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Approach to Equilibrium for Locally Expanding Maps in $\mathbb{R}^k$

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Abstract. By using a well known technique from classical statistical mechanics of one-dimensional lattice spin systems we prove existence of an absolutely continuous invariant asymptotic measure for certain locally expanding maps T of the unit cube in $\mathbb{R}^k$. We generalize herewith in a certain sense the results of Lasota and Yorke on piecewise expanding maps of the unit interval to higher dimensions. We show a Kuzmin-type theorem for these systems from which exponential approach to equilibrium and strong mixing properties follow.

Introduction

There has been quite a lot of interest over the last years in a possible prediction of the long time behaviour of inherent stochastic deterministic systems. Many of these systems depend very sensitively on initial conditions and the smallest uncertainties grow in general exponentially fast. This makes any long term prediction of a single orbit practically impossible. From classical statistical mechanics where one handles systems with a huge number of degrees of freedom, one is familiar with another kind of description which is based on their statistical behaviour. It was quite a surprise that also systems with only a very few degrees of freedom should allow for such a description. But computer calculations, which up to now are the main tool for handling such nonlinear systems, showed that one has in general a well defined mean behaviour: time averages of most observables along single trajectories exist and do not depend on the specific trajectory. This suggests that many of these systems support very special measures for which some sort of generalized ergodic theorem is expected to hold. These measures are the so-called asymptotic measures.

From the physical point of view they just correspond to the well known Gibbs measures of classical statistical mechanics which also describe the long time behaviour of Hamiltonian systems. Contrary to the situation there for general dynamical systems, the asymptotic measures are hardly known and even their eixtence is a rather difficult problem, not to mention explicit analytic expressions.

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So it is not a big surprise that only very few classes of such nonlinear dynamical systems can be treated in a satisfactory way: there are the so-called expanding maps of Shub [1] which expand distances uniformly in all directions, which leads to a smoothing out of a given initial distribution. Krzyzewski and Szlenk [2] proved indeed existence of an invariant absolutely continuous asymptotic measure for these systems. Explicit expressions however are not easy to get.

Another class of well understood systems are Smale's Axiom-A systems [3]. In contrast to Shub's expanding maps they are invertible and can therefore serve as models for time reversible systems. They are characterized by the property that they partly expand and partly contract directions in a uniform way, the contraction being responsible for their in general very singular behaviour. Bowen and Ruelle [4, 5] nevertheless have been able to show existence of an asymptotic measure whose support is often a complicated Cantor-like set called a strange attractor. To get explicit expressions for these measures which are not absolutely continuous with respect to ordinary Lebesque measure is more or less hopeless because already the support of the measure, that is the set on which the long time behaviour of the system takes place, is very difficult to describe analytically. We know only of one non-trivial example, namely Smale's solenoid where a complete discussion can be given. This Axiom-A system has also been used to solve another non-Axiom-A system in [6]. In both cases explicit formulas can be derived and interesting predictions for observables like correlation functions can be made.

As is very often the case, the best understood systems are the one-dimensional ones. Lasota and Yorke [7] discussed piecewise expanding maps of the unit interval generalizing Shub's expanding maps in a certain way. They again showed existence of an absolutely continuous invariant measure for such maps for which, among others, Bowen [8] found conditions for being an asymptotic measure. The most complete discussion of these maps has been given by Hofbauer and Keller [9]. They also showed a Kuzmin-type theorem for the weakly mixing among those systems: smooth enough absolutely continuous measures converge exponentially fast in density to this asymptotic measure, which then implies strong multiple mixing properties, even exactness, and a central limit theorem.

There are quite a lot of examples of piecewise expanding maps of the unit interval where even explicit expressions for the asymptotic measure are known. The most famous are certainly the Gauß-measure for the continued fraction transformation [10] and Parry's measure [11] for the β-transformation introduced originally by Renyi [12]. The main reason why this is possible is the fact that for such piecewise expanding maps the density of the asymptotic measure can be determined as the eigenfunction to the highest eigenvalue of a certain positive operator which is named after Perron and Frobenius [9], and which corresponds to the well known Liouville operator in classical Hamiltonian mechanics. The spectrum of this Perron–Frobenius operator determines whether or not an absolutely continuous invariant measure exists and also if it is an asymptotic measure for this system.

One is then lead immediately to the problem to generalize the whole theory to higher dimensions. There exists indeed a first step to such an extension [13] in dimension two, but the method used there seems to meet serious difficulties in still higher dimensions. These are connected with the problem of finding the appropriate function spaces for the Perron–Frobenius operator where its spectrum can be understood in a satisfactory way. In dimension one and two the space of functions of bounded variation is a good choice and the spectrum of the Perron–Frobenius operator turns out to be quite simple there [9]. In higher dimensions this is no longer the case [13].

The problem is therefore to find a good space for this operator. In this paper we have been inspired by a closely related problem in classical statistical mechanics. The Perron–Frobenius operator has some similarity with the so-called transfer operator or its generalization, the Ruelle–Araki operator [14]. Its spectrum is closely related to certain thermodynamic properties of the underlying one dimensional lattice spin system. In [14] it was shown that a special situation arises if the Ruelle–Araki operator can be defined in certain spaces of holomorphic functions. Then Grothendieck's theory of nuclear operators [15], respectively Krasnoselskii's theory of positive operators on real Banach spaces [16], can be applied. In statistical mechanics this happens if the interaction energy decreases exponentially fast in the distance between lattice points. In this paper we consider therefore a certain class of mappings of the unit cube in $\mathbb{R}^k$ which generalize the concept of piecewise expanding maps to any dimension and which allow the Perron–Frobenius operator to be defined again on certain holomorphic function spaces. To achieve this we have to introduce a generalized Ruelle–Araki operator for these systems. Its spectrum will be closely related to that of the original Perron–Frobenius operator. Using the same techniques as in the classical statistical mechanics case as described in [14], we are able to prove existence of an absolutely continuous asymptotic measure for our class of locally expanding maps in $\mathbb{R}^k$. The method also gives a Kuzmin-type theorem from which exponential approach to equilibrium defined by the above mentioned asymptotic measure and strong multiple mixing properties immediately follow.

Many number theoretic dynamical systems like the continued fraction transformation, certain β-transformations, the Jacobi–Perron algorithm and its generalizations [17, 18], certain higher dimensional F-expansions [19, 20], belong to our class of transformations. Our technique therefore provides a unified treatment of all these systems which were discussed up to now by very different methods in the literature. Our results not only reproduce known results but give to a certain extent also slight improvements in all these cases.

A very interesting problem which we will touch only in passing is connected with the rate of convergence of any smooth enough density to the density of the asymptotic measure in Kuzmin's theorem. It follows from our arguments that this rate is given by minus the logarithm of the absolute value of the second highest eigenvalue of the Ruelle–Araki operator. Unfortunately we were not able to relate this number, except for some trivial cases, to other dynamical invariants of the underlying dynamical systems. Perhaps some recent work by Parry and Tuncel [21] is related to this problem. See in this connection also a recent paper by Keller [22].

In detail the paper is organized as follows: In the first chapter we define a class of locally expanding maps of the unit cube in $\mathbb{R}^k$. We give some examples of transformations belonging to this class. In Chap. II we define a generalized Ruelle–Araki operator and discuss its relation to the regular Perron–Frobenius operator for these transformations. In Chap. III spectral properties of this operator in certain Banach spaces of holomorphic mappings are derived by using the theory of compact positive operators as discussed in Krasnoselskii's book [16]. From these our main theorem on the asymptotic measure for such a transformation then follows immediately.

I. A Class of Locally Expanding Maps in $\mathbb{R}^k$

In the following we denote by I^k the k-dimensional unit cube $I^k=\{x\in\mathbb{R}^k: 0\leqq x_i\leqq 1,\ 1\leqq i\leqq k\}$, and by int I^k its open interior. Let T: $I^k\to I^k$ be a not necessarily everywhere defined map of I^k into itself with the following properties:

($A1$) *Locally Expanding*

There exists a countable partition $I^k=\bigcup\limits_{i\in I}\bar{O}_i$ of I^k into pairwise disjoint open sets O_i with meas $(\partial O_i)=0$ such that $T|O_i:O_i\to TO_i$ is bijective. Denote by ψ_i its inverse ψ_i: $TO_i\to O_i$. Then ψ_i is real analytic and can be continued to a holomorphic mapping of some bounded domain $\Omega\supset I^k$ in $\mathbb{C}^k$ such that $\psi_i(\bar{\Omega})\subset\Omega$ (strictly contained in) for all $i\in I$. On some open domain Ω' with $\bar{\Omega}\subset\Omega'$ we have: det $\psi_i'(z)\neq 0$ and $\sum\limits_{i\in I}|\det\psi_i'(z)|<\infty$ uniformly on Ω'.

($A2$) *Markov Partition*

There exists a countable partition $I^k=\bigcup\limits_{\alpha\in J}\bar{W}_\alpha$ into open disjoint sets W_α with mean$(\partial W_\alpha)=0$, such that to any $\alpha\in J$ and any $i\in I$ there either exists a unique $\beta\in J$ with $\psi_i(W_\alpha)\subset W_\beta\cap O_i$ or $\psi_i(W_\alpha)\subset\mathbb{R}^k\backslash I^k$.

($A3$) *Irreducibility*

a) To any pair $\alpha,\beta\in J$ there exist indices $i_1,\dots,i_r\in I$, $\gamma_1,\dots,\gamma_r\in J$ depending on α,β with $\gamma_1=\beta$ such that

$$\psi_{i_l}\circ\cdots\circ\psi_{i_r}(W_\alpha)\subset W_{\gamma_l}\cap O_{i_l}\quad\text{for all } 1\leqq l\leqq r.$$

We call then $(W_\beta,\psi_{i_1}\circ\cdots\circ\psi_{i_r}W_\alpha)$ an allowed chain.
b) If $|J|=\infty$, then there exists an integer $M>0$ such that for any two allowed chains $(W_\beta,\psi_{i_1}\circ\cdots\circ\psi_{i_r}W_\alpha)$ and $(W_{\beta'},\psi_{i_1}\circ\cdots\circ\psi_{i_r}W_{\alpha'})$, we have $W_\beta=W_{\beta'}$ if $r\geqq M$. Furthermore there exists an $j\in I$ with $\psi_j(W_\alpha)\subset O_j$ for all $\alpha\in J$.

($A4$) *Set of Uniqueness*

For any $x\in I^k$ the set $\lim\limits_{N\to\infty}\{\psi_{i_1}\circ\cdots\circ\psi_{i_N}(x): i_1,\dots,i_N\in I$, such that $\psi_{i_l}\circ\cdots\circ\psi_{i_N}(x)\in I^k$ for all $1\leqq l\leqq N\}$ is a set of uniqueness for any holomorphic function on Ω.

Remarks. 1. Condition (A4) is fulfilled for instance if the set $\mathrm{Per}(T)=\{x\in I^k:\exists n: T^n x = x\}$ is dense in I^k. This follows from the fact that any $x_p\in\mathrm{Per}(T)$ is a fixed point of some mapping $\psi_{i_1}\circ\cdots\circ\psi_{i_r}$, $i_l\in I$. The Earle–Hamilton fixed point theorem [23] for holomorphic mappings shows that $\lim_{N\to\infty}(\psi_{i_1}\circ\cdots\circ\psi_{i_r})^N(z)=x_p$ for all $z\in\Omega$. But $\operatorname{int} I^k$ is certainly a set of uniqueness for any holomorphic f on Ω.

2. The locally expanding property is reflected in the contraction property of the inverse mappings ψ_i which map Ω strictly inside itself.

3. We call the partition $I^k=\bigcup_{i\in I}\bar{O}_i$ proper, if $TO_i=\operatorname{int} I^k$ for all $i\in I$. In this case conditions (A2) and (A3) follow already from (A1): The partition $\{O_i\}$ is already a Markov partition. Because $\operatorname{int} I^k = TO_i$ for all $i\in I$, we get $\psi_i(O_l)\subset\psi_i(\operatorname{int} I^k)=\psi_i(TO_i)=O_i$. On the other hand $(O_l,\psi_{i_1}\circ\cdots\circ\psi_{i_r}O_i)$ is an allowed chain if and only if $l=i_1$.

4. If at least one O_i is proper then $\psi_i(W_\alpha)\subset O_i$ for all $\alpha\in J$.

Definition. We call a transformation $T: I_k\to I_k$ fulfilling conditions (A1) to (A4) a locally expanding map of the unit cube I^k.

Examples of Locally Expanding Maps

Example 1. The continued fraction transformation. Here $k=1$ and $T:I^1\to I^1$ is defined as $Tx=1/x-[1/x]$, where [ ] denotes the largest integer $\leqq$ the argument. Take for O_i the open inverval $O_i=\{x\in I^1: 1/(i+1)<x<1/i\}$, and set $\psi_i(x)=1/(i+x)$ for $i=1,2,3,\ldots$. This defines a proper partition and conditions (A1) to (A4) are easily verified [24].

Example 2. Renyi's β-transformation for β integer or a β-number >1. Again $k=1$ and $T:I^1\to I^1$ is defined as $Tx=\beta x \bmod 1=((\beta x))$. If β is integer, then take $O_i=\{x\in I^1: i/\beta<x<(i+1)/\beta\}$ for $0\leqq i\leqq\beta-1$ and $\psi_i(x)=(x+i)/\beta$. Again this leads to a proper partition and things are very easy.

More interesting is the case where β is not an integer. Parry called β in [11] a β-number if the β-expansion of $((\beta))$ is eventually periodic. Recall that any $x\in I^1$ has an expansion $x=\sum_{i=1}^{\infty}a_i\beta^{-i}$ in powers of β^{-1}, with a_i integer from the set $\{0,\ldots,[\beta]\}$ such that $a_i=[\beta T^{i-1}x]$ for all i. If there exist integers p and N such that $a_{i+p}=a_i$ for all $i\geqq N$, then we call x a β-number. If $((\beta))$ is a β-number, then the orbit $\{T^n((\beta))\}_{n\in\mathbb{N}}$ consists of only finitely many points and vice versa. Call these points $\beta_1,\ldots,\beta_M$, and arrange them in increasing order $0<\beta_1<\cdots<\beta_M<1$. They define a partition $I^1=\bigcup_{j=0}^{M}\bar{W}_j$ with $W_j=\{x\in I^1:\beta_j<x<\beta_{j+1}\}$ with $\beta_0=0$ and $\beta_{M+1}=1$.

If we take for O_i and ψ_i, $0\leqq i\leqq[\beta]-1$ the same definitions as for β an integer and define $O_{[\beta]}$ as

$$O_{[\beta]}=\{x\in I^1:[\beta]/\beta<x<1\},$$

respectively $\psi_{[\beta]}(x) = ([\beta] + x)/\beta$, then the partition $I^1 = \bigcup_{i=0}^{[\beta]} \bar{O}_i$ is no longer proper because $TO_{[\beta]} = \{x \in I^1 : 0 < x < ((\beta))\}$ is a proper subset of int I^1. Nevertheless property (A1) can be checked very easily with an appropriate domain Ω.

We have to show that the partition $I^1 = \bigcup_{i=0}^{M} \bar{W}_i$ is a Markov partition. Because all open sets O_i, $0 \leqq i \leqq [\beta] - 1$ are proper, we get for all these i's: $\psi_i(W_k) \subset O_i$ for all $0 \leqq k \leqq M$. For $i = [\beta]$, on the other hand, all sets W_k with $W_k \subset (0, ((\beta)))$ are mapped under $\psi_{[\beta]}$ into $O_{[\beta]}$, whereas all those W_k's with $W_k \subset (((\beta)), 1)$ are mapped outside I^1. To show now that in the case where $\psi_i(W_k) \subset O_i$ there exists indeed a set W_l with $\psi_i(W_k) \subset W_l$, we only have to prove that there is no solution of the equation

$$\psi_i(x) = \beta_m \tag{1}$$

for any $x \in W_k$ and any $\beta_m \in \{T^n((\beta))\}_{n \in \mathbb{N}}$. Equation (1) implies $i + x = \beta \cdot \beta_m$ or $x = T(\beta_m)$. Therefore $x \in \{T^n((\beta))\}_{n \in \mathbb{N}}$ in contradiction to $x \in W_k$. Therefore $\psi_i(W_k) \subset W_l$ for some l.

Property (A3) follows from ergodicity of T[12] and property (A4) from density of $\mathrm{Per}(T)$ in I^1.

Example 3. The Jacobi–Perron algorithm. In this case k is arbitrary and T: $I^k \to I^k$ is defined as follows [17]:

$$Tx = (x_2/x_1 - [x_2/x_1], x_3/x_1 - [x_3/x_1], \ldots, x_k/x_1 - [x_k/x_1], 1/x_1 - [1/x_1]).$$

This transformation is in a certain sense a generalization of the continued fraction transformation to higher dimensions and plays an interesting role in the characterization of irrational algebraic numbers of higher degree [25]. The ergodic properties of the algorithm in the form of the transformation T above have been discussed by Schweiger [17] and Gordin [26]. We show that this and other related transformations [18] are locally expanding in our sense.

Denote by $O_{\mathbf{n}}$ for $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ with $n_k \geqq 1$ and $n_i \leqq n_k$ the open set

$$O_{\mathbf{n}} = \{x \in I^k : 1/(n_k + 1) < x_1 < 1/n_k, \ldots, n_i x_1 < x_{i+1} < (n_i + 1)x_1 \text{ if } n_i < n_k, \ldots,$$
$$n_j x_1 < x_{j+1} < 1 \text{ if } n_j = n_k\}.$$

They define a partition $I^k = \bigcup_{\mathbf{n}} \bar{O}_{\mathbf{n}}$, with all those $O_{\mathbf{n}}$ proper for which $n_i < n_k$ for $1 \leqq i \leqq k - 1$. The mappings $\psi_{\mathbf{n}} : TO_{\mathbf{n}} \to O_{\mathbf{n}}$ are defined as

$$\psi_{\mathbf{n}}(x) = (1/(x_k + n_k), (x_1 + n_1)/(x_k + n_k), \ldots, (x_{k-1} + n_{k-1})/(x_k + n_k))$$

and have nice holomorphic extensions. The explicit analytic form of the domain Ω which is mapped by all these $\psi_{\mathbf{n}}$ strictly inside itself is known to us only for $k = 2$. For $k \geqq 3$ we used a computer to check the required property.

To find the Markov partition we use a result of Perron [27]. He established some kind of symbolic dynamics for this algorithm: call a sequence $(\mathbf{n}_1, \mathbf{n}_2, \ldots)$ of vectors $\mathbf{n}_i \in \mathbb{N}^k$ with the above mentioned properties allowed if there exists a point $x \in I^k$ with $T^{i-1} x \in O_{\mathbf{n}_i}$ for all i. Obviously not every such sequence is allowed because not all $O_{\mathbf{n}}$'s are proper. Perron showed that there is a $1-1$ correspondence between the allowed sequences and almost all points in I^k. The transformation T in I^k then corresponds simply to the shift operator $\tau(\mathbf{n}_1, \mathbf{n}_2, \ldots) = (\mathbf{n}_2, \mathbf{n}_3, \ldots)$. On the other hand there corresponds to the mapping $\psi_{\mathbf{n}}$ the inverse operator $\tau_{\mathbf{n}}(\mathbf{n}_1, \mathbf{n}_2, \ldots) = (\mathbf{n}, \mathbf{n}_1, \ldots)$ as long as the new sequence is again an allowed sequence. In [17] it was shown that it depends only on a finite segment $(\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{r-1})$ whether $(\mathbf{n}, \mathbf{n}_1, \ldots)$ is allowed if $(\mathbf{n}_1, \mathbf{n}_2, \ldots)$ was allowed.

Let therefore $\boldsymbol{\alpha}$ be the multiindex $\boldsymbol{\alpha} = (\mathbf{n}_1, \ldots, \mathbf{n}_r)$ and define open sets $W_{\alpha} \subset I^k$ as

$$W_{\alpha} = \left\{ x \in I^k : x \in \bigcap_{i=1}^{r} T^{-(i-1)} O_{\mathbf{n}_i} \right\}.$$

Then either $\psi_{\mathbf{n}}(W_{\alpha}) \subset W_{\beta} \cap O_{\mathbf{n}}$ if $\boldsymbol{\beta} = (\mathbf{n}, \mathbf{n}_1, \ldots, \mathbf{n}_{r-1})$ is a segment of some allowed sequence or else $\psi_{\mathbf{n}}(W_{\alpha}) \subset \mathbb{R}^k \backslash I^k$.

The first part of property (A3) follows again from ergodicity of T [17] whereas the second part, remember $|J| = \infty$ in this case, follows from the fact that for $l \geqq r$ $(W_{\beta}, \psi_{\mathbf{n}_1} \circ \cdots \circ \psi_{\mathbf{n}_l} W_{\alpha})$ is an allowed chain only if $\boldsymbol{\beta} = (\mathbf{n}_1, \ldots, \mathbf{n}_r)$. Furthermore $\psi_{\mathbf{n}}(W_{\alpha}) \subset O_{\mathbf{n}}$ for all $\boldsymbol{\alpha}$ if $\mathbf{n}$ is such that $O_{\mathbf{n}}$ is proper.

Property (A4) follows again from density of $\mathrm{Per}(T)$ in I^k. Other higher dimensional number theoretic dynamical systems like Schweiger's modified Jacobi–Perron algorithm [18] or Podsypanin's algorithm [18] all belong to our class of locally expanding maps. They even possess proper partitions and are therefore easier to handle than the original algorithm of Jacobi and Perron.

II. The Ruelle–Araki Operator for Locally Expanding Maps

Let us briefly recall the definition and some properties of the Perron–Frobenius operator which can be defined for all reasonable enough transformations $T: I^k \to I^k$ [7]. Denote by $L_1(I^k)$ the Banach space of absolutely Lebesgue integrable functions on I^k and by $L_\infty(I^k)$ its dual. Then the Perron–Frobenius operator is defined as the operator $\mathscr{L}: L_1(I^k) \to L_1(I^k)$ with the property

$$\int_{I^k} f(x) g(Tx) dx = \int_{I^k} (\mathscr{L} f)(x) g(x) dx \tag{2}$$

for all $f \in L_1(I^k)$ and all $g \in L_\infty(I^k)$.

The following important properties follow immediately from this definition:

$$\text{a)} \int_{I^k} \mathscr{L} f(x) dx = \int_{I^k} f(x) dx \text{ for all } f \in L_1(I^k).$$
$$\text{b)} f_0(x) dx \text{ is } T\text{-invariant measure iff } \mathscr{L} f_0 = f_0. \tag{3}$$

It is just property (b) which makes this operator so interesting.

If the transformation $T: I^k \to I^k$ fulfills condition (A1) then there exists a much more explicit expression for $\mathscr{L}: L_1(I^k) \to L_1(I^k)$:

$$\mathscr{L} f(x) = \sum_{i \in I} |\det \psi_i'(x)| f(\psi_i(x)) \chi_{TO_i}(x), \tag{4}$$

where χ_A denotes the characteristic function of the set $A \subset I^k$. If all open sets O_i are proper, then we can replace this function for all i by the constant 1 and the discussion gets much simpler as we shall soon see.

Because of property b) in (3) we have to investigate the spectrum of the above operator. For this we have to choose the appropriate space where the spectrum or at least the relevant part of it for our purposes can be determined. In dimension $k=1$ or $k=2$ it turned out that the space of functions with bounded variation on I^k is a good choice [9]. For $k \geqq 3$ however this is not any more the case [13].

We will now show that it is possible for locally expanding mappings to define a new operator $\tilde{\mathscr{L}}$ on a space of holomorphic mappings whose spectrum is closely related to that of $\mathscr{L}$ and can fairly well be determined.

To motivate this operator $\tilde{\mathscr{L}}$ we stress the similarity of $\mathscr{L}$ as defined in (4) and the so-called transfer operator, respectively its generalization the Ruelle–Araki operator, for one dimensional spin systems with long range interaction [14]. The mathematical model underlying such a spin system is the one sided subshift of finite type Σ_A over the symbols $F = \{\sigma_1, \ldots, \sigma_n\}$ with transition matrix A indexed by F with entries either zero or one and interaction energy $g \in \mathscr{C}(\Sigma_A)$ [28]. The space Σ_A is the set of all configurations $\xi = (\xi_0, \xi_1, \ldots) \in F^{\mathbb{N}}$ such that $A_{\xi_i, \xi_{i+1}} = 1$ for all $i \in \mathbb{N}$. The shift operator $\tau\colon \Sigma_A \to \Sigma_A$ is defined by $(\tau\xi)_i = \xi_{i+1}$. The thermodynamic properties of such a spin system are completely determined by the spectrum of the Ruelle–Araki operator $\tilde{\mathscr{L}}: \mathscr{C}(\Sigma_A) \to \mathscr{C}(\Sigma_A)$ defined as

$$\tilde{\mathscr{L}} f(\xi) = \sum_{\eta \in \tau^{-1}\xi} \exp g(\eta) f(\eta),$$

which can also be written as an operator on the space $\mathscr{C}(c_F, \Sigma_A)$ of all continuous mappings of Σ_A into the space of sequences indexed by F whose elements we denote by $f(\sigma, \xi)$:

$$\tilde{\mathscr{L}} f(\sigma, \xi) = \sum_{i=1}^{n} A_{\sigma_i, \sigma} \exp g(\xi_{\sigma_i}) f(\sigma_i, \xi_{\sigma_i}), \tag{5}$$

with

$$\xi_{\sigma_i} = (\sigma_i, \xi_0, \xi_1, \ldots) \in F^{\mathbb{N}}.$$

To make a closer connection between operator $\mathscr{L}$ in (4) and this operator $\tilde{\mathscr{L}}$ one has to interpret the characteristic functions χ in (4) as some kind of transition matrices A. That this is possible for locally expanding maps we shall see now.

Using property (A2) we define for any $i \in I$ a transition matrix $A^{(i)}$ indexed by the set J (which in general is infinite) as follows:

$$A^{(i)}_{\beta,\alpha} = \begin{cases} 1 & \text{if } \psi_i(W_\alpha) \subset W_\beta \cap O_i, \\ 0 & \text{if } \psi_i(W_\alpha) \subset \mathbb{R}^k \backslash W_\beta. \end{cases} \tag{6}$$

Next define a mapping $\varphi_i: J \to J$ for any $i \in I$:

$$\varphi_i(\alpha) = \begin{cases} \beta & \text{if there exists } \beta \in J \text{ with } A^{(i)}_{\beta,\alpha} = 1, \\ \alpha & \text{otherwise.} \end{cases} \tag{7}$$

Condition (A3) can then be written as follows:

(A3)′: to any pair α, $\beta \in J$ there exist indices $i_1, \dots, i_r \in I$ such that

$$\varphi_{i_1} \circ \cdots \circ \varphi_{i_r}(\alpha) = \beta, \text{ and } A_\alpha^{(i_1,\dots,i_r)} = 1, \text{ with}$$
$$A_\alpha^{(i_1,\dots,i_r)} := A^{(i_1)}_{\varphi_{i_1}\dots\varphi_{i_r}(\alpha),\varphi_{i_2}\dots\varphi_{i_r}(\alpha)}, \dots, A^{(i_r)}_{\varphi_{i_r}(\alpha),\alpha}.$$

Denote by $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$ the Banach space of all continuous mappings $\tilde{f}$ of $\bar{\Omega}_{\mathbb{R}}$ into the space c_J of bounded sequences indexed by J with norm

$$\|\tilde{f}\| = \sup_{\alpha \in J} \sup_{x \in \bar{\Omega}_{\mathbb{R}}} |\tilde{f}(\alpha, x)|,$$

where $\Omega_{\mathbb{R}} = \Omega \cap \mathbb{R}^k$.

On this space we define a generalized Ruelle–Araki operator $\tilde{\mathscr{L}}$ for the transformation T:

$$\tilde{\mathscr{L}} \tilde{f}(\alpha, x) = \sum_{i \in I} A^{(i)}_{\varphi_i(\alpha),\alpha} \varepsilon_i \det \psi_i'(x) f(\varphi_i(\alpha), \psi_i(x)), \tag{8}$$

where $\varepsilon_i = \operatorname{sign} \det \psi_i'(x)$ is constant on $\bar{\Omega}_{\mathbb{R}}$. Property (A1) shows that $\tilde{\mathscr{L}}$ defines a bounded linear operator on the space $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$.

The spectrum of $\tilde{\mathscr{L}}$ is closely related to that of the Perron–Frobenius operator $\mathscr{L}$ in (4).

To make this clear we define a linear operator $\kappa\colon \mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}}) \to L_1(I^k)$ as

$$\kappa \tilde{f}(x) = \tilde{f}(\alpha, x) \text{ if } x \in W_\alpha. \tag{9}$$

Then

$$\int_{I^k} |\kappa \tilde{f}(x)| dx \leqq \sum_{\alpha \in J} \int_{W_\alpha} |\kappa \tilde{f}(x)| dx \leqq \|\tilde{f}\|$$

and κ is a bounded operator with norm $\leqq 1$. Using this we get

Lemma 1. *For* $\tilde{f} \in \mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$, *we have*

$$\kappa(\tilde{\mathscr{L}} \tilde{f}) = \mathscr{L}(\kappa \tilde{f}).$$

Proof. For almost all $x \in W_\alpha \subset I^k$, we get

$$\kappa(\tilde{\mathscr{L}} \tilde{f})(x) = (\tilde{\mathscr{L}} \tilde{f})(\alpha, x) = \sum_{i \in I} A^{(i)}_{\varphi_i(\alpha),\alpha} \varepsilon_i \det \psi_i'(x) \tilde{f}(\varphi_i(\alpha), \psi_i(x)).$$

There is a contribution only from those i's in I for which $A^{(i)}_{\varphi_i(\alpha),\alpha} = 1$, that means where there exists a $\beta \in J$ with $\psi_i(x) \in W_\beta \cap O_i$ or $x \in TO_i$. Therefore we can write for almost all $x \in W_\alpha$:

$$\begin{aligned} \kappa(\tilde{\mathscr{L}} \tilde{f})(x) &= \sum_{i \in I} \varepsilon_i \det \psi_i'(x) \tilde{f}(\varphi_i(\alpha), \psi_i(x)) \chi_{TO_i}(x) \\ &= \sum_{i \in I} \varepsilon_i \det \psi_i'(x) (\kappa \tilde{f})(\psi_i(x)) \chi_{TO_i}(x). \end{aligned}$$

But then $\kappa(\tilde{\mathscr{L}} \tilde{f})(x) = \mathscr{L}(\kappa \tilde{f})(x)$ for almost all $x \in I^k$, which proves Lemma 1.

A consequence of this lemma is that any eigenfunction $\tilde{f}$ of the operator $\tilde{\mathscr{L}}$ corresponding to an eigenvalue λ determines an eigenfunction $\kappa \tilde{f}$ of the operator $\mathscr{L}$ corresponding to exactly the same eigenvalue. To show existence of an absolutely continuous T invariant measure it is therefore enough to prove that $\lambda = 1$ is an eigenvalue of $\tilde{\mathscr{L}}$ with a positive eigenfunction $\tilde{f}$ in $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$.

Consider first the simpler case where all the O_i are proper. Then $A^{(i)}_{\varphi_i(k),k} = 1$ for all i's in I and all $k \in J = I$: set $\varphi_i(k) = i$ as discussed in Remark 3) in Chap. I. The Ruelle–Araki operator $\tilde{\mathscr{L}}$ then reads

$$\tilde{\mathscr{L}} \tilde{f}(\alpha, x) = \sum_{i \in I} \varepsilon_i \det \psi_i'(x) f(i, \psi_i(x)),$$

and leaves the space $\mathscr{C}(\bar{\Omega}_{\mathbb{R}})$ of all continuous functions on $\bar{\Omega}_{\mathbb{R}}$ regarded as a subspace of $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$ invariant. Any eigenfunction $\tilde{f}$ of $\tilde{\mathscr{L}}$ must belong to this subspace. The Perron–Frobenius and the Ruelle–Araki operators are therefore identical in this case.

To get some insight into the spectrum of the operator $\tilde{\mathscr{L}}$ in (8) we apply the same strategy as we did in [14] for the analogous operator in classical statistical mechanics: we restrict its domain of definition to an appropriate subspace of holomorphic mappings hoping to understand the relevant part of its spectrum better there.

III. The Spectrum of the Ruelle–Araki Operator

Denote by $B(c_J, \Omega)$ the Banach space of all holomorphic mappings $\tilde{f}$ of the domain Ω into the Banach space c_J of all bounded sequences indexed by J with the norm

$$|||\tilde{f}||| : \sup_{\alpha \in J} \sup_{z \in \bar{\Omega}} |\tilde{f}(\alpha, z)|.$$

Property (A1) implies that this space is invariant under $\tilde{\mathscr{L}}$ when considered in a canonical way as subspace of $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$. If furthermore all O_i's are proper, then even the space $B(\Omega)$ of all holomorphic functions on Ω with the sup norm is invariant and every eigenfunction of $\tilde{\mathscr{L}}$ in $B(c_J, \Omega)$ has to belong already to $B(\Omega)$. Let K be a proper cone with interior in the real Banach space B and $\mathscr{M}: B \to B$ a positive operator with respect to K.

Definition [16]. The operator $\mathscr{M}$ is called u_0-positive if there exists an $u_0 \in \operatorname{int} K$ such that for every $f \in K, f \neq 0$ there are numbers $N \in \mathbb{N}$, $\gamma, \delta > 0$ (all depending on f) with $\gamma u_0 \leqq \mathscr{M}^N f \leqq \delta u_0$.

Denote by $B_{\mathbb{R}}(c_J, \Omega)$ the real Banach space $B_{\mathbb{R}}(c_J, \Omega) = \{\tilde{f} \in B(c_J, \Omega): \tilde{f}(\alpha, x) \in \mathbb{R} \, \forall \alpha \in J \text{ and } x \in \bar{\Omega}_{\mathbb{R}}\}$. Then $B(c_J, \Omega) = B_{\mathbb{R}}(c_J, \Omega) + iB_{\mathbb{R}}(c_J, \Omega)$. Furthermore let K be the cone

$$K = \{\tilde{f} \in B_{\mathbb{R}}(c_J, \Omega): f(\alpha, x) \geqq 0 \; \forall \alpha \in J \text{ and } x \in \bar{\Omega}_{\mathbb{R}}\}.$$

An element $\tilde{f} \in K$ belongs to $\operatorname{int} K$ iff $\inf_{\alpha} \min_{x \in \bar{\Omega}_{\mathbb{R}}} \tilde{f}(\alpha, x) > 0$. Trivially the mapping $f_0 \in K$ with $f_0(\alpha, z) = 1$ for all $\alpha \in J$ is an interior element.

Let $\tilde{g}$ be any other interior element of K. Then there exist numbers $\gamma, \delta > 0$ with $\gamma f_0 \leqq \tilde{g} \leqq \delta f_0$. One only has to take $\gamma = \inf_{\alpha} \min_{x \in \bar{\Omega}_{\mathbb{R}}} \tilde{g}(\alpha, x)$ and $\delta = \sup_{\alpha} \sup_{x \in \bar{\Omega}_{\mathbb{R}}} \tilde{g}(\alpha, x)$.

Lemma 2. *The operator* $\tilde{\mathscr{L}}: B_{\mathbb{R}}(c_J, \Omega) \to B_{\mathbb{R}}(c_J, \Omega)$ *is* f_0*-positive with respect to the above cone* K.

Proof. We only have to show that there exists for every $\tilde{f} \in K, \tilde{f} \not\equiv 0$ a number N

with $\tilde{\mathscr{L}}^N \tilde{f} \in \operatorname{int} K$. In a first step we show that there exists N with $\tilde{\mathscr{L}}^N \tilde{f}(\alpha, x) \neq 0$ for all $\alpha \in J$ and $x \in \bar{\Omega}_{\mathbb{R}}$. Let us assume this is not the case. Then there exist for every $n \in \mathbb{N}$ $\alpha = \alpha(n)$ and $x = x(n)$ with $\tilde{\mathscr{L}}^n \tilde{f}(\alpha, x) = 0$. But

$$\tilde{\mathscr{L}}^n \tilde{f}(\alpha, x) = \sum_{i_1, \dots, i_n \in I} A_\alpha^{(i_1, \dots, i_n)} \varepsilon_{i_1} \dots \varepsilon_{i_n} \det(\psi_{i_1} \circ \dots \circ \psi_{i_n})'(x) \cdot \tilde{f}(\varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\alpha), \psi_{i_1} \circ \dots \circ \psi_{i_n}(x)) = 0.$$

From this it follows that all $f(\varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\alpha), \psi_{i_1} \circ \dots \circ \psi_{i_n}(x))$ for which $A_\alpha^{(i_1, \dots, i_n)} = 1$ must vanish. Using the Earle–Hamilton Theorem we can apply in the limit $n \to \infty$ condition (A4) to any $\tilde{f}(\varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\alpha), z)$, where by assumption (A3) $\varphi_{i_1} \circ \dots \circ \varphi_{i_n}(\alpha)$ can be then any $\beta \in J$: $\tilde{f}(\beta, z)$ must vanish on a set of uniqueness and is therefore identically zero for every $\beta \in J$.

The lemma is now proved for $|J| < \infty$ and for the case where the partition is proper. In the case $|J| = \infty$ let us assume that $\inf_\alpha \min_{\bar{\Omega}_{\mathbb{R}}} \tilde{\mathscr{L}}^N \tilde{f}(\alpha, x) = 0$ for the above N and therefore for all $N' > N$. By condition (A3) there exist $j \in I$ with $\psi_j(W_\alpha) \subset O_j$ for all $\alpha \in J$ and $M > 0$ such that $\psi_j^r(W_\alpha) \subset W_\beta$ for some β and all W_α if $r \geqq M$. Choose some $r \geqq M$ and set

$$a := \min_{\bar{\Omega}_{\mathbb{R}}} \tilde{\mathscr{L}}^N \tilde{f}(\beta, x) \text{ and } b := \min_{\bar{\Omega}_{\mathbb{R}}} |\det(\psi_j^r)'(x)|.$$

By assumption there exist to every $\varepsilon > 0$, $\alpha_\varepsilon \in J$ and a $x_\varepsilon \in \bar{\Omega}_{\mathbb{R}}$ with $0 < \tilde{\mathscr{L}}^{N+r} \tilde{f}(\alpha_\varepsilon, x_\varepsilon) < \varepsilon$. But

$$\tilde{\mathscr{L}}^{N+r} \tilde{f}(\alpha_\varepsilon, x_\varepsilon) = \sum_{i_1, \dots, i_r \in I} A_{\alpha_\varepsilon}^{(i_1, \dots, i_r)} \varepsilon_{i_1} \dots \varepsilon_{i_r} \det(\psi_{i_1} \circ \dots \circ \psi_{i_r})'(x_\varepsilon) \cdot \tilde{\mathscr{L}}^N \tilde{f}(\varphi_{i_1} \circ \dots \circ \varphi_{i_r}(\alpha_\varepsilon), \psi_{i_1} \circ \dots \circ \psi_{i_r}(x_\varepsilon)).$$

The right-hand side can be bounded below by the term

$$A_{\alpha_\varepsilon}^{(j, \dots, j)} \varepsilon_j \dots \varepsilon_j \det(\psi_j^r)'(x) \tilde{\mathscr{L}}^N \tilde{f}(\beta, \psi_j^r(x_\varepsilon)),$$

which itself is bounded below by $a \cdot b$. Because $a \cdot b > 0$ this gives a contradiction and $\inf_\alpha \min_{\bar{\Omega}_{\mathbb{R}}} \tilde{\mathscr{L}}^N \tilde{f}(\alpha, x) > 0$. This proves the lemma.

To apply the results of Krasnoselskii [16] for such u_0-positive operators we still have to show that there exists $N \in \mathbb{N}$ with $\tilde{\mathscr{L}}^N$ compact.

Lemma 3. *There exists $N \geqq 1$ such that the operator $\tilde{\mathscr{L}}^N$: $B(c_J, \Omega) \to B(c_J, \Omega)$ is compact.*

Proof. Consider first the case $|J| < \infty$ or those T's with a proper partition. Then the operator can be extended as a bounded operator to the nuclear spaces $\mathscr{H}(c_J, \Omega)$ [29], respectively $\mathscr{H}(\Omega)$, such that

$$\tilde{\mathscr{L}} : \mathscr{H}(c_J, \Omega) \to B(c_J, \Omega), \text{ respectively } \tilde{\mathscr{L}} : \mathscr{H}(\Omega) \to B(\Omega).$$

$\tilde{\mathscr{L}}$ is therefore automatically nuclear of order zero [15]. The same is then true for $\tilde{\mathscr{L}} : B(c_J, \Omega) \to B(c_J, \Omega)$ and analogous in the other case.

In the case $|J| = \infty$ this argument does not work any more because the space $\mathscr{H}(c_J, \Omega)$ is no longer nuclear [29]. Using condition $(\mathrm{A3})_b$ we know that there exists $M > 0$ such that $\varphi_{i_1} \circ \dots \circ \varphi_{i_r}(\alpha) = \varphi_{i_1} \circ \dots \circ \varphi_{i_r}(\beta)$ if $r \geqq M$ and $A_\alpha^{(i_1, \dots, i_r)} = A_\beta^{(i_1, \dots, i_r)} = 1$.

Consider then the expression for $\tilde{\mathscr{L}}^M$:

$$\tilde{\mathscr{L}}^M \tilde{f}(\alpha, z) = \sum_{i_1,\dots,i_M \in I} \varepsilon_{i_1} \dots \varepsilon_{i_M} \det(\psi_{i_1} \circ \dots \circ \psi_{i_M})'(z) A_\alpha^{(i_1,\dots,i_M)} \cdot \tilde{f}(\varphi_{i_1} \circ \dots \circ \varphi_{i_M}(\alpha), \psi_{i_1} \circ \dots \circ \psi_{i_M}(z)).$$

Because of the above remark the term $\tilde{f}(\dots)$ does not depend on α as long as $A_\alpha^{(i_1,\dots,i_M)} = 1$. For fixed $z \in \bar{\Omega}$ define then a linear mapping π: $B(c_J, \Omega) \to c_{I^M}$, where c_{I^M} denotes the Banach space of all bounded sequences indexed by I^M as follows:

$$\pi \tilde{f}_{i_1,\dots,i_M} = \begin{cases} \tilde{f}(\varphi_{i_1} \circ \dots \circ \varphi_{i_M}(\alpha), \psi_{i_1} \circ \dots \circ \psi_{i_M}(z)) & \text{if } \exists \alpha \in J \text{ with } A_\alpha^{(i_1,\dots,i_M)} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The mapping π is trivially bounded with norm $\|\pi\| \leqq 1$. The operator $l: c_{I^M} \to c_J$ defined for fixed $z \in \bar{\Omega}$ as

$$l(c)_\alpha = \sum_{i_1,\dots,i_M \in I} A_\alpha^{(i_1,\dots,i_M)} \varepsilon_{i_1} \dots \varepsilon_{i_M} \det(\psi_{i_1} \circ \dots \circ \psi_{i_M})'(z) c_{i_1 \dots i_M}$$

on the other hand is a compact operator. It can namely be written as

$$l = \sum_{i_1,\dots,i_M \in I} \lambda_{i_1 \dots i_M} u^*_{i_1 \dots i_M} \otimes v_{i_1 \dots i_M}$$

with $u^*_{i_1 \dots i_M} \in c^*_{I^M}$ and $\|u^*_{i_1 \dots i_M}\| \leqq 1$, $v_{i_1 \dots i_M} \in c_J$ and $\|v_{i_1 \dots i_M}\| \leqq 1$,

and

$$\sum_{i_1,\dots,i_M \in I} |\lambda_{i_1 \dots i_M}| < \infty.$$

To see this one only has to set

$$\begin{aligned} u^*_{i_1 \dots i_M}(c) &= c_{i_1 \dots i_M}, \\ (v_{i_1 \dots i_M})_\alpha &= A_\alpha^{(i_1,\dots,i_M)} \\ \lambda_{i_1 \dots i_M} &= \varepsilon_{i_1} \dots \varepsilon_{i_M} \det(\psi_{i_1} \circ \dots \circ \psi_{i_M})'(z). \end{aligned}$$

This shows that for fixed $z \in \bar{\Omega}$, the set $\{\tilde{\mathscr{L}}^M \tilde{f}(.,z) \colon |||f||| \leqq 1\}$ is compact in c_J. Montel's Theorem as stated and proved in [29] then shows that $\{\tilde{\mathscr{L}}^M \tilde{f} \colon |||\tilde{f}||| \leqq 1\}$ is compact in $B(c_J, \Omega)$ and $\tilde{\mathscr{L}}^M$ therefore compact, because this set is automatically equicontinuous. Taking the above results together we now get

Theorem. *Let T: $I^k \to I^k$ be locally expanding. Then there exists a T-invariant absolutely continuous measure μ whose density f is locally holomorphic. For T a Kuzmin-type theorem is valid and the dynamical system (I^k, T, μ) has strong multiple mixing properties. The correlation functions of smooth enough observables decay exponentially fast.*

Proof. Because $\tilde{\mathscr{L}}^M$ is a compact and $\tilde{\mathscr{L}}$ u_0-positive this operator has a simple positive eigenvalue λ_1 with an eigenfunction $\tilde{f}_1 \in \operatorname{int} K$, whereas all other eigenvalues are contained in a disc of radius strictly smaller than λ_1 [16]. To show that

$\lambda_1 = 1$ we use Lemma 1 which tells us that $\kappa\tilde{f}_1$ is a strictly positive eigenfunction of the Perron-Frobenius operator $\mathscr{L}$ in the space $L_1(I^k)$ with the same eigenvalue λ_1. Property a, in (3) then gives immediately $\lambda_1 = 1$.

Let us normalize $\tilde{f}_1$ in such a way that $\int_{I^k} (\kappa\tilde{f}_1)(x)dx = 1$ and call $\kappa\tilde{f}_1 = f_1$.

The spectral properties permit the following representation for the operator $\tilde{\mathscr{L}} : \tilde{\mathscr{L}} = P_{\tilde{f}_1} + \tilde{\mathscr{N}}$, where $P_{\tilde{f}_1}$ is a projection operator onto $\tilde{f}_1$ defined by

$$P_{\tilde{f}_1}(\tilde{g}) = \int_{I^k} (\kappa\tilde{g})(x)\,dx\,\tilde{f}_1,$$

and $\tilde{\mathscr{N}}$ is some bounded operator with spectral radius $\rho = |\lambda_2|$, where λ_2 is the second highest eigenvalue of $\tilde{\mathscr{L}}$. Furthermore $\tilde{\mathscr{N}}\ P_{\tilde{f}_1} = P_{\tilde{f}_1}\tilde{\mathscr{N}} = 0$. This last property implies that $\tilde{\mathscr{L}}^n = P_{\tilde{f}_1} + \tilde{\mathscr{N}}^n$ for all $n \geqq 1$, and therefore $|||\tilde{\mathscr{L}}^n - P_{\tilde{f}_1}||| \leqq C\rho^n$ for some constant C and n large enough.

Consider any $g \in L_1(I^k)$ such that there exists $\tilde{g}$ in $B(c_J, \Omega)$ with $g = \kappa\tilde{g}$. Then we get

$$\begin{aligned}\|\mathscr{L}^n g - \int_{I^k} g(x)\,dx\,f_1\| &= \|\mathscr{L}^n\kappa\tilde{g} - \int_{I^k}\kappa\tilde{g}(x)\,dx\,\kappa\tilde{f}_1\| \\ &= \|\kappa(\tilde{\mathscr{L}}^n\tilde{g} - \int_{I^k}\kappa\tilde{g}\,dx\,\tilde{f}_1)\| \leqq |||(\tilde{\mathscr{L}}^n - P_{\tilde{f}_1})\tilde{g}||| \leqq C\rho^n|||\tilde{g}|||,\end{aligned}$$

for n large enough. But this is just a Kuzmin-type theorem. By using a simple approximation argument one derives from this that for any $g \in L_1(I^k)$:

$$\lim_{n\to\infty} \|\mathscr{L}^n g - \int_{I^k} g(x)\,dx\,f_1\| = 0. \tag{11}$$

To show that the system (I^k, T, μ) has strong multiple mixing properties one has to consider for $g_0, \ldots, g_1 \in L_1(I^k)$ the multiple corrleation function

$$\int_{I^k} g_0(x)g_1(T^{n_1}x)g_2(T^{n_1+n_2}x)\ldots g_l(T^{n_1+\cdots+n_l}x)f_1(x)\,dx$$

in the limit $n_i \to \infty$ for all $1 \leqq i \leqq l$. But

$$\begin{aligned}\lim_{n_1\to\infty}\int_{I^k} g_0(x)g_1(T^{n_1}x)\ldots g_1(T^{n_1+\cdots+n_l}x)f_1(x)\,dx = \lim_{n_1\to\infty}\int_{I^k}\mathscr{L}^n(g_0(x)f_1(x))\cdot \\ \cdot g_1(x)\ldots g_l(T^{n_2+\cdots+n_l}(x))\,dx.\end{aligned}$$

which because of (11) is equal to

$$\int_{I^k} g_0(x)f_1(x)\,dx \int_{I^k} g_1(x)g_2(T^{n_2}x)\ldots g_l(T^{n_2+\cdots+n_l}x)f_1(x)\,dx.$$

Repeating this argument several times we get finally

$$\lim_{n_1\to\infty}\ldots\lim_{n_l\to\infty}\int_{I^k}\prod_{i=1}^{l} g_i(T^{n_1+\cdots+n_i}x)g_0(x)f_1(x)\,dx = \prod_{i=0}^{l}\int_{I^k} g_i(x)f_1(x)\,dx.$$

It is clear that for all $g_i \in L_1(I^k)$ which can be continued to $\tilde{g}_i \in B(c_J, \Omega)$ the above multiple mixing property is exponentially fast. This proves our theorem.

Remarks. 1) With arguments similar to those used in [9] it should be possible to prove even exactness [30] of the system (I^k, T, μ) and a central limit theorem.

2) The rate of decay of the above correlation functions as well as the rate of convergence of a smooth enough measure to the equilibrium measure μ in Kuzmin's theorem is determined by $-\log \rho$, where ρ is the spectral radius of $\tilde{\mathcal{N}}$ or in our case the second highest eigenvalue λ_2 of $\tilde{\mathcal{L}}$. In the case of the β-transformation ρ can easily be determined and turns out to be β^{-1} and the decay rate is therefore just the Kolmogorov–Sinai entropy $h(T)$ for this transformation. Unfortunately this is not true in general as one can see already from the following simple one dimensional locally expanding map $T: I^1 \to I^1$ defined as

$$Tx = \begin{cases} 1/p & x & 0 \leqq x < p \\ 1/q & (x-p) & p \leqq x \leqq 1 \end{cases} \qquad \text{for } p+q=1, p \neq q.$$

In this case $\rho = p^2 + q^2$, whereas $h(T) = -p \ln p - q \ln q$. Nevertheless one expects some relation between ρ and dynamical invariants of the underlying system.

Such invariants have been discussed in a different problem in a recent paper by Parry and Tuncel in [21] in the form of a function $P(\zeta)$ which in our case would be the highest eigenvalue of the following Ruelle–Araki operator $\tilde{\mathcal{L}}_\zeta$: $B(c_J, \Omega) \to B(c_J, \Omega)$:

$$\tilde{\mathcal{L}}_\zeta \tilde{f}(\alpha, z) = \sum_{i \in I} A^{(i)}_{\varphi_i(\alpha),\alpha} (\varepsilon_i \det \psi_i'(z))^{1+\zeta} \tilde{f}(\varphi_i(\alpha), \psi_i(z)),$$

which is well defined for instance for $\operatorname{Re} \zeta \geqq 0$. It would be interesting to understand this function $P(\zeta)$ also in our case and especially its connection with the decay rate ρ. (See in this connection also [22] and [30]).

3) Our method gives a new proof of Kuzmin's original theorem on the continued fraction transformation [31]. It also allows a unified treatment of many other number theoretic systems which have been discussed in the literature using quite different techniques in every case. The analyticity properties of the invariant density have not been known up to now.

4) It should be possible to weaken finally the holomorphy properties of the inverse mappings ψ_i in condition (A1). A careful study of the generalized Ruelle–Araki operator in an appropriate subspace of $\mathscr{C}(c_J, \bar{\Omega}_{\mathbb{R}})$ should give results very similar to ours.

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References

1. Shub, M.: Am. J. Math. **91**, 175–199 (1969)
2. Krzyzewski, K., Szlenk, W.: Studia Math. **33**, 83–92 (1969)
3. Smale, S.: Bull. Am. Math. Soc. **73**, 747–817 (1967)
4. Ruelle, D.: Am. J. Math. **98**, 619–654 (1976)
5. Bowen, R., Ruelle, D.: Invent. Math. **29**, 181–202 (1975)

6. Mayer, D., Roepstorff, G.: Strange attractors and asymptotic measures of discrete-time dissipative systems, J. Stat. Phys. **31**, 309–326 (1983)
7. Lasota, A., Yorke, J.: Trans. Am. Math. Soc. **186**, 481–488 (1973)
8. Bowen, R.: Israel J. Math. **28**, 298–314 (1978)
9. Hofbauer, F., Keller, G.: Math. Z. **180**, 119–140 (1982)
10. Billingsley, P.: Ergodic theory and information. New York: J. Wiley, 1965, pp. 40
11. Parry, W.: Acta Math. Acad. Sci. Hung. **11**, 401–416 (1960)
12. Renyi, A.: ibid. **8**, 472–493 (1957)
13. Keller, G.: Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une region bornée du plan. C.R. Acad. Sci. Paris **289A**, 625–627 (1979)
14. Mayer, D.: The Ruelle-Araki transfer operator in classical statistical mechanics. In: Lecture Notes in Physics Vol. **123**, Berlin Heidelberg, New York: Springer 1980
15. Grothendieck, A.: Produits tensoriels topologiques et espaces nucleaires. Mem. Am. Math. Soc. **16**, Providence, R. I.: AMS 1955
16. Krasnoselskii, M.: Positive solutions of operator equations, chap. 2. Groningen: P. Noordhoff, 1964
17. Schweiger, F.: The metrical theory of the Jacobi-Perron algorithm. Lecture Notes in Mathematics, Vol. **334**, Berlin Heidelberg, New York: Springer 1973
18. Schweiger, F.: On a modified Jacobi-Perron algorithm. In: Ergodic theory, Lecture Notes in Mathematics, Vol. **729**, pp. 199–202. Berlin Heidelberg, New York: Springer 1979
19. Waterman, S.: Z. Wahrsch. Verw. Geb. **16**, 77–103 (1970)
20. Waterman, S.: J. Math. Anal. Appl. **59**, 288–300 (1977)
21. Parry, W., Tuncel, S.: Ergod. Th. Dynam. Syst. **1**, 303–335 (1981)
22. Keller, G.: On the rate of convergence to equilibrium in one dimensional systems. University of Heidelberg Preprint 1983
23. Earle, C., Hamilton, R.: A fixed point theorem for holomorphic mappings. In: Global Analysis, Proc. Symp. Pure Math. XVI, Providence, R. I.: Amer. Math. Soc. 1970
24. Mayer, D.: Bull. Soc. Math. France **104**, 195–203 (1976)
25. Bernstein, L.: The Jacobi-Perron algorithm: Its theory and applications. In: Lecture Notes in Mathematics, Vol. **207**, Berlin Heidelberg, New York: Springer 1971
26. Gordin, M.: Sov. Math. Dokl. **12**, 331–335 (1971)
27. Perron, O.: Math. Ann. **64**, 1–76 (1907)
28. Mayer, D.: The transfer-matrix of a one-sided subshift of finite type with exponential interaction. Lett. Math. Phys. **1**, 335–343 (1976)
29. Grothendieck, A.: J. Reine Angew. Math. **192**, 35–64, 77–95 (1953)
30. Goldstein, S., Penrose, O.: A nonequilibrium entropy for dynamical systems. J. Stat. Phys. **24**, 325–343 (1981)
31. Kuzmin, R.: Atti de Congresso Internazionale de Matematici (Bologna) **6**, 83–89 (1928)

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