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On the Positive Energy Theorem Involving Mass and Electromagnetic Charges

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Abstract. A new theorem relating mass and charges is deduced, which can be applied to more general physical systems than those covered by the theorem of Gibbons and Hull [1].

1. Introduction

In the Newtonian description of gravitational interactions the masses in consideration are positive. Since the beginnings of general relativity the question of the positivity of mass has been in the mind of physicists. One of the reasons why positivity of the energy is of fundamental importance in general relativity, is the fact that the energy carried by gravitational waves [2] is positive [3–5]. And so one would like to know if the Bondi [6] mass is positive for any retarded time u.

Schoen and Yau [7] gave a proof of the positive energy theorem at spacelike infinity involving extremal surfaces.

Earlier Deser and Teitelboim [8] gave a proof that the total energy in supergravity theory is nonnegative. Grisaru [9] presented an argument, based on Deser's and Teitelboim's result, in which it was stated that the energy functional in classical Einstein theory is positive.

Finally Witten [10], perhaps inspired by these results, gave a proof that the ADM mass [5] is positive, based on a technique using spinors. The result of the positive energy theorem using spinors has been generalized by Gibbons and Hull [1] to 4-dimensional space-times admitting electromagnetic charges. The inequalities they obtain involve the gravitational mass and the charges mentioned above. As in Witten's version they impose an elliptic differential equation on the spinor field, which involves the electromagnetic fields, together with a local inequality which is responsible for the resulting global inequality.

The purpose of this work is to show that the local inequality used by Gibbons and Hull [1] is actually a member of a 1-parameter family of local inequalities for which the same philosophy holds. In fact the parameter here called r must satisfy $0 \leqq |r| \leqq 1$ in order to produce the positivity condition, and the results of Witten and

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Gibbons and Hull are obtained for the extreme values $|r|=0$ and $|r|=1$ respectively.

In Sect. 2 one can find a brief review of the relevant aspects of the positive energy theorem, in which the form suggested by E. Witten is compared with the one suggested by Nester.

In Sect. 3 a 2-form is introduced which depends linearly on the parameter r, and it is found that the positivity argument goes through provided $|r| \leqq 1$. Finally a discussion of the physical significance of this inequality is presented.

2. Positive Energy Theorem

The original idea has been to apply Stokes' theorem to an hypersurface N which stretches out to spacelike infinity. To N one attaches an asymptotic boundary ∂N.

From Witten's work, the form to be integrated on N is

$$W_{ab} \equiv \tfrac{1}{2} \varepsilon_{ab}{}^{cd} D_c(\bar{\Psi}\gamma_d \Psi), \tag{2.1}$$

where $a, b, c, \ldots$ are abstract indices [11] in the space-time M, D is the Riemannian connection in M, and the spinor Ψ is assumed to satisfy

$$\gamma^{\underline{\underline{a}}} D_{\underline{\underline{a}}} \Psi = 0, \tag{2.2}$$

where $\underline{\underline{a}} = 1, 2, 3$ runs through the spacelike indices of an orthonormal basis in which the timelike vector is orthogonal to N. Originally Witten worked with a spinor just defined on the hypersurface N, but here ψ is thought to be a spinor field in the space-time M.

The idea now is to relate the integral of this form to the ADM mass. Under condition (2,2) one has

$$\int_{\partial N} W = 8\pi P^a_{\mathrm{ADM}} \bar{\Psi}_0 \gamma_a \Psi_0, \tag{2.3}$$

where Ψ_0 is the asymptotic value of Ψ, assumed constant.

By applying Stoke's theorem,

$$\int_{\partial N} W = \int_N dW, \tag{2.4}$$

and proving that the right-hand side is positive under the dominant energy conditions, one completes the proof of the theorem.

Existence of the solution to the equation imposed on ψ has been established [12]. Nester [13] has suggested instead integrating the form

$$E_{ab} \equiv \bar{\Psi}\gamma_{[a} D_{b]} \gamma_5 \Psi + \text{c.c.}, \tag{2.5}$$

where c.c. means complex conjugate.

One can find the following relation between these forms:

$$E_{ab} = -W_{ab} + \tfrac{1}{4} \varepsilon_{ab}{}^{cd} (\not{D}(\bar{\Psi})\gamma_c \gamma_d \bar{\Psi} - \bar{\Psi}\gamma_c \gamma_d \not{D}\Psi), \tag{2.6}$$

where $\not{D} \equiv \gamma^a D_a$ is the standard Dirac operator on M.

One observes that if

$$\not{D}\Psi = 0 \tag{2.7}$$

(which, by demanding $e_0{}^a D_a \Psi \equiv D_0 \Psi = 0$, agrees with Witten's condition), then

$$E_{ab} = -W_{ab}\,. \tag{2.8}$$

Working with an orthogonal tetrad in which the timelike vector is orthogonal to N, one studies the spacelike components of dE

$$dE_{\underline{\underline{abc}}} = \frac{1}{3!}\,\varepsilon_{0\underline{\underline{abc}}}[2D^{\underline{\underline{e}}}(\bar{\Psi})\gamma_0 D_{\underline{\underline{e}}}(\Psi) + 2D_{\underline{\underline{e}}}(\bar{\Psi})\gamma^{\underline{\underline{e}}}\gamma^0\gamma^{\underline{\underline{d}}} D_{\underline{\underline{d}}}\Psi + G^{0\underline{\underline{d}}}\bar{\Psi}\gamma_{\underline{\underline{d}}}\Psi]\,, \tag{2.9}$$

which one notices is independent of $D_0\Psi$, so the integral in Eq. (2.10) is independent of it. Similarly one can see that the integral in Eq. (2.3) is independent of $D_0\Psi$.

Nester has also pointed out that as long as Ψ behaves asymptotically as $\Psi_0 + O(1/r)$, where Ψ_0 is a constant spinor with respect to the cartesian coordinates at spacelike infinity, then

$$\int_{\partial N} E = -8\pi P^a_{\mathrm{ADM}} \bar{\Psi}_0 \gamma_a \Psi_0\,. \tag{2.10}$$

If the dominant energy condition holds, Eqs. (2.2), (2.9), and (2.10) immediately give the positivity of energy. It is worthwhile to remark that if Eq. (2.2) is dropped, then one cannot use W_{ab} in Eq. (2.3) to give the ADM momentum. However, as Nester has shown, Eq. (2.10) remains valid independently of the imposition of the Witten equation (2.2). For example, in the case of the Reissner-Nordström solution one can check that the form E_{ab} correctly gives the mass even when Ψ satisfies a condition of the form $\not{D}\Psi = S\Psi$, where S is an operator linear in F_{ab}. This fact has already been used by Gibbons and Hull [1], who chose S such that one gets a desired inequality involving the charges of the electromagnetic fields. Their theorem relates mass and electromagnetic charges. They assume the spacetime M is asymptotically flat and satisfies the Einstein field equation, where the energy momentum tensor is expressed as the sum of two parts, the matter tensor T_{ab} and the electromagnetic stress tensor. It is also assumed that the matter tensor satisfies

$$T_{ab} u^a v^b \geqq [(J^a_{\mathrm{E}} u_a)^2 + (J^a_{\mathrm{M}} u_a)^2]^{1/2} \tag{2.11}$$

for all pairs of future directed timelike unit vectors u^a and v^a, where J_{E} and J_{M} are the electric and magnetic vector current, respectively. The main reason for imposing the last condition on the matter tensor is that it assures negativity of the integral of the exterior derivative of the appropriate generalization of E; also one can note that it is the local expression of the result expressed by inequality (2.12). Then it is proved that associated to a proper non-singular hypersurface N the ADM [14] or the Bond [6] mass m satisfies

$$m \geqq (Q_{\mathrm{E}}{}^2 + Q_{\mathrm{M}}{}^2)^{1/2}\,, \tag{2.12}$$

where Q_{E} and Q_{M} are the electric and magnetic charges respectively associated with that hypersurface. The proof is similar to the version of the positive energy theorem given by Nester [13], with the addition of the introduction of a supercovariant derivative [15] acting on spinors.

One notes that the condition (2.11) offers some difficulties when one tries to apply it to many physical systems. This comes from the fact that the elementary charge (the electron charge) is huge[1] when compared with the mass, let us say, of any atom. So, for a substance made out of atoms with typical mass of 50 atomic units, one needs only to have an excess of one electron charge every 10^{16} atoms to produce a situation in which the condition (2.11) does not hold. In particular one can see that by using a battery from a flashlight to hold two average coins at a potential difference of 1.5 V, and placing them parallely at a distance of 1 mm, one produces a situation in which condition (2.11) does not hold.

The result appearing in the next section will not suffer from this difficulty.

3. A 1-Parameter Family of Theorems

In the study of the positive energy theorem in a 5-dimensional spacetime [16] in the context of Kaluza-Klein theory [17], one makes use of a 3-form which is the natural extension of the 2-form used by Nester. This 3-form is Lie derived in the direction of the extra dimension, and so defines a 2-form intrinsic to the 4-dimensional space-time M, which is given by

$$\alpha_{ab} = E_{ab} + r\bar{\Psi}{}^*F_{ab}\gamma_5\Psi + r\bar{\Psi}F_{ab}\Psi\,, \tag{3.1}$$

where r is a parameter and F_{ab} the Faraday tensor. The integration of α gives

$$\int_{\partial N} \alpha = -8\pi\Psi_0^\dagger(p^a\gamma_0\gamma_a + Q_{\mathrm{E}}r\gamma_0\gamma_5 - Q_{\mathrm{M}}r\gamma_0)\Psi_0\,.$$

In spite of the fact that α is defined in the space-time M, it is more convenient for the following calculation to use the notation borrowed from the treatment of 5-dimensional Kaluza-Klein theory [16, 17]. So $\mathscr{D}$ below refers to the Riemannian connection of a Lorentzian metric with signature $(+----)$ and having a Killing vector χ_4 satisfying $g(\chi_4, \chi_4) = -1$. In the integration of $d\alpha$ one will be concerned with

$$\begin{aligned} d\alpha_{\underline{\underline{abc}}} = \tfrac{1}{3!}\,\varepsilon_{0\underline{\underline{abc}}}[&2\mathscr{D}^{\underline{\underline{e}}}(\bar{\Psi})\gamma_0\mathscr{D}_{\underline{\underline{e}}}(\Psi) + 2\mathscr{D}_{\underline{\underline{e}}}(\Psi)\gamma^{\underline{\underline{e}}}\gamma^0\gamma^{\underline{\underline{d}}}\mathscr{D}_d(\Psi) \\ &+ 2\mathscr{D}_{\underline{\underline{e}}}(\Psi)\gamma^{\underline{\underline{e}}}\gamma^0\gamma^4\mathscr{D}_4(\Psi) + 2\mathscr{D}_4(\Psi)\gamma^4\gamma^0\gamma^{\underline{\underline{e}}}\mathscr{D}_{\underline{\underline{e}}}(\Psi) - 8\pi T^{0e}\Psi\gamma_e\Psi \\ &- 8\pi r J^0_{\mathrm{M}}\bar{\Psi}\Psi - 8\pi r J^0_{\mathrm{E}}\Psi\gamma_5\Psi + (2-8r^2)H^{0e}\Psi\gamma_e\Psi]\,, \end{aligned} \tag{3.2}$$

where the Einstein field equation, $G_{ab} = -8\pi T_{ab} + 2H_{ab}$ and Maxwell equations[2] have been used, and where

$$H_{ab} \equiv F_{ac}F_b{}^c - \frac{g_{ab}}{4}F_{ed}F^{ed}\,. \tag{3.3}$$

Here T_{ab} is the matter tensor with no contribution from electromagnetic fields. Note that one is using $\gamma_4 = \gamma_5$, so in particular $\gamma^4 = -\gamma_5$.

One knows that

$$H^{0e}\bar{\Psi}\gamma_e\Psi \leqq 0\,, \tag{3.4}$$

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1 Here one is using units in which the velocity of light and the gravitational constant are one. The relation between the electric charge of the electron and its mass is $e/m_e = 2\times10^{21}$

2 For generality one is considering a non-zero magnetic current

so the idea is to require

$$T_{ab}u^a v^b \geqq [(rJ^a_{\mathrm{E}}u_a)^2 + (rJ^a_{\mathrm{M}}u_a)^2]^{1/2} \tag{3.5}$$

for any pair of future directed timelike unit vectors u and v in M, and some condition on the spinors such that the integral of $d\alpha$ is negative. $r \neq 0$ is to be chosen such that the inequality (3.5) has the widest possible physical applicability (see comment at the end).

The following equation is assumed to hold

$$\gamma^{\underline{\underline{e}}}\mathscr{D}_{\underline{\underline{e}}}\Psi = vrF^{ab}\gamma_a\gamma_b\gamma_5\Psi\,. \tag{3.6}$$

It is useful to rearrange terms by defining

$$\tilde{D}_e\Psi \equiv \mathscr{D}_e\Psi + \gamma_e\mu rF^{ab}\gamma_a\gamma_b\gamma_5\Psi\,,$$

where μ and v are constants to be determined by maximizing the allowable values of r. Then one has

$$\begin{aligned} d\alpha_{\underline{\underline{abc}}} = \tfrac{1}{3!}\,\varepsilon_{0\underline{\underline{abc}}}[&2\tilde{D}^{\underline{\underline{e}}}(\overline{\Psi})\gamma^0\tilde{D}_{\underline{\underline{e}}}(\Psi) - 8\pi T^{0e}\overline{\Psi}\gamma_e\Psi - 8\pi rJ^0_{\mathrm{E}}\overline{\Psi}\gamma_5\Psi \\ &- 8\pi rJ^0_{\mathrm{M}}\overline{\Psi}\Psi + (2-r^2z)H^{0e}\overline{\Psi}\gamma_e\Psi]\,, \end{aligned} \tag{3.7}$$

where $z = 16(v^2 + v + \frac{1}{2} + 2\mu v + 3\mu^2)$.

Here v and μ are chosen such that z is minimum, which gives $v = -\frac{3}{4}$, $\mu = \frac{1}{4}$; $z_{\min} = 2$. Note then that Eq. (3.6), to be satisfied by the spinor, can now be expressed by $\gamma^{\underline{\underline{e}}}\tilde{D}_{\underline{\underline{e}}}\Psi = 0$.

From Eq. (3.7) it follows that after putting $z = z_{\min} = 2$, the integral $\int_N d\alpha$ will be negative if $r^2 \leqq 1$. And so the following result is deduced:

Theorem. *Let M be a space-time asymptotically flat at spatial infinity in which the Einstein field equations are satisfied. Let N be a non-singular spacelike hypersurface extending to spatial infinity. Assume that on N the following condition is satisfied*

$$T_{ab}u^a v^b \geqq [(rJ^a_{\mathrm{E}}u_a)^2 + (rJ^a_{\mathrm{M}}u_a)^2]^{1/2}\,, \tag{3.8}$$

for $|r| \leqq 1$, and where u and v are two future directed timelike unit vectors in M; then the ADM mass and the electromagnetic charges satisfy

$$m \geqq [(Q_{\mathrm{E}}r)^2 + (Q_{\mathrm{M}}r)^2]^{1/2}\,. \tag{3.9}$$

Equation (3.6) with $|r| = 1$ is essentially the same[3] as Eq. (22) of Gibbons et al. [18][4], so one can apply their argument to the case of black holes. In particular if one has a black hole and no matter outside, then condition (3.8) is satisfied for all $|r| \leqq 1$, and so one proves inequality (3.9) for $|r| = 1$, which also agrees with the result of M. Ludvigsen and Vickers [19]. A discussuion of the existence of solutions of Eq. (3.6) can be found in [18].

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3 The terms are the same modulo constants

4 Here one is using the opposite signature with respect to the one used in [18]

So from this one observes that to get a black hole with the charge of the electron "e" one needs a mass greater than $m_c = e = 1.86 \times 10^{-6}\,\mathrm{g} = \sqrt{\alpha} m_K = m_K/11.7$, where this α is the fine-structure constant and m_K is the Planck mass.

In particular, it is improper to consider a collapsed object with the mass and charge of the electron in the framework only of general relativity; that is, for example, as the source for the Reissner-Nordström solution, since in that case one will have a naked singularity [20].

All the above mentioned results are also valid for the Bondi mass. This follows from the statement of Israel and Nester [21] that the form E of Eq. (2.5) gives the Bondi mass as long as the spinor Ψ behaves asymptotically as $\Psi_{\mathrm{const}} + O(1/r)$ on an asymptotically null nontimelike hypersurface N [22].

The idea of considering space-time as a Riemannian manifold, not necessarily flat, is known to be successful for the study of physical systems of astronomical sizes; so for a comoving coordinate the mass density $\varrho \equiv T_{00}$ and charge densities ϱ_E and ϱ_M are measured in macroscopic volumes.

Experimentally one finds that $\varrho_{\mathrm{E}}/\varrho < e/m_e \equiv 1/r_1$, where "$e$" and "$m_e$" are the charge and mass of the electron. On the other hand from the present understanding of magnetic monopoles [23] one also has that $\varrho_{\mathrm{M}}/\varrho < 1/r_1$. Note that $r_1 = 4.90 \times 10^{-22}$. To study the consequences of this in the inequality of the form (3.8), it is only necessary to consider the extreme case in which the velocity associated to the movement of the gravitational matter is parallel to that of electrically charged matter. This will represent a system of charged particles of the same sign.

First note that in Eq. (3.2) there is only need to require inequality (3.8) for which $u = e_0$; e_0 being part of a tetrad associated to the spacelike hypersurface N, that is e_1, e_2, and e_3 are tangent to N. In terms of this tetrad the following notation will be used:

$$(T_a{}^b e^a{}_0) = (T_0{}^b) = (\varrho_c \cosh\beta_c, \varrho_c \bar{v}_c \cosh\beta_c),$$
$$(J_E) = (\varrho_E \cosh\beta_c, \varrho_E \bar{v}_v \cosh\beta_c),$$

where bar denotes a 3-vector and

$$\cosh\beta_c \equiv \frac{1}{[1 - v_c^2]^{1/2}}, \quad \text{with} \quad v_c^2 \equiv -\bar{v}_c \cdot \bar{v}_c < 1\,.$$

ϱ_c refers to the mass density of the charges and v_c to their velocity with respect to this tetrad. Also an arbitrary timelike future directed unit vector v will be denoted by $v = (\cosh\beta, \bar{v}\cosh\beta)$, where

$$\cosh\beta \equiv \frac{1}{[1 - v^2]^{1/2}}, \quad \text{and} \quad v^2 \equiv -\bar{v} \cdot \bar{v} < 1\,.$$

Then one has

$$\begin{aligned} T_{ab} e_0{}^a v^b &= \varrho_c \cosh\beta_c \cosh(1 - \bar{v} \cdot \bar{v}_c) \\ &\geqq \varrho_c \cosh\beta_c \cosh\beta(1 - v \cdot v_c) \\ &= \varrho_c(\cosh\beta_c \cosh\beta - \sinh\beta_c \sinh\beta) \\ &= \varrho_c \cosh(\beta_c - \beta)\,, \end{aligned}$$

where

$$\sinh\beta_c \equiv \frac{v_c}{[1-v_c^2]^{1/2}}, \qquad \sin\beta \equiv \frac{v}{[1-v^2]^{1/2}}.$$

So it is observed that $T_{ab}e_0{}^a v^b \geqq \varrho_c$.

Now note that $rJ_E{}^{\underline{a}}e_0{}^{\underline{b}}\eta_{\underline{ab}} = r\varrho_E \cosh\beta_c$, where $\underline{a}$, $\underline{b}=0, 1, 2, 3$, and recall that

$$\varrho_c > r_1 \varrho_E. \tag{3.10}$$

Assume now that β_c has a maximum β_c^* (which is associated to the maximum velocity measured by an observer moving orthogonally to the hypersurface N); then it is easy to see that Eq. (3.10) implies

$$T_{ab}e_0{}^a v^b \geqq r_0 J_E{}^{\underline{a}} e_0{}^{\underline{b}} \eta_{\underline{ab}} \tag{3.11}$$

with

$$r_0 \equiv \frac{r_1}{\cosh\beta_c^*}. \tag{3.12}$$

The same argument applies if electric charges are replaced by magnetic charges, and also if both kind of charges are allowed to appear. The consideration of a more general situation, in which there is in addition neutral matter moving in an arbitrary direction, will introduce an extra positive term on the left-hand side of inequality (3.11). From all this, then, it is deduced that the inequality

$$T_{ab}e_0{}^a v^b \geqq r_0[(J_E{}^{\underline{a}} e_0{}^{\underline{b}} \eta_{\underline{ab}})^2 + (J_M{}^{\underline{a}} e_0{}^{\underline{b}} \eta_{\underline{ab}})^2]^{1/2} \tag{3.13}$$

is always satisfied.

So one concludes from one's knowledge of the properties of matter that the following result is true:

Let M be a space-time asymptotically flat at spatial (null) infinity in which the Einstein field equations are satisfied, and which may contain bounded matter and black holes in the interior. Then, associated to a spacelike (asymptotically null) hypersurface N, (for which β_c^* exists), the ADM (Bondi) mass satisfies $m \geqq r_0[(Q_E)^2 + (Q_M)^2]^{1/2}$, where r_0 is defined by Eq. (3.12), and Q_E and Q_M are the electric and magnetic charges respectively.

Of course, the difference here with former results is that the condition on the matter fields is automatically satisfied.

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References

1. Gibbons, G.W., Hull, C.M.: A Bogomolny bound for general relativity and solitons in $N=2$ supergravity. Phys. Lett. **109**B, 190 (1982)
2. Weber, J., Wheeler, J.A.: Reality of the cylindrical gravitational waves of Einstein and Rosen. Rev. Mod. Phys. **29**, 509 (1957)
3. Araki, H.: On weak time-symmetric gravitational waves. Ann. Phys. (N.Y.) **7**, 456 (1959)

4. Brill, D.: On the positive definite mass of the Bondi-Weber-Wheeler time-symmetric gravitational waves. Ann. Phys. (N.Y.) **7**, 466 (1959)
5. Arnowitt, R., Deser, S., Misner, C.: Note on positive-definiteness of the energy of the gravitational field. Ann. Phys. (N.Y.) **11**, 116 (1960)
6. Bondi, H., Burg, M.G.J. van der, Metzner, A.W.K.: Gravitational waves in general relativity VII. Waves from axi-symmetric isolated systems. Proc. R. Soc. London Ser. A **269**, 21 (1962)
7. Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**, 45 (1979)
8. Deser, S., Teitelboim, C.: Supergravity has positive energy. Phys. Rev. Lett. **39**, 249 (1977)
9. Grisaru, M.T.: Positivity of the energy in Einstein theory. Phys. Lett. **73**B, 207 (1978)
10. Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. **80**, 381 (1981)
11. Penrose, R.: In: Structure of the space-time, in Battelle Rencontres, DeWitt, C.M., Wheeler, J.A. (eds.). New York: W. A. Benjamin, Inc. 1968
12. Parker, T., Taubes, C.H.: On Witten's proof of the positive energy theorem. Commun. Math. Phys. **84**, 223 (1982)
Reula, O.: Existence theorem for solutions of Witten's equation and nonnegativity of total mass. J. Math. Phys. **23**, 810 (1982)
13. Nester, J.M.: A new gravitational energy expression with a simple positivity proof. Phys. Lett. **83**A, 241 (1981)
14. Arnowitt, R., Deser, S., Misner, C.W.: Coordinate invariance and energy expressions in general relativity. Phys. Rev. **122**, No. 3, 997 (1961)
15. See also: Tod, K.P.: All metrics admitting super-covariantly constant spinors. Phys. Lett. **121**B, 241 (1983)
16. Moreschi, O.: Pittsburgh Ph. D. Thesis (1983)
17. Moreschi, O.M., Sparling, G.A.J.: On Riemannian spaces with conformal symmetries or a tool for the study of generalized Kaluza-Klein theories. J. Math. Phys. **24**, No. 2, 303 (1983)
18. Gibbons, G.W., Hawking, S.W., Horowitz, G.T., Perry, M.J.: Positive mass theorems for blackholes. Commun. Math. Phys. **88**, 295 (1983)
19. Ludvigsen, M., Vickers, J.A.G.: J. Phys. A: Math. Gen. **16**, 1169 (1983)
20. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-time. Cambridge: Cambridge University Press 1973
21. Israel, W., Nester, J.M.: Positivity of the Bondi gravitational mass. Phys. Lett. **85**A, 259 (1981)
22. Existence of solutions to Witten's equation for the case of an asymptotically null non-timelike hypersurface has been studied by O. Reula and K. P. Tod (preprint). See also: Horowitz, G.T., Perry, M.J.: Gravitational energy cannot become negative. Phys. Rev. Lett. **48**, 371 (1982), Ludvigsen, M., Vickers, J.A.G.: A simple proof of the positivity of the Bondi mass. J. Phys. A: Math. Gen. **15**, L67 (1982)
23. 't Hooft, G.: Magnetic monopoles in unified gauge theories. Nucl. Phys. B**79**, 276 (1974)

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