M_p}{2n}.$$

This shows that r_1 exists and that (21) holds.   □

We apply this lemma to the following situation. Suppose that $f(0) < 0$, $f(u) < 0$, $0 \leq u < \alpha$, and $f(\alpha) = 0$; see Fig. 5.

Let $\varepsilon > 0$ be any positive number. Consider the orbit of (18) which starts out at $u = p$, $v = 0$, $r = 0$. By our lemma, this orbit crosses the line $u = \alpha + \varepsilon$, at some point which we call q_ε; cf Fig. 5. We let T_ε be defined by $u(T_\varepsilon, p) = \alpha + \varepsilon$. Then we have the following lemma.

Lemma 7. *If* $f(u) \geq \theta$ *for* $u \geq \alpha + \varepsilon$, *then*

$$-q_\varepsilon / T_\varepsilon \geq \theta/n.$$

Proof. If we integrate $-(vr^{n-1})' = r^{n-1} f(u)$, we get

$$-q_\varepsilon T_\varepsilon^{n-1} = \int_0^{T_\varepsilon} r^{n-1} f(u) dr \geq \theta T_\varepsilon^n / n,$$

from which the result follows.

We can now state our first theorem.

Theorem 8. *Suppose that $f(u) \geqq m > 0$ for $u \geqq A$. Then*

(1) *for any $p > A$, there is a $T > 0$ such that $u(T, p) = A$.*

(2) *Let $q = v(T, p)$; if $qT \to -\infty$ as $p \to \infty$, then the problem (12), (13) has a solution, with $R = T(p)$.*

Proof. Part (1) is a consequence of Lemma 6. For the second part, consider the system (20). We shall show that if $w(r_1) = 0$, then $z(r_1) < 0$, provided that $-qT$ is sufficiently large. Then Theorem 5 implies that $u(T(p), p) = 0$ for some $p > 0$. (In the applications, we usually take $A = \alpha + \varepsilon$, $q = q_\varepsilon$, $T = T_\varepsilon$; cf. Fig. 5.)

Suppose $f(u) \geqq B$, $B < 0$, and set

$$\beta = (n-1)/T, \qquad \delta = (n-1)q/TB = \beta q/B.$$

Equation (20) can be explicitly integrated as

$$w(r) = e^{-\beta r}q - \frac{B}{\beta}(1 - e^{-\beta r}), \quad \text{and thus} \quad 1 + \delta = e^{\beta r_1}. \tag{22}$$

Furthermore,

$$z(r_1) - A = \int_0^{r_1} w(s)\,ds = -\frac{q}{\beta}(e^{-\beta r_1} - 1) - \frac{Br_1}{\beta} - \frac{B}{\beta^2}(e^{-\beta r_1} - 1),$$

and using (22), we find

$$z(r_1) = A + \frac{qT}{n-1}\left[1 - \frac{\ln(1+\delta)}{\delta}\right].$$

If $\phi(\delta) = 1 - \delta^{-1}\ln(1 + \delta)$, then $\phi(0) = 0$, and $\phi(\delta) > 0$ for $\delta > 0$. Since $\delta = (n-1)B^{-1}(q/T)$, it follows from Lemma 4 that ϕ is bounded away from zero. Since $qT \to \infty$ as $p \to \infty$, we see that $z(r_1) < 0$ for large p. This completes the proof. $\square$

Our next result gives a condition under which $q \to -\infty$ as $p \to \infty$; before stating it, we need a little notation. Thus, referring to Fig. 5, let $A = \alpha + \varepsilon$, $F' = f$, and

$$M_p = \sup_{A \leqq u \leqq p} f(u).$$

Proposition 9. *Suppose that*

$$\lim_{p \to \infty}\left[F(p) + \frac{n-1}{n}M_p(A - p)\right] = +\infty; \tag{23}$$

then $q \to -\infty$ as $p \to \infty$.

Proof. The equation $-(r^{n-1}v)' = f(u)r^{n-1}$ gives $-(r^{n-1}v)' \leqq M_p r^{n-1}$, and by integration we get

$$\frac{v}{r} \geqq -\frac{M_p}{n}.$$

Then if $u(T) = A$, $(T = T_\varepsilon)$, we have

$$q^2/2 + F(A) - F(p) = \int_0^T H' dr = \int_0^T -(n-1)\frac{v^2}{r} dr$$

$$\geqq \int_0^T \frac{n-1}{n} M_p v dr = \frac{n-1}{n} M_p (A-p),$$

and (23) implies the desired result.

We next give a class of functions f for which T_ε is bounded away from zero.

Proposition 10. *Suppose that $(f(u)/u)' > 0$ and that $f'(u)$ is bounded from above.[7] Then T_ε is bounded away from zero.*

Proof. The equation $-(r^{n-1}v)' = r^{n-1}f(u)$, gives $-(r^{n-1}v)' \leqq r^{n-1}f(p)$, and if we integrate this from 0 to r, we get $-v \leqq rf(p)/n$. Integrating again gives, for $p > 2(\alpha + \varepsilon)$,

$$\frac{p}{2} < p - (\alpha + \varepsilon) \leqq p - u(r) = \int_0^{T_\varepsilon} -v dr \leqq \frac{T_\varepsilon^2 f(p)}{2n}.$$

Thus if $f'(u) < k$, we have

$$T_\varepsilon^2 \geqq np/f(p) \geqq n/f'(p) \geqq n/k, \tag{24}$$

and the proof is complete. □

Corollary 11. *Suppose that the hypotheses of the last proposition hold. Then $T(p)$ is bounded away from zero.*

Proof. Replace T_ε by $T(p)$ in the above proof. □

As a consequence of the last three results, we have the following theorem.

Theorem 12. *Suppose that (23) holds, and that the hypotheses of Proposition 10 are valid. Then problem (12), (13) has a positive solution, for some $R = R(n)$.*

We pause here to give some examples. First, if $f(u) = u - e^{-u}$, and $n = 2$, then it is easy to check that all of the above hypotheses hold. If $f(u) = 2u - 2 + e^{-u}$, and $n = 2$, then again all of the above hypotheses hold.

We remark that an existence theorem for any function $f(u)$ which satisfies $(f(u)/u)' > 0$, *cannot* be obtained by the usual method of upper and lower solutions. This holds since the positive solution being unique (by Theorem 4), it must necessarily be a stable solution of the associated time dependent parabolic problem.

$$u_t = \Delta u + f(u), \quad (x, t) \in D_R^n \times \mathbb{R}_+$$

$$u = 0, \quad x \in \partial D_r^n \times \mathbb{R}_+;$$

see [7, Theorem 10.5]. But, as we shall show in Sect. 6, solutions which satisfy the above condition cannot be stable.

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7   In particular, if (15) holds, this will be the case

We shall now give some general conditions under which $q_\varepsilon T_\varepsilon \to -\infty$ (cf. Theorem 8). These will be applied to yield existence theorems for certain classes of f's.

We assume that $f(u) = O(u^k)$, $k > 0$, as $u \to \infty$, or more precisely, we assume that [8]

$$f(u)/u^k \to 1 \quad \text{as} \quad u \to +\infty. \tag{25}$$

We fix α, choose $p > \alpha$ and let T be the "time" that the orbit starting at $u = p$, $v = 0$, $t = 0$, takes to get to the line $u = \bar{u}$, where $\alpha \le \bar{u} < p$. (We think of $\bar{u}$ as $\alpha + \varepsilon$ in our above earlier discussion.)

We have $-(vr)' = (n-2)v + rf(u)$. Thus, if $q = v(T)$, then integrating this equation from $r = 0$ to $r = T$ gives

$$-qT = -(n-2)(p-\bar{u}) + \int_0^T rf(u(r))dr. \tag{26}$$

Now choose points $1 > a_1 > a_2 > \ldots > a_s > 0$, and let u be successively: $p = a_0 p$, $a_1 p, \ldots, a_s p$, $\bar{u}$. If T_j denotes the "time" the orbit takes to go from $u = a_{j-1}p$ to $u = a_j p$, then setting $T_0 = 0$, we have

$$\int_0^T rf(u(r))dr = \sum_{j=0}^s \int_{T_j}^{T_{j+1}} rf(u(r))dr + \int_{T_s}^T rf(u(r))dr$$

$$\ge \sum_{j=0}^s \int_{T_j}^{T_{j+1}} rf(u(r))dr.$$

But

$$\int_{T_j}^{T_{j+1}} rf(u(r))dr \ge c' \int_{T_j}^{T_{j+1}} (a_{j+1}p)^k rdr = c'p^k a_{j+1}^k \frac{T_{j+1}^2 - T_j^2}{2},$$

so that

$$\int_0^T rf(u(r))dr \ge c'\frac{p^k}{2}\{a_1^k T_1^2 + a_2^k(T_2^2 - T_1^2) + \ldots + a_s^k(T_s^2 - T_{s-1}^2)\}$$

$$= c'\frac{p^k}{2}\{T_1^2(a_1^k - a_2^k) + \ldots + T_{s-1}^2(a_{s-1}^k - a_s^k) + T_s^2 a_s^k\},$$

$$\ge c'\frac{p^k}{2}\{T_1^2(a_1^k - a_s^k) + \ldots + T_{s-1}^2(a_{s-1}^k - a_s^k)\},$$

and this latter sum converges, as $s \to \infty$, to the Riemann-Stieltjes integral

$$c'\frac{p^k}{2}\int_0^1 T_a^2 da^k,$$

where T_a is the "time" the orbit takes to go from p to ap. Thus from (26), we get

$$-qT \ge -(n-2)(p-\bar{u}) + \frac{p^k}{2}\int_0^1 T_a^2 da^k. \tag{27}$$

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8   It is not difficult to consider the more general case $f(u)/u^k \to c > 0$ as $u \to \infty$

Next, $-(r^{n-1}v)' = r^{n-1}f(u) \leqq r^{n-1}cp^k$, and so if we integrate this from 0 to r, we get

$$-v \leqq rcp^k/n.$$

Integrating again from $r=0$ to $r=T_a$, gives

$$p - ap \leqq \frac{T_a^2}{2n}cp^k,$$

so that $T_a^2 \geqq (1-a)p^{1-k}\dfrac{2n}{c}$. If we put this in (27) we obtain

$$-qT \geqq -(n-2)(p-\bar{u}) + \frac{pnc'}{c}\int_0^1 (1-a)da^k = -(n-2)(p-\bar{u}) + \frac{pnc'}{c}\left(1 - \frac{k}{k+1}\right).$$

Thus $-qT \to \infty$ as $p \to \infty$, provided that $\dfrac{nc'}{c}\left(1 - \dfrac{k}{k+1}\right) > n-2$. Since we may take c/c' arbitrarily close to 1, we want $n/(n-2) > k+1$, or $2/(n-2) > k$. We have thus proved the following theorem.

Theorem 13. *Suppose that $f(u)$ satisfies (25). Then the problem (12), (13) has a positive solution for some R, provided that*

$$\frac{2}{n-2} > k. \tag{28}$$

Notice that (28) holds if $n=3$ and $0 < k < 2$, and it also holds for all $k > 0$ if $n=2$.

We shall prove one last theorem which together with Theorem 13 will imply, in particular, an existence theorem for (12), (13), for all n, provided that (15) holds. To this end, note that we have shown above that in order that $-q_\varepsilon T_\varepsilon \to \infty$, we need [cf. (27)]

$$\frac{p^{k-1}}{2}\int_0^1 T_a^2 da^k > n-2. \tag{29}$$

For $n=3$, we have just seen that we can estimate T_a^2 from below, and this gives us an existence theorem for $k < 2$. We shall now obtain different estimates on T_a which will enable us to get a different existence theorem. We again assume that (25) holds.

If $0 < p$, we have

$$T_a = \int_p^{ap}\frac{du}{v} = \int_{ap}^p \frac{du}{-v}, \tag{30}$$

and if $\tau < r$, $f(u(\tau)) \geqq u(\tau)^k \geqq u(r)^k$. Thus as above

$$-r^{n-1}v = \int_0^r \tau^{n-1}f(u(\tau))d\tau \geqq c\int_0^r \tau^{n-1}u(r)^k d\tau,$$

so that $-v/r \geqq u^k/n$, where we are using the notation $u = u(r)$. From this we obtain

$$\frac{-v^2(\tau)}{\tau} \leqq c\frac{u(r)^k v(\tau)}{n}. \tag{31}$$

Next, if $F' = f$, then

$$\frac{v^2}{2} + F(u) = F(p) - (n-1) \int\limits_0^r \frac{v^2}{\tau} d\tau,$$

or

$$\frac{v^2}{2} = F(p) - F(u) - (n-1) \int\limits_0^r \frac{v^2}{\tau} d\tau = \int\limits_u^p f(s) ds - (n-1) \int\limits_0^r \frac{v^2}{\tau} d\tau.$$

From (25),

$$\int\limits_u^p f(s) ds \leqq c \frac{s^{k+1}}{k+1} \Big|_u^p = c \frac{(p^{k+1} - u^{k+1})}{k+1},$$

and from (31)

$$-(n-1) \int\limits_0^r \frac{v^2}{t} dt \leqq c \frac{n-1}{n} u^k (u-p).$$

Therefore

$$\frac{v^2}{2} \leqq \frac{c}{k+1} (p^{k+1} - u^{k+1}) - c \frac{n-1}{n} u^k (p-u).$$

If we use this in (30) we obtain

$$T_a \geqq \int\limits_{ap}^p \frac{du}{\sqrt{2 \left\{ \dfrac{c}{k+1} (p^{k+1} - u^{k+1}) - c \dfrac{n-1}{n} u^k (p-u) \right\}}}.$$

Setting $u = ps$, gives

$$T_a \geqq \int\limits_a^1 \frac{p^{(1-k)/2} ds}{\sqrt{2 \left\{ \dfrac{c}{k+1} (1 - s^{k+1}) - c \dfrac{n-1}{n} s^k (1-s) \right\}}}. \tag{32}$$

Now let's consider the case where $k = 1$; then from (32)

$$T_a \geqq \int\limits_a^1 \frac{ds}{\sqrt{c(1-s^2) - 2c \left(\dfrac{n-1}{n} \right) (s - s^2)}}. \tag{33}$$

Likewise, for $k = 1$, (29) becomes

$$\int\limits_0^1 T_a^2 da > 2(n-2). \tag{34}$$

Since T_a depends continuously on c and c can be taken arbitrarily close to 1, we can assume $c = 1$ in (33), and use the resulting expression for T_a in order to prove (34).

Thus if $c = 1$,

$$T_a \geqq \int_a^1 \frac{ds}{\sqrt{\frac{n-2}{n}\left[s^2 - 2\frac{n-1}{n-2}s + \left(\frac{n-1}{n-2}\right)^2\right] - \frac{1}{(n-2)^2}}}$$

$$= \sqrt{\frac{n}{n-2}} \int_a^1 \frac{ds}{\sqrt{\left(s - \frac{n-1}{n-2}\right)^2 - \frac{1}{(n-1)^2}}}$$

$$= \sqrt{\frac{n}{n-2}} \ln \left| \left(\frac{n-1}{n-2} - s\right) + \sqrt{\left(s - \frac{n-1}{n-2}\right)^2 - \frac{1}{(n-2)^2}} \right|_a^1$$

$$= \frac{n}{n-2} \ln |(n-1) - a(n-2) + \sqrt{[(n-2)a - (n-1)]^2 - 1}|.$$

Using this in (34) gives

$$\int_0^1 T_a^2 \, da \geqq \int_0^1 \frac{n}{n-2}[\ln|(n-1) - a(n-2) + \sqrt{[(n-2)a - (n-1)]^2 - 1}|]^2 \, da.$$

If we let $y = (n-1) - (n-2)a$, we have

$$\int_0^1 T_a^2 \, da \geqq \int_1^{n-1} n[\ln(y + \sqrt{y^2 - 1})]^2 \, dy.$$

Since we want (34) we must show

$$\int_1^{n-1} [\ln(y + \sqrt{y^2 - 1})]^2 \, dy > \frac{2(n-2)}{n}. \tag{35}$$

Note now that $2 > 2(n-2)/n$, and that the left-hand side of (35) is monotone in n. But when $n = 3$, it is easy to show that

$$\int_1^2 [\ln(y + \sqrt{y^2 - 1})]^2 \, dy > 2/3;$$

hence (35) holds for $n = 3$. If $n = 4$, we have that

$$\int_1^3 [\ln(y + \sqrt{y^2 - 1})]^2 \, dy > 2 > 2(n-2)/n,$$

and so (35) holds for $n \geqq 3$. It follows from Theorem 12 that for $k = 1$, (12), (13) has a solution for all n. We have thus proved the following theorem, and Corollary 15.

Theorem 14. *Suppose that $f(u)$ satisfies (25). Then the problem (12), (13) has a positive solution for all sufficiently large p, for some $R = R(p)$, if*

$$\int_0^1 \left(\int_a^1 \frac{ds}{\sqrt{2\left\{\frac{1 - s^{k+1}}{k+1} - \frac{n-1}{n}s^k(1-s)\right\}}} \right)^2 da^k > 2(n-2). \tag{36}$$

Corollary 15. *If $f(u)$ satisfies (25) with $k = 1$, then (12), (13) has a positive solution for all sufficiently large p, for all $n \geqq 1$, for some R.*

Corollary 16. *Suppose that* (15) *holds and* $f(u) > 0$ *for some* $u > 0$. *Then* (12), (13) *has a positive solution for all sufficiently large* p, *for all* $n \geq 1$, *for some* R.

Proof. Since $f'' \leq 0$, we have

$$\frac{f(u)}{u} \leq f'(u) \leq f'(0),$$

and since $f(u)/u$ is an increasing function,

$$\lim_{u \to \infty} f(u)/u = A$$

exists. Thus given $\varepsilon > 0$, we have

$$(A - \varepsilon)u \leq f(u) \leq Au \tag{37}$$

for $u \geq u_\varepsilon$.

Now in Eq. (1), make the change of variables $y = x\sqrt{A - \varepsilon}$; then (1) goes over into

$$\Delta_y u + \frac{1}{A - \varepsilon} f(u) = 0, \qquad |y| < R\sqrt{A - \varepsilon}. \tag{38}$$

We thus see that there is a one-one correspondence between solutions of the Dirichlet problem for (1) and (38). Thus, if we define $g(u) = (A - \varepsilon)^{-1} f(u)$, we have the estimate

$$u \leq g(u) \leq \frac{A}{A - \varepsilon} u$$

if $u \geq u_\varepsilon$. We may now apply Corollary 15 to the equation $\Delta u + g(u) = 0$, to conclude the existence of a solution to the Dirichlet problem for every n, for (38), and hence for (1).

As a consequence of this last result, we have the following theorem.

Theorem 17. *Suppose that* (15) *holds, and* $f(0) < 0$. *Then there are real numbers* $R_1 > R_2 > 0$ *(depending on* f *and* n), *such that the problem* (1), (2) *has positive solutions if and only if* $R_1 \geq R > R_2$. *Moreover if* u *is the positive solution of* (1), (2) *for* $R = R_1$ *then* $u(R_1) = u'(R_1) = 0$.

Proof. In the proof of Theorem 4, we have shown that such an R_1 exists, while the existence of the lower bound R_2 follows from Corollary 11. The actual existence of solutions is a consequence of Corollary 15.

Notice that for $f(u) = u - e^{-u}$, the bound $T^2(p) \geq 2n/f'(0)$ gives $R_2 \geq \sqrt{n}$.

Concerning the case where $(f(u)/u)' < 0$, we have the following existence theorem, which is essentially known, and follows from degree theory arguments and results about positive operators; see [1]. We show here how existence also follows from Theorem 14. Recall that in this case f can have at most one positive root.

Theorem 18. *Suppose that* $(f(u)/u)' < 0$ *in* $u > 0$. *Then the following statements concerning the problem* (1), (2) *are valid:*

   A. $0 \leq R_0 < T(p) < R_1 \leq \infty$, *where* $R_0 = 0$ *if and only if* $f(0) > 0$, *while* $R_1 < \infty$ *if and only if* $\lim_{u \to \infty} f(u)/u = \lambda > 0$.

   B. $\text{dom}(T) = \{p : f(p) > 0\}$.

We shall first prove the assertions concerning R_0. If $f(0) > 0$ and $f(p) > 0, p > 0$, then $f(u) \geq m > 0$ on $0 \leq u \leq p$ for some $m > 0$. Then integrating $-(r^{n-1}v)'$ $= f(u)r^{n-1} \geq mr^{n-1}$ yields $-v \geq mr/n$, and integrating again gives $u(r) \leq p$ $- mr^2/2n$. Thus $u(r) < 0$ if $r^2 > 2np/m$ [so $p \in \text{dom}(T)$!] and $T(p)^2 < 2np/m$, so $T(p)$ $\rightarrow 0$ as $p \rightarrow 0$; thus $R_0 = 0$. If $f(0) = 0$, then since $f(u) > 0$ for some $u > 0$ (in order for positive solutions to exist), and (11) holds, we see that $f'(0) > 0$. Let $A = f'(0)$ and note that $f(u) \leq (A + \varepsilon)u$, for small $u > 0$, for some $\varepsilon > 0$. For such u, along a solution we have

$$(r^{n-1}v)' = -r^{n-1}f(u) \geq -r^{n-1}(A + \varepsilon)u \geq -r^{n-1}(A + \varepsilon)p,$$

so $v \geq -r(A + \varepsilon)p/n$, and thus $-T(p)^2(A + \varepsilon)p/2n \leq -p$; hence $T(p)^2 \geq 2n/(A + \varepsilon)$, and thus $R_0 \geq 0$.

The equation $v' = -(n-1)vr^{-1} - f(u)$ implies that $(r^{n-1}v)' = -r^n f(u)$. Integrating this from 0 to r gives $r^{n-1}v(r) = -\int_0^r s^n f(u(s))ds$. Hence, if $u(0) = p$, we get

$$\max_{0 \leq u \leq p} (-v(r)) = \max_{0 \leq r \leq T(p)} \frac{1}{r^{n-1}} \int_0^r s^n f(u(s))ds$$

$$\leq \max_{0 \leq r \leq T(p)} \frac{1}{r^{n-1}} \int_0^r s^n \left[\max_{0 \leq u \leq p} f(u) \right] ds$$

$$= \left[\max_{0 \leq u \leq p} f(u) \right] \frac{T(p)^2}{n+1} = \frac{M_p}{n+1} T(p)^2,$$

where $M_p = \max\{f(u) : 0 \leq u \leq p\}$. Since $u' = v$ implies that $T(p) \geq p \Big/ \max\limits_{0 \leq u \leq p} (-v(r))$, we obtain $T(p)^3 \geq (n+1)p/M_p$. We claim that $p/M_p \rightarrow \infty$ if and only if $f(p)/p \rightarrow 0$, as $p \rightarrow \infty$.

In order to prove the claim, first note that $M_p \geq f(p)$ implies that if $M_p/p \rightarrow 0$, then $f(p)/p \rightarrow 0$. Conversely, suppose that $f(p)/p \rightarrow 0$. If f is bounded, say $f(p) \leq k$, then $k/p \geq M_p/p$, so $M_p/p \rightarrow 0$. We may thus assume that f is unbounded. Let $N > 0$ be given; since $f(p)/p \rightarrow \infty$ and $(f(p)/p)' < 0$, we see that there is a $p_0 > 0$ such that $p/f(p) \geq N$ for $p \geq p_0$. On the closed interval $[0, p_0]$, f is bounded, say $f(p) \leq k$. Since f is unbounded, there is a $q > p_0$ such that $f(q) = k + 1$. Let $f(\xi)$ $= \max\{f(p) : 0 \leq p \leq q\}$. Then $\xi > p_0$, $\xi/f(\xi) > p_0/f(p_0) \geq N$ so that $\xi/M_\xi = \xi/f(\xi)$ $> N$. Since N was arbitrary and $u/f(u)$ is an increasing function, we see that $M_p/p \rightarrow 0$. This proves our claim.

Suppose that $f(u)/u \geq \lambda > 0$. We write (1) in the form

$$u'' + \frac{n-1}{r}u' + \lambda^2 u = \lambda^2 u - f(u) \equiv h(t) \leq 0.$$

For the linear system

$$w'' + \frac{n-1}{r}w' + \lambda^2 w = 0,$$

the associated time map T is constant, $T(p) \equiv T_L$; this follows since ϕ, as defined by (10) is identically zero. If $T(p)$ is the time map associated to (1), (2), we shall show that $T(p) \leq T_L$, and this will give the desired conclusion.

Let $\varepsilon > 0$ be a given small number, and let v_1 and v_2 be linearly independent solutions for the above linear (homogeneous) equation, where

$$v_1(\varepsilon) = 1, \quad v_1'(\varepsilon) < 0, \quad \text{and} \quad v_2(\varepsilon) = 0, \quad v_2'(\varepsilon) > 0.$$

Using variation of parameters, we can write the solution of (1), (2) as

$$u(t) = av_1(t) + bv_2(t) + \int_\varepsilon^t \frac{v_2(t)v_1(s) - v_1(t)v_2(s)}{W(s)} h(s)ds,$$

where a and b are constants, and $W(s)$ is the associated Wronskian; i.e.,

$$W(s) = \det \begin{pmatrix} v_1(s) & v_1'(s) \\ v_2(s) & v_2'(s) \end{pmatrix}.$$

Since $W(\varepsilon) > 0$, we see $W(s) > 0$ for all s. Also $0 < W(T_L) = -v_1'(T_L)v_2(T_L)$, $v_1'(T_L) < 0$ imply that $v_2(T_L) > 0$. Since

$$u(T_L) = bv_2(T_L) + \int_\varepsilon^{T_L} \frac{v_2(T_L)v_1(s)}{W(s)} h(s)ds,$$

we see that $u(T_L) \leqq 0$ if $b \leqq 0$. Thus, if $b \leqq 0$, we will have proved that $T(p) \leqq T_L$.
It remains to show $b \leqq 0$. We have

$$u'(\varepsilon) = av_1'(\varepsilon) + b, \quad u(\varepsilon) = av_1(\varepsilon)$$

so that

$$b = \left[\frac{u'(\varepsilon)}{u(\varepsilon)} - \frac{v_1'(\varepsilon)}{v_1(\varepsilon)} \right] \Bigg/ v_1(\varepsilon)u(\varepsilon),$$

and thus if the numerator is not zero,

$$\operatorname{sgn} b = \operatorname{sgn} \left[\frac{u'(\varepsilon)}{u(\varepsilon)} - \frac{v_1'(\varepsilon)}{v_1(\varepsilon)} \right].$$

In order to compute this sign, we define $\theta = \arctan(u'/u)$, $\psi = \arctan(v_1'/v_1)$. Then

$$\dot\theta(r) = - \frac{v^2 + uf(u) - (n-1)uvr^{-1}}{u^2 + v^2},$$

and as we have observed in the proof of Theorem 3, $\lim vr^{-1} = -f(u(0))/n$ as $r \to 0$, so that

$$\dot\theta(0) = - \frac{1}{n} f(u(0))/u(0).$$

Similarly, $\dot\psi(0) = -\lambda^2/n$. Thus $\dot\psi(0) \geqq \dot\theta(0)$, and so $v_1'(\varepsilon)/v_1(\varepsilon) \geqq u'(\varepsilon)/u(\varepsilon)$, for small $\varepsilon > 0$. This implies that $b \leqq 0$.

Finally, assume that $f(\bar p) = 0$ for some $\bar p > 0$. Then since $T' > 0$ and $u = 0$, $v = \bar p$ is a "rest point," $T(p) \to \infty$ as $p \nearrow \bar p$. This completes the proof of A.

To prove B, we first suppose that $f(\bar p) = 0$ for some $\bar p > 0$. Then $f(p) < 0$ for $p > \bar p$, as solutions cannot exist if $p \geqq \bar p$. Let $0 < p < \bar p$, then as noted above, $p \in \operatorname{dom}(T)$ if $f(0) > 0$. Suppose $0 < p < \bar p$ and $f(0) = 0$. As noted above, $f'(0) > 0$.

Define a new positive smooth function $g(u)$ by $g(u) = f(u)$ if $0 \le u \le p$, and $g(u)/u \searrow 0$ as $u \to \infty$ (see the depicted figure). Then since the orbit through $v = 0$, $u = p$ depends only on the values of $f(u)$ for $u \le p$, we may apply a remark after Theorem 3 to conclude that $p \in \text{dom}(T)$. Thus B holds if $f(\tilde{p}) = 0$ for some $\tilde{p} > 0$. If $f(u) > 0$ for all $u > 0$, then if $f(0) > 0$, as above $\text{dom}(T) = \{u > 0\}$, while if $f(0) = 0$, Corollary 15 implies $p \in \text{dom}(T)$ if $p \gg 1$. But Theorem 3 shows that $p \in \text{dom}(T)$ for p near 0, and the assertion follows since $\text{dom}(T)$ is connected (cf. Theorem 3). The proof is complete.
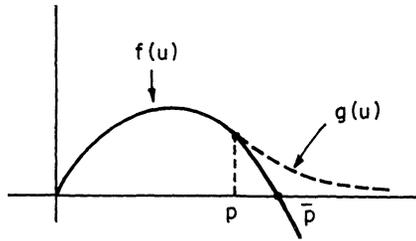
5. Nondegenerate Solutions

In this section we shall show that the ideas in the previous sections can be used to prove the non-degeneracy of solutions. We assume that $f \in C^3$.

We begin with the following theorem.

Theorem 19. *Let $u(\cdot, \tilde{p})$ be a solution of (3), (4), where $g \in C^3$. Then $u(\cdot, \tilde{p})$ is non-degenerate if and only if $u_p(L, \tilde{p}) \ne 0$.*

Proof. Differentiating (3) with respect to p gives for $w = u_p$[9],

$$w'' + g_1(\tilde{u}, \tilde{u}', t)w + g_2(\tilde{u}, \tilde{u}', t)w' = 0, \qquad (39)$$

where $\tilde{u}(r) \equiv u(r, \tilde{p})$, and g_i denotes differentiation of g with respect to its ith argument. Also, $u'(0, p) = 0$ gives $u_p'(0, p) = 0$. Thus, if $\tilde{u}_p(L, \tilde{p}) = 0$, then $\tilde{u}_p$ satisfies (39) and the correct boundary conditions. Moreover, $u(0, p) = p$, $u_p(0, p) = 1$, so $u_p(0, \tilde{p}) \ne 0$; thus the solution $u(\cdot, \tilde{p})$ is degenerate.

Conversely, if $u(\cdot, \tilde{p})$ is degenerate; i.e., if there exists a non-trivial solution w of (39) which satisfies $w'(0) = w(L) = 0$, then $w(0) \ne 0$ and $u_p(t, \tilde{p}) = w(t)/w(0)$, since both satisfy (39) and the same initial conditions. $\square$

We remark that the condition $u_p(L, \tilde{p}) \ne 0$ is generally very difficult to verify directly. However, if $u'(L, \tilde{p}) \ne 0$, then the equation $u(t, p) = 0$ defines t implicitly as a smooth function of p near $(L, \tilde{p})$ (see the appendix); namely $T(p)$. Differentiating the equation $u(T(p), p) = 0$, with respect to p, gives, at $\tilde{p}$, $u_p(L, \tilde{p}) = -u'(L, \tilde{p})T'(\tilde{p})$. Thus $u_p(L, \tilde{p}) = 0$ if and only if $T'(p) = 0$. Theorems 2–4 give conditions under which $T'(p) \ne 0$. The condition $u'(L, \tilde{p}) \ne 0$ is automatically satisfied if $g(0, 0, L) > 0$. One sees this latter fact from the "phase portrait." Namely, since $u' = v$, $v' = -g(u, v, t)$, We find that if $u(L, \tilde{p}) = u'(L, \tilde{p}) = 0$, then $-g(0, 0, L) = v'(L) > 0$, since $v(L - \varepsilon) < 0$, and $u'(L) = 0$. Similarly, if $g(0, 0, t) = 0$ for all $t \ge 0$, then no orbit reaches the origin in finite time since the line $u = v = 0$ in $\mathbb{R}^3$ is invariant; thus

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9   The fact that u depends smoothly on p is shown in the appendix; $f \in C^3$ is needed to ensure w'' is continuous

$u'(L, \tilde{p}) \neq 0$ in this case too. The case where $u'(L, \tilde{p}) = 0$ is quite interesting and will be considered in a forthcoming publication.

We now come to the main theorem in this section.

Theorem 20. *Let $u = u(\cdot, p)$ be a non-negative solution of (1), (2) on $\Omega = D_R^n$. If both*

$$T'(p) \neq 0, \quad and \quad u'(R, p) \neq 0,$$

then u is non-degenerate, and conversely. In particular, u is non-degenerate if (11) holds.

Proof. First, recall from [4] that $u'(r) < 0$ if $r > 0$. We want to show that 0 is not in the spectrum of the linearized equations if $T'(p) \neq 0$ and $u'(R, p) \neq 0$; that is, we want to show that $v \equiv 0$ is the only solution of the problem

$$\Delta v + f'(u)v = 0, \quad x \in \Omega, \quad v(\partial\Omega) = 0. \tag{40}$$

Now it is a standard result that every solution of (40) can be written in the form

$$v(r, \theta) = \sum_{N=0}^{\infty} a_N(r)\Phi_N(\theta), \quad \theta \in S^{n-1}, \quad r \geq 0, \tag{41}$$

where $\Phi_0 = \text{const.}$ and for $N \geq 1$, Φ_N is an eigenfunction of the Laplacian on the $(n-1)$-sphere S^{n-1} (see, e.g., [2] for the case $n = 3$). If we use this in (40), we obtain the equation

$$\sum_{N=0}^{\infty} \left(a_N'' + \frac{n-1}{r} a_N' + \frac{\lambda_N}{r^2} a_N + f'(u)a_N \right) \Phi_N = 0, \tag{42}$$

where

$$-\lambda_N = N(N+n-2), \quad N \geq 0. \tag{43}$$

see [2, p. 161].

In view of (42) we have, for $N \geq 0$,

$$a_N'' + \frac{n-1}{r} a_N' + \frac{\lambda_N}{r^2} a_N + f'(u)a_N = 0, \tag{44}$$

and[10] for $N \geq 1$,

$$a_N(R) = a_N(0) = 0. \tag{45}$$

We shall show that (44), and (45) imply that $a_N(r) \equiv 0$, $0 \leq r \leq R$, if $N \geq 1$. Then we shall show that Theorem 19 implies that $a_0(r) \equiv 0$, and so we will have that $v \equiv 0$, and the non-degeneracy of u will be proved. To this end, let $w = du/dr$; then w satisfies the equation

$$w'' + \frac{n-1}{r} w' - \frac{n-1}{r^2} w + f'(u)w = 0.$$

---

10 $a_N(r)\Phi_N(\theta)$ is undefined at $r = 0$ unless $a_N(0) = 0$. Formally, $a_N(r) = \int_{S^{n-1}} v(r, \theta)\Phi_N(\theta)d\theta$, and since v is continuous at $r = 0$, $a_N(0) = v(0) \int_{S^{n-1}} \Phi_N(\theta)d\theta = \dfrac{v(0)}{\lambda_N} \int_{S^{n-1}} \Delta\Phi_N(\theta)d\theta = 0$

Assume $N \geq 1$; multiply this last equation by $-a_N r^{n-1}$, multiply (44) by wr^{n-1}, add the two resulting equations and integrate from $r=0$ to $r=\tilde{R} \leq R$, where $\tilde{R}$ is the first zero of a_N. This gives

$$r^{n-1}[w(r)a_N'(r) - a_N(r)w'(r)]\Big|_0^{\tilde{R}} = \int_0^{\tilde{R}} r^{n-3}[-\lambda_N - (n-1)]a_N(r)w(r)dr,$$

or, in view of the boundary conditions (45),

$$\tilde{R}^{n-1}w(\tilde{R})a_N'(\tilde{R}) = \int_0^{\tilde{R}} r^{n-3}[-\lambda_N - (n-1)]a_N(r)w(r)dr. \tag{46}$$

Note that $-\lambda_N - (n-1) > 0$ if $N > 1$, while $-\lambda_1 = n-1$. Thus (46) implies that $a_N(r) \equiv 0$ if $N \geq 2$. Since $w(R) \neq 0$, $u(R) = 0$ (and $u \not\equiv 0$), then setting $\tilde{R} = R$ in (46) gives $a_1(R) \equiv 0$.

It remains to prove that $a_0(r) \equiv 0$. Since $u'(R, p) \neq 0$, it follows from the remark which we made after the proof of Theorem 18, that

$$u_p(R, p) = -u'(R, p)T'(p) \neq 0.$$

Hence from Theorem 19, $a_0(r) \equiv 0$.

Conversely, if the solution $u(\cdot, p)$ is non-degenerate, then $u'(R, p) \neq 0$, for otherwise, the function $u'(r, p)\Phi_1(\theta)$ would be a non-zero solution of (44), (45). Furthermore, using the fact that $u'(R, p) \neq 0$, we see as above, that if $T'(p) = 0$, then $u_p(\cdot, p)$ would be a non-zero solution of (40); thus $T'(p) \neq 0$. $\square$

It is useful to summarize our results for solutions of (1), (2) in the cases where $\Omega = D_R^n$. We define $\psi(u) = f(u)/u$, and consider two cases:
1) $\psi'(u) < 0$ in $u > 0$, and 2) both $\psi'(u) > 0$ and $f''(0) < 0$ in $u > 0$.

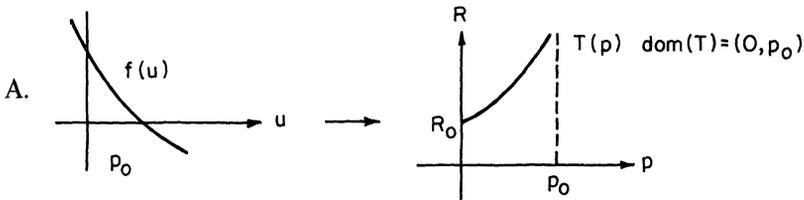*Case 1.* $\psi'(u) < 0$. [If $f(0) = 0$, then there are no positive solutions.]

Fig. 6. Solutions exists only for $R > R_0 > 0$

B. $f(u) > 0$ in $u > 0$. Here there are two subcases:
i) if $f(u)/u \to 0$ as $u \to \infty$, then T has the following form:

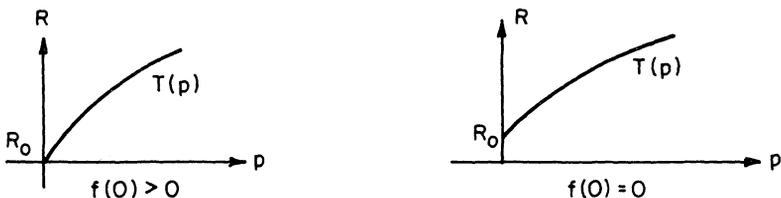**Fig. 7.** $T(p) \to \infty$ as $p \to \infty$

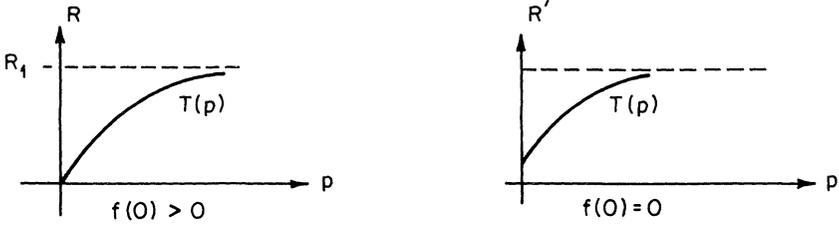ii) if $f(u)/u \geqq \lambda^2 > 0$, then T has the following form:

Fig. 8. $T(p) \leqq R_1$ for all $p > 0$

All of the above positive solutions are unique, stable and non-degenerate.

Case 2. $\psi'(u) > 0$, $f(0) < 0$, and $f''(u) \leqq 0$.

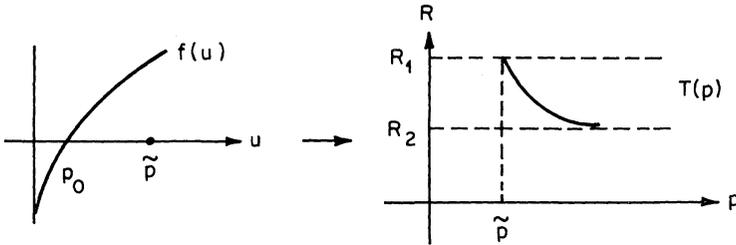**Fig. 9**

Solutions exist for $R_2 < R \leqq R_1$; all positive solutions are unique, and non-degenerate if $p > \tilde{p}$. If $u(0) = \tilde{p}$, the corresponding solution is degenerate and in this case $u'(R_1, \tilde{p}) = 0$.

6. Concluding Remarks

If u is a positive solution of (1), (2) in any domain $\Omega \subset \mathbb{R}^n$ (Ω is need not be an n-ball), and $\sigma(u)$ denotes the spectrum of the linearized operator about u, then we shall show that $\sigma(u) \subset \{x > 0\}$ if (11) holds, and $\sigma(u) \cap \{x > 0\}$ is non-void if $(f(u)/u)' > 0$. In fact, we have the following somewhat more general result.

Theorem 21. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let u be a non-negative solution of the problem*

$$\Delta u(x) + f(u(x)) = 0, \quad x \in \Omega, \tag{46--47}$$

$$a(x)u(x) + b(x)\frac{du(x)}{dn} = 0, \quad x \in \partial\Omega, \tag{48}$$

where $a^2 + b^2 = 1$, and d/dn denotes the outward-pointing normal derivative on $\partial\Omega$. Then if $\psi(u) \equiv f(u) - uf'(u)$, we have
   a) *if $\psi > 0$ in $u > 0$, $\sigma(u) \subset \{x \in \mathbb{R}; x < 0\}$,*
   b) *if $\psi < 0$ in $u > 0$, $\sigma(u) \cap \{x \in \mathbb{R}; x > 0\}$ is non-void.*

Proof. Let u be a non-negative solution of (47), (48), and consider the eigenvalue equation

$$\Delta v + f'(u)v = \lambda v, \quad x \in \Omega, \tag{49}$$

with boundary conditions (48). Suppose that v is a solution of (49), (48), and λ is the principal eigenvalue of $\Delta + f'(u)$ on Ω, together with the boundary conditions (48). From a well-known result, (see [8]), we may assume that $v > 0$ on Ω.

We multiply (49) by u, (47) by v, add and integrate over Ω to get

$$\int_\Omega v[\Delta u + uf'(u)] + \int_{\partial\Omega} \left(u\frac{dv}{dn} - v\frac{dn}{dn} \right) = \lambda \int_\Omega uv .$$

Since the equations $au + b\dfrac{du}{dn} = 0$ and $av + b\dfrac{dv}{dn} = 0$ have a non-zero solution (a, b), we see that the above boundary term vanishes. Then using (47), we find

$$\int_\Omega -v\psi(u) = \lambda \int_\Omega uv . \tag{50}$$

This shows that if $\psi > 0$ in $u > 0$, then $\lambda < 0$, while if $\psi < 0$ in $u > 0$, the $\lambda > 0$. $\quad\square$

As we have remarked earlier, this together with Theorem 4 implies that if (15) holds, then one cannot prove existence theorems for (1), (2) via the method of upper and lower solutions (nor by a variational approach, in which u is a minimum of a functional).

The same argument yields still a more general result. Namely, consider the equation with self-adjoint boundary conditions

$$Lu(x) + f(u(x)) = 0, \quad x \in \Omega .$$

Here Ω is a bounded domain (not necessarily a ball), and L is a linear 2nd-order elliptic operator. Then the following theorem holds (cf. [8, Chap. 11, Appendix]).

Theorem 22. *Let u be a positive solution of the above problem. Then conclusions a) and b) of the last theorem are valid.*

Appendix

We shall show here that if u is a solution of (12) and (13), then u is a smooth function of p (and r)[11]. The usual theorems are not applicable here since the coefficient of u' in (12) is not continuous in any neighborhood of $r = 0$. Note that as a consequence of the smoothness of u, it follows that T is a smooth function of p.

Theorem. *Let $f \in C^k$ and let $u = u(r, p)$ be a solution of (12), (13) with $u(0, p) = p$. Then u is a C^k function[12] of r and p.*

Proof. It suffices to show that u is a C^k function in any interval $0 \leq r \leq \delta$, since we may then apply the standard theorems.

Now the equation $(r^{n-1}u')' = -r^{n-1}f(u)$ gives

$$u'(r, p) = -\frac{1}{r^{n-1}} \int_0^r s^{n-1}f(u(s, p))ds ,$$

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11   A similar result is given in a preprint by Ni and Nussbaum, Uniqueness and Nonuniqueness for Positive Radial Solutions of $\Delta u + f(u, r) = 0$

12   In the applications to the non-degeneracy problem, we need $k \geq 3$ since u_p'' must be defined

and so if we take first the case $n > 2$, we have

$$u(r,p) = p + \int_0^r -\frac{1}{t^{n-1}} \left(\int_0^t s^{n-1} f(u(s,p)) ds \right) dt$$

$$= p - \int_0^r \int_r^s (s/t)^{n-1} f(u(s,p)) dt ds$$

$$= p + \int_0^r (r^{2-n} - s^{2-n}) s^{n-1} f(u(s,p)) ds$$

$$= p + \frac{1}{n-2} \int_0^r \left[r \left(\frac{s}{r} \right)^{n-1} - s \right] f(u(s,p)) ds.$$

Thus, if $n > 2$,

$$u(r,p) = p + \frac{1}{n-2} \int_0^r \left[r \left(\frac{s}{r} \right)^{n-1} - s \right] f(u(s,p)) ds, \tag{51}$$

whiel if $n = 2$, a similar calculation gives

$$u(r,s) = p + \int_0^s s \ln(r/s) f(u(s,p)) ds. \tag{52}$$

We shall assume $n > 2$, and prove the theorem in this case; the case $n = 2$ is similar and will be left to the reader. Thus, motivated by (51), we let $\mathscr{J}$ be the class of C^k functions of r and p with the C^k-norm which satisfy the conditions

$$h(0,p) = p, \qquad h_r(0,p) = 0.$$

We define a mapping T on $\mathscr{J}$ by

$$Th(r,p) = p + \frac{1}{n-2} \int_0^r \left[r \left(\frac{s}{r} \right)^{n-1} - s \right] f(h(s,p)) ds.$$

It is straightforward to check that Th is in $\mathscr{J}$, and that T is a contraction mapping for small $r \geq 0$. It follows that T has a unique fixed point h in $\mathscr{J}$. We set $u(r,p) \equiv h(r,p)$; then $u'(0,p) = 0$, $u(0,p) = p$, and by direct calculation, u satisfies (12). $\square$

References

1. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**, 620–709 (1976)
2. Berger, M., Gauduchon, P., Mazet, E.: Le spectre d'une variété Riemannienne. In: Lecture Notes in Mathematics, Vol. 194. Berlin, Heidelberg, New York: Springer 1971

3. Courant, R., Hilbert, D.: Methods of mathematical physics, Vol. 1. New York: Wiley 1962
4. Gidas, B., Ni, W.N., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. Commun. Math. Phys. **68**, 209–243 (1979)
5. Hirsch, M., Smale, S.: Differential equations, dynamical systems, and linear algebra. New York: Academic Press 1974
6. Lions, P.L.: On the existence of positive solutions of semilinear elliptic equations. SIAM Rev. **24**, 441–467 (1982)
7. Peletier, L.A., Serrin, J.: Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$. Arch. Rat. Mech. Anal. **81**, 181–197 (1983)
8. Smoller, J.: Shock waves and reaction-diffusion equations. Berlin, Heidelberg, New York, Tokyo: Springer 1983
9. Smoller, J., Tromba, A., Wasserman, A.: Nondegenerate solutions of boundary-valued problems. Nonlinear Anal. **4**, 207–215 (1980)
10. Smoller, J., Wasserman, A.: Global bifurcation of steady-state solutions. J. Diff. Eqs. **39**, 269–290 (1981)

Note added in proof. If $f \in C^0$, and $u(r, p)$ solves (12), (13), with $u(0, p) = p$, then u is a C^2 function of r, and u, u_r, u_{rr} are C^1 in p. This holds since u is a fixed point of the above map T, now considered as a map from $C^0[0, a] \to C^2[0, a]$; hence u is C^2 in r. Now we may follow the usual proof of differentiability with respect to initial values to see that u, u_r, u_{rr} are C^1 in p. Thus in Sect. 5, we need only assume $f \in C^1$.