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Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and Their Mixing Properties*

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Abstract. We consider one parameter analytic hamiltonian perturbations of the geodesic flows on surfaces of constant negative curvature. We find two different necessary and sufficient conditions for the canonical equivalence of the perturbed flows and the non-perturbed ones. One condition says that the "Hamilton-Jacobi equation" (introduced in this work) for the conjugation problem should admit a solution as a formal power series (not necessarily convergent) in the perturbation parameter. The alternative condition is based on the identification of a complete set of invariants for the canonical conjugation problem. The relation with the similar problems arising in the KAM theory of the perturbations of quasi periodic hamiltonian motions is briefly discussed. As a byproduct of our analysis we obtain some results on the Livscic, Guillemin, Kazhdan equation and on the Fourier series for the $SL(2,\mathbb{R})$ group. We also prove that the analytic functions on the phase space for the geodesic flow of unit speed have a mixing property (with respect to the geodesic flow and to the invariant volume measure) which is exponential with a universal exponent, independent on the particular function, equal to the curvature of the surface divided by 2. This result is contrasted with the slow mixing rates that the same functions show under the horocyclic flow: in this case we find that the decay rate is the inverse of the time ("up to logarithms").

1. The Integrability Problem and its Invariants

Integrable hamiltonian systems are important in mechanics because they provide classes of systems whose dynamical behaviour is well understood and which can be used as a "reference behaviour" for systems close to integrable ones.

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There are, however, other dynamical systems whose behaviour is also well understood, although very different from that of integrable systems. One should naturally think to use such systems, also, as a reference for the behaviour of other classes of mechanical systems. Therefore we shall extend the notion of integrability as follows: Let Σ be an l-dimensional compact analytic manifold and let $T^*\Sigma$ be the phase space for the hamiltonian flows on Σ. As usual we shall denote a point in $T^*\Sigma$ by $(\underline{p}, \underline{q})$, $\underline{q}$ being the coordinates of a point of Σ and $\underline{p}$ being a conjugate momentum in the same system of coordinates. Geometrically p is a cotangent vector.

An analytic hamiltonian system on $T^*\Sigma$ will be a pair (W, H) with $W \subset T^*\Sigma$ being the closure of an open set and with H being an analytic function on W[1] and such that, for every $(\underline{p}, \underline{q}) \in W$, the solution $S_t(\underline{p}_0, \underline{q}_0)$ to the equations:

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}(\underline{p}, \underline{q}), \dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}(\underline{p}, \underline{q}), \quad (\underline{p}, \underline{q}) \in W, \tag{1.1}$$

with initial datum $(\underline{p}_0, \underline{q}_0) \in W$, exists for all $t \in R$.

Then the following definition extends the well known notion of integrability:

Definition 1. Let (W, H), (W', H') be two analytic hamiltonian systems on two compact analytic surfaces Σ and Σ'. We say that "(W', H') is (W, H)-integrable," or simply "H' is H-integrable" if there is a C^∞ canonical map $\mathscr{C}$ mapping W onto W' and an analytic function defined on $H(W)$, denoted F and such that $F' = (dF/dE) \neq 0$, and

$$H'(\mathscr{C}(\underline{p}, \underline{q})) = F(H(\underline{p}, \underline{q})), \quad (\underline{p}, \underline{q}) \in W. \tag{1.2}$$

If $\mathscr{C}$ is also analytic we say that H' is analytically H-integrable.

The possibility that $\mathscr{C}$ is C^∞ but not analytic leaves us more flexibility in the formulation of the results that we are able to prove.

The above definition says, in other words, that H' is H-integrable if the flow generated by H on W is canonically conjugate, up to a time scale change given by $F'(H)$, to the flow generated by H' on W'.

In our terminology a map $\mathscr{C}$, $(\underline{p}', \underline{q}') = \mathscr{C}(\underline{p}, \underline{q})$, of W onto W', is canonical if it is at least C^∞ with C^∞ inverse and, if calling

$$G(\mathscr{C}) = \{(\underline{p}, \underline{q}, \underline{p}', \underline{q}') | (\underline{p}, \underline{q}) \in W, (\underline{p}', \underline{q}') \in W', (\underline{p}', \underline{q}') = \mathscr{C}(\underline{p}, \underline{q})\}, \tag{1.3}$$

there is a $\Psi \in C^\infty(G(\mathscr{C}))$ such that:

$$\underline{p} \cdot d\underline{q} = \underline{p}' \cdot d\underline{q}' + d\Psi. \tag{1.4}$$

A more precise name for such $\mathscr{C}$'s could be "action preserving global canonical transformations": if λ is a closed curve in W and $\lambda' = \mathscr{C}(\lambda)$, the "actions" of λ and λ' are the same:

$$\oint_\lambda \underline{p} \cdot d\underline{q} = \oint_{\lambda'} \underline{p}' \cdot d\underline{q}'. \tag{1.5}$$

1 As usual f will be said analytic (or C^∞) on a closed set W if it is the restriction to W of a function analytic (or C^∞) in its vicinity

More generally, in the literature one calls "locally canonical" a transformation $\mathscr{C}$ such that $d\underline{p} \wedge d\underline{q} = d\underline{p}' \wedge d\underline{q}'$. This last relation implies a relation like (1.4) in small neighborhoods of phase space, but Ψ need not be "uniform", i.e. exist globally as a single valued function on $G(\mathscr{C})$. We will not use this more general notion.

The simplest integrable systems are those which are part of a one parameter family of integrable systems.

Definition 2. Let $H_\varepsilon = H_0 + f_\varepsilon$ be an analytic function on $W \times [-\theta, \theta]$, $\theta > 0$, with $(\underline{p}, \underline{q}, \varepsilon)$ analytic in $W \times [-\theta, \theta]$ and divisible by ε.

We shall say that the family of hamiltonians H_ε, $\varepsilon \in [-\theta, \theta]$, is H_0-integrable if:

i) There is a family $\mathscr{C}_\varepsilon$, $\varepsilon \in [-\theta, \theta]$, of canonical maps of W into $T^*\Sigma$ such that $\mathscr{C}_\varepsilon(\underline{p}, \underline{q})$ is C^∞ in $(p, q, \varepsilon) \in W \times [-\theta, \theta]$, analytic in ε and differs from the identity map by a quantity which is divisible by ε.

ii)

$$H_\varepsilon(\mathscr{C}_\varepsilon(\underline{p}, \underline{q})) = F_\varepsilon(H_0(\underline{p}, \underline{q})), \quad (\underline{p}, \underline{q}) \in W \times [-\theta, \theta] \tag{1.6}$$

with $F_\varepsilon(E)$ being analytic in (E, ε) on the set $\mathscr{E} \times [-\theta, \theta]$, $\mathscr{E} = \{E | E = H_0(\underline{p}, \underline{q}), (\underline{p}, \underline{q}) \in W\} \equiv H_0(W)$, and such that $F_\varepsilon(E) - E$ is divisible by ε.

Equation (1.6) can be regarded, given H_ε, as an equation for $\mathscr{C}_\varepsilon$, F_ε, and we shall call it the "Hamilton-Jacobi" equation for the integration of H_ε with respect to H_0. Similarly we call (1.2) the "Hamilton-Jacobi" equation for the integration of H' with respect to H.

Families of integrable perturbations with respect to the system

$$W = V \times T^l, \; V \subset \mathbb{R}^l, \; H_0(\underline{A}, \underline{\phi}) = \underline{\omega}_0 \cdot \underline{A}, \tag{1.7}$$

where $(\underline{A}, \underline{\phi})$ denotes a point on $V \times T^l$, T^l being the l-dimensional torus, $\underline{\omega}_0 \in \mathbb{R}^l$, ("harmonic oscillators"), have been studied recently and enjoy remarkable properties [9].

The case when W is as in (1.7) and $H_0(\underline{A}, \underline{\phi}) = h_0(\underline{A})$ is $\underline{\phi}$-independent is the problem studied by the well known KAM theory.

Definition 3. If in Definition 2 we replace the requirement on $\mathscr{C}_\varepsilon$ to be of class C^∞ with that of being analytic we obtain the notion of "analytically H_0-integrable" family of perturbations.

In this paper we analyse the case when H_0 is the hamiltonian for the geodesic flow on a Riemannian surface of constant negative curvature, equal to -1 (say), and:

$$W = \{(\underline{p}, \underline{q}) | H_0(\underline{p}, \underline{q}) \in [1/2, 3/2]\}. \tag{1.8}$$

Our objective is to find necessary and sufficient conditions for recognizing the H_0-integrability of a family of perturbations.

As in the classical case, arising in the KAM theory, it will generally be impossible to conjugate two hamiltonian flows. The obstacles may lie in the existence of "invariants," i.e. of quantities associated with the H_ε-flow that must assume values determined by H_0, just because the H_ε-flow and the H_0-flow are canonically conjugate.

We describe some of them here. Suppose that, for simplicity, the H_0 and H_ε flows, for all $\varepsilon\in[-\theta,\theta]$, have only a denumerable family of closed orbits on each energy surface H_0 or $H_\varepsilon=E$, $E\in[1/2,3/4]$, and also that for all $\varepsilon\in[-\theta,\theta]$ such orbits stay pairwise distinct, i.e. they can be labeled by (n,E,ε), $n=1,2,\ldots$, $E\in[1/2,3/4]$, $\varepsilon\in[-\theta,\theta]$, in such a way that they depend continously (hence analytically) on (E,ε) at fixed n and are pairwise distinct for all n, at fixed (E,ε).

This assumption holds if (W,H_0) is the hamiltonian for the geodesic flow on a surface of constant negative curvature: this is a simple consequence of Anosov's structural stability theorem and of the isomorphism between geodesic flows of different energies (but, of course, on the same surface).

Then let:

$$T(E,n,\varepsilon)=\{\text{period of the orbit } (E,n,\varepsilon)\},$$

$$-\lambda_-(E,n,\varepsilon)=\lambda_+(E,n,\varepsilon)=\{\text{Lyapunov exponents of } (E,n,\varepsilon),\ -\lambda_-=\lambda_+>0\},$$

$$A(E,n,\varepsilon)=\{\text{action of } (E,n,\varepsilon)\}=\left\{\oint_{(E,n,\varepsilon)}\underline{p}\cdot d\underline{q}\right\}. \tag{1.9}$$

If H_ε is H_0-integrable in the sense (1.6), clearly:

$$\frac{T(E,n,\varepsilon)}{T(E,m,\varepsilon)}=\frac{T(F_\varepsilon^{-1}(E),n,0)}{T(F_\varepsilon^{-1}(E),m,0)}=t_{nm}, \tag{1.10}$$

$$\frac{\lambda_+(E,n,\varepsilon)}{\lambda_+(E,m,\varepsilon)}=\frac{\lambda_+(F_\varepsilon^{-1}(E),n,0)}{\lambda_+(F_\varepsilon^{-1}(E),m,0)}=l_{mn}, \tag{1.11}$$

$$\frac{A(E,n,\varepsilon)}{A(E,m,\varepsilon)}=\frac{A(F_\varepsilon^{-1}(E),n,0)}{A(F_\varepsilon^{-1}(E),m,0)}=a_{mn}, \tag{1.12}$$

where in the right-hand side there is no (E,ε)-dependence because for $\varepsilon=0$ the flow is geodesic and the intermediate ratios do not depend on E.

Clearly the identities (1.10) through (1.12) are necessary conditions for the integrability of the family H_ε. It is easy to see that (1.10), (1.11) are not a complete set of invariants for the canonical integrability of a family of perturbations of the geodesic flow: see Appendix E for an example.

However we shall prove:

Proposition 1. i) *The numbers* (1.12) *are a complete set of invariants for the canonical conjugation ("H_0-integrability") problem on $W\times[-\bar{\theta},\bar{\theta}]$ if H_0 is the above described geodesic flow and $\bar{\theta}$ is small enough (given f_ε).*

ii) *Furthermore the integrability in* i) *is in fact analytic in the sense of Definition 3, whenever it exists.*

We shall call the left-hand side of (1.12), the "action invariants;" Proposition 1 shows their completeness.

A related result that Proposition 1 extends is the following: Let g^0 be a smooth Riemaniann metric of negative curvature on a compact surface. Then the hamiltonian for the geodesic flow has the form:

$$H_0(\underline{p},\underline{q})=\frac{1}{2}\sum_{i,j=1,2}(g^0(\underline{q})^{-1})_{ij}p_ip_j\,. \tag{1.13}$$

Consider the following "geometric perturbations":

$$f_\varepsilon(\underline{p}, \underline{q}) = \frac{\varepsilon}{2} \sum_{i,j=1,2} s_{ij}(\underline{q}, \varepsilon) p_i p_j \tag{1.14}$$

with s analytic in $(q, \varepsilon) \in \sum \times [-\theta, \theta]$; they correspond to changes in the metric $g^0 \to g_\varepsilon = ((g^0)^{-1} + \varepsilon s)^{-1}$.

Then one can ask when $H_\varepsilon = H_0 + f_\varepsilon$, which is still a geodesic flow, is H_0-integrable with $W = T^*\Sigma$ and $F_\varepsilon(E) = E$ and $\mathscr{C}_\varepsilon$ of the form:

$$\mathscr{C}_\varepsilon(\underline{p}, \underline{q}) = (J_\varepsilon(\underline{q})^{-1}\underline{p}, \hat{\mathscr{C}}_\varepsilon(\underline{q})), \tag{1.15}$$

where $\hat{\mathscr{C}}_\varepsilon$ is a diffeomorphism of Σ and J_ε is its jacobian matrix.

A complete set of invariants for this type of conjugation, "geometric conjugation" or "trivial conjugation," is in some cases known to be the set of the eigenvalues of the Laplace-Beltrami operator [2].

Our methods have on one hand some resemblance with those of [2], particularly in the use of the key result [3]: however we make strong use of the group theoretic structures provided by the assumption of constant curvature and obtain results which probably do not hold for the theory of perturbations of the geodesic flows on surfaces of non-constant negative curvature (while [2] deals mainly with such general manifolds). But the main difference is that we do not deal with the geometrical conjugacy problem and consider, rather, the general action preserving canonical conjugacy using techniques developed in the context of the KAM theory. On the other hand Propositions 1 and 2 (see Sect. 2) are, technically, an extension of a nice criterion of convergence for the Birkhoff series due to Rüssmann [12].

2. The Flows on Constant Negative Curvature Surfaces. Good Coordinates. Integrability and Perturbation Theory

The surfaces of constant negative curvature are constructed as follows: If $z = x + iy$, $x \in \mathbb{R}$, $y > 0$, is regarded as a point in the upper half plane $\mathbb{C}_+$, the action of the group PSL(2, $\mathbb{R}$) on $\mathbb{C}_+$ is

$$z \to zg = \frac{az + c}{bz + d} \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

The most general compact analytic surface of constant negative curvature is:

$$\Sigma = \mathbb{C}_+ / \Gamma, \tag{2.2}$$

where Γ is a hyperbolic fuchsian group (i.e. a fuchsian group without parabolic or elliptic elements [7]). It is endowed with the PSL(2, $\mathbb{R}$) invariant metric $ds^2 = (dx^2 + dy^2)/y^2$. The surface Σ can be identified with a fundamental domain Σ_0 of Γ with "opposite sides" identified modulo Γ.

On Σ_0 the geodesic flow is described, by definition, by the hamiltonian:

$$H_0 = y^2(p_x^2 + p_y^2)/2. \tag{2.3}$$

Any motion with non-zero velocity will reach $\partial\Sigma_0$ in a finite time t_0 at the point (x_0, y_0) with speed $(\dot{x}_0, \dot{y}_0)$: if γ is an element of Γ reflecting (x_0, y_0) to another element of $\partial\Sigma_0$, the motion will continue after t_0 by reappearing at $z_0'=z_0\gamma$, if $z_0=x_0+iy_0$, with velocity $\dot{z}_0'=\dot{x}_0'+i\dot{y}_0'$ given by:

$$\dot{z}_0'=\dot{z}_0(bz_0+d)^{-2}, \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4}$$

This is a somewhat complicated description of the geodesic flow.

There is, however a better representation: it is inspired by ref. [4]. Let:

$$j(z,g)=(bz+d). \tag{2.5}$$

It is easy to verify that the transformation $\mathscr{K}$ of $T^*\mathbb{C}_+\backslash\{H_0=0\}$ onto PGL(2, $\mathbb{R}$):

$$\mathscr{J}\mathscr{K}: (p_x, p_y, x, y) \leftrightarrow \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix}=g, \tag{2.6}$$

defined by

$$\begin{aligned} p_x+ip_y &= i(\det g)^2 j(i, g^{-1})^2/2, \\ x+iy &= ig^{-1}, \end{aligned} \tag{2.7}$$

is such that (see Appendix D):

$$p_x dx+p_y dy=p_1 dq_1+p_2 dq_2-d(\det g)/2, \tag{2.8}$$

therefore it is canonical. Furthermore $\mathscr{K}((p_x+ip_y,\ x+iy)\gamma^{-1})=\gamma g$, so that $\mathscr{K}$ defines a map from $T^*\Sigma\backslash\{H_0=0\}$ onto $\Gamma\backslash$PGL(2, $\mathbb{R}$).

The map $\mathscr{K}$ transforms the hamiltonian (2.3) into (see Appendix D):

$$H_0(g)=(\det g)^2/8. \tag{2.9}$$

Therefore if $f_\varepsilon(g)$ is a function of (g, ε) analytic on $W\times[-\theta, \theta]$, divisible by ε, the hamiltonian equations for $H_\varepsilon=H_0+f_\varepsilon$ are in the new coordinates:

$$\dot{g}=-(\det g)g\sigma_z/4+\frac{\sigma\partial f}{\partial g}(g)\sigma_x, \tag{2.10}$$

where

$$\sigma=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \left(\frac{\partial f}{\partial g}(g)\right)_{ij} \equiv \frac{\partial f}{\partial g_{ij}}(g), \quad ij=1,2.$$

The Liouville measure on $T^*\Sigma\backslash\{H_0=0\}$, realized as $\Gamma\backslash$PGL(2,$\mathbb{R}$), is just $dp_1 dp_2 dq_1 dq_2$, i.e. it is the Haar measure of PGL(2, $\mathbb{R}$) considered as a measure on the homogeneous space $\Gamma\backslash$PGL(2, $\mathbb{R}$).

We do not discuss in more detail why Eqs. (2.9), (2.10) induce a flow on $\Gamma\backslash$PGL(2, $\mathbb{R}$): it is obvious that they do so on the whole PGL(2, $\mathbb{R}$) and the fact that it can be regarded as a flow on PGL(2, $\mathbb{R}$) stems from the observation that if $t \to g(t)$ is a solution to (2.10), then so is $t\to\gamma g(t)$. This observation can be used to say that the entries of $g \in \mathrm{PGL}(2,\mathbb{R})$ provide a global system of coordinates on the phase space of the geodesic flow deprived of the points with zero velocity and provided one identifies the $g \in \mathrm{PGL}(2,\mathbb{R})$ modulo Γ.

We shall use the above remarks quite extensively. Their main usefulness is in providing a simple characterization of the C^∞-canonical maps $\mathscr{C}$ defined on a set of the form $W=\{g|H_0(g)\in[1/2,3/4]\}$. Suppose that $\mathscr{C}$ is close to the identity and that it is defined on an open set W' containing W, large enough so that if $(\underline{p}',\underline{q}')=\mathscr{C}(\underline{p},\underline{q})$, $(\underline{p},\underline{q})\in W$, then $(\underline{p}',\underline{q})$, $(\underline{p},\underline{q}')\in W'$. Then $\mathscr{C}$ can be described by a generating function Φ of class C^∞ on the set $\{(\underline{p}',\underline{q})|(\underline{p}',\underline{q})\in W', \exists(\underline{p},\underline{q}')\in W'$ such that $(\underline{p}',\underline{q}')=\mathscr{C}(\underline{p},\underline{q})\}\equiv\tilde{W}$ via the relation:

$$\left.\begin{aligned} \underline{p}&=\underline{p}'+\frac{\partial\Phi}{\partial\underline{q}}(\underline{p}',\underline{q}) \\ \underline{q}'&=\underline{q}+\frac{\partial\Phi}{\partial\underline{p}'}(\underline{p}',\underline{q}) \end{aligned}\right\} \quad \forall(\underline{p}',\underline{q})\in\tilde{W}, (W\subset\tilde{W}\subset W'), \tag{2.11}$$

which can be written in a better form in terms of matrices,

$$g=\begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix}, g'=\begin{pmatrix} p_1' & q_2' \\ -p_2' & q_1' \end{pmatrix}, \hat{g}=\begin{pmatrix} p_1' & q_2 \\ -p_2' & q_1 \end{pmatrix}, \hat{g}'=\begin{pmatrix} p_1 & q_2' \\ -p_2 & q_1' \end{pmatrix}. \tag{2.12}$$

Then (2.11) becomes:

$$\hat{g}'=\hat{g}-\sigma\frac{\partial\Phi}{\partial g}(\hat{g})\sigma\,. \tag{2.13}$$

The whole point of the above discussion is the *globality of* (2.13) *on* W: it follows from the easily checked property that if $g'=\mathscr{C}(g)$, $\gamma\in\Gamma$, then $\gamma g'=\mathscr{C}(\gamma g)$, because $\gamma\hat{g}'$ and $\gamma\hat{g}$ solve (2.13) if $\hat{g}$, $\hat{g}'$ do. What is slightly less clear is that conversely any canonical transformation $\mathscr{C}$ close enough to the identity can be generated by a relation of the form (2.13). This remarkable fact can be seen by observing that if $\mathscr{C}$ has the property:

$$\underline{p}\cdot d\underline{q}=\underline{p}'\cdot d\underline{q}'+d\Psi \tag{2.14}$$

with Ψ a C^∞-function on the graph of $\mathscr{C}$, then (2.14) can be rewritten as:

$$\underline{p}\cdot d\underline{q}+\underline{q}'\cdot d\underline{p}'=d(\Psi+\underline{p}'\cdot\underline{q}')\,, \tag{2.15}$$

and $\Psi+(\underline{p}'\cdot\underline{q}')=\Psi+\det g'$ is a C^∞ function on the graph of $\mathscr{C}$ because Ψ is such by assumption and $\det g'$ is single valued and C^∞ on $\Gamma\backslash\mathrm{PGL}(2,\mathbb{R})$, and therefore it can be thought as a function on the graph of $\mathscr{C}$.

Furthermore the function on the graph of $\mathscr{C}$ defined by $(g,g')\to(\det\hat{g})$ is C^∞ when $\mathscr{C}$ is so close to the identity that the relation $g'=\mathscr{C}(g)$ can be put in the form $\hat{g}'=\hat{\mathscr{C}}(\hat{g})$, using the implicit function theorem, and $\mathscr{C}(\gamma g)=\gamma\mathscr{C}(g)$ when g and γg belong to the boundary of a fundamental domain in $\mathrm{PGL}(2,\mathbb{R})$ with respect to the action of Γ. Therefore the function $\Phi=\Psi+(\det g')-(\det\hat{g})$ is C^∞ on the graph of $\mathscr{C}$ and, if $\mathscr{C}$ is close enough to the identity (in the C^1-sense at least), it can be regarded as a function of $\hat{g}$ on $\{\hat{g}|g\in W\}$. But then (2.15) becomes $\underline{p}\cdot d\underline{q}+\underline{q}'\cdot d\underline{p}' =d(\det\hat{g}+\Phi(\hat{g}))$, which is (2.14).

The situation is very similar to the one that arises in the change of coordinates in systems described in action angle variables $(\underline{A}, \underline{\theta}) \in V \times T^l$: in that case a change of coordinates $(\underline{A}', \underline{\theta}') = \mathscr{C}(\underline{A}, \underline{\theta})$ defined on a set of the form $V \times T^l$, $V \subset R^l$, $T^l = \{l\text{-dimensional torus}\}$ is canonical (in our action-preserving sense) if and only if it can be generated by a relation of the form:

$$\underline{A} = \underline{A}' + \frac{\partial \Phi}{\partial \underline{\theta}}(\underline{A}', \underline{\theta}), \qquad \underline{\theta}' = \underline{\theta} + \frac{\partial \Phi}{\partial \underline{A}'}(\underline{A}', \underline{\theta}), \tag{2.16}$$

where Φ is periodic in the $\underline{\theta}$'s.

The outcome of the above discussion is the fact that it allows us to replace the search for the solutions $\mathscr{C}_\varepsilon$, F_ε of (1.6) by the search for a function Φ, C^∞ on $W \times [-\bar{\theta}, \bar{\theta}]$, $\bar{\theta} \leqq \theta$, analytic in ε:

$$\Phi(g, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k \Phi^{(k)}(g), \tag{2.17}$$

which generates $\mathscr{C}$ near W for sufficiently small ε.

Analogously we set

$$F_\varepsilon(E) = E + \sum_{k=1}^{\infty} \varepsilon^k F^{(k)}(E), \tag{2.18}$$

and rewrite (1.6) as

$$H_\varepsilon\left(\underline{p}', \underline{q} + \frac{\partial \Phi}{\partial \underline{p}'}(\underline{p}', \underline{q})\right) = F_\varepsilon\left(H_0\left(\underline{p}' + \frac{\partial \Phi}{\partial \underline{q}}(\underline{p}', \underline{q}), \underline{q}\right)\right). \tag{2.19}$$

Expanding everything in powers of ε and denoting by $\{\cdot, \cdot\}$ the Poisson bracket, one easily finds:

$$\begin{aligned}
&\{H_0, \Phi^{(1)}\}(\underline{p}', \underline{q}) = f^{(1)}(\underline{p}', \underline{q}) - F^{(1)}(H_0(\underline{p}', \underline{q})), \\
&\{H_0, \Phi^{(2)}\}(\underline{p}', \underline{q}) = f^{(2)}(\underline{p}', \underline{q}) - F^{(2)}(H_0(\underline{p}', \underline{q})) + \frac{\partial f^{(1)}}{\partial \underline{p}'}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial \underline{q}}(\underline{p}', \underline{q}) \\
&\quad + \frac{dF^{(1)}}{dE}(H_0(\underline{p}', \underline{q})) \frac{\partial H_0}{\partial \underline{q}}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial \underline{p}'}(\underline{p}', \underline{q}) - \frac{1}{2} \sum_{i,j=1}^{2} \\
&\quad \cdot \left\{ \frac{\partial^2 H_0}{\partial p_i' \partial p_j'}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial q_i}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial q_j}(\underline{p}', \underline{q}) - \frac{\partial H_0}{\partial q_i \partial q_j}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial p_i'}(\underline{p}', \underline{q}) \frac{\partial \Phi^{(1)}}{\partial p_j'}(\underline{p}', \underline{q}) \right\} \\
&\quad \vdots \\
&\{H_0, \Phi^{(k)}\}(\underline{p}', \underline{q}) = f^{(k)}(\underline{p}', \underline{q}) + \Delta^{(k)}(f^{(k-1)}, \ldots, f^{(1)}, \Phi^{(k-1)}, \ldots, \Phi^{(1)}, F^{(k-1)}, \ldots, F^{(1)}),
\end{aligned} \tag{2.20}$$

where $\Delta^{(k)}$ is a differential polynomial in its variables: very complicated, as the case $k=2$ already shows.

Let us study the first Equation (2.20): it means that $\Phi^{(1)}$ is a function whose derivative along the H_0-flow is $f^{(1)}$ up to a constant $F^{(1)}(H_0)$. Therefore integrating both sides with respect to the Liouville measure:

$$\mu_E(d\underline{p}d\underline{q}) = \text{const}\, \delta(H_0(\underline{p}, \underline{q}) - E) d\underline{p} d\underline{q}, \tag{2.21}$$

$$\text{const} = \left(\int \delta(H_0(\underline{p}', \underline{q}') - E) d\underline{p}' d\underline{q}'\right)^{-1}, \tag{2.22}$$

one finds the condition:

$$F^{(1)}(E) = \int f^{(1)}(\underline{p}, \underline{q})\mu_E(d\underline{p}\,d\underline{q}) \equiv \bar{f}^{(1)}(E), \tag{2.23}$$

determining $F^{(1)}$ uniquely. Then we shall say that the first order of perturbation theory is "well defined" if the equation:

$$\{H_0, \Phi^{(1)}\} = f^{(1)} - \bar{f}^{(1)}(H_0) \tag{2.24}$$

admits a unique $C^\infty(W)$-solution up to an (arbitrary) function of H_0. In this case $\Phi^{(1)}$ will be defined uniquely by imposing:

$$\int \mu_E(d\underline{p}\,d\underline{q})\Phi^{(1)}(\underline{p}, \underline{q}) = 0. \tag{2.25}$$

Inductively we can say that the k^{th} order of perturbation theory is well defined if the equation:

$$\{H_0, \Phi^{(k)}\} = f^{(k)} - \bar{f}^{(k)} + \Delta^{(k)} - \bar{\Delta}^{(k)} \tag{2.26}$$

(with obvious notations) admits a unique solution $\Phi^{(k)}$ on W with $\int \mu_E(d\underline{p}\,d\underline{q})\Phi^{(k)}(\underline{p}, \underline{q}) = 0$.

More generally one might like to say that a hamiltonian admits perturbation theory to order k if (2.20) can be recursively solved up to order k by suitably choosing at each step the arbitrary function of H_0 which can be added to each $\Phi^{(k)}$. It is, however, not surprising that this notion is not really more general: a hamiltonian admits perturbation theory to order k in the weaker sense just proposed if and only if it admits perturbation theory to order k in the former stronger sense.

The simplest way to understand this is to remark that (1.6) obviously admits many solutions if it admits one. Let in fact $(E, \varepsilon) \to R_\varepsilon(E)$ be defined and C^∞ on $W \times [-\theta, \theta]$, ε-analytic and divisible by ε, and consider the canonical transformation generated by $(\det \hat{g}) + R_\varepsilon(H_0(\hat{g}))$:

$$\left.\begin{aligned} \underline{p} &= \underline{p}'(1 + R'_\varepsilon(H_0(\hat{g}))) \\ \underline{q}' &= \underline{q}(1 + R'_\varepsilon(H_0(\hat{g}))) \end{aligned}\right\} \Rightarrow \underline{p} \cdot \underline{q} \equiv \underline{p}' \cdot \underline{q}'. \tag{2.27}$$

But $(\det g) = \underline{p} \cdot \underline{q}$, $H_0(\underline{p}, \underline{q}) = H_0(\underline{p}', \underline{q}')$: so that if (1.6) admits a solution $\mathscr{C}_\varepsilon$, F_ε, then also $\mathscr{C}_\varepsilon \mathscr{C}_\varepsilon^{(0)}$, F_ε, with $\mathscr{C}_\varepsilon^{(0)}$ given by (2.27) is a solution. This ambiguity of the solutions to (1.6) can easily be related to the ambiguity in the choice of $\Phi^{(k)}$ and used to prove that the k^{th} order of perturbation theory exists or does not exist independently of the arbitrary choices which one has to make in order to build it. Therefore the following definition is very natural:

Definition 4. Let H_0 be the hamiltonian for the geodesic flow on a surface of constant negative curvature and let f_ε be analytic in $W \times [-\theta, \theta]$, $W = \{\underline{p}, \underline{q} | H_0(\underline{p}, \underline{q}) = E \in [1/2, 3/2]\}$.

We say that f_ε "admits a finite perturbation theory" around H_0 if it admits k^{th} order perturbation theory for all $k = 1, 2, \ldots$.

Of course "having a finite perturbation theory" is a canonical invariant. This means that if $\mathscr{C}_\varepsilon$ is defined and analytic on $W \times [-\theta, \theta]$, canonically maps W into

$T^*\Sigma$, and differs from the identity map by an ε-divisible quantity, then defining $\hat{f}_\varepsilon$ by

$$H_0(\mathscr{C}_\varepsilon(\underline{p},\underline{q}))+f_\varepsilon(\mathscr{C}_\varepsilon(\underline{p},\underline{q}))=H_0(\underline{p},\underline{q})+\hat{f}_\varepsilon(\underline{p},\underline{q}) \tag{2.28}$$

on $W\times[-\theta',\theta']$, θ' small enough, $\hat{f}_\varepsilon$ admits a finite perturbation theory around H_0.

The perturbation theory can be easily extended to cover the case when

$$H_\varepsilon=h_\varepsilon(H_0)+f_\varepsilon, \tag{2.29}$$

where $h_\varepsilon(E)=E+\sum_{k=1}^{\infty}\varepsilon^k h^{(k)}(E)$ is analytic in $H_0(W)\times[-\theta,\theta]$, and one can give a similar definition of finiteness of the perturbation theory (canonically invariant in the same sense as above). We do not discuss the (trivial) details. One could reduce this case to the preceding by putting the ε-dependent part of h_ε into f_ε or, alternatively, repeat the arguments leading to (2.20).

It is then remarkable that:

Proposition 2. *Let f_ε be analytic on $W\times[-\theta,\theta]$ and suppose that the perturbation theory for $H_\varepsilon=H_0+f_\varepsilon$ is finite. Then:*

i) *the family H_ε is H_0-integrable for ε small enough,*

ii) *the theory of perturbations yields a convergent series for Φ_ε and F_ε for ε small enough and Φ_ε generates the integrating map $\mathscr{C}_\varepsilon$,*

iii) *the family H_ε is analytically integrable (with respect to H_0) for ε small enough.*

It is important to remark that the condition of finiteness of perturbation theory only involves the derivatives of f_ε at $\varepsilon=0$. So does the constancy of the action invariants: in fact the closed periodic orbits depend analytically on ε and therefore their actions are also analytic in ε, and are determined by their derivatives at $\varepsilon=0$. These derivatives, in turn, can be computed, (without having to solve the perturbed differential equations), in terms of the unperturbed motions, knowing only the derivatives at $\varepsilon=0$ of f_ε.

It will turn out that we shall prove first Proposition 2 and then we shall show that the integrability conditions of the k^{th} order of perturbation theory are equivalent to the conditions that the Taylor coefficients of order $1,2,\ldots,k$, about $\varepsilon=0$, of the action invariants vanish identically in E, for all m, n [see (1.13)].

In Sect. 3 we prove only the statements i) of Propositions 1 and 2, and statement ii) of Proposition 2. Statement iii) of Proposition 2 is proven in Appendix G.

Other results are presented in Sects. 4, 6: in Sect. 6 we discuss the relevance of our treatment of the Fourier analysis on $L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$ for the analysis of the mixing properties of the geodesic and horocyclic flows on the surfaces of constant negative curvature.

In this paper we shall often bound the n^{th} derivative of a function, holomorphic in some variable w as it varies in some complex domain in $\mathbb{C}$, by $n!$ times the maximum of the function (in the given domain) divided by the n^{th} power of the distance of the point to the boundary of the domain: we call such an estimate a "dimensional estimate."

Usually our domains will be parametrized by parameters called ϱ, ξ or θ and they will have the property that the distance between the boundaries of the domains parametrized by ϱ', ξ', θ', and ϱ, ξ, θ is bounded below by one of the three numbers $(\varrho-\varrho')/2$, $(\xi-\xi')/2$, $(\theta-\theta')/2$. If $\varrho'=\varrho e^{-\sigma}$, $\xi'=\xi e^{-\delta}$, $\theta'=\theta e^{-\tau}$ as will often be the case, the above numbers become $\varrho\hat{\sigma}/2$, $\xi\hat{\delta}/2$, $\theta\hat{\tau}/2$, where, to shorten the notations, we set:

$$\hat{x}=(1-e^{-x}) \quad \text{for} \quad x>0. \tag{2.30}$$

3. Proof of Proposition 2

We shall regard f_ε as a function on PGL(2, $\mathbb{R}$), parametrized by ε, and such that $f_\varepsilon(g)=f_\varepsilon(\gamma g)$, for all $\gamma\in\Gamma$ ("Γ-periodic function"). For convenience of notation we write $f_0(g,\varepsilon)=f_\varepsilon(g)$. The analyticity of f_0 will be imposed by requiring that f_0 admits a holomorphic extension to a suitable complex neighborhood of $W\times[-\theta_0,\theta_0]$. We shall look at $W=\{g|g\in\Gamma\backslash\mathrm{PGL}(2,\mathbb{R}),\ H_0(g)\in[1/2,\ 3/2]\}$ as consisting of points:

$$g=\sqrt{D}\,\phi \tag{3.1}$$

with $D>0$, $\phi\in\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})\equiv T$; thus we can write, see (2, 11):

$$W=\mathscr{D}\times T \quad \text{with} \quad \mathscr{D}=\{D|D\in\mathbb{R}_+, D^2/8\in[1/2,3/4]\}. \tag{3.2}$$

We shall use the following sets:

$$\begin{aligned}
&\mathscr{D}(\varrho_0)=\{D|D\in\mathbb{C}, \exists D'\in\mathscr{D} \text{ such that } |D'-D|<\varrho_0\}, \qquad \varrho_0<1,\\
&\tilde{H}(\xi_0)=\left\{g|g\in\mathrm{PSL}(2,\mathbb{C}), g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}, |d-1|, |b|, |c|<\xi_0\right\}, \qquad \xi_0<1,\\
&C(\theta_0)=\{\varepsilon|\varepsilon\in\mathbb{C}, |\varepsilon|<\theta_0\}, \qquad \theta_0<1,\\
&H(\xi_0)=\{g|g\in\tilde{H}(\xi_0/(1-\xi_0^2)^{1/2}), \mathbb{C}_+g \text{ is outside the circle } B(\xi_0)\},\\
&B(\xi_0)=\{z|z\in\mathbb{C}, |z+i(\xi_0+\xi_0^{-1})/2|<(\xi_0^{-1}-\xi_0)/2\}, \qquad \text{(see Fig. 1)},\\
&T(\xi_0)=TH(\xi_0),\ \tilde{T}(\xi_0)=T\tilde{H}(\xi_0).
\end{aligned} \tag{3.3}$$

For convenience we shall only consider small values for ξ_0, namely $\xi_0<1/10$. In terms of the above sets we can introduce several notions:

1) We say that a function f_0 on $W\times[-\theta_0,\theta_0]$ is $(\varrho_0,\xi_0,\theta_0)$-analytic if the function $f_0(\sqrt{D}\phi,\varepsilon)$ can be extended to a holomorphic function of (D,ϕ,ε) in $\mathscr{D}(\varrho_0)\times T(\xi_0)\times C(\theta_0)$ or, equivalently, if the function $(D, h, \varepsilon)\to(U(h)f_0)(\sqrt{D}\phi,\varepsilon)=f(\sqrt{D}\phi h,\varepsilon)$ can be holomorphically extended to $\mathscr{D}(\varrho_0)\times H(\xi_0)\times C(\theta_0)$. If in the above definition we replace $T(\xi_0)$ by $\tilde{T}(\xi_0)$, we say that f_0 is strongly $(\varrho_0,\xi_0,\theta_0)$-analytic.

2) Similarly we can define the "ξ-analytic" or the "strongly ξ-analytic" functions on T as those f such that the function $f(\phi)$ admits a holomorphic

extension to $T(\xi_0)$ [or to $\tilde{T}(\xi_0)$] or, equivalently, such that the function of h, $h \to (U(h)f)(\phi) \equiv f(\phi h)$, can be holomorphically extended to $H(\xi_0)$ or to $\tilde{H}(\xi_0)$ (respectively).

3) A function f on W, see (3.2), will be called (ϱ_0, ξ_0)-analytic or strongly (ϱ_0, ξ_0)-analytic if it is defined on W and if the function $(D, \phi) \to f(\sqrt{D}\phi)$ can be extended holomorphically to $\mathscr{D}(\varrho_0) \times T(\xi_0)$ or $\mathscr{D}(\varrho_0) \times \tilde{T}(\xi_0)$ respectively; equivalently: if the function $(D, h) \to f(\sqrt{D}\phi h)$ admits a holomorphic extension to $\mathscr{D}(\varrho_0) \times H(\xi_0)$ or to $\mathscr{D}(\varrho_0) \times \tilde{H}(\xi_0)$ respectively.

4) If f is analytic on T in either of the above senses we shall set

$$\|f\|_\xi = \sup_{g \in T(\xi)} |f(g)|, \quad \|f\|_{2,\xi} = \sup_{h \in H(\xi)} \left(\int_T |f(\phi h)|^2 d\phi\right)^{1/2},$$
$$\|\tilde{f}\|_\xi = \sup_{g \in \tilde{T}(\xi)} |f(g)|, \quad \|\tilde{f}\|_{2,\xi} = \sup_{h \in \tilde{H}(\xi)} \left(\int_T |f(\phi h)|^2 d\phi\right)^{1/2}. \tag{3.5}$$

It is convenient to regard (3.5) as defined for any function on T: whenever the function is not ξ-analytic in the sense necessary to make some of the right-hand side of (3.5) meaningful we interpret it as being $+\infty$. Our proof will rely on general results about the linearized Hamilton Jacobi Equation (2.24).

Since the H_0-flow conserves the value of H_0 and the Poisson bracket is nothing but the derivative along the H_0-flow (i.e. $\sqrt{D}\,\phi \to \sqrt{D}\,\phi\, e^{-D\sigma_z t/4}$, $t \in \mathbb{R}$) Eq. (2.24) can be written as

$$\frac{d}{dt} \Phi(\sqrt{D}\,\phi\, e^{-D\sigma_z t/4})\Big|_{t=0} = f(\sqrt{D}\,\phi). \tag{3.6}$$

The theory of (3.6) with f (ϱ, ξ)-analytic (respectively strongly ξ-analytic) will be reduced to the theory of the equation:

$$(\mathscr{L}\Phi)(\phi) \equiv \frac{d}{dt} \Phi(\phi\, e^{-\sigma_z t/2})\Big|_{t=0} = f(\phi) \tag{3.7}$$

with f ξ-analytic on T (respectively strongly ξ-analytic).

The first theorem on the theory of Eq. (3.7) is the following [2, 3]:

Proposition 3. *Consider the equation*

$$\mathscr{L}\Phi = f \tag{3.8}$$

with $f \in C^1(T)$, $T \equiv \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$. *Suppose that for every periodic orbit p of the* H_0*-flow on T (corresponding to a closed geodesic on* Σ*) and for all* $\phi \in p$:

$$\int_0^{\tau(p)} f(\phi\, e^{-\sigma_z t/2}) dt = 0, \tag{3.9}$$

where $\tau(p)$ *is the period of p. Then:*

i) *there exists a unique* $\Phi \in C^1(T)$ *satisfying* (3.8) *and such that* $\int_T \Phi(\phi) d\phi = 0$.

ii) *If* $f \in C^\infty(T)$ *then* $\Phi \in C^\infty(T)$.

We shall need the following strengthened version of Proposition 3:

Proposition 4. *Let f be analytic on T and suppose that the equation* (3.8) *is solvable, i.e.* (3.9) *holds. There are three constants $C, q, \bar{\delta}>0$, independent of f and such that for all $\xi \in (0,\xi_0)$:*

$$\text{i)}\quad \|\Phi\|_{\xi e^{-\bar{\delta}}} \leqq C\xi^{-q}\|f\|_{\xi}. \tag{3.10}$$

$$\text{ii)}\quad \widetilde{\|\Phi\|}_{\xi e^{-\delta}} \leqq C(\xi\hat{\delta})^{-q}\widetilde{\|f\|}_{\xi}, \qquad \forall\delta>0, \tag{3.11}$$

where $\hat{\delta} \equiv (1-e^{-\delta})$ and the notation (3.5) *is used.*

iii) *Suppose that f depends also on n parameters $(x_1, \ldots, x_n) \in S \subset \mathbb{C}^n$ and that f is holomorphic on $T(\xi)\times S$. Then Φ also is holomorphic in $T(\xi e^{-\bar{\delta}})\times S$ and satisfies* (3.10) *for all $x\in S$.*

iv) *In the same situation as in* iii) *suppose that f is holomorphic on $\widetilde{T(\xi)}\times S$, then Φ is also holomorphic on $\widetilde{T(\xi)}\times S$ and satisfies* (3.11).

Of course i), iii) follow from the much stronger (3.11).

The statements i), ii) of Proposition 2 depend only on the statements i), iii) of Proposition 4 while the stronger result iii) of Proposition 2 follows from ii) and iv) of Proposition 4.

In this section we show how i), iii) of Proposition 4 can be used to prove i), ii) of Proposition 2. Actually we have written the proof in such a way that replacing everywhere the words "ξ-analytic" by "strongly ξ-analytic", "(ξ,ϱ,θ)-analytic" by "strongly (ξ,ϱ,θ)-analytic" and the sets $H(\xi)$, $T(\xi)$ by $\widetilde{H(\xi)}$, $\widetilde{T(\xi)}$, one obtains the proof of iii) of Proposition 2 from ii), iv) of Proposition 4).

The proof of Proposition 4, ii), iv), is much more intricate than that of Proposition 4, i), so we provide independent proofs of i), iii) and of ii), iv). The reader will easily see why the scheme of proof for i), iii) falls short of proving ii), iv): actually it motivates the interesting conjecture that ii), iv) could hold in the form obtained by deleting all the tildas. Furthermore it brings up some interesting properties of the Fourier transforms of analytic functions on T.

In Sects. 4 and 5 we develop the proof of Proposition 4, i), iii) and in Appendix G we develop the proof of ii), iv) which also proves (independently) i), iii) again.

For simplicity of notation let $W(\varrho_0,\xi_0,\theta_0)=\mathscr{D}(\varrho_0)\times T(\xi_0)\times C(\theta_0)$ and suppose:

$$0<\xi_0<\varrho_0<1; \qquad \theta_0<1; \qquad \xi_0<1/10. \tag{3.12}$$

We shall consider hamiltonians of the form $H_0(g,\varepsilon)$:

$$H_0(g,\varepsilon)=h_0(H_0(g),\varepsilon)+f_0(g,\varepsilon), \tag{3.13}$$

where f_0 is divisible by ε and

$$h_0(E,\varepsilon)=E+\sum_{k=1}^{\infty}\varepsilon^k h^{(k)}(E), \qquad E=D^2/8, \tag{3.14}$$

is holomorphic as a function of $(D,\varepsilon)\in\mathscr{D}(\varrho_0)\times C(\theta_0)$. Equation (3.13) is slightly more general than H_0+f_0, which is the hamiltonian we want to study.

We now ask under which conditions there is a canonical map $\mathscr{C}_\varepsilon$ analytic in ε and C^∞ on $W\times[-\theta_0,\theta_0]$, with ($\mathscr{C}_\varepsilon$-identity) divisible by ε, and such that:

$$h_0(H_0(\mathscr{C}_\varepsilon(g)),\varepsilon)+f_0(\mathscr{C}_\varepsilon(g),\varepsilon)=F_\varepsilon(H_0(g)) \tag{3.15}$$

for some F_ε which is C^∞ on $\mathscr{D}\times[-\theta_0,\theta_0]$ and analytic in ε with $F_\varepsilon(E)-E$ divisible by ε.

As already mentioned in Sect. 2 there is a perturbation theory for this problem: it is obtained from the one discussed at length in Sect. 2 by considering the family of hamiltonians $H_0+\hat{f}_0$ with $\hat{f}_0=f_0+(h_0(H_0,\varepsilon)-H_0)$.

We shall suppose that the perturbation theory for (3.15) is finite and then we shall prove that $\mathscr{C}_\varepsilon$ and F_ε do indeed exist for ε small and are analytic in ε: so Proposition 2 will be a special case of this slightly more general case.

To proceed we introduce the following three sequences of positive numbers:

$$\varrho_k=\varrho_0\exp-5\sum_{j=1}^{k}\sigma_j,\qquad \xi_k=\xi_0\exp-5\sum_{j=1}^{k}\delta_j,\qquad \theta_k=\theta_0\exp-5\sum_{j=1}^{k}\tau_j, \tag{3.16}$$

with (see Proposition 4 for the meaning of $\bar{\delta}$):

$$\begin{aligned}&\sigma_k=\tau_k=(1+k^2)^{-1},\\ &\delta_k=\begin{cases}\bar{\delta} & \text{if we wish to prove only i), ii) of Proposition 2},\\ (1+k^2)^{-1} & \text{to prove iii) of Proposition 2}.\end{cases}\end{aligned} \tag{3.17}$$

There will be no formal difference in the proof of i), ii) or of iii) in Proposition 2 if one does not substitute the explicit expressions for δ_k: however many inequalities will be true only for the first choice of δ_k when we only suppose valid i), iii) of Proposition 4, while they will be true also with the second choice if we suppose ii), iv) of Proposition 4.

We shall use a recursive algorithm whose steps will be indexed by an integer $n=1,2,\dots$. The purpose of the algorithm is the construction of a sequence of canonical transformations parametrized by ε, $\mathscr{C}^{(0)}$, $\mathscr{C}^{(1)}$, ... such that:

i) $\mathscr{C}^{(n-1)}$ is holomorphic on $W(\varrho_n,\xi_n,\theta_n)$ and, as a map $(\phi,\varepsilon)\to(\phi',\varepsilon')$ with $\varepsilon'\equiv\varepsilon$, then

$$\mathscr{C}^{(n-1)}W(\varrho_n,\xi_n,\theta_n)\subset W(\varrho_{n-1},\xi_{n-1},\theta_{n-1}), \tag{3.18}$$

ii) $$\|\mathscr{C}^{(n-1)}\text{-identity}\|_{\varrho_n,\xi_n,\theta_n}\leqq C\theta_0\,e^{-n^2}, \tag{3.19}$$

where $\|\ \|_{\varrho,\xi,\theta}$ denotes the supremum norm $W(\varrho,\xi,\theta)$.

Note that it immediately follows from (3.18), (3.19), (3.16) that the composition:

$$\mathscr{C}_\varepsilon=\lim_{n\to\infty}\mathscr{C}^{(0)}\dots\mathscr{C}^{(n-1)} \tag{3.20}$$

exists and is C^∞ on $W(\varrho_\infty,\xi_\infty,\theta_\infty)$ and analytic in (D,ε) and even in ϕ if $\xi_\infty>0$, i.e. if δ_k is given by the second formula of (3.17).

The map $\mathscr{C}^{(n-1)}$ will be constructed inductively. $\mathscr{C}^{(0)}$ is obtained by requiring that on $W(\varrho_1,\xi_1,\theta_1)$:

$$h_0(H_0(\mathscr{C}^{(0)}(g)),\varepsilon)+f_0(\mathscr{C}^{(0)}(g),\varepsilon)=h_1(H_0(g),\varepsilon)+f_1(g,\varepsilon), \tag{3.21}$$

with h_1 analytic on $\mathscr{D}(\varrho_1)\times C(\theta_1)\equiv W(\varrho_1,\theta_1)$ and differing from h_0 by a polynomial of order 1. Then one builds successively $\mathscr{C}^{(1)}, \mathscr{C}^{(2)}, \dots$ by requiring that on $W(\varrho_n,\xi_n,\theta_n)$:

$$h_{n-1}(H_0(\mathscr{C}^{(n-1)}(g)),\varepsilon)+f_{n-1}(\mathscr{C}^{(n-1)}(g),\varepsilon)=h_n(H_0(g),\varepsilon)+f_n(g,\varepsilon) \qquad (3.22)$$

with f_n divisible by ε^{2^n} and that h_n-h_{n-1} be a polynomial in ε of order ε^{2^n-1} divisible by $\varepsilon^{2^{n-1}}$.

If we define $E_k, \varepsilon_k, \lambda_k$ such that:

$$\begin{aligned} E_k &> \sup_{\mathscr{D}(\varrho_k)\times C(\theta_k)} \left|\frac{\partial h_k}{\partial E}(E,\varepsilon)-1\right|, \\ \varepsilon_k &> \sup_{W(\varrho_k,\xi_k,\theta_k)} \left|\frac{\partial f_k}{\partial g}(g)\right|, \\ \lambda_k &> \sup_{W(\varrho_k,\xi_k,\theta_k)} |f_k|, \end{aligned} \qquad (3.23)$$

we shall also require that, for ε_0 small enough, and for suitable constants B, b, for all the considered choices of ϱ_0, ξ_0, θ_0 the following hold:

$$E_k<B\varepsilon_0\xi_0^{-b}; \qquad \varepsilon_k<(B\varepsilon_0\xi_0^{-b})^{(3/2)^k}; \qquad \lambda_k<(B\varepsilon_0\xi_0^{-b})^{(3/2)^k}. \qquad (3.24)$$

It is clear that if (3.16), (3.18), (3.19), (3.24) hold, the limit $h_\infty(E,\varepsilon)=\lim_{n\to\infty} h_n(E,\varepsilon)$ exists and is holomorphic on $\mathscr{D}(\varrho_\infty)\times C(\theta_\infty)$ and:

$$H_\varepsilon(\mathscr{C}_\varepsilon(g))=h_\infty(H_0(g),\varepsilon). \qquad (3.25)$$

It "remains" therefore to check that, with the definitions (3.16), (3.17), (3.23) it is possible to define $\mathscr{C}^{(0)}, \mathscr{C}^{(1)}, \dots, h_1, h_2, \dots, f_1, f_2, \dots$ so that (3.18), (3.19), (3.24) hold for ε_0 small enough: in fact since f_0 is divisible by ε one can always reduce the value of ε_0 by redefining θ_0 (which does not appear explicitly in (3.24)).

The remaining parts of the proof will be organized in several steps:

1) Definition of the Generating Function of $\mathscr{C}^{(n)}$

Assume inductively we have constructed $\mathscr{C}^{(0)}, \mathscr{C}^{(1)}, \dots, \mathscr{C}^{(n-1)}, f_0, f_1. f_n, h_0, h_1, \dots, h_n$ verifying (3.18), (3.19), (3.24) with the f_i's and the h_i's verifying the properties mentioned after (3.22). Then $\mathscr{C}^{(n)}$ will be defined via a generating function Φ. The function Φ will be the solution to the equation:

$$\{H_0,\Phi\}=\left\{\frac{f_n(g,\varepsilon)-\bar f_n(H_0(g),\varepsilon)}{h_n'(H_0(g))}\right\}^{[\leqq 2^{n+1}-1]}, \qquad (3.26)$$

where $[\leqq p]$ denotes the truncation of a Taylor series in ε to order p. Eq. (3.26) arises from the requirement that:

$$\begin{aligned} &h_n(\mathscr{C}^{(n)}(g),\varepsilon)=h_{n+1}(H_0(g),\varepsilon)+O(\varepsilon^{2^{n+1}}), \\ &h_{n+1}-h_n=\{\text{polynomial in } \varepsilon \text{ of order } 2^{n+1}-1 \text{ divisible by } \varepsilon^{2^n}\}. \end{aligned} \qquad (3.27)$$

In fact the above requirement leads, via a simple calculation similar to the one needed to define the first order of perturbation theory (discussed in Sect. 2), to the requirement that:

$$h_n'(H_0(g),\varepsilon)\{H_0,\Phi\}(g)=f_n(g,\varepsilon)+G_n(H_0(g),\varepsilon)+O(\varepsilon^{2^{n+1}}) \tag{3.28}$$

with G_n a polynomial of order $\varepsilon^{2^{n+1}-1}$ divisible by ε^{2^n}. Equation (3.28) is equivalent to Eq. (3.26).

The solvability of (3.26) has to be proved. It follows from the invariance of the finiteness of perturbation theory under canonical transformations (discussed in Sect. 2) that h_n+f_n admits a finite perturbation theory. However the perturbation theory for h_n+f_n yields a series for the generating function of the integrating map which starts from the order ε^{2^n}, and it is easily seen by power counting that the sum of the orders between ε^{2^n} and $\varepsilon^{2^{n+1}-1}$ is a function Φ verifying (3.28) so that (3.28) and hence (3.26) do have a solution.

Applying Proposition 3, since (3.26) has the form (3.6) and can be regarded as an equation of the form (3.8) parametrized by E (or D), we see that Φ can be bounded by:

$$\|\Phi\|_{\varrho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}}<C(\hat{\xi}_n\hat{\delta}_n)^{-q}\left\|\left[\frac{f_n-\bar{f}_n}{h_n'(H_0,\varepsilon)}\right]^{[\leqq 2^{n+1}-1]}\right\|_{\varrho_n,\xi_n,\theta_n e^{-\tau_n}}, \tag{3.29}$$

and using dimensional bounds:

$$\begin{aligned}\|\Phi\|_{\varrho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}}&<C(\hat{\xi}_n\hat{\delta}_n)^{-q}\hat{\tau}_n^{-1}\left\|\frac{f_n-\bar{f}_n}{h_n'(H_0,\varepsilon)}\right\|_{\varrho_n,\xi_n,\theta_n}\\&<RC(\hat{\xi}_n\hat{\delta}_n)^{-q}\hat{\tau}_n^{-1}(1-E_n)^{-1}\left\|\frac{\partial f_n}{\partial g}\right\|_{\varrho_n,\xi_n,\theta_n},\end{aligned} \tag{3.30}$$

where R is an estimate of the length of the maximal distance of two points in $W(\varrho_0,\xi_0)$. Hence:

$$\|\Phi\|_{\varrho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}}<B_1(\hat{\xi}_n\hat{\delta}_n)^{-q}\hat{\tau}_n^{-1}\varepsilon_n \quad \text{if} \quad E_n<1/2, \tag{3.31}$$

where B_1 is a suitable constant and we recall the notation $\hat{x}=(1-e^{-x})$, for $x>0$.

2) Definition of $\mathscr{C}^{(n)}$

Recalling (2.11), (2.12) we define $\mathscr{C}^{(n)}$ by inverting (2.13) in the form:

$$g=g'+\Delta(g')=\mathscr{C}^{(n)}(g'),\qquad g'=g+\Delta'(g)=\tilde{\mathscr{C}}^{(n)}(g), \tag{3.32}$$

the first being obtained by solving the second of (2.11) with respect to $\underline{q}$ and substituting in the first while the second of (3.32) is obtained by solving the first of (2.11) with respect to $\underline{p}'$ and substituting in the second. We want to have Δ, Δ' defined on a domain as large as possible, say $W(\varrho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$. This can be reasonably well achieved by using an analytic implicit function theorem, (see for instance [10] where a similar theorem is proved when $W(\varrho',\xi',\theta')$ is replaced by a different multiply connected set).

It is, in any case, easy to see that inversions in (3.32) can be made under the (very strong, but dimensionally natural) conditions:

$$B_2' \left\| \frac{\partial \Phi}{\partial g} \right\| < \xi_n \hat{\delta}_n ; \quad B_2'' \left\| \frac{\partial^2 \Phi}{\partial g \partial g} \right\| < 1 , \tag{3.33}$$

where $\|\cdot\|$ denotes the supremum over, say, $W(\varrho_n e^{-2\sigma_n}, \xi_n e^{-2\delta_n}, \theta_n e^{-2\tau_n})$ and B_2', B_2'' are suitably large positive numbers [note that in (3.33), the second condition guarantees the local analytic invertibility, while the first provides the globality of this local inversion of (2.11)]. Therefore, under the conditions (3.33) the functions Δ, Δ' can be defined on $W(\varrho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$.

The conditions (3.33) can be implemented using (3.31) and dimensional estimates by the conditions:

$$B_3(\xi_n \hat{\delta}_n)^{-q} \hat{\tau}_n^{-1} \varepsilon_n (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-2} < 1 , \qquad E_n < 1/2 , \tag{3.34}$$

where B_3 is a suitably larger number. Here the inequality $\varrho_n > \xi_n$ has been used to replace ϱ_n^{-1} by ξ_n^{-1}.

Since Δ and Δ' are equal to some derivatives of Φ computed at suitable points [see (2.11) or (2.13)], we infer from (3.31), (3.34) that, by dimensional estimates and if $\|\cdot\|$ denotes the sup on $W(\varrho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$,

$$\|\Delta\|, \|\Delta'\| < B_1(\xi_n \hat{\delta}_n)^{-q} \varepsilon_n \hat{\tau}_n^{-1} (\xi_n \hat{\sigma}_n \hat{\delta}_n)^{-1} < \frac{\xi_n \hat{\sigma}_n \hat{\delta}_n}{B_4} , \tag{3.35}$$

where B_4 can be given any arbitrary value provided we readjust the constant B_3 in (3.34): for later use we fix the constants so that $e^{-4\delta_0} + \hat{\delta}_0 / B_4 < e^{-3\delta_0}$.

So under the conditions (3.34) Δ and Δ' are defined on $W(\varrho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$, and by the last inequality of (3.35) and by the choice of B_4:

$$\mathscr{C}^{(n)}, \tilde{\mathscr{C}}^{(n)} : W(\varrho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n}) \to W(\varrho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n}) . \tag{3.36}$$

Here as well as above we keep changing the coefficients of ϱ_n, ξ_n, θ_n simultaneously even when this is not really necessary (e.g. in all the above inequalities $\theta_n e^{-p\tau_n}$, $p > 1$, can be replaced simply by $\theta_n e^{-\tau_n}$); this is done to make the notations uniform.

Furthermore on $W(\varrho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n})$ the maps $\mathscr{C}^{(n)}$, $\tilde{\mathscr{C}}^{(n)}$ differ from the identity map by less than:

$$B_1(\xi_n \hat{\delta}_n)^{-q} \hat{\tau}_n^{-1} (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-1} \varepsilon_n . \tag{3.37}$$

Finally a remarkable property of Δ, Δ' is related to the very definition of Φ as solution of (3.26), (3.27), i.e. setting $H_0 \equiv H_0(g)$:

$$h_n'(H_0, \varepsilon) \{H_0, \Phi\} (g) = f_n^{[<2^{n+1}]}(g, \varepsilon) - \bar{f}_n^{[<2^{n+1}]}(H_0, \varepsilon) + r_n(g, \varepsilon) , \tag{3.38}$$

$$r_n(g, \varepsilon) = - \left[\frac{f_n(g, \varepsilon) - \bar{f}_n(H_0, \varepsilon)}{h_n'(H_0, \varepsilon)} \right]^{[\geqq 2^{n+1}]} h_n'(H_0, \varepsilon) + [f_n(g, \varepsilon) - \bar{f}_n(H_0, \varepsilon)]^{[\geqq 2^{n+1}]} \tag{3.39}$$

[just multiply both sides of (3.26) by h_n'].

Therefore, again by dimensional estimates,

$$\|r_n\|_{\varrho_n e^{-\sigma_n}, \xi_n e^{-\delta_n}, \theta_n e^{-\tau_n}} < B_5 e^{-\delta_n 2^{n+1}} \hat{\tau}_n^{-1} \hat{\varepsilon}_n R , \tag{3.40}$$

with B_5 a suitably large number, and R arises when, as in (3.30), we bound $f_n - \bar{f}_n$ by $\partial f_n/\partial g$ (expressing the first as a path integral of the latter).

Equation (3.38) could be expressed as relations involving Δ, Δ' by expressing $\partial\Phi/\partial g$ in terms of them; we do not need to do this explicitly.

3) Definition of h_{n+1}, f_{n+1}

We shall naturally define:

$$\begin{aligned} &h_{n+1}(E,\varepsilon) = h_n(E,\varepsilon) + [\bar{f}_n(E,\varepsilon)]^{[<2^{n+1}]} \\ &f_{n+1}(g',\varepsilon) = f_n(g'+\Delta(g'),\varepsilon) + h_n(H_0(g'+\Delta(g')),\varepsilon) - h_{n+1}(H_0(g),\varepsilon), \end{aligned} \tag{3.41}$$

and we proceed to estimate ε_{n+1}, E_{n+1}, λ_{n+1} on $W(\varrho_{n+1}, \xi_{n+1}, \theta_{n+1})$.

The estimate, based again on dimensional considerations, is straight-forward but quite technical and it is developed in detail in Appendix A, where the following bounds are proved: if (3.34) holds one can take for E_{n+1}, ε_{n+1}, λ_{n+1}:

$$\begin{aligned} &E_{n+1} = E_n + B_6 \varepsilon_n \hat{\tau}_n^{-1}, \\ &\varepsilon_{n+1} = B_7(\varepsilon_n^2 + \varepsilon_n \exp - \tau_n 2^{n+1})(\hat{\tau}_n \hat{\delta}_n \xi_n)^{-b_7}, \\ &\lambda_{n+1} = B_8((\varepsilon_n^2 + \varepsilon_n \exp - \tau_n 2^{n+1})(\hat{\tau}_n \hat{\delta}_n \xi_n)^{-b_8} + \lambda_n \hat{\tau}_n^{-1} \exp - \tau_n 2^{n+1}). \end{aligned} \tag{3.42}$$

A simple inductive argument shows that there are B, $b > 0$ such that (3.24) [as well as (3.34) for all n] hold.

This completes the reduction of the proof of Proposition 2 to that of Proposition 4.

4. Fourier Analysis on $L_2 \equiv L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$

In this section we develop some tools for the proof of Proposition 4. We suppose the reader is familiar with the chapter on $\mathrm{SL}(2,\mathbb{R})$ of [5], as well as with the first chapter of [6], where the theory of the unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and the related Fourier analysis are developed.

Let f be ξ-analytic (respectively strongly ξ-analytic) on $T \equiv \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$. It is easy to prove by using the Cauchy theorem that there is a constant C_1 such that, using the notations of Sect. 3 and (2.30),

$$\begin{gathered} \|f\|_{2,\xi} < \|f\|_\xi, \ \|f\|_{\xi e^{-\tau}} < C_1 \hat{\tau}^{-3} \|f\|_{2,\xi}, \\ (\text{respectively } \widetilde{\|f\|}_{2,\xi} < \widetilde{\|f\|}_\xi, \ \widetilde{\|f\|}_{\xi e^{-\tau}} < C_1 \hat{\tau}^{-3} \widetilde{\|f\|}_{2,\xi}) \end{gathered} \tag{4.1}$$

(this is basically the well known procedure of bounding a holomorphic function by some integral norm).

Let us define the unitary representation U of $\mathrm{PSL}(2,\mathbb{R})$ on $L_2(T) \equiv L_2$ induced by the action of $\mathrm{PSL}(2,\mathbb{R})$ on the homogeneous space $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$:

$$(U(g)f)(\phi) = f(\phi g). \tag{4.2}$$

Here the scalar product considered in L_2 is $(f,f') = \int_T \overline{f(g)} f'(g)\,dg$, with dg denoting the normalized Haar measure on T.

Let

$$L_2 = \bigoplus_{a \in A} Y^{(\alpha)} \tag{4.3}$$

be a decomposition of L_2 into U-invariant, pairwise orthogonal subspaces on which U acts irreducibly: such a decomposition is always possible, [6].

We denote $U^{(a)}$ the restriction of U to $Y^{(a)}$ and we briefly recall some known facts about the above decomposition and a few of their developments that we shall need in the proof.

One associates with a hyperbolic fuchsian group $\Gamma \subset G_0 \equiv \mathrm{PSL}(2, \mathbb{R})$ the following four classes of entities:

1) The automorphic forms of order $n = 1, 3, 5, \ldots$, i.e. the functions which are holomorphic in the upper half plane $\mathbb{C}_+$ and such that $\forall \gamma \in \Gamma$,

$$\phi(z\gamma) = \phi(z) j(z, \gamma)^{n+1} \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad j(z, \gamma) = (bz + d). \tag{4.4}$$

They form a $n(g-1)$-dimensional linear space, g being the genus of the compact surface associated with the given fuchsian group.

We shall, once and for all, choose a basis in the above linear space and we shall label its elements as $\phi^{(n,j,+)}$, $j = 1, \ldots, (n-1)g$; we shall also suppose that $\phi^{(n,j,+)}$ are orthonormal with respect to a convenient scalar product:

$$\delta_{jk} = \int_T \overline{\phi^{(n,j,+)}(ig^{-1})} \phi^{(n,k,+)}(ig^{-1}) |j(i, g^{-1})|^{-2(n+1)} dg. \tag{4.5}$$

2) The antiautomorphic forms of order $n = 1, 3, 5, \ldots$: they are just the complex conjugates of the corresponding automorphic forms. We shall put:

$$\overline{\phi^{(n,j,-)}} = \phi^{(n,j,+)} \tag{4.6}$$

and $(n, j, +)$, $(n, j, -)$ will be often denoted by the symbol a.

3) The eigenfunctions of the Laplace-Beltrami operator relative to the eigenvalue $(1-u^2)/4 \in (0, +\infty)$. The normalization that we choose for the operator Δ on $L_2(\Sigma)$ is such that, in the ordinary cartesian coordinates on the upper half plane $\mathbb{C}_+$, it is

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{4.7}$$

The variable u will then only take finitely many values in $(0, 1)$ and countably many of the form is, $s \in (0, +\infty)$ (this is a consequence of the general properties of the spectrum of the Laplace operator on a compact Riemann surface).

The elements of a basis in the space of such eigenvectors will be labeled with an index $a = (u, j)$, where the first number fixes the eigenvalue and the second distinguishes the independent eigenvectors associated with the same eigenvalue $(1-u^2)/4$ whenever the latter is degenerate, otherwise $j = 1$.

We fix once and for all the basis and we also suppose that it has the property:

$$\int_T \overline{\phi^{(u,j)}(ig^{-1})} \phi^{(u,k)}(ig^{-1}) dg = \delta_{jk}. \tag{4.8}$$

It follows from the general theory of the PSL(2, $\mathbb{R}$)-induced representations that the multiplicity of the eigenvalue $(1-u^2)/4$ cannot exceed the value $g|u^2|$, g being the genus of the surface. We shall also denote (u,j) by a.

4) The function identically 1 on $\mathbb{C}_+$ will be denoted $\phi^{(0)}$. We shall denote by A the set of values that, according to the above classification, the index a can take, plus the index 0. Set $\bar{a}=(n,j,-)$ if $a=(n,j,+)$. Then we can define the functions on T

$$\begin{aligned} &E^{(a)}(g)=\phi^{(a)}(ig^{-1}),\ a\in A,\ a=0 \quad \text{or} \quad a=(u,j), \\ &E^{(a)}(g)=\overline{E^{(\bar{a})}(g)}=\overline{\phi^{(a)}(ig^{-1})j(i,g^{-1})^{-n-1}}, \qquad a=(n,j,+). \end{aligned} \tag{4.9}$$

We shall define, for all a in A:

$$Y^{(a)}=\{\text{subspace of } L_2 \text{ spanned by } U(g)E^{(a)} \text{ as } g \text{ varies in } G_0\}. \tag{4.10}$$

The "duality theorem" [6] on the induced representations says:

$$L_2=\bigoplus_{a\in A} Y^{(a)}, \tag{4.11}$$

and $Y^{(a)}$ is orthogonal to $Y^{(a')}$ if $a\neq a'$; furthermore U acts irreducibly on $Y^{(a)}$. Each $Y^{(a)}$ can be realized in a "standard way" as a space of functions $\hat{Y}^{(a)}$ defined either on the line $\mathbb{R}$ or on the upper half plane $\mathbb{C}_+$ as follows:

i) If $a=(n,j,+)$ then $Y^{(a)}$ can be realized as a subspace $\hat{Y}^{(a)}\subset L_2\left(\mathbb{C}_+,\ y^{n+1}\dfrac{dx\,dy}{y^2}\right)$ consisting in the functions of (x,y) which are holomorphic in the variable $z=x+iy$. If $\hat{a}=(n,j,-)=\bar{a}$ the space $Y^{(\hat{a})}$ can be realized as a subspace of $L_2(\mathbb{C}_+, y^{n-1}dx\,dy)$ containing the functions which are obtained from those in $\hat{Y}^{(a)}$ by complex conjugation (i.e. the antiholomorphic functions on $\mathbb{C}_+$), [6].

ii) If $a=(u,j)$, $u=is$, $s\in\mathbb{R}_+$, then $Y^{(a)}$ can be realized as $\hat{Y}^{(a)}=L_2(\mathbb{R},dx)$ while if $a=(s,j)$, $0<s<1$, the $Y^{(a)}$ can be realized as $\hat{Y}^{(a)}=L_2(\mathbb{R};s)$, where we denote by $L_2(\mathbb{R};s)$ the completion of $C_0^1(\mathbb{R})$ with respect to the scalar product:

$$(f,g)=\int \overline{f(x)}g(y)|x-y|^{-1+s}dx\,dy=\int \overline{\tilde{f}(p)}\tilde{g}(p)C(s)|p|^s dp, \tag{4.12}$$

where $\tilde{f},\tilde{g}$ denote the ordinary Fourier transform, [6].

iii) Finally the identification of $Y^{(a)}$ and $\hat{Y}^{(a)}$ and the realization of the restriction of U to $Y^{(a)}$ as a unitary representation $\hat{U}^{(a)}$ of G_0 acting on $\hat{Y}^{(a)}$ are implemented as follows. First one defines an irreducible representation $\hat{U}^{(a)}$ of G_0 on $\hat{Y}^{(a)}$, secondly one prescribes the representative $\hat{\phi}^{(a)}$ of $\phi^{(a)}$ in $\hat{Y}^{(a)}$, and finally one uses the cyclicity of the vector $\hat{\phi}^{(a)}$ under the action of $\hat{U}^{(a)}$ to associate with every vector of $\hat{Y}^{(a)}$ a (unique) vector in $Y^{(a)}$.

We set:

$$\begin{aligned} &(\hat{U}^{(a)}(g)f)(z)=f(zg)j(z,g)^{-n-1}, && f\in\hat{Y}^{(a)},\ a=(n,j,+), \\ &(\hat{U}^{(a)}(g)f)(z)=f(zg)\overline{j(z,g)^{-n-1}}, && f\in\hat{Y}^{(a)},\ a=(n,j,-), \\ &(\hat{U}^{(a)}(g)f)(x)=f(xg)\,(j(x,g)^2)^{-(1+u)/2}, && f\in\hat{Y}^{(a)},\ a=(u,j). \end{aligned} \tag{4.13}$$

Since the space $\hat{Y}^{(a)}$ never depends on the index j in a we shall sometimes denote it as $\hat{Y}^{(n,+)}$ or $\hat{Y}^{(n,-)}$ or $\hat{Y}^{(u)}$. Then the following identification of $\phi^{(a)}$ as vectors $\hat{\phi}^{(a)} \in \hat{Y}^{(a)}$ is possible, [6]:

$$\begin{aligned} \phi^{(n,j,+)}(g) &\leftrightarrow (z+i)^{-n-1} M_n, \\ \phi^{(n,j,-)}(g) &\leftrightarrow (\bar{z}-i)^{-n-1} M_n, \end{aligned} \tag{4.14}$$

and

$$\phi^{(u,j)}(g) \leftrightarrow |x+i|^{-u-1} M_u, \tag{4.15}$$

where the constants M_a are the normalization constants $(M_n = (n\pi 2^{-2n})^{1/2}$, $M_s = (2^{s-1}(-s!))^{-1/2}$, $M_{is} = (\sqrt{\pi})^{-1})$.

The above associations (4.14), (4.15) and the irreducibility of $U^{(a)}$ uniquely associate with $f \in L_2$ "its Fourier transform $(\hat{f}^{(a)})_{a\in A}$", where

$$\hat{f}^{(a)} \in \hat{Y}^{(a)} \tag{4.16}$$

is the function corresponding to the component $f^{(a)}$ of f on $Y^{(a)}$ via the basic relations (4.14), (4.15).

To discuss the remarkable properties of the Fourier transforms of functions on T which are ξ-analytic, let us recall the notation:

$$\|f\|_\xi = \sup_{g\in G_0,\, h\in H(\xi)} |f(gh)| \tag{4.17}$$

(already used in Sect. 3). We also introduce the following notation:

$$\begin{aligned} F_f^{(n,j,+)}(z) &= (z+i)^{n+1} \hat{f}^{(n,j,+)}(z), \quad && z \in \mathbb{C}_+, \\ F_f^{(n,j,-)}(z) &= (\bar{z}-i)^{n+1} \hat{f}^{(n,j,-)}(z), \quad && z \in \mathbb{C}_+, \\ F_f^{(u,j)}(x) &= (x^2+1)^{(u+1)/2} \hat{f}^{(u,j)}(x), \quad && x \in \mathbb{R}, \end{aligned} \tag{4.18}$$

and, for $z \in \mathbb{C}_+ H(\xi)$, $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\xi)$, and $\tilde{a} = (n,j,+)$, $(n,j,-)$:

$$A^{(\tilde{a})}(z,\xi) = \inf (n/4\pi)^{1/2} \left| \frac{(a\zeta + c) + i(b\zeta + d)}{\sqrt{(\operatorname{Im}\zeta)}} \right|^{n+1}, \tag{4.19}$$

where the inf is taken over the $\zeta \in \mathbb{C}_+$, $h \in H(\xi)$, $\zeta h = z$; or for $z \in \mathbb{R}H(\xi)$, $\tilde{a} = (u,j)$, $\hat{\delta} = (1 - e^{-\delta})$:

$$A^{(\tilde{a})}(z,\xi) = \inf C_3 (\xi\hat{\delta})^{-1} \left| \left(\frac{(ax+c)^2 + (bx+d)^2}{1+x^2} \right)^{(1+u)/2} \right|, \tag{4.20}$$

where the inf is over all $\delta > 0$ and the pairs $x \in \mathbb{R}$, $h \in H(\xi e^{-\delta})$, $xh = z$, and the powers $(1+u)/2$ are defined by cutting the complex plane on the negative real axis (as we shall see the number in brackets never takes a negative value) and, finally, the constant C_3 has a value which will be conveniently fixed later.

Finally define

$$Q^{(a)}(\xi) = \begin{cases} \mathbb{C}_+ H(\xi) & \text{if} \quad a = (n,j,\pm) \\ \mathbb{R} H(\xi) & \text{if} \quad a = (u,j), \end{cases} \tag{4.21}$$

and $Q^{(a)}$ will be called the "ξ-domain of the representation a". Its form is simple:

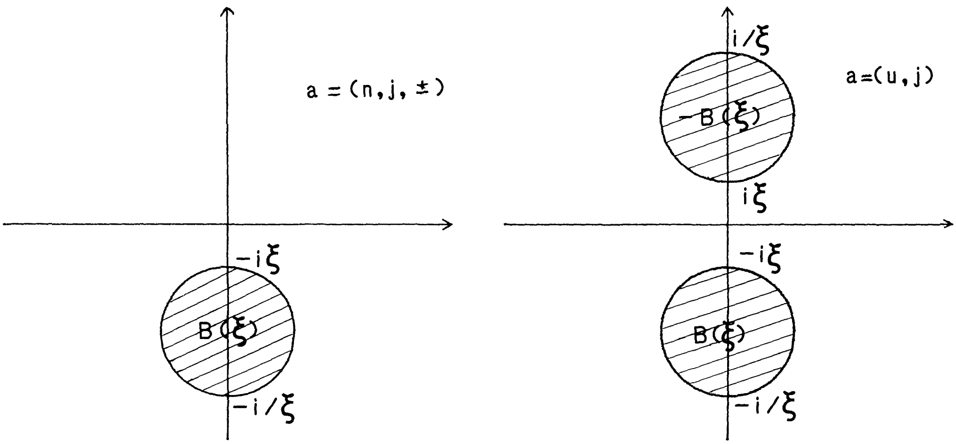

Fig. 1. The domains $Q^{(a)}(\xi)$ (unshaded)

i.e. $Q^{(a)}(\xi)$ is either $\mathbb{C}\backslash B(\xi)$ or $\mathbb{C}\backslash(-B(\xi)\cup B(\xi))$, see Fig. 1, and (3.3).

In Appendix B we shall prove the following structural theorem on the Fourier transform of ξ-analytic functions.

Lemma 1. *Let* $f\in Y^{(a)}$, $a\in A$, $a\neq 0$.

i) *The function* f *is* ξ*-analytic if and only if the functions* $F_f^{(a)}$ *can be holomorphically extended to the* ξ*-domain* $Q^{(a)}(\xi)$ *of the representation* a.

ii) *If* f *is* ξ*-analytic:*

$$|F_f^{(a)}(z)|<A^{(a)}(z,\xi)\,\|f\|_{\xi}\,. \tag{4.22}$$

iii) *If* f *is* ξ*-analytic and there is* $N>0$ *such that*

$$|F_f^{(a)}(z)|<A^{(a)}(z,\xi)N\,,\qquad z\in Q^{(a)}(\xi)\,, \tag{4.23}$$

then

$$\|f\|_{\xi e^{-\delta}}<M_1(\xi\delta)^{-m_1}N\sqrt{|a|}\,,\qquad \delta>0\,, \tag{4.24}$$

where $|a|=n$ *if* $a=(n,j,\pm)$, $|a|=|u|$ *if* $a=(u,j)$ *and* M_1, m_1 *are independent on* a, f, *or* ξ, δ.

The proof of the above lemma also provides the proof of:

Lemma 2. *If* $f\in L_2$ *is* ξ*-analytic and depends on some complex parameters* $\underline{w}\in W\subset\mathbb{C}^q$, *and it is holomorphic in* $(g,\underline{w})\in T\times W$, *then* $F_f^{(a)}$ *are holomorphic in* $Q^{(a)}(\xi)\times W$ *and* (4.22) *holds for each* $\underline{w}\in W$. *Conversely if* f *is such that* $f\in Y^{(a)}$ *for all* $\underline{w}\in W$ *and* $F_f^{(a)}$ *is holomorphic on* $Q^{(a)}(\xi)\times W$, *then* f *is* ξ*-holomorphic in* $T\times W$ *and verifies* (4.24) *for each* $\underline{w}$ *for which* $F_f^{(a)}$ *verifies* (4.23).

The ξ-analytic functions also turn out to have an "exponential decay" of their Fourier transform, expressed by the following lemma:

Lemma 3. *If* $f \in L_2$ *is* ξ*-analytic:*

$$\|f^{(a)}\|_{\xi e^{-\delta}} < M_2(\xi\delta)^{-m_2} e^{-v_2\xi\hat{\delta}|a|} \|f\|_{\xi}, \qquad \delta > 0, \tag{4.25}$$

where M_2, m_2, v_2 *do not depend on* f, ξ, δ, a.

Finally we shall need, for the proof of Proposition 4, the following estimates for the functions $A^{(a)}(z, \xi)$: Let, for $z \in \mathbb{R}H(\xi)$:

$$d(z, \xi) = \sup\{\delta | \delta > 0 \quad \text{such that} \quad z \in \mathbb{R}H(\xi e^{-\delta})\}, \tag{4.26}$$

then:

Lemma 4. *There are constants* M_3, m_3, $v_3 > 0$ *such that*

$$A^{(n,j,\pm)}(z, \xi) = \sqrt{(n/4\pi)} \left[|i+z|^2 \frac{1-\xi^2}{\xi} \frac{1}{(|z + i(\xi^{-1}+\xi)/2|^2 - (\xi^{-1}-\xi)/2)^2)} \right]^{(n+1)/2} \tag{4.27}$$

and for all $z \in \mathbb{R}H(\xi)$:

$$M_3^{-1}(\hat{d}(z, \xi))^{-m_3} e^{-v_3\xi|u|} < A^{(u,j)}(z, \xi) < M_3(\hat{d}(z, \xi))^{-m_3}. \tag{4.28}$$

Lemmas 1–4 are proved in Appendix B, C.

In Sect. 5 we shall show how Proposition 4), i), iii) can be deduced from the theory of the Fourier transform developed in the above lemmas. The proof of ii), iv) can be done along the same lines but one needs a refinement of the bounds (4.28), see Appendix G.

5. Proof of Propositions 4 and 1

Consider Eqs. (3.8) or (3.7). Since

$$\Phi(g e^{-\sigma_z t/2}) \equiv (U(e^{-\sigma_z t/2})\Phi)(g), \tag{5.1}$$

it is clear that (3.7) can be reduced by the reduction of the representation U – one just considers its projections on the spaces $Y^{(a)}$.

If $\hat{\Phi}^{(a)}$, $\hat{f}^{(a)}$ are the components of order a of the Fourier transforms of Φ, f and if one uses (4.13) one finds that the Fourier transforms obey the following equations:

$$\left.\frac{d}{dt} \hat{\Phi}^{(a)}(e^{-t}z) e^{-(1+a)t/2}\right|_{t=0} = \hat{f}^{(a)}(z), \tag{5.2}$$

where z varies in $\mathbb{C}_+$ or $\mathbb{R}$ according to the value of a and we have shortened the notations by writing $(1+a)$ for $(1+n)$ or $(1+u)$ when $a = (n, j, \pm)$ or, respectively, $a = (u, j)$. More explicitly:

$$z \frac{d}{dz} \hat{\Phi}^{(a)}(z) + \frac{(1+a)}{2} \hat{\Phi}^{(a)}(z) = -\hat{f}^{(a)}(z). \tag{5.3}$$

The solution of (5.3) is, for $a = (u, j)$:

$$\hat{\Phi}^{(a)}(x) = -\int_0^x \left(\frac{y}{x}\right)^{(1+u)/2} \hat{f}^{(a)}(y) dy/y + K_a x^{-(1+u)/2}, \tag{5.4}$$

and for $a=(n,j,+)$:

$$\hat{\Phi}^{(a)}(z)=-\int_0^z\left(\frac{\zeta}{z}\right)^{(1+n)/2}\hat{f}^{(a)}(\zeta)\,d\zeta/\zeta+K_a z^{-(1+n)/2}\,, \tag{5.5}$$

and a similar expression holds for $a=(n,j,-)$. K_a are arbitrary integration constants.

Recalling that $F_f^{(a)}$ are holomorphic functions in $Q^{(a)}(\xi)$, it is convenient to try to write (5.4), (5.5) in terms of $F_f^{(a)}$. Since we assume that Φ exists it follows that $\hat{\Phi}^{(a)}$ must have some square integrability properties (i.e. $\hat{\Phi}^{(a)}\in\hat{Y}^{(a)}$) and the analyticity of $f^{(a)}$ together with the bounds in Lemma 1 immediately imply that in order that $\hat{\Phi}^{(a)}$ be square integrable it is necessary that

$$\begin{aligned}&\int_0^{\pm\infty}|y|^{(1+u)/2}\hat{f}^{(u,j)}(y)\,dy/y=0\,, \qquad K_{(u,j)}=0\,,\\ &\int_0^{\infty}\zeta^{(1+n)/2}\hat{f}^{(n,j,+)}(\zeta)\,d\zeta/\zeta=0\,, \qquad K_{(a,n,+)}=0\,,\end{aligned} \tag{5.6}$$

and similar conditions must hold for $a=(n,j,-)$.

In fact the ξ-analyticity of f implies that $F_f^{(a)}$ are bounded at ∞ and therefore the $\hat{f}^{(a)}$ have simple decay properties at ∞ which show that the integrals (5.6) converge. Since they provide the coefficient of the leading term in the decay of $\hat{\Phi}^{(a)}$ at ∞ and this leading term is not square integrable, they must vanish.

Assuming (5.6) and defining the powers of the complex numbers by cutting the plane $\mathbb{C}$ so that $\arg z\in(-\pi,\pi]$, say, (5.4), (5.5) can be written in a form which is suited to see the analyticity properties of $\Phi^{(a)}$. If Λ, Λ' are contours in $Q^{(a)}(\xi)$ linking 0 or ∞ with z and staying in the same quadrant as z itself (if $z>0$ it belongs to the first quadrant, if $z<0$ it belongs to the second) then, if $a=(u,j)$:

$$\begin{aligned}F_\Phi^{(a)}(z)&=-((1+z^2)/z)^{(1+u)/2}\int_0^z(\zeta/(1+\zeta^2))^{(1+u)/2}F_f^{(u,j)}(\zeta)\,d\zeta/\zeta\\ &=((1+z^2)/z)^{(1+u)/2}\int_z^\infty(\zeta/(1+\zeta^2))^{(1+u)/2}F_f^{(u,j)}(\zeta)\,d\zeta/\zeta\,,\end{aligned} \tag{5.7}$$

and if $a=(n,j,+)$:

$$\begin{aligned}F_\Phi^{(a)}(z)&=-((i+z)^2/z)^{(1+n)/2}\int_0^z(\zeta/(i+\zeta)^2)^{(1+n)/2}F_f^{(n,j,+)}(\zeta)\,d\zeta/\zeta\\ &=((i+z)^2/z)^{(1+n)/2}\int_z^\infty(\zeta/(i+\zeta)^2)^{(1+n)/2}F_f^{(n,j,+)}(\zeta)\,d\zeta/\zeta\,,\end{aligned} \tag{5.8}$$

where the integrals are along the paths Λ or Λ'; a similar expression holds for $a=(n,j,-)$.

It is easy to see that Eqs. (5.7), (5.8) define functions on $(\mathbb{C}\backslash[(-B(\xi)\cup B(\xi)\cup\{\text{real and imaginary axes}\})])$ or on $(\mathbb{C}\backslash[B(\xi)\cup\{\text{real and imaginary axes}\}])$, by analytically continuing their values on the real axis or on the first quadrant. However as a consequence of (5.6) and of the boundedness at infinity of the F-functions, it is easily seen that there are no discontinuities in the values that the functions take at the two sides of the cuts. So (5.7), (5.8) actually define

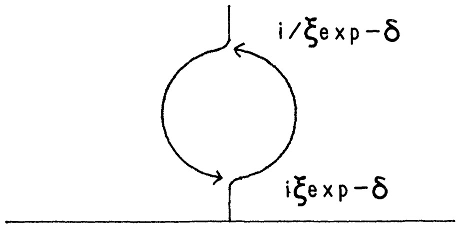

Fig. 2. The contours Λ_∞ and Λ_0

holomorphic functions in $Q^{(a)}(\xi)$, (i.e. their values on the real axis or on the first quadrant can be continued in a single valued way).

Therefore $\Phi^{(a)}$ are ξ-analytic for all a. The same argument can be applied to prove the strong ξ-analyticity of $\Phi^{(a)}$ if f is supposed strong ξ-analytic. However we need bounds on the size of $\Phi^{(a)}$ to conclude something about the analyticity of Φ.

Consider first $a=(u,j)$. Let $z\in Q^{(a)}(\xi e^{-\delta})$, and observe that since $F_\Phi^{(a)}$ is holomorphic in $Q^{(a)}(\xi)$, the value of $F_\Phi^{(a)}(z)$ will be bounded by the maximum of $F_\Phi^{(a)}(z)$ on the two circles forming the boundary of $Q^{(a)}(\xi e^{-\delta})$, i.e. $-\partial B(\xi e^{-\delta})\cup\partial B(\xi e^{-\delta})$. Let z_0 be a point in, say, the upper circle. We define a contour from $+i\infty$ to $i\xi e^{-\delta}$ going down the imaginary axis to $i\xi^{-1}e^{\delta}$ and then going around the circle $-B(\xi e^{-\delta})$ counterclockwise to $i\xi e^{-\delta}$: we call it Λ_∞.

Define Λ_0 as the contour from 0 to $i\xi^{-1}e^{\delta}$ going up along the imaginary axis to $i\xi e^{-\delta}$ and then following counterclockwise the boundary of $-B(\xi e^{-\delta})$ up to $i\xi^{-1}e^{\delta}$.

There are two cases: either $z_0\in\Lambda_\infty$ or $z_0\in\Lambda_0$. In the first case we write [by (5.7)]:

$$F_\Phi^{(a)}(z_0)=(\Lambda_\infty)\int_{z_0}^{\infty}((1+z_0)^2/z_0)^{(1+u)/2}(\zeta/(1+\zeta^2))^{(1+u)/2}F_f^{(a)}(\zeta)d\zeta/\zeta\,. \tag{5.10}$$

In the second case

$$F_\Phi^{(a)}(z_0)=(\Lambda_0)\int_0^{z_0}[\text{same function as in (5.10)}]\;d\zeta/\zeta\,, \tag{5.11}$$

where the integrals are along the contours marked in parentheses before the integral signs.

Along the integration paths the argument $\theta(\zeta)$ of $\zeta/(1+\zeta^2)$ relevant for the u dependent powers in (5.10), (5.11) is a monotonically decreasing function of ζ so that the u-dependent part verifies the inequality,

$$|((1+z_0^2)/z_0)^{(1+u)/2}(\zeta/(1+\zeta^2))^{(1+u)/2}|=\left|\frac{(1+z_0^2)\zeta}{z_0(1+\zeta^2)}\right|^{(1+\mathrm{Re}\,u)/2}$$
$$\cdot\exp-(\mathrm{Im}\,u)(\theta(\zeta)-\theta(z_0))/2<\left|\frac{1+z_0^2}{1+\zeta^2}\frac{\zeta}{z_0}\right|^{(1+\mathrm{Re}\,u)/2}. \tag{5.12}$$

Therefore estimating the integrals,

$$\int\left|\frac{1+z_0^2}{1+\zeta^2}\frac{\zeta}{z_0}\right|^{(1+\mathrm{Re}\,u)/2}d|\zeta|/|\zeta|\quad\text{along }\Lambda_0\text{ or }\Lambda_\infty\,, \tag{5.13}$$

by $M_4(\xi\hat{\delta})^{-(1+\mathrm{Re}u)/2}$, we find, using (4.28):

$$|F_\Phi^{(a)}(z_0)| < M_3(\xi\hat{\delta})^{-m_3} M_4(\hat{\delta}\xi)^{-(1+\mathrm{Re}u)/2} \|f^{(a)}\|_\xi, \tag{5.14}$$

$$|F_\Phi^{(a)}(z)| < M_5(\xi\hat{\delta})^{-m_5} e^{v_3\xi} A^{(a)}(z, \xi e^{-\delta}) \|f^{(a)}\|_\xi, \tag{5.15}$$

where the second inequality follows from the arbitrariness of δ provided $z \in \mathbb{R}H(\xi e^{-\delta})$; and the constants are u-independent because $\mathrm{Re}u \neq 0$ only for finitely many values of u.

The inequality (5.15) implies, by Lemma 1, that

$$\|\hat{\Phi}^{(a)}\|_{\xi e^{-\delta}} < M_6(\xi\hat{\delta})^{-m_6} e^{v_6\xi|u|} \|f^{(a)}\|_\xi, \tag{5.16}$$

where the constants M_6, v_6, m_6 are suitable numerical constants.

Consider now $a=(n,j,+)$. We shall prove a better inequality:

$$|F_\Phi^{(a)}(z)| < M_7(\log\xi^{-1}) A^{(a)}(z,\xi) \|f^{(a)}\|_\xi, \tag{5.17}$$

which, by Lemma 1, implies:

$$\|\hat{\Phi}^{(a)}\|_{\xi e^{-\delta}} < M_8(\xi\hat{\delta})^{-m_8} \|f^{(a)}\|_\xi, \qquad \delta > 0. \tag{5.18}$$

Then (5.18) and (5.16), together with Lemma 3 and the estimate on the multiplicities of the automorphic forms of given order and of the degeneracy of the eigenvalues of the Laplace-Beltrami operator, imply (3.10) with $\bar{\delta}$ chosen so that $v_2\bar{\delta} < v_6 e^{-\bar{\delta}}$.

To derive (5.17), we note that since $(n+1)/2$ is an integer, it is equivalent to

$$|\hat{\Phi}^{(a)}(z)| < M_7(\log\xi^{-1}) \|f^{(a)}\|_\xi A'(z,a,\xi), \tag{5.19}$$

where

$$\begin{aligned} A'(z,a,\xi) &= A^{(a)}(z,\xi)|(i+z)|^{-(n+1)} \\ &= \{((1-\xi^2)/\xi[|z+(\xi^{-1}+\xi)/2|^2 \\ &\quad -((\xi^{-1}-\xi)/2)^2]\}^{(n+1)/2}(n/4\pi)^{1/2}, \end{aligned}$$

and

$$\hat{\Phi}^{(a)}(z) = -\int_0^z (\zeta/z)^{(n+1)/2} \hat{f}^{(a)}(\zeta)\, d\zeta/\zeta = \int_z^\infty (\zeta/z)^{(n+1)/2} \hat{f}^{(a)}(\zeta)\, d\zeta/\zeta \tag{5.20}$$

along contours in $Q^{(a)}(\xi)$.

Let $c=(\xi^{-1}+\xi)/2$, $r=(\xi^{-1}-\xi)/2$. We distinguish four cases.

Case 1: $\mathrm{Im}z \leqq -c$. We choose, in (5.20), to integrate along the path:

$$\zeta(t) = t(z+ic) - ic, \qquad t \in \mathbb{R}_+ + 1. \tag{5.21}$$

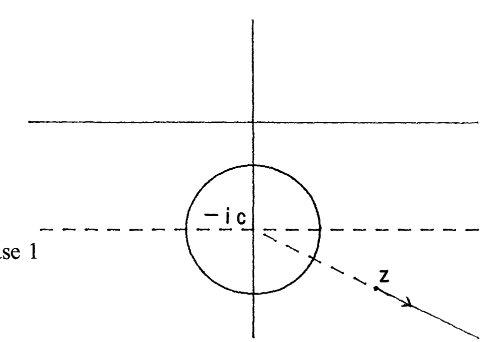

Fig. 3. Contour in case 1

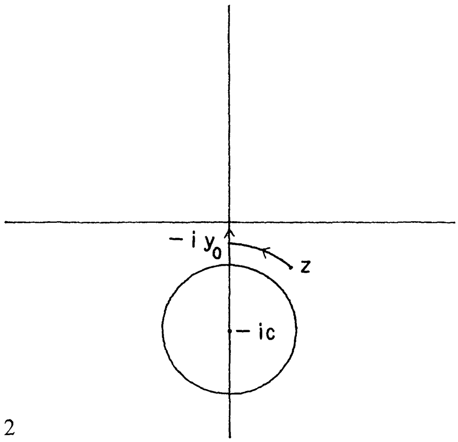

Fig. 4. Contour in case 2

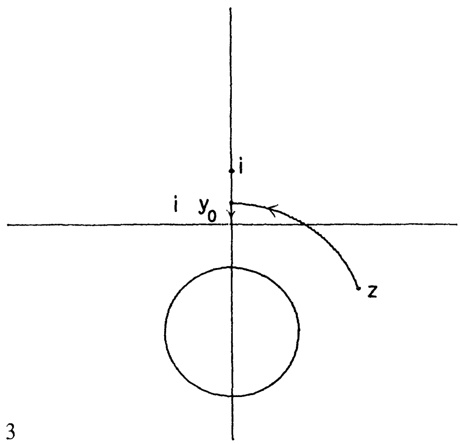

Fig. 5. Contour in case 3

Setting $x=|z+ic|$ and using the bounds given by (4.22), (4.27), and $(x^2t^2-r^2)^{-1}<(x^2-r^2)^{-1}t^{-2}$ and $|\zeta(t)/z|<t$, for all $t>1$:

$$\begin{aligned}|\hat{\Phi}^{(a)}(z)| &\leqq \int_1^\infty |\zeta(t)/z|^{(n+1)/2}((1-\xi^2)/\xi)^{(n+1)/2}(x^2t^2-r^2)^{-(n+1)/2}dt/t\sqrt{n/4\pi}\,\|f^{(a)}\|_\xi \\ &\leqq \|f^{(a)}\|_\xi((1-\xi^2)/\xi(x^2-r^2))^{-(n+1)/2}\left(\int_1^\infty t^{-(n+1)/2}dt/t\right)\sqrt{n/4\pi} \\ &\leqq (2/n+1)\,\|f^{(a)}\|_\xi A'(z,a,\xi)\,. \end{aligned} \tag{5.22}$$

Case 2: $-c<\operatorname{Im}z$, $|z+ic|\leqq c$. In this case we choose the path leading to the imaginary axis above $-ic$ along a circle with center $-ic$ and ending at $-iy_0$, then continue to zero along the imaginary axis. Then (4.22), (4.27) imply:

$$\begin{aligned}|\hat{\Phi}^{(a)}(z)| &\leqq \pi A'(z,a,\xi)\,\|f^{(a)}\|_\xi + \|f^{(a)}\|_\xi A'(z,a,\xi)\int_0^{y_0}(y/y_0)^{(n+1)/2}dy/y \\ &\leqq (\pi+2/(n+1))\,A'(z,a,\xi)\,\|f^{(a)}\|_\xi\,. \end{aligned} \tag{5.23}$$

Case 3: $-c<\operatorname{Im}z$, $c<|z+ic|\leqq c+1$. We move on a circle centered at $-ic$ and leading upwards from z to the imaginary axis at iy_0 and then we go down to 0 along the imaginary axis.

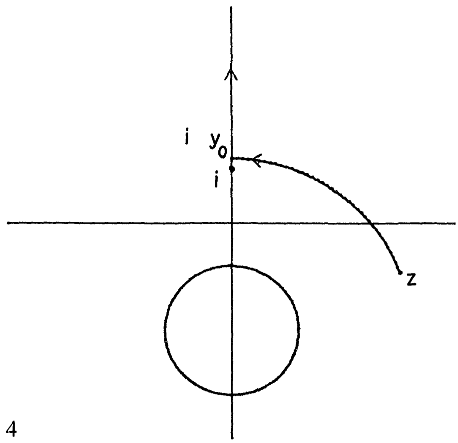

Fig. 6. Contour in case 4

As before the part contributed by the first piece of the contour to $\hat{\Phi}^{(a)}(z)$ will be bounded by $\pi A'(z, a, \xi) \|f^{(a)}\|_\xi$. The second part contributes

$$\int_0^{y_0} (y/|z|)^{(n+1)/2} A'(iy, a, \xi) \|f^{(a)}\|_\xi dy/y, \tag{5.24}$$

and using the inequality $y^{(n+1)/2} A'(iy, a, \xi) < y_0^{(n+1)/2} A'(iy_0, a, \xi)$, valid if $0 < y < y_0 < 1$, and $|z| > y_0$ and $A'(iy_0, a, \xi) = A'(z, a, \xi)$, we see that for all p between 0 and $(n+1)/2$:

$$\begin{aligned}
|\hat{\Phi}^{(a)}(z)| &\leqq \|f^{(a)}\|_\xi A'(z, a, \xi)\left(\pi + \int_0^{y_0} \left(\frac{y}{y_0}\right)^{(n+1)/2} \frac{A'(iy, a, \xi)}{A'(iy_0, a, \xi)} dy/y\right) \\
&\leqq \|f^{(a)}\|_\xi A'(z, a, \xi)\left(\pi + \int_0^{y_0} \left[\frac{y}{y_0} \frac{1+2cy_0+y_0^2}{1+2cy+y^2}\right]^p \frac{dy}{y}\right) \\
&\leqq \|f^{(a)}\|_\xi A'(z, a, \xi)\left(\pi + \min_{0<p<\frac{n+1}{2}} (2(1+c))^p \int_0^{y_0} \left(\frac{y}{y_0}\right)^p \frac{dy}{y}\right) \\
&\leqq \|f^{(a)}\|_\xi A'(z, a, \xi)\,(\pi + e\log(2+2c)).
\end{aligned} \tag{5.25}$$

Case 4: $|z+ic| > c+1$, $\operatorname{Im} z > -c$. In this case we draw a path as in Case 3 except that from iy_0 we proceed to $+i\infty$ along the imaginary axis. We find, using $A'(iy_0, a, \xi) = A'(z, a, \xi)$ and the inequality $y^{(n+1)/2} A'(iy, a, \xi) < y_0^{(n+1)/2} A'(iy_0, a, \xi)$ if $1 < y_0 < y$, and using also the remark that in the same region of values of y,

$$y^{(n+1)/2} A'(iy, a, \xi)/y_0^{(n+1)/2} A'(iy_0, a, \xi) < 2(1+c)\, y_0/y, \tag{5.26}$$

we find:

$$\begin{aligned}
|\hat{\Phi}^{(a)}(z)| &< \|f^{(a)}\|_\xi A'(z, a, \xi)\left(\pi + \min_{0<p<\frac{n+1}{2}} \int_{y_0}^{\infty} \left(\frac{y(1+2cy_0+y_0^2)}{y_0(1+2cy+y^2)}\right)^p \frac{dy}{y}\right) \\
&< \|f^{(a)}\|_\xi A'(z, a, \xi)\,(\pi + e\log 2(1+c)).
\end{aligned} \tag{5.28}$$

So that, collecting (5.28), (5.25), (5.23), (5.22), we obtain (5.18) and complete the proof of Proposition 4.

We now prove Proposition 1. The action invariants are the same for h_0+f_0 and for h_n+f_n, of course. Recalling that f_n is divisible by ε^{2^n}, we see that the coefficients of the Taylor expansions in ε of the individual actions are explicitly computable up to order $O(\varepsilon^{2^{n+1}})$ by a simple calculation: the condition of constancy simply means that if (E,k,ε) denotes a closed periodic orbit of energy E for the hamiltonian H_ε, then

$$T(E,k,\varepsilon)^{-1}\int_{(E,k,\varepsilon)}\left[\frac{f_n(g\,e^{-\sigma_z(\det g)t})}{h_n'(E,\varepsilon)}\right]dt=\{\text{function of } E,\varepsilon\}+O(\varepsilon^{2^{n+1}}) \quad (5.29)$$

for $g\in(E,k,\varepsilon)$, i.e. the right-hand side of (2.19) does not depend on k up to terms of order 2^{n+1} in ε. Therefore the right-hand side of (5.29) can be computed by letting k tend to ∞ and by using some general results [11] of ergodic theory which tell us that the limit must be the average of the function in square brackets of (5.29), up to terms of order $\varepsilon^{2^{n+1}}$, i.e.

$$\int\left[\frac{f_n(g)}{h'(E,\varepsilon)}\right]^{[<2^{n+1}]}\mu_E(dg). \quad (5.30)$$

Therefore the constancy of the action invariants can be written, by (5.29), (5.30), as

$$\int\left[\frac{f_n(g)-\bar{f}_n(E)}{h'(E,\varepsilon)}\right]^{[<2^{n+1}]}\mu_E(dg)=0. \quad (5.31)$$

Therefore the constancy of the action invariants just yields the conditions necessary for the integrability of the equations defining the n^{th} step in the construction described in Sect. 3 to build the canonical transformation conjugating H_ε with a function of H_0, i.e. it permits, in the language of the proof of Sect. 3 to define h_{n+1}, f_{n+1} in terms of h_n, f_n and, therefore, Proposition 1 appears as a corollary of the proof of Proposition 2.

6. Mixing Rates for the Geodesic and the Horocyclic Flows. Concluding Remarks

Consider the geodesic and the horocyclic flows on the unit cotangent bundle of our surface of constant negative curvature. In the preceding sections we have seen that the geodesic flow can be described as a flow on the space $T=\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ using the matrices $g_1(t)=\exp-\sigma_z t/2$; the flow is:

$$t:g\to gg_1(t),\qquad g\in T. \quad (6.1)$$

Similarly the horocyclic flow can be described on T in terms of the matrices $g_0(t)=\exp t\sigma_+$, where $\sigma_+=\begin{pmatrix}0&0\\1&0\end{pmatrix}$: the flow is

$$t:g\to gg_0(t),\qquad g\in T. \quad (6.2)$$

Let f be a ξ-analytic function on T with zero average with respect to the natural invariant measure dg. We want to study the quantity:

$$M(f,t;j)=\int_T\overline{f(g)}f(gg_j(t))(dg),\qquad j=0,1, \quad (6.3)$$

when $t \to \infty$. By general results of ergodic theory it is known to tend to zero. We prove:

Proposition 5. *There exist two functions $C(\xi)$, $b(\xi)$ such that for all ξ-analytic functions f:*

$$|M(f,t,j)| < \|f\|_\xi^2 C(\xi) \cdot \begin{cases} t^{b(\xi)} \exp - t/2\,, & j=1 \\ (\log t)^{b(\xi)} t^{-1}\,, & j=0 \end{cases} \tag{6.4}$$

for all $t > 2$. The above bounds are optimal to leading order in the t-dependence.

Observation. Note that even in the geodesic flow case this decay is not exp- (exponential in t) as it is in the case in the toral automorphisms of the Anosov type. In other words the mixing is "pretty weak."

Proof. Let $f = \sum_{a \in A} f^{(a)}$. Clearly we have to study the functions $M(f^{(a)}, t) = M(t)$ for each fixed $a \neq 0$. We use the Fourier transform and the bounds (4.22), (4.28) to obtain when $a = (u, j)$, $u = is$:

$$\begin{aligned} M(t) &= \left| \int_{-\infty}^{+\infty} \overline{\hat{f}^{(a)}(x)}\, \hat{f}^{(a)}(x\,e^{-t})\, e^{-t(1+is)/2}\, dx \right| \\ &\leq (M_3(\xi/2)^{-m_3})^2\, \|f^{(a)}\|^2\, \xi/2 \int_{-\infty}^{+\infty} (1+x^2)^{-1/2}(1+x^2 e^{-2t})^{-1/2}\, e^{-t/2}\, dx \\ &\leq M_3' \xi^{-m_3'}\, t\, e^{-t/2}\, \|f^{(a)}\|_{\xi/2}\,. \end{aligned} \tag{6.5}$$

A similar calculation yields the same results in the case of the discrete and of the supplementary series.

Lemma 3, Sect. 4, is then used to perform the sum over the indices a, taking of course into account the multiplicity estimates given by the duality theorem, and one finds the first (6.4).

The horocyclic flow can be studied in the same fashion and we leave the details to the reader.

The optimality statement relies on the consideration of special examples. Consider the function $E^{(a)}$ introduced in (4.9); then using (4.14) we can express the function $M(E^{(a)}, t)$ in terms of elementary integrals which can be studied explicitly. The slowest decay is given by the elements with $a = (is, j)$, i.e. by the functions associated with the principal series, and their M functions decay as $\exp - t/2$ times an oscillating function of t.

For instance:

$$M(E^{(is,j)}, t) = \int_{-\infty}^{+\infty} ((1+x^2)^{-(1-is)} (\sqrt{(1+x^2 e^{-2t})})^{-(1+is)}\, e^{-(1+is)t})^{1/2}\, dx\,. \tag{6.6}$$

A first concluding remark is that Proposition 4 can be regarded as a regularity theorem for the solutions of (3.8), which is the equation studied by Livscic, Guillemin and Kazhdan (whose results become, in our case, Proposition 3).

A second remark is that the question that we have studied about the canonical conjugability is the same question that, when asked for the perturbations of the integrable systems in the classical sense, leads to the KAM theory.

We see that the perturbations of the geodesic flows on surfaces of constant negative curvature are in some sense better than the ones of integrable systems. There are cases ("Birkhoff series") when the latter have a well defined perturbation series which, however, is known to be divergent. This phenomenon cannot happen in the cases studied in this paper.

Appendix A: Estimates (3.42)

The estimate for E_{n+1} obviously follows from the first Eq. (3.41). To estimate ε_{n+1} we write (with $\Delta=\Delta(g')$):

$$f_{n+1}(g',\varepsilon)\equiv h_n(H_0(g'+\Delta),\varepsilon)-h_n(H_0(g'),\varepsilon)+f_n(g'+\Delta,\varepsilon)-[\bar{f}_n(H_0(g'),\varepsilon)]^{[<2^{n+1}]}.$$

By (3.38), this is equal to:

$$\begin{aligned}&h_n(H_0(g'+\Delta),\varepsilon)-h_n(H_0(g'),\varepsilon)-h_n'(H_0(g'),\varepsilon)\{H_0,\Phi\}(g)\\&\quad+f_n(g'+\Delta,\varepsilon)-f_n^{[<2^{n+1}]}(g',\varepsilon)+\bar{f}_n^{[<2^{n+1}]}(H_0(g'),\varepsilon)\\&\quad+r_n(g',\varepsilon)-\bar{f}_n(H_0(g'),\varepsilon)^{[<2^{n+1}]}\\&=\{h_n(H_0(g'+\Delta),\varepsilon)-h_n(H_0(g'),\varepsilon)-h_n'(H_0(g'),\varepsilon)\{H_0,\Phi\}(g')\}\\&\quad+\{[f_n(g'+\Delta,\varepsilon)-f_n(g',\varepsilon)]^{[<2^{n+1}]^*}\}+\{f_n(g'+\Delta,\varepsilon)^{[\geqq 2^{n+1}]^*}\}+\{r_n(g',\varepsilon)\},\end{aligned}$$

where $[\cdot]^*$ denotes the truncation with respect to the powers of ε disregarding the ε-dependence of Δ itself; the last four terms in curly brackets will be respectively denoted $\tilde{f}$, f^{III}, f^{IV}, f^{V}, so that $f_{n+1}(g',\varepsilon)=\tilde{f}+f^{\mathrm{III}}+f^{\mathrm{IV}}+f^{\mathrm{V}}$.

The function $\tilde{f}$ has to be discussed in more detail.

We introduce the following temporary notations: $g'=(\underline{p}',\underline{q}')$, $g=(\underline{p},\underline{q})$, $\underline{\Delta}=(\underline{\Delta}_1,\underline{\Delta}_2)$, so that $\mathscr{C}^{(n)}$ becomes:

$$(\underline{p},\underline{q})=(\underline{p}',\underline{q}')+(\underline{\Delta}_1,\underline{\Delta}_2)\,. \tag{A.3}$$

Then, using $\underline{\Delta}_1(\underline{p}',\underline{q}')=\partial\Phi(\underline{p}',\underline{q})/\partial\underline{q}$, $\underline{\Delta}_2(\underline{p}',\underline{q}')=-\partial\Phi(\underline{p}',\underline{q})/\partial\underline{p}'$:

$$\begin{aligned}\{H_0,\Phi\}(\underline{p}',\underline{q}')&=\left(\frac{\partial H_0}{\partial\underline{p}'}\frac{\partial\Phi}{\partial\underline{q}'}-\frac{\partial H_0}{\partial\underline{q}'}\frac{\partial\Phi}{\partial\underline{p}'}\right)\\&=\left(\frac{\partial H_0}{\partial\underline{p}'}\underline{\Delta}_1+\frac{\partial H_0}{\partial\underline{q}'}\underline{\Delta}_2\right)+\left[\frac{\partial H_0}{\partial\underline{p}'}\left(\frac{\partial\Phi}{\partial\underline{q}'}-\frac{\partial\Phi}{\partial\underline{q}'}(\underline{p}',\underline{q}'+\underline{\Delta}_2)\right)\right.\\&\quad\left.-\frac{\partial H_0}{\partial\underline{q}'}\left(\frac{\partial\Phi}{\partial\underline{p}'}+\frac{\partial\Phi}{\partial\underline{p}'}(\underline{p}',\underline{q}'+\underline{\Delta}_2)\right)\right]\equiv(K_1)+[K_2]\,,\end{aligned} \tag{A.4}$$

where the functions whose arguments are not explicitly written are to be thought of as computed at $(\underline{p}',\underline{q}')$ and the functions K_1, K_2, introduced at the last step, to shorten the notation, are identified with the two main brackets in the intermediate term of (A.4). So

$$\begin{aligned}\tilde{f}(g',\varepsilon)=&\{h_n(H_0(\underline{p}'+\underline{\Delta}_1,\underline{q}'+\underline{\Delta}_2),\varepsilon)-h_n(H_0(\underline{p}',\underline{q}'),\varepsilon)\\&-h_n'(H_0(\underline{p}',\underline{q}'),\varepsilon)K_1\}-\{h_n'(H_0(\underline{p}',\underline{q}'),\varepsilon)K_2\}\equiv\{f^{\mathrm{I}}\}+\{f^{\mathrm{II}}\}\,.\end{aligned} \tag{A.5}$$

We now proceed successively to estimate f^{I} to f^{V} in $W(\varrho_n e^{-4\sigma_n},\ \xi_n e^{-4\delta_n},\ \theta_n e^{-4\tau_n})$.

By Eq. (3.40),

$$\|f^{\mathrm{V}}\| < A_1 e^{-\tau_n 2^{n+1}} \hat{\varepsilon}_n \hat{\tau}_n^{-1} . \tag{A.6}$$

By a dimensional estimate,

$$\|f^{\mathrm{IV}}\| < A_2 \hat{\tau}_n^{-1} \lambda_n e^{-\tau_n 2^{n+1}} . \tag{A.7}$$

Again by a dimensional estimate, using also (3.35):

$$\|f^{\mathrm{III}}\| < A_3' \varepsilon_n \hat{\tau}_n^{-1} (\hat{\delta}_n^{-1} \hat{\sigma}_n^{-1}) \|\Delta\| < A_3 \varepsilon_n^2 (\hat{\sigma}_n \hat{\tau}_n \hat{\delta}_n \xi_n)^{-a_3} , \tag{A.8}$$

where A_3, a_3 are suitable constants and the exponents have been made uniform to simplify the notations.

Another dimensional estimate, plus (3.35), (3.31), shows that:

$$\|f^{\mathrm{II}}\| < A_4' E_n B_1 (\xi_n \hat{\delta}_n)^{-q} \hat{\tau}_n^{-1} \varepsilon_n (\xi_n \hat{\delta}_n \hat{\tau}_n)^{-2} \|\Delta\| < A_4 \varepsilon_n^2 (\hat{\tau}_n \hat{\sigma}_n \hat{\delta}_n \xi_n)^{-a_4} . \tag{A.8}$$

Lastly, applying the Taylor formula to second order to exhibit the $\|\Delta\|^2$ factor

$$\|f^{\mathrm{I}}\| < A_5' E_n (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-2} \|\Delta\|^2 , \tag{A.9}$$

hence

$$\|f^{\mathrm{I}}\| < A_5 (\xi_n \hat{\delta}_n \hat{\tau}_n \hat{\sigma}_n)^{-a_5} \varepsilon_n^2 . \tag{A.9}$$

To estimate the derivatives we observe that the above estimates imply on $W(\varrho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n})$:

$$\|f^{\mathrm{I}} + f^{\mathrm{II}} + f^{\mathrm{III}} + f^{\mathrm{V}}\| < A_6 (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-a_6} (\varepsilon_n^2 + \varepsilon_n e^{-\tau_n 2^{n+1}}) . \tag{A.10}$$

We have on $W(\varrho_n e^{-5\sigma_n}, \xi_n e^{-5\delta_n}, \theta_n e^{-5\tau_n})$:

$$\left| \frac{\partial (f^{\mathrm{I}} + f^{\mathrm{II}} + f^{\mathrm{III}} + f^{\mathrm{V}})}{\partial g} \right| < A_7 (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-a_7} (\varepsilon_n^2 + \varepsilon_n e^{-\tau_n 2^{n+1}}) \tag{A.11}$$

with $a_7 = a_6 + 1$. Also

$$\frac{\partial}{\partial g_{ij}'} (f^{\mathrm{IV}}(g' + \Delta, \varepsilon)) = \frac{\partial f^{\mathrm{IV}}}{\partial g_{ij}'} (g' + \Delta, \varepsilon) + \sum_{h,k=1}^{2} \frac{\partial f^{\mathrm{IV}}}{\partial g_{hk}'} (g' + \Delta, \varepsilon) \frac{\partial \Delta_{hk}}{\partial g_{ij}'} , \tag{A.13}$$

so that, using (3.35) and the fact that its right-hand side is bounded by 1,

$$\left\| \frac{\partial f^{\mathrm{IV}}}{\partial g'} \right\| < A_8 \varepsilon_n e^{-\tau_n 2^{n+1}} \hat{\tau}_n^{-1} . \tag{A.14}$$

Clearly (A.14) and (A.11) prove the second Eq. (3.42) while the third follows from (A.7).

Appendix B: Proof of Lemmas 1, 2, 4

Consider first the case $a = (u, j)$. By assumption the functions $h \to (\phi, U(h) f)$ defined for $h \in \operatorname{Re} H(\xi)$ and a given $\phi \in L_2$, can be holomorphically extended to $H(\xi)$ and, by Schwartz' inequality, verify:

$$|(\phi, U(h) f)| < \|\phi\|_2 \|f\|_\xi . \tag{B.1}$$

We choose ϕ to be an element $\phi_{x_0}\in Y^{(a)}$ such that for all $x\in\mathbb{R}$:

$$\hat{\phi}_{x_0}(x)=e^{-(x-x_0)}\chi(x-x_0) \quad \text{if} \quad u=is, \tag{B.2}$$

where $\chi(x)=0$ if $x<0$ and $\chi(x)=1$ otherwise, or such that:

$$\int_{-\infty}^{+\infty}\hat{\phi}_{x_0}(y)|x-y|^{s-1}dy=e^{-(x-x_0)}\chi(x-x_0) \quad \text{if} \quad u=s, \tag{B.3}$$

i.e.

$$\hat{\phi}_{x_0}(y)=\frac{1}{2\pi}\int e^{-i(y-x_0)p}\frac{|p|^s dp}{(1-ip)(2\cos\pi s/2)(s-1)!}. \tag{B.4}$$

A simple calculation yields the values of the L_2 norms in $Y^{(a)}$ of the above functions

$$1/\sqrt{2} \quad \text{if} \quad u=is,\ (4(\cos\pi s/2)^2(s-1)!)^{-1/2} \quad \text{if} \quad u=s. \tag{B.5}$$

If $h\in\mathrm{Re}H(\xi)$, $h=\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}$, $\alpha=(1+bc)/d$, it is easy to check the identity:

$$\left(b\frac{\partial}{\partial d}+\alpha\frac{\partial}{\partial c}+1\right)(\phi_{x_0},U(h)f)\equiv\hat{f}^{(a)}(x_0h)\,(j(x_0,h)^2)^{-(1+u)/2}, \tag{B.6}$$

see appendix F. This proves that the right-hand side, i.e. the function $(U(h)f^{(a)})(x_0)$ is a holomorphic function of h for $x_0\in\mathbb{R}$. Equation (B.6) says more: in fact multiplying both sides by $(1+(x_0h)^2)^{(1+u)/2}$, we see that

$$\begin{aligned}F_f^{(a)}(x_0h)&=(\hat{U}^{(a)}(h)\hat{f}^{(a)}(x_0))\,(1+(x_0h)^2)^{(1+u)/2}(j(x_0,h)^2)^{(1+u)/2}\\&=(\hat{U}(h)\hat{f}^{(a)})\,(x_0)\,((\alpha x_0+c)^2+(bx_0+d)^2)^{(1+u)/2}.\end{aligned} \tag{B.7}$$

But since $\alpha\simeq 1$, $d\simeq 1$, $c\simeq 0$, $b\simeq 0$, (recall that we assumed $\xi<1/10$), it is easy to check that the quantity in the last bracket can be bounded, as h varies in $H(\xi)$, by:

$$\begin{aligned}(1+x_0)^2(1-m'\xi)<\mathrm{Re}((\alpha x_0+c)^2+(bx_0+d)^2)<(1+x_0)^2(1+m'\xi),\\ \arg((x_0+c)^2+(bx_0+d)^2)<m''\xi,\end{aligned} \tag{B.8}$$

for all real x_0; for simplicity we may and shall assume $\xi_0<1/2m'$. This means that $F_f^{(a)}(x_0h)$ is holomorphic in h, i.e. in b, c, d, as h varies in $H(\xi)$, for all x_0 in $\mathbb{R}$. If $x>0$ is large enough the point xh covers a neighborhood of ∞ as h varies in $H(\xi)$: hence $F_f^{(a)}$ can be holomorphically extended from the positive real axis to a vicinity of ∞. Under the same circumstances, xh covers a real neighborhood of ∞ as h varies in $\mathrm{Re}H(\xi)$; therefore $F_f^{(a)}(-y)$, where y is large and positive, coincides with the analytic continuation of $F_f^{(a)}$ from the positive real axis through ∞.

On the other hand (B.7), (B.8) also show that $F_f^{(a)}(x)$ can be holomorphically extended to a strip around the whole real axis.

From the above considerations it follows that $F_f^{(a)}$ admits a holomorphic extension to the whole $\mathbb{R}H(\xi)$, single valued in this multiply connected region.

We now use that the distance of $H(\xi e^{-\delta})$ to the boundary of $H(\xi)$ can be estimated to be no less than $\tilde{B}\xi\tilde{\delta}$, where $\tilde{B}$ is a suitably small number, and $\tilde{\delta}=(1-e^{-\delta})$. So (B.6) implies, by a dimensional estimate, the following bound:

$$\begin{aligned}|\hat{f}^{(a)}(xh)\,(j(x,h)^2)^{-(1+u)/2})|&\equiv|F_f^{(a)}(xh)\,((\alpha x+c)^2+(bx+d)^2)^{-(1+u)/2}|\\&\leqq\|\phi_x\|_2\,\|f\|_\xi B(\xi\tilde{\delta})^{-1},\end{aligned} \tag{B.9}$$

if B is a suitably large constant; (B.9) holds for $h\in H(\xi e^{-\delta})$ and $x\in\mathbb{R}$.

Let z be real and choose $h=\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}=(1-\xi^2)^{-1/2}\begin{pmatrix}1 & \pm\xi\\ \pm\xi & 1\end{pmatrix}$, $x=zh^{-1}$, so that (B.9) gives:

$$|F_f^{(a)}(z)|<C_1\xi^{-2-\mathrm{Re}u}\|f\|_\xi, \tag{B.10}$$

because $((\alpha x+c)^2+(bx+d)^2)^{-1}\equiv((1+(xh)^2)j(x,h)^2)^{-1}\equiv((1+z^2)j(z,h^{-1})^{-2})^{-1}\leqq|(1+z^2)/(1+\xi|z|)^2|^{-1}$ provided the arbitrary sign in h is appropriately chosen. The constant C_1 is independent on (u,j) since the norms in (B.5) are uniformly bounded.

Consider now instead of f the function $U(h)f$, $h\in H(\xi e^{-\delta})$. Then $U(h)f$ is $\xi\delta C_2$-analytic, if C_2 is a suitable small constant, for all $\delta>0$ and therefore (B.10) implies:

$$|F_{U(h)f}^{(a)}(x)|<C_3(\xi\delta)^{-2-\mathrm{Re}u}\|f\|_\xi, \tag{B.11}$$

for a suitably large C_3. Hence:

$$|\hat f(xh)(1+(xh)^2)^{(1+u)/2}|<C_3\|f\|_\xi(\xi\delta)^{-2-\mathrm{Re}u}\left|\frac{((\alpha x+c)^2+(bx+d)^2)^{(1+u)/2}}{1+x^2}\right|, \tag{B.12}$$

so if $z\in\mathbb{R}H(\xi)$:

$$|F_f^{(a)}(z)|<C_3\|f\|_\xi\xi^{-2-\mathrm{Re}u}\inf\delta^{-2-\mathrm{Re}u}\left|\frac{((\alpha x+c)^2+(bx+d)^2)^{(1+u)/2}}{1+x^2}\right|, \tag{B.13}$$

where the inf is taken over all the $\delta>0$ such that $z\in\mathbb{R}H(\xi e^{-\delta})$ and over all the pairs $x\in\mathbb{R}$, $h\in H(\xi e^{-\delta})$ such that $xh=z$.

By (B.8) we see that [with the notations of (4.20)]

$$A_\xi^{(a)}(z)>C_3\xi^{-2-\mathrm{Re}u}e^{-m'\xi(\mathrm{Im}u)/2}(1-m'\xi)^{(1+\mathrm{Re}u)/2}\hat d(z,\xi)^{-2-\mathrm{Re}u}. \tag{B.14}$$

To get an upper bound on $A_\xi^{(a)}$ we observe that if $z\in(\partial\mathbb{R}H(\xi e^{-\delta})\cap\mathbb{C}_+)$, then there exists $x_0\in\mathbb{R}$ such that $x_0h_0=z$, with:

$$h_0=(1-\xi^2e^{-2\delta})^{-1/2}\begin{pmatrix}1 & i\xi e^{-\delta}\\ -i\xi e^{-\delta} & 1\end{pmatrix}, \tag{B.15}$$

and for this pair x_0, h_0 the number in the modulus sign of (B.13) is identically 1. Therefore we have also checked (4.28), as a consequence of (B.14), (B.15), (recall here that u can take only finitely many real values so that, in the bound (4.28), m_3 can be chosen u-independent).

We now study the case $a=(n,j,+)$. In this case it follows from the holomorphy properties of the Fourier transforms (in the upper half plane) that for $\zeta\in\mathbb{C}_+$, $h\in\mathrm{Re}H(\xi)$:

$$f(\zeta h)j(\zeta,h)^{-1-n}=(\phi_\zeta^{(a)},U(h)f), \tag{B.16}$$

where

$$\hat\phi_\zeta^{(a)}(z)=\frac{(2i)^n}{-2\pi i}n(z-\bar\zeta)^{-1-n},\qquad \|\phi^{(a)}\|_2=\left(\frac{n}{4\pi}\frac{1}{(\mathrm{Im}\zeta)^{1+n}}\right)^{1/2}. \tag{B.17}$$

Therefore repeating an argument similar to the one given above to discuss the analyticity in the case of the representations with $a=(u,j)$, we find from (3.10) that $\hat{U}^{(a)}(h)\hat{f}(z)$ can be holomorphically extended to $H(\xi)$ as a function of h at $z\in\mathbb{C}_+$ fixed and

$$\begin{aligned} F_f^{(a)}(\zeta h) &= \hat{U}^{(a)}(h)\hat{f}^{(a)}(\zeta)\,((i+\zeta h)j(\zeta,h))^{1+n} \\ &\equiv (\hat{U}^{(a)}(h)\hat{f}^{(a)})(\zeta)\,((\alpha\zeta+c)+i(b\zeta+d))^{1+n}, \end{aligned} \tag{B.18}$$

and this means that $F_f^{(a)}$ can be holomorphically extended to $\mathbb{C}_+H(\zeta)$ [as well as $\hat{f}^{(a)}$ itself since, in these cases, $(n+1)/2$ is an integer].

By arguments similar to the ones used in the cases $a=(u,j)$ we find that (B.17) and (B.18) imply:

$$|F_f^{(a)}(z)| \leqq (n/4\pi)^{1/2}|((i+z)^2 j(\zeta,h)^2/\mathrm{Im}\,\zeta)^{(n+1)/2}|\,\|f\|_\xi \tag{B.19}$$

for all $\zeta\in\mathbb{C}_+$, $h\in H(\xi)$ such that $z=\zeta h$; this proves the first part of I) and ii) [since the discussion of the cases $a=(n,j,-)$ can obviously be reduced to that of the cases $a=(n,j,+)$].

We now prove the second part of I) in Lemma 1, i.e. that if $F_f^{(a)}$ can be holomorphically extended to $Q^{(a)}(\xi)$ then f is ξ-analytic.

If $F_f^{(a)}$ is holomorphic in $Q^{(a)}(\xi)$, then $F_f^{(a)}(z)$ is uniformly bounded in $Q^{(a)}(\xi\,e^{-\delta})$. Calling M_δ a bound for this function it follows that in the cases $a=(u,j)$ [see (B.8)]:

$$|\hat{U}^{(a)}(h)\hat{f}^{(a)}(x)| \leqq M_\delta|((\alpha x+c)^2+(bx+d)^2)^{-(1+u)/2}| \leqq M'_\delta(1+x^2)^{-(1+\mathrm{Re}\,u)/2} \tag{B.20}$$

if $h=\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}$, for a suitable M'_δ.

Expression (B.20) shows, thanks to its uniformity in $h\in H(\xi\,e^{-\delta})$, that $\hat{U}^{(a)}(h)\hat{f}^{(a)}\in\hat{Y}^{(a)}$ and for all $\phi\in\hat{Y}^{(a)}$ the function $h\to(\phi,\hat{U}^{(a)}(h)\hat{f}^{(a)})$ is holomorphic in $H(\xi\,e^{-\delta})$. Therefore f itself is holomorphic on $TH(\xi\,e^{-\delta})$, $\forall\delta>0$.[2]

An identical argument can be given in the cases $a=(n,j,\pm)$. So i) of Lemma 1 is completely proved and it remains to prove iii).

We discuss now the proof of iii) in the case $a=(u,j)$, the others being very similar. Observe that (4.23), (4.28) imply that $U^{(a)}(h)f^{(a)}(x)$ is holomorphic in $h\in H(\xi)$ and, for all $h\in H(\xi\,e^{-\delta})$:

$$|U^{(a)}(h)f^{(a)}(x)| < NC_3\xi^{-2-\mathrm{Re}\,u}\delta^{-2-\mathrm{Re}\,u}(1+x^2)^{-(1+\mathrm{Re}\,u)/2}, \tag{B.21}$$

so that

$$\|U^{(a)}(h)f^{(a)}\|_2 < NC_4(\xi\delta)^{-2-\mathrm{Re}\,u} \quad \text{for all} \quad h\in H(\xi\,e^{-\delta}), \tag{B.22}$$

which by (4.1) implies (4.24) (in this case $\sqrt{|a|}$ can be replaced by 1).

The cases $a=(n,j,\pm)$ are discussed in an identical way and the role of (4.28) is now played by (4.27) which can be checked as follows: Note that the h-image of $\mathbb{C}_+$ is the complement of a circle with center c and radius R, and if $z=\zeta h$, $\zeta\in\mathbb{C}_+$:

$$|(\mathrm{Im}\,\zeta)j(\zeta,h)^{-2}| = ((|z-c|^2-R^2)/2R), \tag{B.23}$$

which can be seen by direct calculation or by suitable geometric arguments.

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2 Here we use a general property explicitly stated in Lemma 5 below

Furthermore the matrix

$$h_\xi=(1-\xi^2)^{-1/2}\begin{pmatrix}1 & i\xi\\ -i\xi & 1\end{pmatrix} \tag{B.24}$$

is on the boundary of $H(\xi)$ and maps $\mathbb{C}_+$ onto the complement of the circle $B(\xi)$ with:

$$c=(\xi^{-1}+\xi)/2\equiv c_0, \qquad R=(\xi^{-1}-\xi)/2\equiv R_0. \tag{B.25}$$

Finally a circle G with center c and radius R which contains a circle with center c' and radius R' is such that for all z outside G:

$$(|z-c'|^2-R'^2)/2R'\geqq(|z-c|^2-R^2)/2R, \tag{B.26}$$

as it is easily seen (e.g. remark that one can always suppose that G is the unit circle centered at the origin and that G' is a smaller circle centered at the point $(0,c')$ and with radius R' such that $c'+R'\leqq 1$, $c'>0$; then the above relation can be checked by simple considerations).

The same inequality holds if G and G' are two circles with disjoint interiors and z is inside G: this can be shown as in the preceding case or, alternatively, by noting that the inequality (B.26) is invariant with respect to the operation of inversion with respect to the circle G.

The above inequalities are exactly saying that the infimum over $H(\xi)$ is obtained by considering $h=h_\xi$ and this yields (4.27).

So Lemma 1 and Lemma 4 are completely proved.

The direct part of Lemma 2 follows from the explicit formulae (B.6), (B.16) and from the uniform boundedness of $F_f^{(a)}$ in every compact subset of $Q^{(a)}(\xi)\times W$. The converse statement, appearing in Lemma 2, follows from the argument after (B.20), guaranteeing that $h\to U(h)f$ regarded as a L_2-valued function defined on $\mathrm{Re}\,H(\xi)$ can be extended holomorphically to $H(\xi)$, combined with the:

Lemma 5. *Let $f\in L_2$ and suppose $h\to U(h)f$, regarded as an L_2-valued function defined for $h\in\mathrm{Re}\,H(\xi)$, can be holomorphically extended to an L_2-valued function on $H(\xi)$ then f is ξ-analytic.*

If f depends parametrically on $w\in W\subset\mathbb{C}^q$ and the L_2-valued function $(h,\underline{w})\to U(h)f$, defined on $(\mathrm{Re}\,H(\xi))\times W$, can be holomorphically extended to $H(\xi)\times W$ then the function $(h,w)\to f(gh)$ can also be holomorphically extended to $H(\xi)\times W$ for all g.

In other words ξ-analyticity of f and holomorphy on $H(\xi)$ of the L_2-valued function $U(h)f$ are equivalent properties [note that the ξ-analyticity obviously implies that $h\to U(h)f$ is holomorphic as a vector valued function]. This is a well known fact.

Appendix C: Proof of Lemma 3

With the matrix notations of Sects. 3, 4 let $\mathscr{E}(\underline{t})=\exp(t_1\sigma+t_2\sigma_x+t_3\sigma_z)$. Let $\mathscr{D}$ be the Casimir operator on $L_2(T)$, for $U(\cdot)$, given by [6]:

$$\mathscr{D}f(g)=-\frac{1}{4}\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial t_2^2}-\frac{\partial^2}{\partial t_3^2}\right)f(g\mathscr{E}(t))\bigg|_{t=0}. \tag{C.1}$$

Observe that $|t_j^{(r)}|<\xi\hat{\delta}/4q$ implies $\mathscr{E}(\underline{t}^{(q)})\dots\mathscr{E}(\underline{t}^{(1)})\in H(\xi\delta/2)$, and that $H(\xi e^{-\delta})\cdot H(\xi\hat{\delta}/2)\subset H(\xi)$, so that the function $U(h)f(g\mathscr{E}(t^{(q)})\dots\mathscr{E}(t^{(1)}))$, is holomorphic in the t variables for $|t_j^{(r)}|<\xi\hat{\delta}/4q$. Therefore by a dimensional estimate and by the Schwartz inequality:

$$|(\phi,\mathscr{D}^q U(h)f)|<4^{-q}3^q 2^q((4q/\xi\hat{\delta})^2)^q\,\|\phi\|_2\,\|f\|_\xi\,, \tag{C.2}$$

i.e. for all $h\in H(\xi e^{-\delta})$:

$$\|\mathscr{D}^q U(h)f\|_2<(6q/\xi\hat{\delta})^{2q}\,\|f\|_\xi\,. \tag{C.3}$$

But by orthogonality and by the fact that $\mathscr{D}$ has on $Y^{(a)}$ the constant value $V(a)=(1-u^2)/4$, if $a=(u,j)$, or $(1-n^2)/4$ if $a=(n,j,\pm)$:

$$V(a)^q\,\|U(h)f^{(a)}\|_2\equiv\|\mathscr{D}^q U(h)f^{(a)}\|_2<\|\mathscr{D}^q U(h)f\|_2<(6q/\xi\hat{\delta})^{2q}\,\|f\|_\xi\,, \tag{C.4}$$

so that for all $h\in H(\xi e^{-\delta})$,

$$\|U(h)f^{(a)}\|_2<(6q/(\xi\hat{\delta}\sqrt{V(a)}))^{2q}\,\|f\|_\xi\,, \tag{C.4}$$

which implies Lemma 3 by optimizing this inequality on q and, finally, using (4.1).

Appendix D: A Canonical Map

We rewrite (2.7), recalling that $g=\begin{pmatrix} p_1 & q_2\\ -p_2 & q_1\end{pmatrix}$:

$$\begin{aligned} p_x+ip_y&=\tfrac{1}{2}i(p_1+iq_2)^2\equiv p\,,\\ x+iy&=\frac{p_2+iq_1}{p_1-iq_2}\equiv q\,, \end{aligned} \tag{D.1}$$

and since $p_xdx+p_ydy=\mathrm{Re}\,p\overline{dq}$, we find:

$$p_xdx+p_ydy=p_1dq_1+p_2dq_2-\tfrac{1}{2}d(\det g)\,, \tag{D.2}$$

and

$$H_0(g)=\tfrac{1}{2}y^2(p_x^2+p_y^2)=\tfrac{1}{2}(\mathrm{Im}\,q)^2p^2=\tfrac{1}{8}(\det g)^2\,. \tag{D.3}$$

Note that the function $\det g$ is single valued on $\Gamma\backslash\mathrm{PGL}(2,\mathbb{R})$, hence on the graph of the map (D.1).

Appendix E: An Example of Non-Global Canonical Map and the Non-Sufficiency of the Period and of the Lyapunov Invariants

Let $\mathscr{C}_\varepsilon(p,z)=(p',z')$ be defined as follows: put $p=p_x+ip_y$, $z=x+iy$, $p'=p'_x+ip'_y$, and let ϕ be an automorphic form of order 1, verifying (4.4), then

$$\mathscr{C}_\varepsilon(p,z)=(p',z')\,,\qquad \begin{aligned} p'&=p+\varepsilon\overline{\phi(z)}\,,\\ z'&=z\,. \end{aligned} \tag{E.1}$$

Then $\mathrm{Re}\,\overline{pdz} = \mathrm{Re}\,p'\overline{dz'} + \varepsilon \mathrm{Re}\,\overline{\phi(z)dz}$, since ϕ is holomorphic the last differential form is closed (i.e. $\mathscr{C}$ is locally canonical), and since $\phi(z)$ is covariant, (4.4), while z is contravariant ($dz' = j(z,\gamma)^{-2} dz$ for all $\gamma \in \Gamma$) the form $\phi(z)dz$ is defined on the whole Σ and (E.1) is well defined as a map of $T^*\Sigma$ into itself, i.e. $\mathscr{C}_\varepsilon$ is globally defined though not globally canonical.

The non-globality of the canonical character follows from the fact that $\phi(z)dz$ is not an integrable form on Σ (because there are no non-singular automorphic functions on Σ) [7]: hence for some closed curve on Σ we have $\oint \phi(z)dz \neq 0$, and we may suppose that we change ϕ by a suitable phase factor (constant on Σ) so that the value of the integral is actually positive. With such a choice (E.1) does not preserve the actions of closed orbits in $T^*\Sigma$.

Consider the hamiltonian:

$$H_0(\mathscr{C}_\varepsilon^{-1}(\underline{p},\underline{q})) \equiv H_\varepsilon(\underline{p},\underline{q}) . \tag{E.2}$$

Clearly H_ε cannot be conjugated with H_0 by a global canonical map: however it is conjugated to it by $(\underline{p},\underline{q}) = \mathscr{C}_\varepsilon(\underline{p}',\underline{q}')$. This suffices for the conservation of the period and of the Lyapunov invariants.

Appendix F: Hint to (B.6)

(B.6) is obtained as a consequence of the following formal identities:

$$\begin{aligned}(\phi_{x_0}, U(h)f) &= \int_{x_0}^{\infty} e^{-(x-x_0)} f(xh)\,|j(x,h)|^{-(1+u)} dx \\ &= \int_{x_0h}^{\infty h} e^{-(yh^{-1}-x_0)} f(y)\,|j(y,h^{-1})|^{(1+u)-2} dy .\end{aligned} \tag{F.1}$$

Then one uses the following relations, valid if $\alpha = (1+bc)/d$:

$$\begin{aligned}&1)\ (b\partial/\partial d + \alpha\partial/\partial c) j(y,h^{-1}) \equiv 0 && \text{for all } y,\\ &2)\ (b\partial/\partial d + \alpha\partial/\partial c + 1)\exp -(yh^{-1}) \equiv 0 && \text{for all } y,\\ &3)\ (b\partial/\partial d + \alpha\partial/\partial c)(x_0h) = j(x,h)^{-2} = j(y,h^{-1})^2 && \text{for } y = xh,\\ &4)\ (b\partial/\partial d + \alpha\partial/\partial c)\infty h \equiv 0 .\end{aligned} \tag{F.2}$$

Appendix G: Proof of i), iv) in Proposition 4

We shall deal throughout this section with the strongly ξ-analytic functions although some of the results hold also for the ξ-analytic functions (as we shall mention).

The basic properties of their Fourier transforms are discussed exactly as in Sect. 4 with the few obvious changes that we list below.

Modify Lemma 1 by replacing $\|f\|_\xi$ by $\|\tilde{f}\|_\xi$ and the words "ξ-analytic" by "strongly ξ-analytic" and the domains $Q^{(a)}(\xi)$ by $\tilde{Q}^{(a)}(\xi)$

$$\tilde{Q}^{(a)}(\xi) = \begin{cases} \mathbb{C}_+ \tilde{H}(\xi), & a = (n,j,\pm) \\ \mathbb{R}\tilde{H}(\xi), & a = (u,j), \end{cases} \tag{G.1}$$

and finally replace $A^{(a)}(z,\xi)$ by the expressions $A^{(a)}(z,\xi)$ obtained by (4.19), (4.20) with infima over the $\zeta\in\mathbb{C}_+$, $h\in\tilde{H}(\xi)$, $\zeta h=z$ or over $\delta>0$, $x\in\mathbb{R}$, $h\in\tilde{H}(\xi e^{-\delta})$ (i.e. replace everywhere $H(\xi)$ by $\tilde{H}(\xi)$). In this way we obtain a lemma we shall call Lemma 1′ which is also true.

Changing Lemmas 2, 3 according to the same prescriptions as above leads to lemmas which we refer as Lemma 2′, Lemma 3′.

The proofs of Lemmas 1′, 2′, 3′ are obtained from those of Lemmas 1, 2, 3 by just replacing everywhere the words ξ-analytic by strongly ξ-analytic and the domain $H(\xi)$ by $\tilde{H}(\xi)$.

The proof of ii), iv) of Proposition 4 uses the following necessary and sufficient condition of strong ξ-analyticity.

Lemma 6. *Let $f\in Y^{(a)}$, $a\in A$, $a\neq 0$. There exist three constants D, D', q independent of f and a such that if f is strongly ξ-analytic then*

i) *the functions $(U(h)f)(z)$ defined by (4.13), for h real and z in the appropriate domain can be extended holomorphically in h, at fixed z, to $\tilde{H}(\xi)$.*

ii) *if $a=(u,j)$ and if $0<\delta<\xi$, $h\in\tilde{H}(\xi-\delta)$, then*

$$|(U(h)\hat{f})(x)|\leqq D\delta^{-1}\,\|\tilde{f}\|_\xi, \tag{G.2}$$

$$|U(h)\hat{f}(x)|\leqq D\delta^{-1}(1+\delta|x|)^{-1-\mathrm{Re}\,u}\,\|\tilde{f}\|_\xi. \tag{G.3}$$

iii) *if $a=(n,j,\pm)$ then for all $\delta\in(0,\xi)$, $h\in H(\xi-\delta)$:*

$$|U(h)\hat{f}(z)|\leqq\sqrt{\frac{n}{4\pi}}\,\frac{D}{(\mathrm{Im}\,z)^{\frac{1+n}{2}}}\,\|\tilde{f}\|_\xi, \tag{G.4}$$

$$|U(h)\hat{f}(z)|\leqq\sqrt{\frac{n}{4\pi}}\,\frac{D}{(\mathrm{Im}\,z)^{\frac{n-1}{2}}\,\delta(1+\delta^2|z|^2)}\,\|\tilde{f}\|_\xi, \tag{G.5}$$

Conversely if $f\in L_2$, $f\in Y^{(a)}$ and $(U(h)\hat{f})(z)$ can be extended holomorphically to $h\in\tilde{H}(\xi)$, and if (G.2)–(G.5) hold with $\|\tilde{f}\|_\xi$ replaced by some N, then f is strongly ξ-analytic and

$$\|\tilde{f}\|_{\xi-\delta}\leqq D'\delta^{-q}N\sqrt{\mathrm{Re}\,a}. \tag{G.6}$$

Identical results hold if strongly ξ-analytic functions are replaced by ξ-analytic functions and if $\|\tilde{f}\|_\xi$ are replaced by $\|f\|_\xi$ and $\widetilde{H(\xi)}$ by $H(\xi)$.

The proof of Lemma 6 is implicit in the proof of Appendix B. In fact i) follows from (B.6) for $a=(u,j)$ (see comment following (B.6)) and from (B.16) for $a=(n,j,\pm)$. Of course the substitutions of $\|f\|_\xi$ by $\|\tilde{f}\|_\xi$ and $H(\xi)$ by $\widetilde{H(\xi)}$ or $A^{(a)}(z,\xi)$ by $\tilde{A}^{(a)}(z,\xi)$, etc., have to be made where met in the text preceding such formulas.

The bound (G.2) follows from (B.6) (with the appropriate insertions of $\sim$, as above) by a dimensional estimate of the left-hand side and by (4.1) and (B.5).

The bound (G.4) follows from (B.16) and (B.17).

The bound (G.3) follows from (G.2) by noting that $U(h)f$ is, for all $h\in\tilde{H}(\xi-\delta)$, still strongly $\frac{\delta}{2}$-analytic so that for $h'\in\tilde{H}\left(\frac{\delta}{2}\right)$ real

$$|U(h')(U(h)\hat{f})(x')|\equiv|U(h)\hat{f}(x'h')|\,|(j(x',h')^2)^{-(1+u)/2}|\leqq D\left(\frac{\delta}{2}\right)^{-1}\|\tilde{f}\|_\xi,$$

i.e.

$$|U(h)\hat{f}(x)| \leqq 2D\delta^{-1} \|\tilde{f}\|_\xi \inf |j(x', h')|^{-(1+n)/2},$$

where the infimum is over all the $x' \in \mathbb{R}$, $h' \in \operatorname{Re}\tilde{H}(\delta/2)$, $x'h' = x$. Use now $j(x', h') = j(x, h'^{-1})$ and pick $h' = \begin{pmatrix} 1 & \pm\delta/2 \\ 0 & 1 \end{pmatrix}$ to obtain an upper bound to the above infimum, proportional to $(1+\delta|x|)^{-1}$. Then redefine D to have it equal in (F.2), (F.3). The bound (G.5) is similarly deduced from (G.4). Proceeding as above one finds

$$|U(h)\hat{f}(z)| \leqq \sqrt{\frac{n}{4\pi}}\, D \|\tilde{f}\|_\xi \inf \frac{1}{(\operatorname{Im} z')^{(n+1)/2}} \frac{1}{|j(z, h'^{-1})|^{1+n}},$$

where the infimum is over the $h' \in \tilde{H}\left(\frac{\delta}{2}\right)$, $z' \in \mathbb{C}_+$, $z'h' = z$ and the infimum is estimated by the choice

$$h' = \left(1 - \left(\frac{\delta}{2}\right)^2\right)^{-1/2} \begin{pmatrix} 1 & -i\delta/2 \\ i\delta/2 & 1 \end{pmatrix}$$

noting that $\operatorname{Im} z' > \delta/2$, so that

$$|U(h)\hat{f}(z)| \leqq \sqrt{\frac{n}{4\pi}}\, D \|\tilde{f}\|_\xi \left(\frac{\delta}{2}\right)^{-(n+1)/2} \left|1 - i\frac{\delta}{2} z\right|^{-(1+n)/2}. \tag{G.7}$$

Then (G.5) follows by raising (G.4) to the power $(n-1)/(n+1)$ and (G.7) to the power $2/(n+1)$ and multiplying them together.

Again one has to redefine D in order to have it identical in (G.2)–(G.5). The statement iii) is a simple consequence of the calculation of the $Y^{(a)}$ norms of the right-hand side of (G.3), (G.5) with $\|\tilde{f}\|_\xi$ replaced by N, and of the remark that such norms can be bounded essentially independently of a.

We now proceed to the proof of the following Lemma 7 which will immediately imply ii), iv) of Proposition 4. In fact from the estimates (G.14), (G.15) below and using Lemma 6, iii), we obtain estimates on $\|\widetilde{\Phi^{(a)}}\|_{\xi e^{-\delta}}$. Such estimates can be combined with Lemmas 3′, 2′ and lead to Proposition 4, ii), iv) (in the same way as (5.16), (5.18) combined with Lemmas 3, 2 yielded Proposition 4, i), iii).

Lemma 7. *Let* $0 < 5\delta < \xi < \xi_0$.

i) *Let* $f \in Y^{(a)}$, $a = (u, j)$, *be strongly* ξ*-analytic and* $\|\tilde{f}\|_\xi = 1$. *Suppose that the equation* $\mathcal{L}\Phi = f$ *admits a solution* $\Phi \in Y^{(a)}$, *then for all* $h \in \tilde{H}(\xi - 4\delta)$

$$|(U(h)\hat{\Phi})(x)| \leqq C\delta^{-7/2}, \qquad \forall x \in \mathbb{R}, \tag{G.8}$$

and for all $h \in \tilde{H}(\xi - 5\delta)$,

$$|(U(h)\hat{\Phi})(x)| \leqq C\delta^{-7/2}(1+\delta|x|)^{-1-\operatorname{Re} u}, \qquad \forall x \in \mathbb{R}. \tag{G.9}$$

ii) *Let* $f \in Y^{(a)}$, $a = (n, j, \pm)$, *and suppose that* $\mathcal{L}\Phi = f$ *has a solution* $\Phi \in Y^{(a)}$, *then for all* $h \in H(\xi - 4\delta)$, *assuming* $\|\tilde{f}\|_\xi = 1$:

$$|U(h)\Phi(z)| \leqq \sqrt{\frac{n}{4\pi}}\, C(\operatorname{Im} z)^{-(n+1)/2}\delta^{-2}, \qquad \forall z \in \mathbb{C}_+, \tag{G.10}$$

and for all $h\in\tilde{H}(\xi-5\delta)$

$$|U(h)\hat{\Phi}(z)|\leqq\sqrt{\frac{n}{4\pi}}\,C(\operatorname{Im}z)^{-(n-1)/2}\delta^{-3}(1+\delta^2|z|^2)^{-1} \tag{G.11}$$

for all $z\in\mathbb{C}_+$.

Remark. As in the proof of Lemma 6, Eqs. (G.9), (G.11) are consequence of (G.8), (G.10), up to a redefinition of the constants. So it suffices to prove (G.8), (G.10).

In the proof we shall set $\tilde{H}(\xi)\equiv\tilde{H}_\xi$ for simplicity of notation, and we shall examine separately the cases $a=(u,j)$, $u=is$ or $a=(u,j)$, $u=s$ or $a=(n,j,\pm)$. Each of these cases will be further subdivided in several subcases. After the analysis of the first few cases the philosophy of the proof should become clear.

Lemma 7 will be an easy consequence of the following lemmas.

Lemma 8.

i) *Let* f *be a strongly* ξ *analytic function such that* $f\in Y^{(u,j)}$, $u\in\mathbb{R}$ *or* $u\in i\mathbb{R}$ *and satisfies condition* (5.6). *Then the equation* $\mathscr{L}\Phi=f$ *has a unique solution* Φ *such that if* $0<4\delta<\xi<0.1$, *then for all* $x\in\mathbb{R}$, *and all* $h\in\tilde{H}_{\xi-4\delta}$, $U(h)\hat{\Phi}(x)$ *is well defined, holomorphic in* h, *and satisfies* $|U(h)\hat{\Phi}(x)|<c'\delta^{-7/2}\|\tilde{f}\|_\xi$.

ii) *Let* f *be a strongly* ξ *analytic function such that* $f\in Y^{(n,j,\pm)}$ *and satisfies condition* (5.6). *Then the equation* $\mathscr{L}\Phi=f$ *has a unique solution* Φ *such that if* $0<4\delta<\xi<0.1$, *then for all* $z\in\mathbb{C}_+$, *and all* $h\in\tilde{H}_{\xi-4\delta}$, $U(h)\hat{\Phi}(z)$ *is well defined, holomorphic in* h, *and satisfies*

$$|U(h)\hat{\Phi}(z)|\leqq\frac{c'\delta^{-2}}{(\operatorname{Im}z)^{(n+1)/2}}\|\tilde{f}\|_\xi\sqrt{\frac{n}{4\pi}}.$$

Lemma 9. i) *Assume* f *is strongly* ξ *analytic, and* $f\in Y^{(u,j)}$. *Assume also that*

$$|U(h)\hat{f}(x)|\leqq 1,\qquad\forall h\in\tilde{H}_\xi,\qquad\forall x\in\mathbb{R}.$$

Then

$$|U(h)\hat{f}(x)|\leqq\frac{5}{2}\frac{1}{(1+\delta|x|)^{1+\operatorname{Re}u}}$$

$$\forall h\in\tilde{H}_{\xi-\delta}\quad\text{and}\quad\forall x\in\mathbb{R}.$$

ii) *Assume* f *is strongly* ξ *analytic, and* $f\in Y^{(n,j,\pm)}$. *Assume also that*

$$|U(h)f(\lambda)|\leqq\frac{1}{(\operatorname{Im}\lambda)^{(n+1)/2}},\qquad\forall h\in\tilde{H}_\xi,\qquad\forall\lambda\in\mathbb{C}_+.$$

Then

$$|U(h)f(\lambda)|\leqq\frac{8\delta^{-1}}{(\operatorname{Im}\lambda)^{(n-1)/2}[(1+\delta\operatorname{Im}\lambda)^2+\delta^2(\operatorname{Re}\lambda)^2]},$$

$$\forall h\in\tilde{H}_{\xi-\delta},\qquad\forall\lambda\in\mathbb{C}_+.$$

Note that the proof of Lemma 9, which we give below for completeness, is essentially, a repetition of the proof of Lemma 6 above.

Proof. Proof of i). Let $g_\kappa = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$, and $h \in \tilde{H}_{\xi-\delta}$. It is easy to verify that if $|\kappa| \leqq \frac{\delta(1-\xi)}{1+\xi^2}$, then $g_\kappa h \in \tilde{H}_\xi$. Therefore for κ real

$$\begin{aligned}|U(h)\hat{f}(x)| &= |U(g_\kappa^{-1})U(g_\kappa h)\hat{f}(x)| \\ &= |U(g_\kappa h)\hat{f}(xg_\kappa^{-1})|\,|j(x, g_\kappa^{-1})|^{1+\mathrm{Re}u}.\end{aligned}$$

We choose $\kappa = -(\operatorname{sign} x)\delta(1-\xi)(1+\xi^2)^{-1}$, and we obtain

$$|U(h)\hat{f}(x)| \leqq |j(x, g_\kappa^{-1})|^{-1-\mathrm{Re}u} = \left(1 + \delta\frac{1-\xi}{1+\xi^2}|x|\right)^{-1-\mathrm{Re}u}.$$

We now observe that $\frac{1-\xi}{1+\xi^2} > \frac{4}{5}$ and the result follows. Proof of ii). Let

$$g^{-1} = \frac{1}{\sqrt{1-\delta^2/4}}\begin{pmatrix} 1 & -i\delta/2 \\ i\delta/2 & 1 \end{pmatrix},$$

it is easy to verify that if $h \in \tilde{H}_{\xi-\delta}$, then $gh \in \tilde{H}_\xi$. We have $U(h)\hat{f}(\lambda) = U(g^{-1})$ $U(gh)\hat{f}(\lambda) = j(\lambda, g^{-1})^{-(n+1)}U(gh)\hat{f}(\lambda g^{-1})$. Therefore

$$|U(h)\hat{f}(\lambda)| \leqq |\mathrm{Im}(\lambda g^{-1})|^{-k/2}|j(\lambda, g^{-1})|^{-k}, \qquad k = n+1,$$

since g^{-1} maps $\mathbb{C}_+$ into $\mathbb{C}_+ + i\delta/2$. If $\lambda = x + iy$, we obtain

$$|U(h)\hat{f}(\lambda)| \leqq [\delta 2^{-1}[(1+\delta y/2)^2 + \delta^2 x^2/4]]^{-k/2}.$$

Multiplying the bound of the hypothesis raised to the power $1 - \frac{2}{k}$, and the above bound raised to the power $2/k$, we obtain the result. Q.E.D.

It is obvious that Lemma 7 follows from Lemmas 8 and 9.

It is easy to derive from formulas (5.4), (5.5), and (5.6) that if $f \in Y^{(a)}$, $U(h)\hat{\Phi}(z)$ is given by

$$\begin{aligned}U(h_0)\Phi(z_0) &= -P(z_0, h_0)^{-k/2}\int_0^{z_0h_0} \zeta^{k/2-1}f(\zeta)\,d\zeta \\ &= P(z_0, h_0)^{-k/2}\int_{z_0h_0}^{+\infty} \zeta^{k/2-1}f(\zeta)\,d\zeta,\end{aligned} \tag{G.12}$$

where h_0 is real,

$$P(z_0, h_0) = (a_0z_0 + c_0)(b_0z_0 + d_0) \quad \text{if} \quad h_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix},$$

$z_0 \in \mathbb{R}$, $k = u+1$ if $a = (u, j)$, and $z_0 \in \mathbb{C}_+$, $k = n+1$ if $a = (n, j, \pm)$. This formula together with the analyticity of $\hat{f}$ allows $U(h)\hat{\Phi}$ to be analytically continued in h in a small complex neighborhood of the identity. In the case of the principal and supplementary series, the non-uniformity of the function $\omega \to \omega^{k/2}$ requires some precautions. Along the paths to be used, ζ will always be of the form $\zeta = zh$, where $z \in \mathbb{C}_+$ (in the case of the principal and supplementary series $z \in \mathbb{R}$), and $h \in \tilde{H}_\xi$, so that the integrand becomes $P(z,h)^{k/2}U(h)\hat{f}(z)\zeta^{-1}$. The rule is that, at the start of the path, $P(z,h)=P(z_0,h_0)$ and the determination of $P(z,h)^{k/2}$ to be chosen is $P(z_0,h_0)^{k/2}$. Along the path it is then determined by analytic continuation. We adopt the following terminology: let $\Theta=\mathbb{C}_+$ in the case of the discrete series, $\Theta=\mathbb{R}$ in the case of the principal or supplementary series. We call "path of type P_0 (respectively P_∞)" a continuous path in the complex plane, starting from some $\zeta_0=z_0h_0$, $z_0\in\Theta$, $h_0\in\tilde{H}_{\xi-\delta}$, ending at 0 (respectively ∞), and such that

a) Each ζ on the path is of the form zh, $z\in\Theta$, $h=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\tilde{H}_{\xi-\delta}$, varying continuously along the path, with $P(z,h)=P(z_0,h_0)$ at the start of the path.

b) $az+c$ never vanishes along the path except, if the path is of type P_0, at the end $\zeta=0$. $(bz+d)$ never vanishes along the path except, if the path is of type P_∞ at the end $\zeta=\infty$.

c) There is a continuous arc $\{\zeta_u\}$ joining ζ_0 to some $\zeta_1=z_1h_1$, $z_1\in\Theta$, h_1 real in $\tilde{H}_{\xi-\delta}$ and a corresponding continuous family of paths starting at ζ_u (and ending always at 0 or always at ∞) satisfying the conditions a) and b) and ending with a real path. (This condition will be non-trivial only in the case of the principal and supplementary series.)

Along any such paths, there is an analytic continuation of f. If the corresponding integrals along all paths of the family [of condition c)] are absolutely convergent, they define an analytic continuation of $\hat{\Phi}$ along the arc $\{\zeta_u\}$ and, consequently, an analytic continuation of $U(h)\hat{\Phi}(z)$ to the point (h_0,z_0). We denote

$$h_0=\begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix}=\begin{pmatrix}1 & \mu_0\\ \sigma_0 & 1+\sigma_0\mu_0\end{pmatrix}\begin{pmatrix}a_0 & 0\\ 0 & a_0^{-1}\end{pmatrix}$$

with $\mu_0=a_0b_0$, $\sigma_0=c_0a_0^{-1}$. In proving the bound for $U(h_0)\hat{\Phi}(z_0)$, it is enough to prove the bound for $\sigma_0=i\eta_0$, $\eta_0\in\mathbb{R}$. Indeed, let h_1 be given by

$$h_1=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}h_0, \qquad t \text{ real, then } |U(h_1)\hat{\Phi}(z)|=|U(h_0)\hat{\Phi}(z+t)|,$$

and we have obviously $\mathrm{Im}(z+t)=\mathrm{Im}(z)$. In order to prove 8, i) for the principal series, we shall first prove the following result.

Lemma 10. *Let $f\in Y^{(is,j)}$ be strongly ξ analytic and satisfy* (5.6). *Then, for all $x_0\in\mathbb{R}$, $U(h_0)\hat{\Phi}(x_0)$ is holomorphic in h_0 in $\tilde{H}_{\xi-3\delta}$ and if $|\mu_0|=|a_0b_0|\geqq\delta/4$, satisfies*

$$|U(h_0)\hat{\Phi}(x_0)|\leqq c(\xi)\delta^{-7/2}\|\tilde{f}\|_\xi.$$

Let us first derive 8i), for the principal series, from Lemma 10. Assume h_0 belongs to $\tilde{H}_{\xi-4\delta}$, $|b_0|<\xi-4\delta$, $|c_0|<\xi-4\delta$. If $|\mu_0|\geqq\delta/4$, the assertion of Lemma 8i), is explicitly in Lemma 10. If $|\mu_0|<\delta/4$, let

$$h_\nu=\begin{pmatrix}1 & \mu_0+\nu\\ \sigma_0 & 1+(\mu_0+\nu)\sigma_0\end{pmatrix}\begin{pmatrix}a_0 & 0\\ 0 & a_0^{-1}\end{pmatrix}.$$

If $|\nu|\leqq\delta/2$ then $h_\nu\in\tilde{H}_{\xi-3\delta}$. By Lemma 10, $U(h_\nu)\hat{\Phi}(x_0)$ is holomorphic in ν in a neighborhood of the disk $|\nu|\leqq\delta/2$ and by the maximum principle, it satisfies

$|U(h_\nu)\hat{\Phi}(x_0)| < c(\xi)\,\|\tilde{f}\|_\xi \delta^{-7/2}$, since the edge of the disk satisfies $|\mu_0+\nu| \geqq |\nu| - |\mu_0| > \frac{\delta}{4}$.

Proof of Lemma 10. Note that it follows from our assumptions that $U(h)\hat{f}(x)$ is simultaneously analytic in h and x. Given $\zeta_0 = x_0 h_0$, any path of the type P_0 or P_∞ originating at ζ_0 such that the corresponding integrals are absolutely convergent, defines an analytic continuation of $U(h)\hat{\Phi}(x)$ to the point (h_0, x_0). To settle the possibility of non-uniformity of such continuations, we shall proceed as follows. The domain of analyticity asserted by Lemma 10 will be divided into several open sets. In each of these we first prove the analyticity, then find integration paths that yield the required bounds. The integrals in (G.12) can be rewritten

$$U(h_0)\hat{\Phi}(x_0) = \int_{\zeta_0}^{(0 \text{ or } \infty)} \left[\frac{P(x,h)}{P(x_0,h_0)}\right]^{k/2} U(h)\hat{f}(x)\frac{d\zeta}{\zeta}, \tag{G.13}$$

where $\zeta = xh$ along the chosen path and $k = 1 + is$. The quantity $[P(x,h)/P(x_0,h_0)]^{k/2}$ will be defined by analytic continuation along the path starting with the value 1. We denote $x+\sigma = re^{i\phi}$, $x_0+\sigma_0 = r_0 e^{i\phi_0}$, $\mu = \varrho e^{i\theta}$, $\mu_0 = \varrho_0 e^{i\theta_0}$, $\theta_0 + \phi_0 = \alpha$, $\theta + \phi = \beta$, $1 + \mu(x+\sigma) = me^{i\psi}$, $1 + \mu_0(x_0+\sigma_0) = m_0 e^{i\psi_0}$. The condition $h_0 \in \tilde{H}_{\xi - 3\delta}$ implies

$$|a_0 - 1| < \xi - 3\delta,\ |a_0|\,|\sigma_0| < \xi - 3\delta,\ |a_0|^{-1}\varrho_0 < \xi - 3\delta,$$
$$\varrho_0 < (\xi - 3\delta)(1+\xi)\,(<2\xi).$$

An easy consequence of Lemma 9 is that for $h \in \tilde{H}_{\xi-\delta}$, and $x \in \mathbb{R}$, we have

$$|U(h)\hat{f}(x)| \leqq \sqrt{2}\,\delta^{-1}, \tag{G.14}$$

$$|U(h)\hat{f}(x)| \leqq \frac{5\delta^{-1}}{\sqrt{2}}(1+\delta|x|)^{-1} \tag{G.15}$$

(we have assumed $\|\tilde{f}\|_\xi = 1$).

I) *Case* $|a_0| r_0 < \xi - \delta$

a) *Analyticity.* We use a path given by $\zeta = xh$, $x = tx_0$, $\mu = \mu_0$, $\sigma = t\sigma_0$ with $0 \leqq t \leqq 1$. This gives

$$U(h_0)\phi(x_0) = \int_0^1 \left[\frac{t(1+t\mu_0(x_0+\sigma_0))}{1+\mu_0(x_0+\sigma_0)}\right]^{k/2} \frac{dt}{t[1+t\mu_0(x_0+\sigma_0)]}\, U(h)\hat{f}(x).$$

This equation holds for real h_0 and extends analytically in $\{h_0 \in \tilde{H}_{\xi-\delta}, |a_0|\,|x_0+\sigma_0| < \xi - \delta\}$. If $\alpha = 0$ or π, $P(x,h)/P(x_0,h_0)$ is positive and

$$|U(h_0)\hat{\Phi}(x_0)| \leqq \int_0^1 \frac{\sqrt{2}}{\delta}[(1 - r_0\varrho_0)t(1 - t\varrho_0 r_0)]^{-1/2}\,dt$$
$$\leqq \frac{2\sqrt{2}}{\delta(1-2\xi^2)}.$$

b) *Bound.* Assume $\alpha \neq 0, \pi$. A first path is given by $\varrho = \varrho_0$, $r = r_0$, $x = r_0\cos\phi$, $\sigma = ir_0\sin\phi$, $\beta = \theta + \phi$ increases to π if $\alpha \in (0,\pi)$, or decreases to $-\pi$ if $\alpha \in (-\pi, 0)$

while ϕ varies so that the argument of $P(x, h)$ remains constant. This path produces a contribution bounded by $\frac{2\pi\sqrt{2}}{\delta(1-2\xi^2)}$. At the end of it we are in the case $\alpha=\pi$, and we can apply the previous estimate since the final value of $|[P(r_0\cos\phi, h)]^{k/2}|$ is inferior to its initial value. We finally obtain

$$|U(h_0)\hat{\Phi}(x_0)| \leqq \frac{2(1+\pi)\sqrt{2}}{\delta(1-2\xi^2)}.$$

II. *Case* $|a_0| r_0 > \xi - 3\delta$

This implies $|r_0\cos\phi_0| > 0$. We treat the case $r_0\cos\phi_0 > 0$, the case $r_0\cos\phi_0 < 0$ being similar. If $1+\mu_0(x_0+\sigma_0) \neq 0$ we define a path of type P_∞ parametrized by $t \in [1, \infty]$ by $\sigma = i\eta_0$, $x = \sqrt{t^2 r_0^2 - \eta_0^2}$ (hence $r = t r_0$), $\varrho = \frac{\varrho_0}{t}$, $\theta = \alpha - \phi$. Defining ϕ, $\phi_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ this implies that ϕ varies monotonically from ϕ_0 to 0, $\zeta = t e^{i(\phi-\phi_0)}\zeta_0$ and $P(x, h) = tP(x_0, h_0)\, e^{i(\phi-\phi_0)}$. We obtain

$$U(h_0)\hat{\Phi}(x_0) = \int_1^\infty [t e^{i(\phi-\phi_0)}]^{(1+is)/2} \frac{r_0 dt\, e^{-i\phi_0}}{t e^{i(\phi-\phi_0)}\sqrt{r_0^2 - \eta_0^2/t^2}} U(h)\hat{f}(x). \tag{G.16}$$

Using (G.15), we obtain

$$|U(h_0)\hat{\Phi}(x_0)| \leqq \int_1^\infty e^{-s(\phi-\phi_0)/2} \frac{5dt}{\sqrt{2}\delta\cos\phi_0\sqrt{t}(1+\delta r_0\cos\phi_0 t)}. \tag{G.17}$$

For fixed x_0, a_0, and σ_0, the left-hand side of (G.16) defines a function $I(h_0, x_0)$ of μ_0 holomorphic in the whole disk $|a_0|\,|\mu_0| < \xi - 3\delta$ including the points, if any, where $1+\mu_0(x_0+\sigma_0)=0$, and C^∞ in all variables. When σ_0 and μ_0 vary at fixed x_0, a_0, avoiding points where $1+\mu_0(x_0+\sigma_0)=0$, I defines a holomorphic function. Hence, by standard properties of analytic functions of several complex variables, I is holomorphic in h_0 in the required domain, coincides with $U(h_0)\hat{\Phi}(x_0)$ at real h_0 and is also analytic in x_0. If $\phi_0 < 0$, $\phi - \phi_0$ increases along the path and we deduce

$$|I(h_0, x_0)| \leqq 5\pi/(2\delta^{3/2}\cos\phi_0\sqrt{r_0\cos\phi_0}).$$

We can assume $|x_0| > \frac{\delta}{4}$, since $|x_0| < \frac{\delta}{2}$ implies $|a_0|\,|x_0 + i\sigma_0| < \xi - \delta$, which falls in case I.

Subcase II.1. $x_0\eta_0 \leqq 0$. From (G.17) we derive, ($\sigma_0 = i\eta_0$)

$$|U(h_0)\hat{\Phi}(x_0))| < \frac{5\pi r_0}{\sqrt{2}(\delta x_0)^{3/2}} < \frac{5\pi(x_0+\xi)}{\sqrt{2}(\delta x_0)^{3/2}} < \frac{5\sqrt{2}\pi\xi}{\delta^3}.$$

Subcase II.2. $r_0\varrho_0 \geqq 1$. We use a path of type P_∞ parametrized by $t \in [0, 1]$ with $x = x_0$, $\sigma = \sigma_0$, $m = t m_0$, $\psi = \psi_0$. Along the path, μ follows a straight segment contained in $|\mu| < \varrho_0$ so that h remains in $H_{\xi-\delta}$. The integral to be computed is

$$\int_0^1 \frac{dt}{t} t^{(1+is)/2} U(h) f(x_0).$$

This is holomorphic in h_0 in the indicated domain, coincides with $U(h_0)\hat{\Phi}(x_0)$ for real h_0 and is bounded by $2\sqrt{2}\delta^{-1}$. It is also analytic in x_0. To connect the point (h_0, x_0) by a continuous arc $\{(h_u, x_u)\}$ satisfying the same conditions ($h_u \in \tilde{H}_{\xi-3\delta}$, $r_u\varrho_u \geqq 1$, $|a_u|r_u > \xi - 3\delta$) to a real point, we vary the angles ϕ_0 and θ_0 so that $|\sin\phi_0|$ decreases to 0 and $|\theta_0|$ decreases to 0, while x_u varies so that $\mathrm{Re}\,\sigma$ remains constant.

Subcase II.3. $r_0\varrho_0 \leqq 1$, $m_0 < \delta$ (hence $r_0\varrho_0 > 1-\delta$). In this case, $r_0\cos\phi_0 = x_0 + \mathrm{Re}\,\sigma_0$ has the same sign as x_0, which we take to be positive, the other case being similar. The path of type P_∞ is defined by $\varrho = \varrho_0$, $\phi = \phi_0$, $\sigma = ir\sin\phi_0$, $x = r\cos\phi_0$, $re^{i\beta}$ varies so that ψ remains fixed and m decreases from m_0 to 0. Along the path, r has at most one minimum (r^2 is a convex function of m) and we have $r \leqq \frac{1}{\varrho_0}$. Moreover, $|a_0|\,|\eta| < \frac{|a_0|\,|\eta_0|}{1-\delta} < \xi - \delta$, so that h remains in $\tilde{H}_{\xi-\delta}$. The integral to be computed is bounded by

$$\left|\int_{r_1}^{r_0}\left(\frac{rm}{r_0m_0}\right)^{1/2}\frac{\sqrt{2}}{\delta}\frac{dr}{r}\right| + \left|\int_{r_1}^{1/\varrho_0}\left(\frac{rm}{r_0m_0}\right)^{1/2}\frac{\sqrt{2}}{\delta}\frac{dr}{r}\right| + \left|\int_{0}^{m_0}\left(\frac{rm}{r_0m_0}\right)^{1/2}\frac{\sqrt{2}}{\delta}\frac{dm}{m}\right| < \frac{6\sqrt{2}}{\delta(1-\delta)}.$$

To end the proof of this subcase we construct an arc joining any point satisfying the preceding condition to a real point while keeping the condition fulfilled. For fixed $x_0 > 0$ the set $D_{x_0,a_0} = \{\mu_0, \sigma_0 : r_0\varrho_0 \leqq 1,\ m_0 < \delta,\ |a_0\sigma_0| < \xi - 3\delta,\ |a_0^{-1}\mu_0| < \xi - 3\delta,\ |a_0|r_0 > \xi - 3\delta\}$ does not depend on the argument of a_0. Therefore we start by moving a_0 along $|a_0| = \text{const}$ until it reaches $|a_0|$. We now move σ and μ keeping $|\mu| = |\mu_0|$ and $r = |x_0 + \sigma| = r_0$. The center of $\{\sigma \,|\, |x_0+\sigma| < r_0\}$ is outside of the disk $|\sigma| \leqq |\sigma_0|$, since $x_0 > 2\xi$. Hence there is an arc of this circle passing through σ_0, contained in the disk $|\sigma| \leqq |\sigma_0|$ and intersecting the real axis at σ_1. We let σ follow this arc from σ_0 to σ_1, while the argument of μ is varied so that $\mu(x_0+\sigma)$ remains constant. Along this arc $(\mu, \sigma) \in D_{x_0,a_0}$. Then we let $\mathrm{Im}\,\mu \to 0$, $\mathrm{Re}\,\mu$ and $\sigma = \sigma_1$ being kept fixed.

Subcase II.4. $\sin\phi_0\cos\phi_0 > 0$, $0 \leqq \alpha \leqq \pi$, $\varrho_0 r_0 < 1$, $m_0 > \frac{\delta}{2}$, $|a_0|r_0 > \xi - 3\delta$. We can assume $0 \leqq \phi_0 \leqq \frac{\pi}{2}$, since the other case is similar. We start by a path such that $\sigma = i\eta_0$, $|\mu| = \varrho_0$, $\theta + \phi = \alpha$, are fixed. We have $\psi \in [0, \pi/2]$. The argument of $P(x,h)$ is $\phi + \psi$, and

$$r\frac{d}{dx}(\phi+\psi) = -\frac{\eta_0}{r} + \frac{\varrho_0 x\sin\alpha}{1+\varrho_0^2r^2+2\varrho_0 r\cos\alpha},$$

and $r\frac{d}{dx}(\phi+\psi)$ is an increasing function of x if $1-\varrho_0^2r^2 \geqq 0$. We distinguish two subcases.

Subcase II.4A. $\frac{d}{dx}(\phi+\psi) \geqq 0$ at $x = r_0\cos\phi_0$. We choose a path where x increases from

$$r_0\cos\phi_0 \quad \text{to} \quad x_1 = \sqrt{\frac{1}{\varrho_0^2} - \eta_0^2},$$

so that r increases from r_0 to $\frac{1}{\varrho_0}$. Along this path, $\phi+\psi$ increases. The contribution of this path is bounded by

$$\frac{\sqrt{2}}{\delta}\int_{r_0}^{1/\varrho_0}\left|\frac{r(1+\varrho_0 re^{i\alpha})}{r_0(1+\varrho_0 r_0 e^{i\alpha})}\right|^{1/2}\left|\frac{d\phi}{dr}\right|dr+\frac{\sqrt{2}}{\delta}\int_{r_0}^{1/\varrho_0}\frac{dr}{|r_0(1+\varrho_0 r_0 e^{i\alpha})r(1+\varrho_0 re^{i\alpha})|^{1/2}}.$$

We now assume that $\varrho_0 \geqq \frac{\delta}{4}$. The first integral is bounded by $4\pi\sqrt{2}\delta^{-2}[(1+\xi)/(\xi-3\delta)]^{1/2}$, since $m_0>\delta/2$. If $\cos\alpha>-1/2$, the second integral is bounded by $8\sqrt{2}\delta^{-3/2}[(1+\xi)/(\xi-3\delta)]^{1/2}$. If $\cos\alpha \leqq -1/2$, the second integral is bounded by $32\delta^{-5/2}[(1+\xi)/(\xi-3\delta)]$. At the end of this path we have $x=x_1$, $h=h_1\in\tilde{H}_{\xi-3\delta}$ and

$$|P(x_1,h_1)|\leqq\frac{2}{\varrho_0}\leqq\frac{4(1+\xi)}{\varrho_0\delta(\xi-3\delta)}|P(x_0,h_0)|,$$

while $\operatorname{Arg}P(x_1,h_1)\geqq\operatorname{Arg}P(x_0,h_0)$. Therefore, finishing the integration as in Subcase II.2, we obtain, using $\varrho_0\geqq\frac{\delta}{4}$, the bound $33\delta^{-5/2}(1+\xi)/(\xi-3\delta)$. We now show how to let a group element,

$$h'=\begin{pmatrix}1 & \mu'\\ \sigma' & 1+\mu'\sigma'\end{pmatrix}\begin{pmatrix}a_0 & 0\\ 0 & a_0^{-1}\end{pmatrix},$$

follow a continuous arc from h_0 to a real value while continuing to satisfy all the conditions of this subcase, notably, with x_0 fixed,

$$-\frac{\eta'}{r'}+\frac{\varrho'\sin\alpha' r'\cos\phi'}{(1+\varrho'^2r'^2+2\varrho' r'\cos\alpha')}\geqq 0.$$

The circle $\{\sigma':|x_0+\sigma'|=r_0\}$, passing through σ_0 and $\overline{\sigma_0}$ intersects the real axis at $\sigma_1=r_0-x_0$ inside the disk $\{\sigma':|\sigma'|\leqq|\sigma_0|\}$, since $|\sigma_0|=|(x_0+\sigma_0)-x_0|\geqq|r_0-x_0|$. Since, by assumption $0<r_0\cos\phi_0=x_0+\operatorname{Re}\sigma_0\leqq r_0$, $\sigma_1\geqq\operatorname{Re}\sigma_0$. We let σ' follow this arc of circle from σ_0 to σ_1. Denoting $x_0+\sigma'=r_0e^{i\phi'}$, ϕ' decreases from ϕ_0 to 0, hence $\sin\phi'=\frac{\eta'}{r'}$ decreases from $\sin\phi_0$ to 0 while $r_0\cos\phi'$ increases. At the same time we keep $\varrho'=\varrho_0$ and $\alpha'=\alpha$ by varying the argument of μ'. Then we let α' vary so that $\cos\alpha'$ increases from $\cos\alpha$ to 1 while $\sin\alpha'$ remains $\geqq 0$.

Subcase II.4B. $\frac{d}{dx}(\phi+\psi)\leqq 0$ at $x=r_0\cos\phi_0$. We let x decrease from $r_0\cos\phi_0$ to 0. Along this path $\phi+\psi$ increases as x decreases. Again the integral has two contributions bounded by:

$$\frac{\sqrt{2}}{\delta}\int_{\eta_0}^{r_0}\left|\frac{r(1+\varrho_0 r\, e^{i\alpha})}{r_0(1+\varrho_0 r_0\, e^{i\alpha})}\right|^{1/2}\left|\frac{d\phi}{dr}\right|dr,$$

which is bounded by $4\pi\sqrt{2}\delta^{-3/2}$, and

$$\frac{\sqrt{2}}{\delta}\int_{\eta_0}^{r_0}\frac{dr}{|r_0(1+\varrho_0 r_0 e^{i\alpha})r(1+\varrho_0 r\, e^{i\alpha})|^{1/2}}. \tag{G.18}$$

If $\cos\alpha \geqq 0$, the contribution of (G.18) is bounded by $2\sqrt{2}\delta^{-1}$. If $\cos\alpha<0$, let $r_1=(1-\delta/2)/\varrho_0|\cos\alpha|$. If $r_0<r_1$, the contribution of (G.18) is bounded by $4\sqrt{2}\delta^{-2}$. Suppose now $\cos\alpha<0$, and $r_0>r_1$. We divide the integration range in (G.18) into $[\eta_0, r_1]\cup[r_1, r_0]$. The contribution of the first part is again bounded by $4\sqrt{2}\delta^{-2}$. That of $[r_1, r_0]$ is bounded by $2\sqrt{2}\delta^{-1}(1-\delta/2)^{-1}$. At the end of the path, $\zeta=0h_1$, $h_1\in\tilde{H}_{\xi-3\delta}$, and

$$|[P(0,h_1)]^{(1+is)/2}| \leqq \frac{2}{\sqrt{\delta}}|P(x_0,h_0)^{(1+is)/2}|.$$

We finish the integration as in Case I and we obtain, for Subcase II.4B, a contribution bounded by $5\sqrt{2}\delta^{-2}$. We now have to describe how to let h' follow a continuous arc from h_0 to a real value while continuing to satisfy all the conditions of this subcase $\left(\text{in particular } \frac{d}{dx}(\phi+\psi)\leqq 0\right)$. We first fix $\sigma'=\sigma_0$, hence $r'=r_0$ and $\phi'=\phi_0$. If $\cos\alpha\leqq 0$, we first vary $\varrho' e^{i\alpha'}$ inside the disk $|\varrho' e^{i\alpha'}|\leqq\varrho_0$ so that $\varrho'\sin\alpha'$ remains constant and $\varrho'\cos\alpha'$ increases from $\varrho_0\cos\alpha$ to 0. If $\cos\alpha\geqq 0$, we omit the preceding step and proceed to the next: ϱ' is kept fixed, α' decreases so that $\sin\alpha'$ decreases to 0 and $\cos\alpha'$ increases to 1. All conditions remain satisfied in this process. Finally we let σ' move from σ_0 to σ_1 as in Subcase II.4A, while varying the argument θ' of μ' so that $\alpha'=\phi'+\theta'$ remains equal to 0.

Subcase II.5. $\sin\phi_0\cos\phi_0>0$, $-\pi<\alpha<0$, $\varrho_0 r_0<1$, $|a_0|r_0>\xi-3\delta$, $m_0>\delta/2$. This case will be reduced to the case $\alpha=0$ or π. The path is defined by $x=x_0$, $\sigma=\sigma_0$, $m=m_0$, $\theta=\theta_0$, ψ increases from its initial value $\psi_0\in\left[-\frac{\pi}{2},0\right]$ to 0. This path gives a contribution bounded by $\frac{\pi}{2}\frac{\sqrt{2}}{\delta}$. The remainder of the integral to be computed is treated as in Subcase II.4B. To find an arc from a point h_0 satisfying the above conditions to the reals, it suffices to let h_0 follow the path described above to the end, then to let σ_0 go to $\sigma_1=r_0-x_0$ as in Subcase II.4A while $\alpha=\theta+\phi$ is kept fixed.

With Cases I and II it has been shown that, for fixed x_0, $U(h_0)\hat{\Phi}(x_0)$ has a uniform analytic continuation in each of the three open sets,

$$\Delta_{\mathrm{I}}(x_0)=\{h_0\in\tilde{H}_{\xi-3\delta}: |a_0|r_0<\xi-\delta\},$$
$$\Delta_{\mathrm{II}}^{\pm}(x_0)=\{h_0\in\tilde{H}_{\xi-3\delta}: |a_0|r_0>\xi-3\delta,\ \pm r_0\cos\phi_0<0\},$$

with $\Delta_{\mathrm{II}}^{+}(x_0)\cap\Delta_{\mathrm{II}}^{-}(x_0)=\emptyset$. To prove that these analytic continuations match, we show that if, e.g. $h_0\in\Delta_{\mathrm{II}}^{+}(x_0)\cap\Delta_{\mathrm{I}}(x_0)$ it can be connected to the reals by a continuous arc not leaving this intersection: first we vary a_0 at constant $|a_0|$ until it reaches $|a_0|$. Then we vary continuously the argument θ of $\mu=\varrho_0 e^{i\theta}$, keeping $|\mu|$ constant, until $\theta=0$. Finally we move σ from σ_0 to $\sigma_1=x_0-r_0$ (as explained at the end of Subcase II.4A) keeping $|x_0+\sigma|=r_0$ constant.

Thus $U(h_0)\hat{\Phi}(x_0)$ extends to a holomorphic function of h_0 in $\tilde{H}_{\xi-3\delta}$ and, if $\varrho_0 \geqq \frac{\delta}{4}$, it is easy to see that this bound for Subcase II.4A majorizes the bounds found for all other subcases. It is itself majorized by $\frac{80}{\delta^{7/2}}$. This concludes the proof of Lemma 10.

We now continue the proof of Lemma 8 by looking at the supplementary series. An easy consequence of Lemma 9 is that if $\|\tilde{f}\|_\xi = 1$, and $f \in Y^{(s,j)}$, there is a positive constant B which does not depend on f such that

$$|U(h)f(x)| \leqq \frac{B}{(1+\delta|x|)^{s+1}} \leqq B \tag{G.19}$$

for all $x \in \mathbb{R}$, and $h \in \tilde{H}_{\xi-\delta}$. We shall now define and estimate $U(h)\hat{\Phi}(x)$. We shall use the notations introduced in the proof of Lemma 9. There are two cases.

Case I. $|a_0|r_0 < \xi - \delta$. We use a path $x = tx_0$, $\sigma = t\sigma_0$, $\mu = \mu_0$, $t \in [0,1]$. As in the case of the principal series,

$$U(h_0)\hat{\Phi}(x_0) = \int_0^1 \left[\frac{t(1+t\mu_0(x_0+\sigma_0))}{1+\mu_0(x_0+\sigma_0)}\right]^{\frac{s+1}{2}} \frac{U(h)f(x)}{t[1+t\mu_0(x_0+\sigma_0)]}\,dt\,.$$

Therefore $|U(h_0)\hat{\Phi}(x_0)| \leqq 2B(1-\xi^2)^{-1}(1+2\xi^2)$, since $|1+t\mu_0(x_0+\sigma_0)| > 1-\xi^2$, and using (G.19).

Case II. $|a_0|r_0 > \xi - 3\delta$. As in the case of the principal series, we note that this implies $|r_0\cos\phi_0| > 0$ and we consider the case $\cos\phi_0 > 0$ (the other case being similar). We use the same contour as in the case of the principal series. We have

$$U(h_0)\hat{\Phi}(x_0) = \int_1^{+\infty} [te^{i(\phi-\phi_0)}]^{\frac{1+s}{2}} \frac{r_0 e^{-i\phi_0}\,dt}{te^{i(\phi-\phi_0)}\sqrt{r_0^2-\eta_0^2/t^2}}\, U(h)\hat{f}(x)\,,$$

hence using (G.19),

$$\begin{aligned} |U(h_0)\hat{\Phi}(x_0)| &\leqq B\int_1^\infty \frac{r_0}{x_0}\,\frac{t^{(s-1)/2}\,dt}{[1+t\delta x_0]^{s+1}} \\ &\leqq \pi B r_0 x_0^{-1}(\delta x_0)^{-(1+s)/2}\,. \end{aligned}$$

As explained in the case of the principal series, (see Case II), it suffices to consider the case $|x_0| > \frac{\delta}{4}$, and using $r_0 \leqq x_0 + \xi$, we obtain

$$|U(h)\hat{\Phi}(x_0)| \leqq 16\pi B(1+\xi)\delta^{-3}\,.$$

This ends the proof of part i) in Lemma 8.

We now show part ii). From Lemma 9 it follows that if $f \in Y^{(n,j,\pm)}$, and if $\|\tilde{f}\|_\xi = \sqrt{\frac{4\pi}{n}}$, we have

$$|U(h)\hat{f}(\lambda)| \leqq (\operatorname{Im}\lambda)^{-k/2}\,, \tag{G.20}$$

and

$$|U(h)\hat{f}(\lambda)| \leqq 8[\delta(\operatorname{Im}\lambda)^{k/2-1}[(1+\delta\operatorname{Im}\lambda)^2+(\delta\operatorname{Re}\lambda)^2]]^{-1} \tag{G.21}$$

for any $\lambda \in \mathbb{C}_+$, and $h \in \tilde{H}_{\xi-\delta}$.

We adopt the following notations.

$$h_0 = \begin{pmatrix} 1 & \mu_0 \\ i\eta_0 & 1+i\mu_0\eta_0 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$$

(it is sufficient to consider the case $\eta_0 \in \mathbb{R}$),

$$\lambda_0 + i\eta_0 = r_0 e^{i\phi_0}, \qquad \mu_0 = \varrho_0 e^{i\theta_0}, \qquad \phi_0 + \theta_0 = \alpha.$$

The integration variable will always be of the form λh, $\lambda \in \mathbb{C}_+$, and h of the form

$$h = \begin{pmatrix} 1 & \mu \\ i\eta & 1+i\mu\eta \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix},$$

and we denote

$$\lambda + i\eta = re^{i\phi}, \; \mu = \varrho e^{i\theta}.$$

There are two cases.

I. Case $\eta_0 \leqq 0$. Our first path is defined by $\eta = \eta_0$, $r = r_0$, $\varrho = \varrho_0$, ϕ varies in such a way that $\sin\phi$ increases from $\sin\phi_0$ to 1, and θ is varied so that $\theta + \phi = \alpha$ remains constant. Along this path, the imaginary part of λ increases. This path gives a contribution bounded by $\frac{\pi}{2}(\operatorname{Im}\lambda_0)^{-k/2}$. At the end of the path we have $\phi = \frac{\pi}{2}$, and let h_1 and $\lambda_1 = i(r_0 - \eta_0)$ be the values assumed by h and λ. Notice that $|P(\lambda_1, h_1)| = |P(\lambda_0, h_0)|$. We now have to treat the case $\phi_0 = \pi/2$, $\lambda_0 = i(r_0 - \eta_0) = iy_0$. We define a path by $r = r_0$, $\varrho = \varrho_0$, $\phi = \phi_0 = \pi/2$, and θ varies in such a way that $\sin\theta$ increases from $\sin\theta_0$ to 1. Along this path, $|1 + i\varrho_0 r_0 e^{i\theta}|^2$ decreases, and the contribution is bounded by $\varrho_0 r_0 \Delta\theta y_0^{-k/2} |1 + i\varrho_0 r_0 e^{i\theta_0}|^{-1}$, where $\Delta\theta = \left|\frac{\pi}{2} - \theta_0\right|$ or $\left|\theta_0 + 3\frac{\pi}{2}\right|$. Since $\Delta\theta \leqq \pi|i - e^{i\theta_0}| = \pi|1 + ie^{i\theta_0}|$, and

$$\varrho_0 r_0 |1 + ie^{i\theta_0}|\,|1 + i\varrho_0 r_0 e^{i\theta_0}|^{-1} \leqq 16,$$

the contribution of this path is bounded by $16\pi y_0^{-k/2}$. Along this path, $|P(\lambda, h)|$ has decreased, while $\operatorname{Im}\lambda$ remained constant. Therefore it is now enough to treat the special case $\lambda_0 = iy_0 = i(v_0 - \eta_0)$, $v_0 > 0$, $\mu_0 = i\varrho_0$, $\varrho_0 > 0$. We use a path where $h = h_0$ is fixed and $\lambda = iy$ varies. Let $v = y + \eta_0$, $v \geqq 0$, we have to estimate

$$U(h_0)\hat{\Phi}(iy_0) = \int_{v_0}^{0 \text{ or } 1/\varrho_0} [v(1-\varrho_0 v)]^{k/2-1}[v_0(1-\varrho_0 v_0)]^{-k/2} dv\, U(h) f(i(v-\eta_0)).$$

Using (G.21) this is bounded in modulus by

$$\int_{v_0}^{0 \text{ or } 1/\varrho_0} 8\delta^{-1}[v_0(1-\varrho_0 v_0)]^{-k/2} |v(1-\varrho_0 v)/(v-\eta_0)|^{k/2-1} |1+\delta(v-\eta_0)|^{-2} dv.$$

Let γ be defined by

$$\gamma = \eta_0 + \sqrt{\eta_0^2 - \eta_0\varrho_0^{-1}}.$$

There are three cases.

a) *If* $0 \leqq v_0 \leqq \gamma$, we integrate from 0 to v_0. The function $v(1-\varrho_0 v)/(v-\eta_0)$ is positive and increasing, and using $\gamma \leqq \frac{1}{2\varrho_0}$, we have $1-\varrho_0 v_0 \geqq 1/2$; we obtain a contribution bounded by $16\delta^{-2} y_0^{-k/2}$.

b) $\gamma \leqq v_0 \leqq \varrho_0^{-1}$. In this case we integrate from v_0 to $\frac{1}{\varrho_0}$. $v(1-\varrho_0 v)(v-\eta_0)$ is positive and decreasing. The contribution is therefore bounded by

$$8\delta^{-1} v_0^{-1} |1-\varrho_0 v_0|^{-1} |v_0-\eta_0|^{1-k/2} \int_{v_0}^{1/\varrho_0} \frac{dv}{[1+\delta(v-\eta_0)]^2},$$

which is less than $32\delta^{-2} y_0^{-k/2}$, since $\gamma > -\eta_0$.

c) *If* $v_0 > \varrho_0^{-1}$, we integrate from ϱ_0^{-1} to v_0. Using the bound $|U(h)\hat{f}(i(v-\eta))| \leqq (v-\eta_0)^{-k/2}$, the contribution of this path is bounded by $2y_0^{-k/2}$.

II. Case $\eta_0 > 0$. In this case we have

$$(U(h_0)\hat{\Phi})(\lambda_0) = (U(h_1)\hat{\Phi})(\lambda_0 + i\eta_0),$$

where

$$h_1 = \begin{pmatrix} 1 & \mu_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \in \tilde{H}_{\xi-\delta}.$$

Therefore applying Case 1, we obtain

$$|U(h_0)\hat{\Phi}(\lambda_0)| \leqq 33\delta^{-2}(y_0+\eta_0)^{-k/2} \leqq 33\delta^{-2} y_0^{-k/2}.$$

This completes the proof of Lemma 8.

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References

1. Arnold, V., Avez, A.: Ergodic problems of classical mechanics. New York: Benjamin 1967
2. Guillemin, V., Kazhdan, I.: On the cohomology of certain dynamical systems. Topology **19**, 301 (1980) and **19**, 291 (1980)
3. Livscic, A.: Some homology properties of U-systems. Mat. Zametki **10**, 555 (1971)
4. Gelfand, I., Fomin, S.: Geodesic flows on manifolds of constant negative curvature. Am. Math. Soc. Transl., Ser. 2, Vol. 1, 1965, p. 49
5. Gelfand, I., Graev, M., Vilenkin, N.: Generalized functions, Vol. 5. New York: Academic Press 1966
6. Gelfand, I., Graev, M., Piateckii-Shapiro, I.: Representation theory and automorphic functions. Philadelphia: Saunders 1969
7. Poincaré, H.: Théorie des groupes Fuchsiens. Acta Math. **1**, 1 (1882)
   Ford, L.: Automorphic functions. New York: Chelsea 1951
8. Arnold, V.: Proof of a theorem of A.N. Kolmogorov on the preservation of conditionally-periodic motions under a small perturbation of the hamiltonian. Russ. Math. Surv. **18**, 9 (1963)
   Poshel, J.: Über differenzierbare Faserung invarianter Tori. Preprint, ETH-Zürich, 1981
   Chierchia, L., Gallavotti, G.: Smooth prime integrals for quasi-integrable Hamilton systems. Il Nuovo Cimento **67** B, 277 (1982)

9. McKean, H., Trubowitz, E.: The spectral class of the quantum-mechanical harmonic oscillator. Commun. Math. Phys. **82**, 471 (1982)
10. Gallavotti, G.: Perturbation theory for classical hamiltonian systems. In: Progress in Physics, PPhI, scaling and self similarity in physics. Fröhlich, J. (ed.). Boston: Birkhäuser 1983
11. Bowen, R.: In: Equilibrium states and the theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470, Berlin, Heidelberg, New York: Springer 1975
12. Rüssmann, H.: Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtlösung. Math. Ann. **169**, 55 (1967); The relationship between Rüssmann's work and the present paper is made clearer if one reads also the paper which one of us (G.G.), A criterion of integrability for perturbed nonresonant harmonic oscillators. "Wick ordering" of the perturbations in classical mechanics and invariance of the frequency spectrum. Commun. Math. Phys. **87**, 365 (1982), wrote after we completed most of this work and where Rüssmann's theorem is derived along the lines of the present paper

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