Commun. Math. Phys. 94, 331–352 (1984)

Communications in
Mathematical
Physics

A Class of Nontrivial Weakly Local Massive Wightman Fields with Interpolating Properties

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Abstract. It is shown that in a quantum field theory satisfying Wightman's axioms with locality replaced by weak locality and cyclicity by a weak irreducibility, every unitary Poincaré invariant and CPT-invariant operator is a scattering operator (in the LSZ-sense). The proof is given by explicit construction of a corresponding class of nontrivial weakly local massive Wightman fields. This result implies Jost's conjecture that only locality leads to nontrivial restrictions for the scattering operator and extends corresponding results of Schneider.

1. Introduction

In an interesting paper Jost [1] gave some arguments for the conjectures that, in the framework of a Wightman theory without locality, first the existence of a scattering operator $S \neq 1$ and the weak locality are compatible, i.e. noncontradicting, conditions and, second, the CPT-invariance of the scattering operator is the only condition for the scattering operator implied by weak locality. As Jost himself says (in this paper), the arguments for these purposes are "formal considerations" (formal construction of models which can be used to prove the assertions just mentioned).

Schneider [2] undertook the attempt to make these arguments in Jost's paper rigorous, i.e. he tried to get the corresponding models rigorous. In fact the results of Schneider are rigorous but there are some deficiencies. The first essential defect is that the test functions for his quantum fields are not the functions from $\mathscr{S}(\mathbb{R}^4)$, as required by the Wightman axioms. Another defect is that his construction of the weakly local quantum fields from the given scattering operator S works only under additional assumptions on S (not only under unitarity, Poincaré invariance and CPT-invariance).

Some remarks of Todorov [3, pp. 666 and 686] suggest the impression that it is easy to construct rigorously quantum fields along the lines sketched by Jost implying the conjectures mentioned above. Perhaps this is true, but we were not able to find any relevant construction in the literature (of course there are several papers where constructions with certain properties are presented, partly with a completely different methodical background; as an example we mention the paper of Yngvason [4] where a class of fields is constructed satisfying translational invariance and spectrality).

The aim of this paper is to present rigorous constructions of quantum fields: for every given unitary Poincaré invariant and CPT-invariant operator S_0 (a candidate for the scattering operator) we construct a quantum field satisfying the Wightman axioms with locality and cyclicity replaced by weak locality and weak irreducibility, respectively, and equipped with a convergent LSZ-scattering theory yielding a scattering operator S which coincides with the given operator S_0. These constructions imply rigorous proofs of the conjectures of Jost [1] (see also Jost [5, pp. 189 and 206]).

Our construction works for massive fields. For simplicity we restrict ourselves to neutral scalar fields in this paper.

The basic method used here is the abstract inverse scattering method (see e.g. [6]) combined with an abstract perturbation-theoretic approach applied to free fields, which apparently has not been taken into consideration so far. The ansatz for our weakly local quantum fields (or "perturbed" free fields) is simply given by $A(0) := V^* A^0(0)\, V$, where $A^0(0)$ is the free field at the point zero. It is the crucial point to express all desired properties of the field $A(\cdot)$, like Poincaré invariance, LSZ-scattering theory and so on, in terms of V. For every given CPT-invariant and Poincaré invariant operator S_0 we have to construct a suitable operator V such that $A(\cdot)$ has a LSZ-scattering theory with the prescribed scattering operator S_0. This approach is useful because the well-developed technique of interpolating asymptotic constants [6] can be applied to the construction of V. An important step in this construction is to find Lorentz invariant asymptotic constants interpolating between zero and the identity. This is done in [7]. We remark that our construction of the field $A(\cdot)$ is a pure operator-theoretical one starting with the mentioned ansatz and using some results of Fredenhagen and Hertel [8] on Wightman fields with a regularity condition. Our assumptions on the operator V imply that the quantum field $A(\cdot)$ is necessarily nonlocal.

The Appendix contains another but related construction yielding fields where Lorentz invariance is replaced by rotational invariance. These models allow a Haag-Ruelle scattering theory leading to a scattering operator which coincides with a prescribed unitary, Poincaré invariant and PCT-invariant operator. The results of the Appendix have been lectured at ISI Delhi (see Baumgärtel [9]).

2. Results

2.1. Preliminaries

For convenience we recall the Wightman properties (axioms) for a massive neutral scalar field.

The system of test functions is given by the Schwartz space $\mathscr{S}(\mathbb{R}^4)$ over $\mathbb{R}^4$. The quantum field is represented by a 5-tuple $\{\mathscr{H}, U_g, \omega; A(\cdot), \mathscr{D}\}$, where $\mathscr{H}$ is a (separable) Hilbert space and U_g is a strongly continuous unitary representation of the restricted Poincare group $\mathscr{P}_+^\uparrow$ whose elements are denoted by $g=\{\Lambda, a\}$, $a \in \mathbb{R}^4$, $\Lambda \in \mathscr{L}_+^\uparrow$, where $\mathscr{L}_+^\uparrow$ denotes the restricted Lorentz group. ω is a normalized vector (vacuum) from $\mathscr{H}$ and $\mathscr{D} \ni \omega$ is a dense linear set in $\mathscr{H}$ with $U_g \mathscr{D} \subseteqq \mathscr{D}$. The operators $A(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, are defined on $\mathscr{D}$ and ima $A(f) \upharpoonright \mathscr{D} \subseteqq \mathscr{D}$. The field $A(\cdot)$ and the representation U_g satisfy the following properties:

1. tempered distribution property,
2. Hermitian structure property (i.e. $A(f^*) \upharpoonright \mathscr{D} = A(f)^* \upharpoonright \mathscr{D}$, where $f^*(x) = \overline{f(x)}$),
3. spectrality inclusive mass gap (i.e. the mass spectrum is given by $\operatorname{spec}^m U_g = \{0\} \cup \{m_0\} \cup \Delta$, $\Delta \subseteqq [m_0+\varepsilon, \infty)$, Δ Borel set, $\varepsilon > 0$, $m_0 > 0$).
4. The vacuum ω is unique.
5. Poincaré invariance property [i.e. $A(\cdot)$ is translationally invariant and Lorentz invariant].
6. The field $A(\cdot)$ is local.

Note that locality implies the existence of an idempotent and anti-unitary operator Θ on $\mathscr{H}$ such that

$$A(\vartheta f) = \Theta A(f) \Theta, \qquad f \in \mathscr{S}(\mathbb{R}^4), \tag{2.1}$$

is valid where $(\vartheta f)(x) = \overline{f(-x)}$ respectively $(\vartheta \hat{f})(p) = \overline{\hat{f}(p)}$, i.e. for momentum space test functions ϑ means simply complex conjugation. The operator Θ is called the CPT-operator of the field $A(\cdot)$ and (2.1) is called the

6′. CPT-invariance property (weak locality).

7. The vacuum ω is cyclic.

Note that cyclicity of the vacuum implies the irreducibility of the field, i.e. if $C \in \mathscr{L}(\mathscr{H})$ and $(u, CA(f)v) = (A(f^*)u, Cv)$ is valid for all $u, v \in \mathscr{D}$, $f \in \mathscr{S}(\mathbb{R}^4)$, then $C = \gamma 1$ follows where γ is a scalar. In particular, if this proposition is true for all $C \in \mathscr{L}(\mathscr{H})$ with the restriction that $C\mathscr{D}_0 \subseteqq \mathscr{D}_0$, $C^*\mathscr{D}_0 \subseteqq \mathscr{D}_0$, where $\mathscr{D}_0 \subset \mathscr{D}$ is a dense linear set from $\mathscr{H}$, given a priori, then we call this property the

7′. weak or $\mathscr{D}_0$-irreducibility.

The asymptotic concepts and properties in connection with Wightman fields are of special interest. For asymptotic purposes one needs a suitable free field for asymptotic comparison. For convenience we collect the corresponding concepts and notations.

First $D_{m_0,0,+}$ denotes the irreducible representation of $\mathscr{P}_+^\uparrow$ labeled by $m_0 > 0$ and $s=0$; $\mathscr{H}_1^0$ is a corresponding representation Hilbert space, the so-called one-particle space. $\mathscr{H}_1^0$ can be realized by $L^2(\mathbb{R}^3, d\mathfrak{p}/(m_0^2+|\mathfrak{p}|^2)^{1/2})$, where $\mathfrak{p}$ denotes the space momentum coordinate, $\mathfrak{p} \in \mathbb{R}^3$. By $\mathscr{H}_n^0 := \mathscr{H}_1^0 \otimes \ldots \otimes \mathscr{H}_1^0$ (n times) we denote the n-particle space, and by $\mathscr{H}^0 := \mathbb{C} \oplus \mathscr{H}_1^0 \oplus S_2 \mathscr{H}_2^0 \oplus \ldots$ (Hilbert sum) the corresponding (symmetric) Fock space; $S_2, S_3, \ldots$ means symmetrization. P_n denotes the orthoprojection from $\mathscr{H}^0$ onto $\mathscr{H}_n^0$ (considered as a subspace of $\mathscr{H}^0$) and $\mathscr{H}_{\text{fin}}^0 \subset \mathscr{H}^0$ the linear submanifold of all finite particle vectors (which is dense in $\mathscr{H}^0$). $\omega_0 = \{1, 0, 0, \ldots\} \in \mathscr{H}^0$ denotes the (normalized) vacuum. By U_g^0 we denote that unitary strongly continuous representation of $\mathscr{P}_+^\uparrow$ on $\mathscr{H}^0$ which is induced by $D_{m_0,0,+}$. The generators $\{P_0, P_1, P_2, P_3\} = P$ of the corresponding translational subrepresentation U_a, $a \in \mathbb{R}^4$, are normalized by the convention

$$U_a := U_{\{1,a\}} = \int_{\mathbb{R}^4} e^{-i(\tilde{a},p)} E(dp) =: e^{-i(\tilde{a},P)}, \qquad a \in \mathbb{R}^4, \tag{2.2}$$

where $\tilde{a}=(a_0, -\mathfrak{a})$ if $a=(a_0, \mathfrak{a})$ (space reflection). Usually one puts $P_0=:H_0$. The absolutely continuous subspace of this representation with respect to the Lebesgue measure on $\mathbb{R}^4$ is given by $\mathscr{H}^0 \ominus (\mathbb{C} \oplus \mathscr{H}_1^0)$. Furthermore, $\mathrm{spec}^m U_g^0 = \mathrm{supp}^m E(\cdot) = \{0\} \cup \{m_0\} \cup [2m_0, \infty)$. The massive neutral scalar field is uniquely defined by its 2-point functional

$$W_2(f \otimes g) = (\omega, A^0(f) A^0(g) \omega) = \int_{H_{m_0}} f(-p) g(p) \mu_{m_0}(dp), \tag{2.3}$$

where H_{m_0} denotes the mass hyperboloid $H_{m_0} = \{p : p_0^2 - |\mathfrak{p}|^2 = m_0^2,\ p_0 > 0\}$, and $\mu_{m_0}(\cdot)$ the corresponding Lorentz invariant measure. As already indicated by (2.3), the corresponding (free) field operators are denoted by $A^0(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$. Note that $A^0(f) \restriction P_n \mathscr{H}^0$ is bounded for all $n = 1, 2, \ldots$. Moreover, for $u \in P_n \mathscr{H}^0$ the vector $A^0(f)u$ is always a finite particle vector, i.e. $\mathscr{H}_{\mathrm{fin}}^0 =: \mathscr{D}^0$ can be taken as the common domain "$\mathscr{D}$" which appears in the definition of Wightman fields. $A^0(f)$ is essentially selfadjoint on $\mathscr{D}^0$ if f is real-valued. The CPT-operator belonging to the free field is denoted by Θ_0. It is simply the complex conjugation of the elements of $\mathscr{H}^0$. $\mathscr{S}(\mathscr{H}_{\mathrm{fin}}^0) \subset \mathscr{H}_{\mathrm{fin}}^0$ means the set of those elements $f = \{f_0, f_1, f_2, \ldots\}$ from $\mathscr{H}_{\mathrm{fin}}^0$ such that $f_j \in \mathscr{S}(\mathbb{R}^{3j})$, $j = 1, 2, \ldots$.

8A. For asymptotic comparison we use the free field $\{\mathscr{H}^0, U_g^0, \omega_0, A^0(\cdot), \mathscr{D}^0\}$, uniquely defined by $m_0 > 0$, $s = 0$. If the field $A(\cdot)$ satisfies the properties 1–7 and if additionally first the one-particle space is irreducible labeled by $m_0 > 0$, $s = 0$, and, second, the vacuum is coupled to the one-particle space via $A(f)\omega$, then one can apply the famous result of Haag-Ruelle (see e.g. Reed and Simon [10, p. 317f.] or Glimm and Jaffe [11, p. 247]), that is, the Haag-Ruelle wave operators,

$$\underset{t \to \pm\infty}{s\text{-lim}}\, e^{itH} K e^{-itH_0} u =: W_\pm u, \tag{2.4}$$

exist where the vectors u form a suitable dense linear set. For the definition of K, see for example [12, 13, 6], see also [10, 11]. The wave operators $W_\pm$ are isometric. Asymptotic completeness of the field $A(\cdot)$ can be defined by clo ima $W_+ =$ clo ima $W_- = \mathscr{H}$ equivalent to the usual definition $\mathscr{H}_{\mathrm{in}} = \mathscr{H}_{\mathrm{out}} = \mathscr{H}$, see [11, p. 239]), the scattering operator is defined by $S = W_+^* W_-$, where $S \restriction \{\lambda \omega_0\} \oplus \mathscr{H}_1^0 = 1$. An asymptotic complete field is called nontrivial if $S \neq 1$.

Note that asymptotic completeness of the field $A(\cdot)$ implies that $\{\mathscr{H}, U_g, \omega\}$ of this field can be identified with $\{\mathscr{H}^0, U_g^0, \omega_0\}$ of the corresponding free field $A^0(\cdot)$ [we omit the simple calculations which transform $A(\cdot)$ unitarily using the wave operators as isometric operators from $\mathscr{H}^0$ onto $\mathscr{H}$].

8B. The second approach for asymptotic comparison is the so-called LSZ-approach. Here the basic idea is to obtain the field $A(\cdot)$ as an interpolating field between two free fields, for $t = -\infty$ (in-field) and $t = +\infty$ (out-field). Therefore, in this approach $\mathscr{H}$ is a priori assumed to be a Hilbert space $\mathscr{H}^0$ of a free field $A^0(\cdot)$. Furthermore, $A(\cdot)$ and $A^0(\cdot)$ are assumed to have the same representation U_g^0 and the same vacuum ω_0. Moreover, $A(\cdot)$ is assumed to be interpolating between $A^0(\cdot)$ and $S^* A^0(\cdot) S$, where S is unitary on $\mathscr{H}$, i.e. one assumes

$$\lim_{t \to \pm\infty} (u, B_t(f) v) = \begin{cases} (u, A^0(\alpha) v), & t \to -\infty, \\ (Su, A^0(\alpha) Sv), & t \to +\infty, \end{cases} \tag{2.5}$$

where $u, v \in \mathcal{D}_0 \subset \mathcal{D}^0$ ($\mathcal{D}_0$ is a fixed dense linear set in $\mathcal{H}$) and where ($\hat{g}$ denotes the Fourier transform of g)

$$B_t(f) = e^{itH} A(f_t) e^{-itH}, \quad \hat{f}_t(\mathfrak{p}, p_0) = \exp(-it(m_0^2 + |\mathfrak{p}|^2)^{1/2}) \hat{\alpha}(\mathfrak{p}, p_0), \quad \alpha \in \mathcal{S}(\mathbb{R}^4). \tag{2.6}$$

Note that it is not required that the operators $A(f_t)$ themselves exist, but the scalar products in (2.5) have to exist. Thus we can consider (2.5) for more general functions f_t, e.g. where $\alpha \in \mathcal{S}(\mathbb{R}^4)$ is replaced by $\alpha(x) = \delta(x_0)\,\alpha_1(\mathfrak{x})$, $\hat{\alpha}_1 \in C_0^\infty(\mathbb{R}^3)$ and $\delta(\cdot)$ is the δ-function, provided that the scalar products of $B_t(f)$ exist. Our way is to prove (2.5) for these functions $\alpha(x) = \alpha_1(\mathfrak{x})\delta(x_0)$. The extension to functions $\alpha(x) = \alpha_1(\mathfrak{x})\beta(x_0)$, $\hat{\alpha} \in C_0^\infty(\mathbb{R}^4)$ is obvious. S is unique up to a constant λ, $|\lambda| = 1$, but S can be normalized by $S \upharpoonright \{c\omega_0\} \oplus \mathcal{H}_1^0 = 1$. For Wightman fields considered in 8A the LSZ-approach is implied by the Haag-Ruelle approach as Hepp has shown [14] (it is to construct a suitable $\mathcal{D}_0$).

In the LSZ-approach in some sense asymptotic completeness is assumed a priori, because the asymptotic in- and out-fields $A^0(\cdot)$ and $S^* A^0(\cdot) S$ need the whole Hilbert space $\mathcal{H}^0$. In any case, asymptotic completeness in this approach means unitarity of S. Therefore, our basic approach is the following: $\{\mathcal{H}, U_g, \omega\}$ will be fixed a priori, namely by setting $\mathcal{H} = \mathcal{H}^0$, $U_g = U_g^0$, $\omega = \omega_0$. Furthermore, the next steps are governed by the *inverse scattering problem:* Let S be given, unitary on $\mathcal{H}$ and equipped with several properties to be explained later. It is to construct $A(f)$, $f \in \mathcal{S}(\mathbb{R}^4)$, and $\mathcal{D}$ such that (most of) the Wightman properties are satisfied and S is realized as the scattering operator of this field, either according to 8A or (at least) according to 8B.

2.2. *Formulation of the Results*

The results are different according to the cases: I) Poincaré invariance (translational invariance together with Lorentz invariance) and II) translational invariance together with rotational invariance. The results with respect to II are described in the Appendix. Here we deal with the results with respect to I.

First we formulate the conditions on our candidates S for scattering operators. Let $S \in \mathcal{L}(\mathcal{H})$ and assume the following conditions to be satisfied:

I. S is unitary,
II. $S \upharpoonright \{\lambda\omega\} \oplus \mathcal{H}_1 = 1$ (normalization property),
III. $SU_g = U_g S$, $g \in \mathcal{P}_+^\uparrow$, (Poincaré invariance),
IV. $\Theta_0 S \Theta_0 = S^*$ (CPT-invariance).

Then we assert that the following theorem is true.

Theorem 1. *Let the triple* $\{\mathcal{H}, U_g, \omega\}$ *be chosen as above and let the bounded operator* S *be equipped with the properties* I–IV. *Then: there exists a quantum field* $\{A(f), f \in \mathcal{S}(\mathbb{R}^4); \mathcal{D}\}$ *belonging to* $\{\mathcal{H}, U_g, \omega\}$ *and a suitable dense linear set* $\mathcal{D}_0 \subset \mathcal{D}$ *such that the properties* 1–5, 6′, 7′, *and* 8B *(with respect to* $\mathcal{D}_0$*) are satisfied and such that* S *is realized as the scattering operator of* $A(\cdot)$ *in the sense of* 8B. *That is, the field* $A(\cdot)$ *has Hermitian structure and it satisfies the tempered distribution property. It is Poincaré invariant, weakly local, weakly irreducible and satisfies the* LSZ-*property with respect to* $\mathcal{D}_0$ *(and to* $\alpha \in \mathcal{S}(\mathbb{R}^4)$*, where* $\hat{\alpha}(p) = \hat{\alpha}_1(\mathfrak{p})\beta(p_0)$ *and* $\hat{\alpha}$ *has compact support) such that* S *is realized as the scattering operator.*

Corollary 1. *There are nontrivial weakly local Wightman fields, i.e. within the framework of the other Wightman axioms (modified by 7′ instead of 7) the conditions of weak locality and nontriviality are non-contradicting.*

Proof. Obvious by Theorem 1. □

Remark 1. Theorem 1 remains true if one drops on the one hand property IV of the operator S and, on the other hand, property 6′ of the field to be constructed. The given proof works essentially also in this case.

Corollary 2. *In the framework of a Wightman theory with the properties* 1–5, 6′, 7′, *and* 8B *at most the condition* IV *of the scattering operator S is an implication of property* 6′.

Proof. As it can be seen from Remark 1, if, within the framework of the properties 1–5, 6′, 7′, and 8B, condition IV is an implication then it is an implication from property 6′ alone (in the sense that only enlarging the system (where property 6′ is dropped) by this property yields IV as an implication). Moreover, the existence of an additional condition on S implied by weak locality contradicts Theorem 1 because this theorem says that to every operator S, equipped only with conditions I–IV, there is a corresponding quantum field which is weakly local. □

Remark 2. The proof of Theorem 1 will be a constructive one. For the models to be constructed we have not been able to prove the cyclicity of the vacuum ω so far. In any case the models are weakly irreducible as asserted in Theorem 1.

Remark 3. The models to be constructed for the proof of Theorem 1 are necessarily nonlocal. For this fact the normalization condition of the operator V on $\{\lambda\omega\}\oplus\mathscr{H}_1$ is mainly responsible. This can be seen easily by the (formal) calculation [see (3.11)]

$$(\omega, A(f)A(g)\omega) = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} f(x)g(y)\,(U_xV^*A^0(0)VU_{-x}\omega, U_yV^*A^0(0)VU_{-y}\omega)\,dx\,dy$$

$$= \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} f(x)g(y)\,(U_xA^0(0)\omega, U_yA^0(0)\omega)\,dx\,dy = (\omega, A^0(f)A^0(g)\omega)\,.$$

Thus the 2-point functionals of the fields $A(\cdot)$ and $A^0(\cdot)$ coincide. Therefore, according to a well-known Theorem of Jost and Schroer (see e.g. Streater and Wightman [15, p. 214]) the field $A(\cdot)$ is necessarily nonlocal. Although one may weaken the normalization condition such that there is no coincidence of the 2-point functionals of $A^0(\cdot)$ and the perturbed field $A(\cdot)$, it seems to be not very likely that one could construct local models by a related ansatz as the one used in this paper.

3. Proof of Theorem 1

3.1. Perturbation Theory for the Free Field

We start with some remarks on (Wightman) fields with regularity condition. This condition was introduced and discussed (among other things) in Fredenhagen and Hertel [8].

Let $\mathscr{H}$ and U_g be given as before. $H\geqq 0$ denotes the Hamiltonian, $U_t=e^{-itH}$. Put $R=(1+H)^{-1}$. Furthermore, let $s(u,v)$ be a Hermitian sesquilinear form defined on $C^\infty(H) := \bigcap_{\varrho=1}^{\infty} \operatorname{dom} H^\varrho$, where s is called Hermitian if $s(u,v)=\overline{s(v,u)}$ for all $u,v \in C^\infty(H)$.

Regularity Condition. There exists a natural $k>0$ such that $s(R^k u, R^k v)$, $u,v \in C^\infty(H)$, is a bounded sesquilinear form.

In other words, there is a bounded selfadjoint operator B such that $s(R^k u, R^k v) = (u, Bv)$ for all $u,v \in C^\infty(H)$. Since $C^\infty(H)$ is invariant with respect to U_a, $a \in \mathbb{R}^4$, one can define a sesquilinear form $s(f)$ on $C^\infty(H)$, for all $f \in \mathscr{S}(\mathbb{R}^4)$, by the formula:

$$s(f)(u,v) = \int_{\mathbb{R}^4} f(x)\, s(U_{-x}u, U_{-x}v)\, dx\,. \tag{3.1}$$

$s(f)$ is also regular, for all $f \in \mathscr{S}(\mathbb{R}^4)$, because of

$$s(U_{-x}R^k u, U_{-x}R^k v) = s(R^k U_{-x}u, R^k U_{-x}v) = (U_{-x}u, BU_{-x}v)\,,$$

which leads to

$$s(f)(R^k u, R^k v) = \int_{\mathbb{R}^4} f(x)\, s(U_{-x}R^k u, U_{-x}R^k v)\, dx = \int_{\mathbb{R}^4} f(x)\,(U_{-x}u, BU_{-x}v)\, dx$$
$$= \left(u, \left(\int_{\mathbb{R}^4} f(x)\, U_x B U_{-x}\, dx\right) v\right),$$

where $\int_{\mathbb{R}^4} f(x) U_x B U_{-x} dx$ is a bounded operator. We need two propositions on regular sesquilinear forms.

Lemma 1. *If s is regular then $s(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, defines uniquely a tempered operator-valued distribution $A(f)$, with Hermitian structure, on $C^\infty(H)$ such that $R^k A(f) R^k$ is bounded, where*

$$R^k A(f) R^k = \int_{\mathbb{R}^4} f(x)\, U_x B U_{-x}\, dx\,. \tag{3.2}$$

Remark 4. For the proof of Lemma 1 see [8]. Here we only note that the operator $A(f)$ is given by the fact that $s(f)(u, R^{2k}v)$ is bounded defining uniquely a bounded operator C. Then $A(f)$ is defined on $C^\infty(H)$ by $A(f)R^{2k} := C$. Note that

$$s(f)(u,v) = (u, A(f)v)\,, \qquad u,v \in C^\infty(H)\,. \tag{3.3}$$

Therefore in the following the form $s(f)(\cdot,\cdot)$ is also denoted by $A(f)[\cdot,\cdot]$ and the starting form s by

$$A(0)[u,v] := s(u,v)\,. \tag{3.4}$$

Correspondingly we put

$$A(x)[u,v] := A(0)[U_{-x}u, U_{-x}v]\,. \tag{3.5}$$

The operator $A(f)$ is closable and, for example, the tempered distribution property is implied by the estimate

$$|(u, A(f)v)| \le \|(1+H)^k u\| \cdot \|(1+H)^k v\| \cdot \|R^k A(0) R^k\| \cdot \int_{\mathbb{R}^4} |f(x)|\, dx\,.$$

Note that Eq. (3.2) means that the field $A(\cdot)$ sandwiched by R^k is defined as a pointwise localized bounded operator; therefore we may write formally

$$R^k A(x) R^k := U_x B U_{-x}, \tag{3.6}$$

in particular

$$R^k A(0) R^k := B. \tag{3.7}$$

This notation is in agreement with the notation given by (3.4) and (3.5). Conversely, one has

Lemma 2. *Let $A(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, be an operator-valued distribution with Hermitian structure, such that $C^\infty(H)$ is an invariant domain and such that $R^k A(f) R^k$ is bounded for some fixed natural k, satisfying the estimate*

$$\|R^k A(f) R^k\| \leqq c \|f\|_{\mathscr{S}}, \quad f \in \mathscr{S}(\mathbb{R}^4), \quad c > 0, \tag{3.8}$$

where $\|\cdot\|_{\mathscr{S}}$ denotes some Schwartz norm. Then there is a sesquilinear form $s(u,v)$ defined on $C^\infty(H)$ satisfying the regularity condition (but possibly with another $k' \geqq k$), and such that

$$s(f)(u,v) = (u, A(f)v), \quad u, v \in C^\infty(H), \tag{3.9}$$

is valid.

Remark 5. For the proof see again [8]. The free field $\{\mathscr{H}, U_g, \omega, A^0(\cdot), \mathscr{D}^0\}$ belonging to $m_0 > 0$, $s = 0$ satisfies the property that $C^\infty(H)$ is an invariant domain of $A^0(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$. In this case by a straightforward calculation one obtains a bound

$$\|A^0(f) R^2 u\| \leqq c \|u\| \cdot \int_{\mathbb{R}^4} |f(x)| dx,$$

hence $A^0(\cdot)$ satisfies the estimate

$$\|R^2 A^0(f) R^2\| \leqq c \int_{\mathbb{R}^4} |f(x)| dx. \tag{3.10}$$

Thus, according to Lemma 2, $R^k A^0(0) R^k$ is bounded for some $k \geqq 2$, but it turns out that in this case $k = 2$ is already possible.

Lemmas 1 and 2, and Remark 5 allow a certain perturbation theory of the free field. According to Lemma 1 one can construct fields with regularity condition by defining $R^k A(0) R^k$ as a bounded operator. Let V be a bounded operator on $\mathscr{H}$. Then a perturbed field can be formally defined by

$$A(0) := V^* A^0(0) V. \tag{3.11}$$

A rigorous definition is given by

$$R^k A(0) R^k := \{(1+H)^k V (1+H)^{-k}\}^* R^k A^0(0) R^k \{(1+H)^k V (1+H)^{-k}\}, \quad k \geqq 2, \tag{3.12}$$

where it is assumed that $(1+H)^k V (1+H)^{-k}$ is also bounded. In this case we call V *k-energetic bounded* (see [7]).

Proposition 1. *Let $V \in \mathscr{L}(\mathscr{H})$ and let V be k-energetic bounded where $k \geqq 2$. Furthermore, let $VU_{\{\Lambda,0\}} = U_{\{\Lambda,0\}}V$ for all $\Lambda \in \mathscr{L}_+^\uparrow$. Then* (3.12) *defines a tempered operator-valued distribution $A(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, with Hermitian structure, satisfying the regularity condition, and which is, additionally, Poincaré invariant.*

Proof. We use the abbreviation

$$\tilde{V} := (1+H)^k V (1+H)^{-k}. \tag{3.13}$$

Furthermore, by

$$A(0)[u, v] := A^0(0)[Vu, Vv], \qquad u, v \in C^\infty(H), \tag{3.14}$$

it is defined a Hermitian sesquilinear form on $C^\infty(H)$ with regularity condition, namely one obtains

$$\begin{aligned} A(0)[R^k u, R^k v] &= A^0(0)[VR^k u, VR^k v] = A^0(0)[R^k \tilde{V}u, R^k \tilde{V}v] \\ &= (\tilde{V}u, R^k A^0(0) R^k \tilde{V}v), \end{aligned}$$

using the notation (3.7) with respect to $A^0(\cdot)$. Hence the sandwiched field operator $R^k A(0) R^k$, corresponding to the sesquilinear form $A(0)[\cdot, \cdot]$ defined by (3.14) is obviously given by formula (3.12). Thus, according to Lemma 1, the corresponding field operators $A(f)$, $f \in \mathscr{S}(\mathbb{R}^4)$, are well-defined on $C^\infty(H)$ as closable operators such that $R^k A(f) R^k$ is bounded. They have Hermitian structure. Now we prove the Poincaré invariance of this field. First, using (3.7) we obtain

$$\begin{aligned} R^k A(f(\cdot - a)) R^k &= \int f(x-a) U_x B U_{-x} dx = \int f(x) U_{x+a} B U_{-x-a} dx \\ &= U_a \int f(x) U_x B U_{-x} dx \, U_{-a} = U_a R^k A(f) R^k U_{-a} \\ &= R^k U_a A(f) U_{-a} R^k, \end{aligned}$$

i.e. translational invariance. Second one has

$$\begin{aligned} (u, A(f(\Lambda^{-1} \cdot))v) &= \int_{\mathbb{R}^4} f(\Lambda^{-1}x) A(0)[U_{-x}u, U_{-x}v] dx \\ &= \int_{\mathbb{R}^4} f(\Lambda^{-1}x) A^0(0)[VU_{-x}u, VU_{-x}v] dx \\ &= \int_{\mathbb{R}^4} f(y) A^0(0)[VU_\Lambda U_{-y} U_\Lambda^{-1} u, VU_\Lambda U_{-y} U_\Lambda^{-1} v] dy \\ &= \int_{\mathbb{R}^4} f(y) A^0(0)[U_\Lambda V U_{-y} U_\Lambda^{-1} u, U_\Lambda V U_{-y} U_\Lambda^{-1} v] dy, \end{aligned}$$

where for brevity $U_\Lambda := U_{\{\Lambda,0\}}$. Now, since the free field is Lorentz invariant, one has

$$A^0(0)[U_\Lambda u, U_\Lambda v] = A^0(0)[u, v]$$

for every $\Lambda \in \mathscr{L}_+^\uparrow$, i.e. these sesquilinear forms coincide. Hence one obtains

$$\begin{aligned} (u, A(f(\Lambda^{-1} \cdot))v) &= \int_{\mathbb{R}^4} f(y) A^0(0)[VU_{-y} U_\Lambda^{-1} u, VU_{-y} U_\Lambda^{-1} v] dy \\ &= \int_{\mathbb{R}^4} f(y) A(0)[U_{-y} U_\Lambda^{-1} u, U_{-y} U_\Lambda^{-1} v] dy = (U_\Lambda^{-1} u, A(f) U_\Lambda^{-1} v) \\ &= (u, U_\Lambda A(f) U_\Lambda^{-1} v) \end{aligned}$$

for all $u, v \in C^\infty(H)$ (note that $C^\infty(H)$ is invariant under U_Λ), that is, $A(\cdot)$ is Lorentz invariant. □

Therefore, in what follows we have to impose on V two conditions, k-energetic boundedness and Lorentz invariance in order to obtain a Poincaré invariant Hermitian tempered operator-valued distribution via the perturbation ansatz (3.11), respectively (3.12).

3.2. An Auxiliary Theorem

This section deals with the construction of a suitable operator V equipped with the two properties from Proposition 1 together with further additional properties connecting this operator with the prescribed operator S.

We assume that $\{\mathscr{H}, U_g, \omega\}$ is given as before and that S satisfies the properties I–IV of Sect. 2.2. Now our aim is to construct an operator V with the following properties:

I. $V \in \mathscr{L}(\mathscr{H})$ (boundedness).

II. $\tilde{V} := (1+H)^m V (1+H)^{-m}$ is bounded for some $m \geqq 2$ (m-energetic boundedness).

III. $VU_\Lambda = U_\Lambda V$, $\Lambda \in \mathscr{L}_+^\uparrow$ (Lorentz invariance).

IV. $V \restriction \{\lambda\omega\} \oplus \mathscr{H}_1 = 1$ (normalization condition).

V. $\Theta_0 V^* \Theta_0 = SV^*$ (CPT-invariance).

VI. The strong limits $\operatorname*{s-lim}_{t \to \pm\infty} e^{itH} V e^{-itH} =: V_\pm$ exist and $V_- = 1$, $V_+ = S$ (interpolating property). Moreover, with respect to a certain dense linear set $\mathscr{D}_0^V$ from $\mathscr{H}$, $\mathscr{D}_0^V \subset C^\infty(H)$, which is given a priori, V satisfies a so-called smoothness estimate

$$\|(V - V_\pm)\, e^{-itH} u\| \leqq c_{n,u}^\pm |t|^{-n}, \; n = 1, 2, 3, \ldots, \; \pm t > 1, \; u \in \mathscr{D}_0^V, \tag{3.15}$$

where $c_{n,u}^\pm > 0$ denote constants which may depend on n and u (*$\mathscr{D}_0^V$-smoothness*).

VII. $\tilde{V}$ is also an interpolating asymptotic constant with the same limits $V_\pm$ as V. $\tilde{V}$ is also $\mathscr{D}_0^V$-smooth and, moreover, the smoothness estimate is uniform with respect to the orbits [defined by $(t, 0)$], i.e.

$$\|(\tilde{V} - V_\pm)\, U_a u\| \leqq c_{n,u}^\pm \langle a, a \rangle^{-n}, \; n = 1, 2, \ldots, \; \langle a, a \rangle > 1, \; \pm t > 1, \; u \in \mathscr{D}_0^V, \tag{3.16}$$

holds where $a = (t, \mathfrak{x})$, hence $\langle a, a \rangle = t^2 - |\mathfrak{x}|^2$ and where $c_{n,u}^\pm > 0$ denote constants which may depend on n and u.

The construction of V is performed by several steps. The first step reduces the construction of V to the construction of a selfadjoint asymptotic constant A interpolating between 0 and 1 with respect to $\mathscr{H} \ominus (\mathbb{C} \oplus \mathscr{H}_1) = E_{\mathrm{ac}} \mathscr{H}$, which is Lorentz invariant and satisfies some other properties given in the next proposition. In the following we put $S = e^{i\eta}$, where η is selfadjoint and bounded.

Proposition 2. *Let $\{\mathscr{H}, U_g, \omega\}$ be defined as before. Let $\mathscr{D}_0 \subset C^\infty(H)$ be a dense linear set in $\mathscr{H}$ and invariant with respect to Θ_0, $\Theta_0 \mathscr{D}_0 \subseteqq \mathscr{D}_0$. Assume that the bounded selfadjoint operator A satisfies the following properties:*

1. *$AE_{\mathrm{ac}} = E_{\mathrm{ac}} A = A$ (normalization).*
2. *A is m-energetic bounded for some $m \geqq 2$.*
3. *A is Lorentz invariant.*

4. *A is an asymptotic constant interpolating between* 0 *and* E_{ac}, *i.e.* $\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH} A e^{-itH} E_{ac} =: A_\pm$ *exists where* $A_- = 0$, $A_+ = E_{ac}$. *Furthermore, A is $\mathscr{D}_0$-smooth.*

5. *$\tilde{A}$ is also a $\mathscr{D}_0$-smooth asymptotic constant with the same limits $A_\pm$. Furthermore, the smoothness estimate is uniform with respect to the orbits defined by $(t,0)$, i.e.*

$$\|(\tilde{A} - A_\pm) U_a u\| \leq c^\pm_{n,u} \langle a, a\rangle^{-n}, \; n = 1, 2, \ldots, \langle a, a\rangle > 1, \; \pm t > 1, \; u \in \mathscr{D}_0$$

holds. Then the operator

$$V: = S^{1/2} A S^{1/2} + S^{-1/2} \Theta_0 A \Theta_0 S^{1/2} + E^\perp_{ac} \tag{3.17}$$

satisfies the properties I–VII, *where $S^{1/2}$ is defined by $S^{1/2} := e^{(i/2)\eta}$, and where $\mathscr{D}^V_0$ is given by*

$$\mathscr{D}^V_0 := S^{-1/2} \mathscr{D}_0 . \tag{3.18}$$

Proof. I is trivial. II is obvious because A is m-energetic bounded. III is obvious because of 3 and $S^{1/2} U_A = U_A S^{1/2}$. IV is obvious because of $AE^\perp_{ac} = E^\perp_{ac} A = 0$ and $S^{1/2} E_{ac} = E_{ac} S^{1/2}$. V. One has

$$\begin{aligned}\Theta_0 V^* \Theta_0 &= \Theta_0 S^{-1/2} A S^{-1/2} \Theta_0 \\ &\quad + \Theta_0 S^{-1/2} \Theta_0 A \Theta_0 S^{1/2} \Theta_0 + E^\perp_{ac} \\ &= S^{1/2} \Theta_0 A \Theta_0 S^{1/2} + S^{1/2} A S^{-1/2} + E^\perp_{ac} \\ &= S(S^{-1/2} A S^{-1/2} + S^{-1/2} \Theta_0 A \Theta_0 S^{1/2} + E^\perp_{ac}) .\end{aligned}$$

Note that $SE^\perp_{ac} = E^\perp_{ac}$ because of the normalization condition of S. VI. The first part follows from 4. The second part follows also from 4: Namely, on the one hand, A is $\mathscr{D}_0$-smooth and, on the other hand, from (3.18) one obtains $S^{1/2} \mathscr{D}^V_0 = \mathscr{D}_0$. Thus from (3.17) the $\mathscr{D}^V_0$-smoothness of V follows. VII follows from 5 and from the fact that $(1+H)^m$ and $(1+H)^{-m}$ commute with $S^{1/2}$ and $S^{-1/2}$. Furthermore, (3.18) is used. □

Remark 6. If $S\mathscr{D}_0 \subseteq \mathscr{D}_0$ then instead of (3.17) the ansatz $V: = AS + \Theta_0 A \Theta_0$ is possible for V yielding for V the same smoothness manifold $\mathscr{D}_0$ as for A.

3.3. Construction of A

The operator A will be constructed separately with respect to the n-particle spaces $\mathscr{H}_n$, $n = 2, 3, \ldots$. Simultaneously, the smoothness manifolds $\mathscr{D}_n$, $n = 2, 3, \ldots$, with respect to $A_n := A \upharpoonright \mathscr{H}_n$ are introduced. We should remark that the essential steps for the construction of the operators A_n and the smoothness manifolds $\mathscr{D}_n$ are already described and developed in [7]. Recall that

$$\mathscr{H}_n = S_n L^2 \Bigg(\underbrace{\mathbb{R}^3 \times \ldots \times \mathbb{R}^3}_{n \text{ times}}, \bigotimes_{\varrho=1}^{n} d\mathfrak{p}_\varrho / (m_0^2 + |\mathfrak{p}_\varrho|^2)^{1/2} \Bigg) \tag{3.19}$$

and, using the abbreviation $\mu(\mathfrak{p}) := (m_0^2 + |\mathfrak{p}|^2)^{1/2}$,

$$(U_{\{A,a\}} f)(\mathfrak{p}_1, \ldots, \mathfrak{p}_n) = e^{-i\left(\tilde{a}, \sum_{\varrho=1}^{n} (\mu(\mathfrak{p}_\varrho), \mathfrak{p}_\varrho)\right)} f(\theta^{-1}\mathfrak{p}_1, \ldots, \theta^{-1}\mathfrak{p}_n) , \tag{3.20}$$

where θ corresponds to Λ via $\theta = \iota^{-1}\Lambda\iota$ (see [7]). We form

$$M^2 = \left(\sum_{\varrho=1}^{n} \mu(\mathfrak{p}_\varrho)\right)^2 - \left|\sum_{\varrho=1}^{n} \mathfrak{p}_\varrho\right|^2 \geqq n^2 m_0^2 . \tag{3.21}$$

The function $M = M(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ is invariant with respect to $\theta : M(\theta\mathfrak{p}_1, \ldots, \theta\mathfrak{p}_n) = M(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$. Equation (3.21) is equivalent

$$\sum_{\varrho \neq \sigma} (\mu(\mathfrak{p}_\varrho)\mu(\mathfrak{p}_\sigma) - (\mathfrak{p}_\varrho, \mathfrak{p}_\sigma)) = (1/2)(M^2 - nm_0^2) \geqq (1/2)n(n-1)m_0^2 . \tag{3.22}$$

Note that each term $\mu(\mathfrak{p}_\varrho)\mu(\mathfrak{p}_\sigma) - (\mathfrak{p}_\varrho, \mathfrak{p}_\sigma) \geqq m_0^2$. If M is fixed one obtains a manifold $\mathcal{O}_M \subset \mathbb{R}^{3n}$. It can be seen easily that $\mathcal{O}_{nm_0} = \{(\mathfrak{p}_1, \ldots, \mathfrak{p}_n): \mathfrak{p}_1 = \mathfrak{p}_2 = \ldots = \mathfrak{p}_n\}$, which is diffeomorphic to $\mathbb{R}^3$. If $M > nm_0$ then $\mathcal{O}_M$ is locally diffeomorphic to $\mathbb{R}^{3n-1}$. Now we consider diffeomorphisms between $\mathbb{R}^{3n} \setminus \mathcal{O}_{nm_0}$ and $(nm_0, \infty) \times \mathcal{O}_{M_1}$, where $M_1 > nm_0$ is fixed. We are interested in diffeomorphisms with special properties.

Lemma 3. *There is a diffeomorphism from $\mathbb{R}^{3n} \setminus \mathcal{O}_{nm_0}$ onto $(nm_0, \infty) \times \mathcal{O}_{M_1}$ with the following properties: Let $(M, \kappa) \in (nm_0, \infty) \times \mathcal{O}_{M_1}$.*

I. $\theta\{\mathfrak{p}_1(M, \kappa), \ldots, \mathfrak{p}_n(M, \kappa) = \{\mathfrak{p}_1(M, \theta\kappa), \mathfrak{p}_2(M, \theta\kappa), \ldots, \mathfrak{p}_n(M, \theta\kappa)$, *where* $\theta\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} := \{\theta\mathfrak{p}_1, \theta\mathfrak{p}_2, \ldots, \theta\mathfrak{p}_n\}$.

II. $\sum_{j=1}^{n} \mathfrak{p}_j(M, \kappa) = M\mathfrak{k}(\kappa)$, $\mathfrak{k}(\kappa) \in \mathbb{R}^3$ *independent of* M.

III. $\bigotimes_{j=1}^{n} d\mathfrak{p}_j/(m_0^2 + |\mathfrak{p}_j|^2)^{1/2} = \varrho(dM) \otimes d\kappa$,

where $d\kappa$ is an invariant measure on $\mathcal{O}_{M_1}$ (with respect to θ) and where $\varrho(\cdot)$ is an absolutely continuous measure on (nm_0, ∞) with $\operatorname{supp}\varrho = (nm_0, \infty)$.

We do not prove this lemma. See for example [7] where the proof is performed for the case $n = 2$. See also Ruijsenaars [16, p. 427] where such diffeomorphisms are explicitly written down by the formulas $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} \leftrightarrow \{\mathfrak{k}, \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-1}\}$ where, as before, $\mathfrak{k}(\mathfrak{p}_1, \ldots, \mathfrak{p}_n) = M^{-1} \sum_{j=1}^{n} \mathfrak{p}_j$, $M^2 = \left(\sum_{j=1}^{n} \mu(\mathfrak{p}_j)\right)^2 - \left|\sum_{j=1}^{n} \mathfrak{p}_j\right|^2$. Furthermore, put $k = (k_0, \mathfrak{k})$, $k_0 = (1 + |\mathfrak{k}|^2)^{1/2}$, and $q_j = B(k)^{-1} p_j$, $j = 1, 2, \ldots, n$, where $B(k)$ denotes the pure Lorentz boost which sends $(1, 0)$ into k. Then one obtains

$$M = \sum_{j=1}^{n} \mu(\mathfrak{q}_j), \ \sum_{j=1}^{n} \mathfrak{q}_j = 0 .$$

This means, for example, that M does not depend on $\mathfrak{k}$. Conversely, $\mathfrak{k}$ does not depend on M, hence $\mathfrak{k}$ can be taken as a "part" of the coordinate κ. To prove III one has to calculate the Jacobian.

According to Lemma 3 the Hilbert space $L^2\left(\mathbb{R}^{3n}, \bigotimes_{j=1}^{n} d\mathfrak{p}_j/\mu(\mathfrak{p}_j)\right)$ is isometrically isomorphic, with respect to a diffeomorphism characterized in Lemma 3, to

$$L^2([nm_0, \infty), \varrho(dM)) \otimes L^2(\mathcal{O}_{M_1}, d\kappa) . \tag{3.23}$$

Therefore, the Hilbert space (3.19) is isometrically isomorphic to $L^2([nm_0, \infty), \varrho(dM)) \otimes S_n L^2(\mathcal{O}_{M_1}, d\kappa)$, where S_n, as before, means symmetrization [note that the points $\kappa \in \mathcal{O}_{M_1}$ are given by $\kappa = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} \in \mathcal{O}_{M_1}$; if π is a permutation of $(1, 2, \ldots, n)$ then $\pi(\kappa) = \{\mathfrak{p}_{\pi(1)}, \mathfrak{p}_{\pi(2)}, \ldots, \mathfrak{p}_{\pi(n)}\} \in \mathcal{O}_{M_1}$]. For convenience in the following we drop the symmetrization, but we take it into account in Proposition 4.

The unitary representation $U_{\{\Lambda,a\}}$ given by (3.20) can be shifted to the space (3.23). It is then denoted by $\hat{U}_{\{\Lambda,a\}}$ and acts by the formula

$$(\hat{U}_{\{\Lambda,a\}}f)(M,\kappa) = e^{-iM(\tilde{a},k(\kappa))} f(M, \theta^{-1}\kappa), \tag{3.24}$$

in particular one has

$$(\hat{U}_\Lambda f)(M,\kappa) = f(M, \theta^{-1}\kappa), \tag{3.25}$$

$$(\hat{U}_t f)(M,\kappa) = e^{-itM\alpha(\kappa)} f(M,\kappa), \tag{3.26}$$

where $\alpha(\kappa) = k_0(\kappa) \geqq 1$. One obtains, for example,

$$(\hat{U}_\Lambda \hat{U}_t \hat{U}_\Lambda^{-1} f)(M,\kappa) = (\hat{U}_{\Lambda(t,0)} f)(M,\kappa) = e^{-itM\alpha(\theta^{-1}\kappa)} f(M,\kappa). \tag{3.27}$$

The next proposition deals with the construction of A_n shifted to the space (3.23). This proposition is the analog of Lemma 7 in [7].

Proposition 3. *Let the Hilbert space be given by* (3.23) *and the representation by* (3.24). *Furthermore, put* $\mathcal{D}_0 := \mathrm{spa}\,\{f \otimes g : f \in C_0^\infty((nm_0, \infty)),\ g \in C_0^\infty(\mathcal{O}_{M_1})$. *Then there exists a bounded operator B with the following properties:*

I. *B is selfadjoint.*

II. *B is m-energetic bounded,* $m \geqq 2$.

III. $B\hat{U}_\Lambda = \hat{U}_\Lambda B$, $\Lambda \in \mathcal{L}_+^\uparrow$.

IV. *B is a* $\mathcal{D}_0$*-smooth asymptotic constant, interpolating between* 0 *and* 1, *i.e.* $\operatorname*{s-lim}_{t\to\pm\infty} e^{itH} B e^{-itH} =: B_\pm$ *exists where* $B_- = 0$, $B_+ = 1$ *and*

$$\|(B - B_\pm)e^{-itH}v\| \leqq c_{n,v}^\pm |t|^{-n},\ n = 1, 2, \ldots,\ \pm t > 1, \qquad v \in \mathcal{D}_0.$$

V. $\tilde{B}$ *is also an interpolating asymptotic constant with the same limits and it is* $\mathcal{D}_0$*-smooth where the smoothness estimate is uniform with respect to the orbits defined by* $(t, 0)$, *i.e.*

$$\|(\tilde{B} - B_\pm)U_a v\| \leqq c_{n,v}^\pm \langle a, a\rangle^{-n},\ n = 1, 2, \ldots,\ \langle a, a\rangle > 1,\ \pm t > 1, \qquad v \in \mathcal{D}_0.$$

Proof. We use the ansatz

$$B := X \otimes 1_\kappa, \tag{3.28}$$

where X is bounded acting on $L^2([nm_0, \infty), \varrho(dM)) = L^2([nm_0, \infty), \varrho'(M)dM)$, which is isometrically isomorphic to $L^2([nm_0, \infty), dM)$, and where 1_κ denotes the identity on $L^2(\mathcal{O}_{M_1}, d\kappa)$. The space $L^2([nm_0, \infty), dM)$ can be enlarged to $\mathcal{K} := L^2(\mathbb{R}, dM)$. The corresponding projection $\chi_{[nm_0, \infty)}(M)$ is denoted by P. On the space $\mathcal{K}$ the representation $\hat{U}_t$ acts as multiplication by $e^{-itM\alpha(\kappa)}$, where $\kappa \in \mathcal{O}_{M_1}$ appears as a parameter running through $\mathcal{O}_{M_1}$. This representation (where κ is fixed) we denote by $\hat{U}_t^\kappa$. Now we use the operator $X := PFQF^*P \restriction P\mathcal{K}$ of Proposition 2 of [7]. Recall that F denotes Fourier transformation and that Q acts as multiplication by $q(x) \in C^\infty(\mathbb{R})$, where $0 \leqq q(x) \leqq 1$, $q(x) = 0$ for $x < 0$ and $q(x) = 1$

for $x>1$. Furthermore, recall that H acts as $-id/dx$ in $L^2(\mathbb{R}, dx) = FL^2(\mathbb{R}, dM)$. Hence one easily calculates

$$\tilde{Q} := (i+H)^m Q(i+H)^{-m} = Q + \sum_{\varrho=1}^{m} \binom{m}{\varrho} (H^\varrho q)(i+H)^{-\varrho},$$

which is bounded. Moreover one obtains $\operatorname*{s-lim}_{t\to\pm\infty} e^{itH} C_\varrho e^{-itH} = 0$ for $C_\varrho = H^\varrho q$, $\varrho = 1, 2, \ldots$. Thus $\tilde{X} := (i+H)^m X(i+H)^{-m}$ is given by $\tilde{X} = PF\tilde{Q}F^*P \upharpoonright P\mathcal{K}$, which is bounded. Because the projection P appears in this formula for $\tilde{X}$ one can choose also $(1+H)$ instead of $(i+H)$ in this formula (see Remark 1 in the paper [7]). This proves II. I is trivial. According to Proposition 2 of [7] X and $\tilde{X}$ are smooth asymptotic constants with the same limits $X_- = 0$, $X_+ = 1$, where the corresponding smoothness manifold in $L^2([nm_0, \infty), dM)$ can be taken as $C_0^\infty((nm_0, \infty))$. This proves IV and the first part of V.

Now we emphasize the fact that, since the representations $\hat{U}_t$ and $\hat{U}_a$ have a special structure, the smoothness estimates do not depend on κ and, moreover, they are uniform with respect to the orbit belonging to $(t, 0)$. More precisely, let $a = \Lambda(t, 0)$, $\Lambda \in \mathcal{L}_+^\uparrow$, and let f, g be defined as in the assumptions of Proposition 3. Then one has

$$\begin{aligned}\|(\tilde{B} - B_\pm) U_a(f \otimes g)\| &= \|(\tilde{B} - B_\pm) U_{\Lambda(t,0)}(f \otimes g)\| \\ &= \|(\tilde{B} - B_\pm) U_\Lambda U_t U_\Lambda^{-1}(f \otimes g)\| = \|(\tilde{B} - B_\pm) e^{-itM\alpha(\theta^{-1}\kappa)} f(M) \otimes g(\kappa)\|.\end{aligned}$$

Now $\tilde{B} - B_\pm = (\tilde{X} - X_\pm) \otimes 1_\kappa$, hence

$$\left\|(\tilde{B} - B_\pm) U_a(f \otimes g)\right\|^2 = \int_{\mathcal{O}_{M_1}} \|(\tilde{X} - X_\pm) U_t^{\theta^{-1}\kappa} f(M)\|^2_{L^2([nm_0, \infty), \varrho(dM))} |g(\kappa)|^2 d\kappa$$

follows. But

$$\begin{aligned}\|(\tilde{X} - X_\pm) U_t^{\theta^{-1}\kappa} f(M)\|_{L^2([nm_0, \infty), \varrho(dM))} &\leqq c_{n,f}^\pm |t\alpha(\theta^{-1}\kappa)|^{-n} \\ &\leqq c_{n,f}^\pm |t|^{-n},\end{aligned}$$

and this estimate is obviously independent of κ and also independent of θ (respectively Λ). Hence

$$\|(\tilde{B} - B_\pm) U_a(f \otimes g)\| \leqq \left(\int_{\mathcal{O}_{M_1}} |g(\kappa)|^2 d\kappa\right)^{1/2} c_{n,f}^\pm \langle a, a\rangle^{-n/2} \tag{3.29}$$

follows because $\langle a, a\rangle = t^2$ on the orbit defined by $(t, 0)$. This proves the last part of V. Note that III follows immediately from (3.28). □

Remark 7. Proposition 3 can be proved also by other arguments using first the result for $n=2$ proved in Lemma 7 of [7]. Then one uses Lemma 6 of [7] and the arguments given in the proof of this lemma which are founded on Proposition 6 of [7] (inner tensor product representation). Also the uniformity of the smoothness estimates with respect to the orbits can be proved in this way.

Note that it is possible to enlarge the smoothness manifold of Proposition 3. More precisely, we have the following result.

Corollary 3. *Assume the assumptions of Proposition 3 to be valid. Then the smoothness manifold $\mathcal{D}_0$ in* IV *and* V *of Proposition 3 can be replaced by the larger manifold $\mathcal{D}_0' := C_0^\infty((nm_0, \infty) \times \mathcal{O}_{M_1})$.*

Proof. Let $f \in C_0^\infty((nm_0, \infty))$, $g \in C_0^\infty(\mathcal{O}_{M_1})$. Obviously,

$$\sup_{t \geqq 1} t^n \|(X - X_+) e^{-itH} f\| = : [f] \tag{3.30}$$

defines a seminorm on $C_0^\infty((nm_0, \infty))$. This seminorm $[\cdot]$ turns out to be continuous with respect to the C_0^∞-topology (the corresponding calculation is omitted). Furthermore, according to (3.29) one has an estimate of the form

$$\sup_{t \geqq 1} t^n \|(B - B_+) e^{-itH}(f \otimes g)\| \leqq [f] \cdot \|g\|, \tag{3.31}$$

where $\|g\| = \left(\int_{\mathcal{O}_{M_1}} |g(\kappa)|^2 d\kappa \right)^{1/2}$. Now for finite sums $v := \sum_{j=1}^{N} f_j \otimes g_j$ one defines a so-called cross norm by

$$|v| := \inf_{v = \Sigma f_j \otimes g_j} \sum_{j=1}^{N} [f_j] \cdot \|g_j\|.$$

It turns out that the norm $|\cdot|$ is also continuous with respect to the C_0^∞-topology of $C_0^\infty((nm_0, \infty) \times \mathcal{O}_{M_1})$. Moreover, an arbitrary function $h \in C_0^\infty((nm_0, \infty) \times \mathcal{O}_{M_1})$ can be approximated by finite sums $\sum f_j \otimes g_j$ with respect to this topology. Furthermore, the estimate

$$\sup_{t \geqq 1} t^n \|(B - B_+) e^{-itH} v\| \leqq |v|, \qquad v = \sum_{j=1}^{N} f_j \otimes g_j, \tag{3.32}$$

is valid. Therefore, also each $h \in C_0^\infty((nm_0, \infty) \times \mathcal{O}_{M_1})$ satisfies (3.32). Similarly one proceeds for $t \to -\infty$ and for the proof of the uniformness of the estimates. □

Finally one has to rewrite Proposition 3, i.e. one has to pull back the diffeomorphism. Moreover, one has to take into account the symmetrization.

Proposition 4. *Let $\mathcal{H}_n$, $n \geqq 2$, be as in* (3.19) *and $U_{\{A,a\}}$ as in* (3.20). *Furthermore, put $\mathcal{D}_0 := S_n C_0^\infty(\mathbb{R}^{3n} \backslash \mathcal{O}_{nm_0})$. Then there exists an operator A_n with properties which are completely analogous to the properties* I–V *for the operator B of Proposition 3.*

Proof. As already noted before, a diffeomorphism φ satisfying Lemma 3,

$$\varphi : \mathbb{R}^{3n} \backslash \mathcal{O}_{nm_0} \mapsto (nm_0, \infty) \times \mathcal{O}_{M_1}$$

implements an isometric isomorphism

$$\Phi : S_n L^2(\mathbb{R}^{3n}, \bigotimes_{j=1}^{n} d\mathfrak{p}_j / (m_0^2 + |\mathfrak{p}_j|^2)^{1/2}) \to L^2([nm_0, \infty), \varrho(dM)) \otimes S_n L^2(\mathcal{O}_{M_1}, d\kappa).$$

The symmetrization $S_n \mathcal{D}'_0$ of the smoothness manifold $\mathcal{D}'_0$ from Corollary 3 is mapped onto $S_n C_0^\infty(\mathbb{R}^{3n} \backslash \mathcal{O}_{nm_0})$ under φ^{-1}. Now one has $U_g = \Phi^{-1} \hat{U}_g \Phi$, and one can pull back the operator B of Proposition 3 by defining $A_n := \Phi^{-1} B \Phi$. A_n satisfies all the asserted properties. □

The next step is to put together all operators A_n, $n = 2, 3, \ldots$, constructed before, into the direct sum $A := 0 \oplus 0 \oplus A_2 \oplus A_3 \oplus \ldots$ and to prove that A satisfies all the required properties.

Proposition 5. *The operator $A := 0 \oplus 0 \oplus A_2 \oplus A_3 \oplus \ldots$ satisfies all properties of the assumption of Proposition 2, where $\mathscr{D}_0 := \{f \in \mathscr{H}_{\text{fin}};\ f_0 \in \mathbb{C},\ f_1 \in C_0^\infty(\mathbb{R}^3),\ f_n \in C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{O}_{nm_0}$ for $n \neq 0, 1\}$ ($\mathscr{D}_0$ consists of finite particle vectors only).*

Proof. Only the boundedness and the m-energetic boundedness of A has to be proved. That is, we have to prove $\sup_n \|A_n\| < \infty$ and $\sup_n \|\tilde{A}_n\| < \infty$. One has

$$\|A_n\| = \|X_n \otimes 1_\kappa\| = \|X_n\| = \|PFQ_nF^*P \restriction P\mathscr{K}\| \leqq \|Q_n\|,$$

but $\|Q_n\| \leqq \sup_{x \in \mathbb{R}} |q(x)| = 1$. Correspondingly one obtains

$$\|\tilde{A}_n\| \leqq \|\tilde{Q}_n\| \leqq \|Q_n\| + \sum_{\varrho=1}^{m} \binom{m}{\varrho} \|H^\varrho q\| \cdot \|(i+H)^{-\varrho}\| \leqq 1 + k_m.$$

3.4. Final Step of the Proof of Theorem 1

Now we choose the operator A from Sect. 3.3 which is defined in Proposition 5. Then, according to Proposition 2, by (3.17) the operator V is defined. With the help of this operator V we form the perturbed field $A(\cdot)$, formally defined by (3.11). Next we have to check all the properties of Theorem 1: 3 and 4 are obvious because of the choice of $\{\mathscr{H}, U_g, \omega\}$. 1, 2, and 5 follow immediately from Proposition 1, 6′: To prove CPT-invariance, first we have to define an appropriate CPT-operator Θ. It is defined by

$$\Theta := S^* \Theta_0. \tag{3.33}$$

Note that the smoothness manifold $\mathscr{D}_0^V$ for V from Proposition 2 is invariant under Θ: Let $u \in \mathscr{D}_0^V$, then $u = S^{-1/2}v$, where $v \in \mathscr{D}_0$. Now

$$\Theta u = S^* \Theta_0 S^{-1/2} v = S^* S^{1/2} \Theta_0 v = S^{-1/2} \Theta_0 v.$$

Since $\Theta_0 v \in \mathscr{D}_0$ the assertion $\Theta u \in \mathscr{D}_0^V$ follows.

Now we are able to prove the CPT-invariance of the field. It is sufficient to do the calculation for the sandwiched field operators $R^m A(x) R^m$. Then one obtains

$$\begin{aligned} \Theta R^m A(x) R^m \Theta &= \Theta U_x \tilde{V}^* R^m A^0(0) R^m \tilde{V} U_{-x} \Theta \\ &= U_{-x} \Theta \tilde{V}^* \Theta_0 R^m A^0(0) R^m \Theta_0 \tilde{V} \Theta U_x. \end{aligned}$$

Because of (3.33) and $\Theta_0 \tilde{V}^* \Theta_0 = S\tilde{V}^*$, $\Theta_0 \tilde{V} \Theta_0 = \tilde{V} S^*$ one obtains

$$\Theta R^m A(x) R^m \Theta = U_{-x} \tilde{V}^* R^m A^0(0) R^m \tilde{V} U_x = R^m A(-x) R^m.$$

But this implies CPT-invariance, i.e. from $R^m A(f) R^m = \int f(x) R^m A(x) R^m dx$ one gets, using $(\vartheta f)(x) = \overline{f(-x)}$,

$$\begin{aligned} R^m A(\vartheta f) R^m &= \int \overline{f(-x)} R^m A(x) R^m dx = \int \overline{f(x)} R^m A(-x) R^m dx \\ &= \int \overline{f(x)} \Theta R^m A(x) R^m \Theta dx = \Theta \{\int f(x) R^m A(x) R^m dx\} \Theta \\ &= R^m \Theta A(f) \Theta R^m. \end{aligned}$$

8B. We choose $u, v \in \mathscr{D}_0^V$ and $\alpha(x) = \delta(x_0)\alpha_1(\mathfrak{x})$, $\hat{\alpha}_1 \in C_0^\infty(\mathbb{R}^3)$, i.e. $\hat{\alpha}_1$ has compact support. Then we have to prove the asymptotic relations (2.5). We put $S_+ = S$,

$S_- = 1$. Recall that we formally have $(f_t(x) = \delta(x_0)\tilde{f}_t(\mathfrak{x}))$

$$B_t(f) = e^{itH} A(f_t) e^{-itH} = e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} V^* A^0(0) V U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH}, \tag{3.34}$$

and this formula becomes rigorous if we multiply by R^m from the right and from the left, i.e. $R^m B_t(f) R^m$ is well-defined and we consider

$$(u', B_t(f) v') = (R^m u, B_t(f) R^m v) = (u, R^m B_t(f) R^m v),$$

where $u' = R^m u$, $v' = R^m v$, $u', v' \in \mathscr{D}_0^V$ and $u, v \in \mathscr{D}_0^V$ (note that $\mathscr{D}_0^V$ is invariant with respect to H). Then we obtain

$$R^m B_t(f) R^m = e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} \tilde{V}^* R^m A^0(0) R^m \tilde{V} U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH}.$$

On the other hand, it is easy to show that

$$R^m S^* A^0(\alpha) S R^m = e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} S^* R^m A^0(0) R^m S U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH} \tag{3.35}$$

is valid as an identity with respect to t. Therefore, in the case $t \to +\infty$ we have to estimate the expression

$$\begin{aligned}(u, R^m B_t(f) R^m v) &- (u, R^m S^* A^0(\alpha) S R^m v)\\ &= \left(u, e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} \{\tilde{V}^* R^m A^0(0) R^m \tilde{V} - S^* R^m A^0(0) R^m S\} U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH} v\right).\end{aligned} \tag{3.36}$$

Using that

$$\begin{aligned}\tilde{V}^* R^m A^0(0) R^m \tilde{V} - S^* R^m A^0(0) R^m S = \tilde{V}^* R^m A^0(0) R^m (\tilde{V} - S)\\ + (\tilde{V}^* - S^*) R^m A^0(0) R^m S,\end{aligned}$$

the right-hand side of (3.36) equals

$$\begin{aligned}&\left(u, e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} \{\tilde{V}^* R^m A^0(0) R^m (\tilde{V} - S)\} U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH} v\right)\\ &+ \left(u, e^{itH} \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{\mathfrak{x}} \{(\tilde{V}^* - S^*) R^m A^0(0) R^m S\} U_{-\mathfrak{x}} d\mathfrak{x}\, e^{-itH} v\right).\end{aligned} \tag{3.37}$$

The first term in this sum can be written in the form

$$\begin{aligned}T_1 &= \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) (u, (U_{-a} \tilde{V}^* R^m A^0(0) R^m U_a U_{-a} (\tilde{V} - S) U_a) v) d\mathfrak{x}\\ &= \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) (U_{-a} R^m A^0(0) R^m \tilde{V} U_a u, U_{-a} (\tilde{V} - S) U_a v) d\mathfrak{x},\end{aligned}$$

where $a = (t, -\mathfrak{x})$, hence one obtains the estimate

$$\begin{aligned}|T_1| &\leqq \int_{\mathbb{R}^3} |\tilde{f}_t(\mathfrak{x})| \cdot \|R^m A^0(0) R^m\| \cdot \|\tilde{V}\| \cdot \|u\| \cdot \|U_{-a}(\tilde{V} - S) U_a v\| d\mathfrak{x}\\ &= C_1 \int_{\mathbb{R}^3} |\tilde{f}_t(\mathfrak{x})| \cdot \|(\tilde{V} - S) U_a v\| d\mathfrak{x}.\end{aligned} \tag{3.38}$$

The second term in the sum can be written similarly as

$$\begin{aligned}T_2 &= \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) (u, (U_{-a} (\tilde{V}^* - S^*) R^m A^0(0) R^m S U_a) v) d\mathfrak{x}\\ &= \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) (U_{-a} (\tilde{V} - S) U_a u, U_{-a} R^m A^0(0) R^m S U_a v) d\mathfrak{x}.\end{aligned}$$

Therefore, in this case we obtain the estimate

$$|T_2| \leqq \int_{\mathbb{R}^3} |\tilde{f}_t(\mathfrak{x})| \cdot \|U_{-a}(\tilde{V}-S)U_a u\| \cdot \|R^m A^0(0) R^m\| \cdot \|v\| \, d\mathfrak{x}$$
$$= C_2 \int_{\mathbb{R}^3} |\tilde{f}_t(\mathfrak{x})| \cdot \|(\tilde{V}-S)U_a u\| \, d\mathfrak{x}. \tag{3.39}$$

That is, the crucial point is to estimate integrals of the form

$$\int_{\mathbb{R}^3} |\tilde{f}_t(\mathfrak{x})| \cdot \|(\tilde{V}-S)U_a v\| \, d\mathfrak{x}, \quad a=(t,-\mathfrak{x}) \in \mathbb{R}^4, \quad v \in \mathcal{D}_0^V. \tag{3.40}$$

To do this we recall the following two facts:

1. The asymptotic constant $\tilde{V}$ is $\mathcal{D}_0^V$-smooth where the smoothness estimate is uniform on the orbits $\langle a,a\rangle = \text{const} > 1$, $t>1$, i.e. we have

$$\|(\tilde{V}-S)U_a v\| \leqq c_{n,v}^{+} \langle a,a\rangle^{-n}, \quad n=1,2,\ldots, \langle a,a\rangle > 1, t>1, \quad v \in \mathcal{D}_0^V,$$

where $\langle a,a\rangle = t^2 - |\mathfrak{x}|^2$.

2. $\operatorname{supp}\hat{\alpha}_1$ is compact. This implies the existence of an estimate for the corresponding solution $\tilde{f}_t(\mathfrak{x})$ of the Klein-Gordon equation of the form

$$|\tilde{f}_t(\mathfrak{x})| \leqq c_N(1+t+|\mathfrak{x}|)^{-N}, \quad |\mathfrak{x}| \geqq \gamma t, 0<\gamma<1, \tag{3.41}$$

for all $N=1,2,3,\ldots$, where γ is a constant and where the constant $c_N>0$ may depend on N (see Reed and Simon [10, p. 43]).

According to these facts we split the integral (3.40) into two terms

$$\int_{\mathbb{R}^3} \ldots d\mathfrak{x} = \int_{|\mathfrak{x}| \leqq \gamma t} \ldots d\mathfrak{x} + \int_{|\mathfrak{x}| \geqq \gamma t} \ldots d\mathfrak{x}, \quad 0<\gamma<1. \tag{3.42}$$

(i) In the case of the second term we use a rough estimate

$$\|(\tilde{V}-S)U_a v\| \leqq \|\tilde{V}\| \cdot \|v\| + \|v\| =: C$$

by a constant, and we obtain

$$\int_{|\mathfrak{x}| \geqq \gamma t} \ldots d\mathfrak{x} \leqq C \cdot c_N \int_{|\mathfrak{x}| \geqq \gamma t} (1+t+|\mathfrak{x}|)^{-N} d\mathfrak{x} < C_N \cdot t^{-N+3}.$$

(ii) In the case of the first term we use the uniform estimate

$$|\tilde{f}_t(\mathfrak{x})| \leqq c t^{-3/2}, \quad \mathfrak{x} \in \mathbb{R}^3,$$

(see Reed and Simon [10, p. 43]). Furthermore, for $|\mathfrak{x}| \leqq \gamma t$ one obtains $\langle a,a\rangle = t^2 - |\mathfrak{x}|^2 \geqq t^2(1-\gamma^2)$. Hence, using the smoothness estimate, we get

$$\int_{|\mathfrak{x}| \leqq \gamma t} \ldots d\mathfrak{x} \leqq c t^{-3/2} c_{n,v}^{+} (1-\gamma^2)^{-n} t^{-2n} (4\pi/3)(\gamma t)^3 < C_n' \cdot t^{-2n+3/2}.$$

Thus the relation (2.5) is proved for $t \to +\infty$. The case $t \to -\infty$ can be treated similarly.

Now it remains to show that 7′ is true. But this follows easily from 8B: Let $C \in \mathcal{L}(\mathcal{H})$, and assume that $u \in \mathcal{D}_0^V$ implies $Cu \in \mathcal{D}_0^V$ and $C^*u \in \mathcal{D}_0^V$. Now assume $(u, CA(f)v) = (A(f^*)u, Cv)$ to be valid for all $u, v \in \mathcal{D}_0^V$ and for all test functions f. This means

$$(C^*u, A(f)v) = (u, A(f)Cv), u, v \in \mathcal{D}_0^V, f \in \mathcal{S}(\mathbb{R}^4).$$

Now we choose test functions appearing in the LSZ-limit process discussed in 8B. Then, for $t \to -\infty$, one obtains immediately

$$(C^* u, A^0(\alpha)v) = (u, A^0(\alpha)Cv), \tag{3.43}$$

where $A^0(\alpha)$ is the free time-zero field for the test function $\hat{\alpha}_1 \in C_0^\infty(\mathbb{R}^3)$, which can be chosen arbitrarily. That is, one has $(u, CA^0(\alpha)v) = (A^0(\bar{\alpha})u, Cv)$ for $u, v \in \mathcal{D}_0^V$. Then this is also true for all $u, v \in \mathcal{D} \supset \mathcal{D}_0^V$ (where $\mathcal{D}$ denotes the domain for the free field operators) and for all $\alpha_1 \in \mathcal{S}(\mathbb{R}^3)$. Note that it is easy to generalize this formula to

$$(u, CA^0(f)v) = (A^0(\bar{f})u, Cv), \qquad u, v \in \mathcal{D}, \tag{3.44}$$

for all $f \in \mathcal{S}(\mathbb{R}^4)$. Namely, first one has to choose test functions of the form $\alpha_1(\mathfrak{x})\beta(x_0)$ instead of $\alpha_1(\mathfrak{x})\delta(x_0)$ in formula (2.6) or in (3.34), where $\hat{\beta}$ has compact support. With $\hat{f}(p) = \hat{\alpha}_1(\mathfrak{p})\hat{\beta}(p_0)$, i.e. $\hat{f}_t(p) = e^{-it\mu(\mathfrak{p})}\hat{\alpha}_1(\mathfrak{p})\hat{\beta}(p_0)$, one obtains formally

$$\begin{aligned} B_t(f) &= e^{itH} \int_{\mathbb{R}^4} \tilde{f}_t(\mathfrak{x})\beta(x_0) U_x V^* A^0(0) V U_{-x} dx\, e^{-itH} \\ &= \int_{\mathbb{R}} \beta(x_0) dx_0 \int_{\mathbb{R}^3} \tilde{f}_t(\mathfrak{x}) U_{-a} V^* A^0(0) V U_a d\mathfrak{x}, \end{aligned}$$

where now $a = (t - x_0, -\mathfrak{x})$ and the estimates can be performed similarly as before. Second, one has to extend the set of test functions by linearity and continuity. But since the free field is cyclic, hence irreducible, from (3.44) the equation $C = \gamma 1$ follows where γ is a scalar. □

4. Appendix

In this appendix we describe briefly a corresponding result in the case where the Lorentz invariance of the field is replaced by rotational invariance, i.e. Property 2.1.5 is replaced by

5′. The field $A(\cdot)$ is translationally invariant (with respect to $\mathbb{R}^4$) and rotationally invariant.

The triple $\{\mathcal{H}, U_g, \omega\}$ is chosen as above. We assume that the prescribed operator S satisfies the conditions I–IV of Sect. 2.2. For convenience we add a further property of S. As before, let $P_1, P_2, \ldots$ denote the projections onto the n-particle spaces. Define Q_j by

$$Q_j := \sum_{\text{finite sum}} P_{n_j}, \qquad j = 1, 2, \ldots,$$

and such that the system $Q_1, Q_2, \ldots$ is disjoint and complete, i.e. $\sum_j Q_j = 1$. Now the additional property can be formulated as follows.

V. $SQ_j = Q_j S$, $\quad j = 1, 2, \ldots$.

Then we can prove the following result.

Theorem 2. *Let the triple* $\{\mathcal{H}, U_g, \omega\}$ *be given as above and let the bounded operator* S *be equipped with the properties* I–V. *Then: there exists a quantum field* $\{A(f), f \in \mathcal{S}(\mathbb{R}^4), \mathcal{D}\}$ *belonging to* $\{\mathcal{H}, U_g, \omega\}$ *and a suitable dense linear set* $\mathcal{D}_0 \subset \mathcal{D}$

such that the properties 1–4, 5′, 6′, 7, *and* 8A *(with respect to* $\mathscr{D}_0$*) are satisfied and such that S is realized as the scattering operator of* $A(\cdot)$ *in the Haag-Ruelle sense* 8A. *That is, the field* $A(\cdot)$ *has Hermitian structure and it satisfies the tempered distribution property. It is translationally and rotationally invariant, weakly local, cyclic and the Haag-Ruelle wave operators exist on* $\mathscr{D}_0$ *realizing S as the corresponding Haag-Ruelle scattering operator.*

Proof. Also in this case one solves first an auxiliary problem, namely to construct an operator V with the following properties:

I. V is unitary.

II. $V \upharpoonright \{\lambda\omega\} \oplus \mathscr{H}_1 = 1$.

III. $VU_g = U_g V$, $g \in \mathscr{E}_+$ (the Euclidean group consisting of the space translations and space rotations).

IV. $\Theta_0 V^* \Theta_0 = SV^*$.

V. $VQ_j = Q_j V$, $j = 1, 2, \ldots$.

VI. The limits $\operatorname{s-lim}_{t \to \pm\infty} e^{itH} V e^{-itH} =: V_\pm$ exist where $V_- = 1$, $V_+ = S$, correspondingly for V^*.

An ansatz for the solution of this auxiliary problem is given by the following formula (put $S = e^{i\eta}$, where η is selfadjoint and bounded):

$$V := \exp\{(i/2) B\eta B - (i/2)\Theta_0 B\eta B\Theta_0\} \exp\{(i/2)\eta\},$$

where B is a selfadjoint and bounded asymptotic constant with limits $B_\pm$, where $B_- \upharpoonright E_{ac}\mathscr{H} = 0$, $B_+ \upharpoonright E_{ac}\mathscr{H} = 1$, $B \upharpoonright \{\lambda\omega\} \oplus \mathscr{H}_1 = 1$ and $BU_g = U_g B$ for all $g \in \mathscr{E}_+$ (the construction of such a B is discussed and performed in [7]).

With the help of V the field can be constructed as a time-sharp field. First we define the time-zero field. Let $\gamma_1 \in \mathscr{S}(\mathbb{R}^3)$, $\gamma(x) = \gamma_1(\mathfrak{x})\delta(x_0)$, $\mathfrak{x} \in \mathbb{R}^3$, where $\mathfrak{x}$ is the space coordinate.

$$A(\gamma) := V^* A^0(\gamma) V, \qquad \gamma_1 \in \mathscr{S}(\mathbb{R}^3). \tag{4.1}$$

This ansatz is in some sense a counterpart to that of Glimm and Jaffe. In the Glimm-Jaffe approach the time-zero field remains unchanged (that is free) but the Hamiltonian will be changed. In our approach the Hamiltonian remains unchanged but the time-zero field is transformed unitarily. Note that $A(\gamma) \upharpoonright Q_j\mathscr{H}$, $j = 1, 2, \ldots$ is bounded. The full field can be defined by time smearing:

$$A(\alpha \otimes \gamma_1) = \int_{-\infty}^{\infty} \alpha(x_0) U_{x_0} V^* A^0(\gamma) V U_{-x_0} dx_0 . \tag{4.2}$$

The corresponding domain is simply given by $\mathscr{D} = \mathscr{H}_{\text{fin}}$ (the set of all finite particle vectors). The CPT-operator Θ is defined as before: $\Theta := S^*\Theta_0$.

Now one has to verify all properties of $A(\cdot)$, expressed in Theorem 2. We drop most of these verifications. The most interesting property is 8A. To verify this property, first we choose a so-called Haag-Ruelle h-function $h(p)$ with the usual well-known properties. Furthermore, let the function $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ be multiplicative-generating. Then one obtains, $\gamma(x) = \gamma_0(\mathfrak{x})\delta(x_0)$,

$$A(\beta * \gamma) = \int_{\mathbb{R}^4} \beta(x) U_x V^* A^0(\gamma) V U_{-x} dx ,$$

where $\hat{\beta}(p)=h(p)\hat{\alpha}(\mathrm{p})$, and where in this case $\hat{\beta}$ means the 4-dimensional Fourier transformation. Now the identification operator $K_n := K \upharpoonright \mathscr{H}_n$ (see [12, 13, 6]) where the symmetrization is dropped is given by

$$K_n\left\{\bigotimes_{j=1}^{n}\{\hat{\alpha}_j\hat{\gamma}_0\}\right\}=\prod_{j=1}^{n}\left\{\int_{\mathbb{R}^4}\beta_j(x)U_xV^*A^0(\gamma)VU_{-x}dx\right\}\omega,$$

and the pre-wave operator by

$$\begin{aligned}&e^{itH}K_n e^{-itH}\left\{\bigotimes_{j=1}^{n}\hat{\alpha}_j\hat{\gamma}_0\right\}\\&\quad=\prod_{j=1}^{n}\left\{\int_{\mathbb{R}^4}\beta_j(x)U_x e^{itH}V^*e^{-itH}A^0(\gamma)e^{itH}Ve^{-itH}U_{-x}dx\right\}\omega.\end{aligned}$$

But the strong limits of this expression for $t\to\pm\infty$ exist, being equal to

$$\begin{aligned}&\prod_{j=1}^{n}\left\{\int_{\mathbb{R}^4}\beta_j(x)U_xV_\pm^*A^0(\gamma)V_\pm U_{-x}dx\right\}\omega=V_\pm^*\prod_{j=1}^{n}\left\{\int_{\mathbb{R}^4}\beta_j(x)U_xA^0(\gamma)U_{-x}dx\right\}\omega\\&\quad=V_\pm^*\prod_{j=1}^{n}\{A^0(\hat{\beta}_j\hat{\gamma})\}\omega=V_\pm^*\left\{\bigotimes_{j=1}^{n}\hat{\alpha}_j\hat{\gamma}_0\right\},\end{aligned}$$

where $V_+=S$, $V_-=1$. That is, the Haag-Ruelle wave operators coincide with V_+^*, V_-^*, i.e. $W_+=V_+^*$, $W_-=V_-^*$ and $W_+^*W_-=V_+V_-^*=S$. Note that the cyclicity is an easy implication of 8A. □

References

1. Jost, R.: TCP-Invarianz der Streumatrix und interpolierende Felder. Helv. Phys. Acta **36**, 77–82 (1963)
2. Schneider, W.: S-Matrix und interpolierende Felder. Helv. Phys. Acta **39**, 81–106 (1966)
3. Todorov, I.T.: Der axiomatische Zugang zur Quantenfeldtheorie. Fortschr. Phys. **13**, 649–700 (1965)
4. Yngvason, J.: Translationally invariant states and the spectrum ideal in the algebra of test functions for quantum fields. Commun. Math. Phys. **81**, 401–418 (1981)
5. Jost, R.: The general theory of quantized fields. Providence, R.I.: Am. Math. Soc. 1965 (quoted according to the Russian translation: Moscow: Izd. Mir 1967)
6. Baumgärtel, H., Wollenberg, M.: Mathematical scattering theory. Berlin: Akademie-Verlag 1983 and Boston: Birkhäuser 1983
7. Baumgärtel, H., Wollenberg, M.: Interpolating asymptotic constants for the Poincaré group, in particular on Fock space. Math. Nachr. **119** (1984) (to appear)
8. Fredenhagen, K., Hertel, J.: Local algebras of observables and pointlike localized fields. Commun. Math. Phys. **80**, 555–561 (1981)
9. Baumgärtel, H.: The inverse problem of scattering theory and quantum field theory. Technical Report ISI Delhi (to appear)
10. Reed, M., Simon, B.: Methods of modern mathematical physics III. Scattering theory. New York: Academic Press 1979
11. Glimm, J., Jaffe, A.: Quantum physics. A functional integral point of view. Berlin, Heidelberg, New York: Springer 1981

12. Baumgärtel, H., Neidhardt, H., Rehberg, J.: On identification operators between free and interacting quantum fields. Math. Nachr. **116**, 75–88 (1984)
13. Baumgärtel, H.: On the structure of relative identification operators for quantum fields and their connection with the Haag-Ruelle scattering theory. Ann. Inst. Henri Poincaré. Phys. Théor. **40**, 235–243 (1984)
14. Hepp, K.: On the connection between the LSZ and Wightman quantum field theory. Commun. Math. Phys. **1**, 95–111 (1965)
15. Streater, R., Wightman, A.: PCT, spin and statistics, and all that. New York: Benjamin 1964 (quoted according to the German translation: Mannheim: Bibliographisches Institut 1969)
16. Ruijsenaars, S.M.M.: A positive energy dynamics and scattering theory for directly interacting relativistic particles. Ann. Phys. **126**, 393–449 (1980)

Communicated by K. Osterwalder

Received December 5, 1983; in revised form March 26, 1984