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Uniqueness and a Priori Bounds for Certain Homoclinic Orbits of a Boussinesq System Modelling Solitary Water Waves

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Abstract. This paper establishes surprisingly precise a priori bounds on the L_∞-norm of certain singular solutions of a system of two nonlinear Sturm-Liouville equations which model solitary water waves.

These solutions can be interpreted as homoclinic orbits for a system of four first order ordinary differential equations. The uniqueness of these homoclinic orbits is established for certain choices of a parameter c, the phase speed of the waves. These observations do not result from perturbation of linear theory, but are global.

I. Introduction

The present paper sets out to further analyse the set of solitary wave solutions of the equations of Boussinesq type which Bona and Smith [1] introduced to model long water waves in a channel. In earlier papers [3–5] it was shown that these equations,

$$\eta_t + u_x + (u\eta)_x - \tfrac{1}{3}\eta_{xxt} = 0\,,$$

$$u_t + \eta_x + uu_x - \tfrac{1}{3}(u_t + \eta_x)_{xx} = 0\,,$$

have travelling solitary wave solutions $(u(x-ct), \eta(x-ct))$ for each value of the wave speed c with $c > 1$. These solutions satisfy the boundary-value problem

$$\left.\begin{array}{ll} c(u - \tfrac{1}{3}u'') = \eta - \tfrac{1}{3}\eta'' + \tfrac{1}{2}u^2 & \text{on } \mathbb{R}\,, \\ c(\eta - \tfrac{1}{3}\eta'') = u + u\eta & \text{on } \mathbb{R}\,, \\ u' < 0\,, \quad \eta' < 0 & \text{on } (0, \infty)\,, \\ \lim\limits_{x\to\infty} u(x) = \lim\limits_{x\to\infty} \eta(x) = 0\,, & u, \eta \text{ even on } \mathbb{R}\,. \end{array}\right\} \qquad (*)$$

For fixed c they correspond to certain homoclinic orbits joining the rest point (0, 0) to itself.

In [3] it was shown that these solutions (c, u, η) of this system of ordinary differential equations contains a set which is connected in $\mathbb{R} \times H^s(\mathbb{R}) \times H^s(\mathbb{R})$, which joins the point $(1,0,0)$ to infinity and whose projection onto $c \in \mathbb{R}$ covers the interval $(1, \infty)$. (See [3, Theorem 3.5] for a precise statement of this existence theory.) The purpose of the present investigation is to establish uniform a priori bounds for solutions of the solitary wave problem and to make some statement about uniqueness. It is shown in Sect. II that any solution (c, u, η) of $(*)$ with $c > 1$ has

$$c - 1/c < |u|_{L^\infty(\mathbb{R})} = u(0) < 2(c-1), \tag{1.1}$$

and

$$2(c-1) < |\eta|_{L^\infty(\mathbb{R})} = \eta(0) < c^2 - 1. \tag{1.2}$$

Moreover,

$$\eta(x) < cu(x) < \left\{1 + \frac{\eta(0)}{2+\eta(0)}\right\} \eta(x), \qquad x \in \mathbb{R}. \tag{1.3}$$

A linearized analysis in a neighbourhood of the rest point $(0,0) = (u, \eta)$ shows that the left-hand inequality in (1.3) is sharp (see the remark after the proof of Lemma 2.3). It is shown in Sect. IV that the right-hand inequality is also sharp for large c (see the remark after the proof of Lemma 4.1).

In fact for large values of c, a rather precise statement can be made; namely that

$$\left.\begin{aligned} u(0) &= \tfrac{3}{2}c + O\left(\frac{1}{c}\right), \\ \eta(0) &= \tfrac{3}{4}c^2 + O(1), \end{aligned}\right\} \quad \text{as} \quad c \to \infty.$$

This suggests a rescaling of the equations as follows:

$$u = cU, \qquad \eta = c^2 V,$$

whence

$$U - \tfrac{1}{3}U'' = V - \tfrac{1}{3}V'' + \tfrac{1}{2}U^2, \tag{1.4a}$$

$$V - \tfrac{1}{3}V'' = \lambda U + UV, \tag{1.4b}$$

where $\lambda = c^{-2}$. In Sect. IV it is shown that the limiting equations, $\lambda = 0$, has a unique solution satisfying suitable boundary conditions, which is given by $U = 2\phi$, $V = \phi$, where ϕ is a solution of the rescaled solitary wave equation for K.dV.

$$\phi - \tfrac{1}{3}\phi'' = 2\phi^2. \tag{1.5}$$

An application of the Inverse Function Theorem in Sect. V yields the existence of a smooth curve of solutions of the system (1.4) for $\lambda > 0$ sufficiently small, i.e. for c sufficiently large. Thus, for large values of c the solutions of $(*)$, when appropriately scaled, are *unique*, and look like solutions of (1.5), the travelling wave solutions of Kd.V. [This follows once it is shown in Sect. IV that all solutions (U, V) of (1.4) converge to $(2\phi, \phi)$ as $\lambda \to 0$.]

It might appear that uniqueness for $(*)$ would follow by writing the pair of second order equations as a system of four first order equations and analyzing the rest point $(0,0,0,0) \in \mathbb{R}^4$. It turns out that both stable and unstable manifold have dimension two, so orbital uniqueness does not follow and the method fails to yield sufficient information to infer the required uniqueness for $(*)$.

However, whatever interest the system has as a model for water waves depends upon assumptions about the smallness of the amplitude of the waves. In this case also we were able to obtain a uniqueness result. The main conclusion of Sect. III is that if for some $c>1$ there exists a solution (u,η) of $(*)$ with $3u(0)/2c \leqq 1$, then that solution is the unique solution of $(*)$ for that choice of c. Clearly, this result implies that near to the bifurcation point $(1,0,0)$ the set of nontrivial solutions of $(*)$ comprises a curve of solutions parametrized by $c>1$. Because of the a priori bounds already mentioned, we can infer the following uniqueness statements:

(i) if (c,u,η) is a solution of $(*)$ and $u(0) \leqq 1$, then (u,η) is the unique solution of $(*)$ for this value of c;

(ii) if $1 < c \leqq \frac{3}{2}$ or if $c \gg 1$ the solution of $(*)$ is unique.

These statements are given here to emphasize that this uniqueness theorem in Sect. III is not about infinitesimally small solutions, but is in a certain sense global. We only regret that so far the method does not give uniqueness for all $c>1$.

II. A Priori Estimates

2.1. Bounds for $(u(0), \eta(0))$

It has been established (Toland [3]) that there exist solutions (u,η) of the following problem for every $c>1$:

$$c(u-\tfrac{1}{3}u'')=(\eta-\tfrac{1}{3}\eta'')+\tfrac{1}{2}u^2, \tag{2.1}$$

$$c(\eta-\tfrac{1}{3}\eta'')=u+u\eta, \tag{2.2}$$

$$u,\ \eta \text{ positive and even on } \mathbb{R}, \tag{2.3}$$

$$u'<0, \quad \eta'<0 \quad \text{on} \quad (0,\infty), \tag{2.4}$$

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} \eta(x) = 0. \tag{2.5}$$

Since (2.1), (2.2) is the system of Euler-Lagrange equations for the functional

$$\tfrac{1}{2}\int_{\mathbb{R}} \{(1+\eta)u^2+\eta^2+\tfrac{1}{3}\eta'^2\} - c\int_{\mathbb{R}} \{u\eta+\tfrac{1}{3}u'\eta'\},$$

it follows by Noether's theorem (or by direct verification) that if (u,η) satisfies (2.1)–(2.5), then the Hamiltonian

$$H(\eta,u)=\tfrac{1}{2}((1+\eta)u^2+\eta^2)-cu\eta-\tfrac{1}{6}\eta'^2+\frac{c}{3}u'\eta'=0 \quad \text{on } \mathbb{R}. \tag{2.6}$$

Now because u and η are even functions it follows that $u'(0)=\eta'(0)=0$, and so by (2.6)

$$\tfrac{1}{2}((1+\eta(0))u(0)^2+\eta(0)^2)-cu(0)\eta(0)=0 \tag{2.7}$$

if (u, η) satisfies (2.1)–(2.5) for some $c>1$. (Henceforth, we will suppose $c>1$ is fixed.) The implicit relation (2.7) gives a locus $\mathscr{C}$ of possible points, where $(u(0), \eta(0))$ might lie. A uniqueness theorem for the system would then say that there exists exactly one point (u_0, η_0) on this locus such that the solution of the initial value problem for (2.1), (2.2) with $u(0)=u_0$, $\eta(0)=\eta_0$, $u'(0)=\eta'(0)=0$ yields a solution of (2.1)–(2.5).

In the first quadrant $\mathscr{C}$ is a simple closed curve which is smooth everywhere except at the origin. If points on $\mathscr{C}$ are denoted by (u_0, η_0), then

$$u_0 = \frac{\{c \pm \sqrt{(c^2-1-\eta_0)}\}\eta_0}{1+\eta_0}. \tag{2.8}$$

A sketch of $\mathscr{C}$ is given in Fig. 1. The following *a priori* bounds are now obvious.

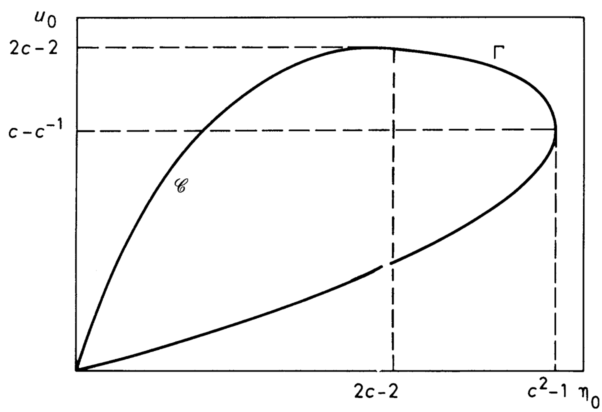

Fig. 1. The locus of possible locations of $(u(0), \eta(0))$

Lemma 2.1. *If* (u, η) *satisfies* (2.1)–(2.5), *then*

$$0 < u(x) \leqq u(0) \leqq 2(c-1) \tag{2.9}$$

and

$$0 < \eta(x) \leqq \eta(0) \leqq c^2-1 \tag{2.10}$$

for all $x \in \mathbb{R}$.

Proof. An examination of Fig. 1 yields the result. q.e.d.

In the next section we will see how global consideration of the solutions make it possible to exclude a large portion of $\mathscr{C}$ from further consideration.

2.2. *A Priori Bounds on* (u, η)

In this section we relate the global behaviour of u and η on $\mathbb{R}$.

Lemma 2.2. *If* (u, η) *satisfies* (2.1)–(2.5), *then*

$$cu(x) > \eta(x) > 0, \qquad x \in \mathbb{R}, \tag{2.11}$$

$$cu'(x) < \eta'(x) < 0, \qquad x \in (0, \infty). \tag{2.12}$$

Proof. From (2.1) it follows that $(cu-\eta)-\frac{1}{3}(cu-\eta)''>0$ on $\mathbb{R}$, whence by (2.5) and the Maximum Principle (2.11) follows. By (2.1) and (2.2)

$$-\frac{c}{3}u''=\left(\frac{1}{c}-c\right)u+\frac{1}{c}u\eta+\tfrac{1}{2}u^2,$$

and now (2.5) implies that $u'\to 0$ as $x\to\infty$. After differentiating (2.1), an application of the Maximum Principle on $(0,\infty)$ yields (2.12). q.e.d.

The next result is a strengthening of a result which was proved in [3] for certain solutions of (2.1)–(2.5).

Lemma 2.3. *If* (u,η) *satisfies* (2.1)–(2.5), *then*

$$cu(x)\leqq\left\{1+\frac{\eta(0)}{2+\eta(0)}\right\}\eta(x)<2\eta(x),\qquad x\in\mathbb{R},\tag{2.13}$$

and

$$0>cu'(x)\geqq 2\eta'(x),\qquad x\in(0,\infty).\tag{2.14}$$

Proof. For any $\alpha>0$ it follows from (2.1) and (2.2) that

$$\begin{aligned}((1+\alpha)\eta-cu)-\tfrac{1}{3}((1+\alpha)\eta-cu)''&=\frac{\alpha}{c}(u+u\eta)-\frac{u^2}{2}\\&=\frac{u}{2c}(2\alpha+2\alpha\eta-cu\}\qquad\qquad(2.15)\\&=\frac{u}{2c}((1+\alpha)\eta-cu)+\frac{u}{2c}(2\alpha(1+\eta)-(1+\alpha)\eta).\end{aligned}$$

Now observe that, by (2.9), $(u/2c)<1$ on $\mathbb{R}$, and that

$$\left(1-\frac{u}{2c}\right)((1+\alpha)\eta-cu)-\tfrac{1}{3}((1+\alpha)\eta-cu)''=\frac{u}{2c}(2\alpha(1+\eta)-(1+\alpha)\eta).\tag{2.16}$$

Now we put $\alpha=\dfrac{\eta(0)}{2+\eta(0)}$, and note that the right-hand side of (2.16) becomes

$$\frac{u}{c}\left(\frac{\eta(0)-\eta}{2+\eta(0)}\right)\geqq 0\quad\text{on }\mathbb{R}.$$

Equation (2.16) and the Maximum Principle then yield (2.13).

Now putting $\alpha=1$ in (2.16) yields

$$\left(1-\frac{u}{2c}\right)(2\eta-cu)-\tfrac{1}{3}(2\eta-cu)''=\frac{u}{c}.$$

Since $u'<0$ on $(0,\infty)$ and $2\eta-cu>0$ by (2.13) it follows by the result of Gidas et al. [2, Theorems 3, p. 222] that

$$(2\eta-cu)'<0\quad\text{on}\quad(0,\infty).\quad\text{q.e.d.}$$

Remark 1. An analysis of the system (2.1), (2.2) in a neighbourhood of the rest point (0, 0) shows that no eigenvalue of the linearization is complex or zero. The Stable Manifold Theorem then ensures that for solutions of (2.1)–(2.5)

$$\lim_{x\to\infty}\frac{u'(x)}{\eta'(x)}=\lim_{x\to\infty}\frac{u(x)}{\eta(x)}=c^{-1}.$$

Hence the result of Lemma 2.2 is sharp. In Sect. IV we will see that as $c\to\infty$ the conclusion of Lemma 2.3 is also sharp. Hence these results are optimal uniformly in $x\in\mathbb{R}$ and $c>1$.

Remark 2. According to Lemma 2.2 the expression $\left(\frac{c}{3}u'-\frac{1}{6}\eta'\right)<0$ on $(0,\infty)$, and so $\frac{c}{3}u'\eta'-\frac{1}{6}\eta'^2>0$ on $\mathbb{R}$. It follows from (2.6) that $\frac{1}{2}((1+\eta)u^2+\eta^2)-cu\eta<0$ on $\mathbb{R}$; in other words, a trajectory lies entirely inside $\mathscr{C}$.

2.3. Further Bounds on $(u(0),\eta(0))$

The following curious observation is the key to further development. The curve $\mathscr{C}$ of possible values of $(u(0),\eta(0))$ is given by

$$\tfrac{1}{2}(1+\eta_0)u_0^2+\tfrac{1}{2}\eta_0^2-cu_0\eta_0=0,$$

and so its slope is given locally by

$$\frac{du_0}{d\eta_0}=\frac{cu_0-\eta_0-\frac{1}{2}u_0^2}{u_0+u_0\eta_0-c\eta_0}. \tag{2.17}$$

However, if for some solution of (2.1)–(2.5), $(u(0),\eta(0))=(u_0,\eta_0)\in\mathscr{C}$, then

$$\frac{cu_0-\eta_0-\frac{1}{2}u_0^2}{u_0+u_0\eta_0-c\eta_0}=\frac{cu''(0)-\eta''(0)}{-c\eta''(0)}=\frac{1}{c}-\frac{u''(0)}{\eta''(0)}. \tag{2.18}$$

However, since by L'Hôpital's rule

$$\lim_{x\to 0}\frac{u'(x)}{\eta'(x)}=\frac{u''(0)}{\eta''(0)},$$

it follows from Lemmas 2.2 and 2.3 that

$$\frac{2}{c}\geqq\frac{u''(0)}{\eta''(0)}\geqq\frac{1}{c}. \tag{2.19}$$

The following result is now immediate.

Lemma 2.4. *If* (u,η) *satisfies* (2.1)–(2.5), *then* $(u(0),\eta(0))\in\mathscr{C}$ *at a point, where the slope* $\frac{du_0}{d\eta_0}$ *of* $\mathscr{C}$ *satisfies the inequalities*

$$-\frac{1}{c}\leqq\left.\frac{du_0}{d\eta_0}\right|_{(u(0),\eta(0))}\leqq 0. \tag{2.20}$$

Proof. This is immediate from (2.17)–(2.19). q.e.d.

Remark. The significance of this observation is that only a small portion of the locus $\mathscr{C}$ can contain $(u(0), \eta(0))$ for solutions of (2.1)–(2.5). For emphasis we express (2.20) in words: only the portion Γ of $\mathscr{C}$, where u_0 is *decreasing* with respect to η_0 *at a rate not greater than* $1/c$ are possible locations for $(u(0), \eta(0))$. In particular, $(u(0), \eta(0))$ cannot be very close to the point S of vertical tangency of $\mathscr{C}$. Now we will see that it cannot be too close to R, the point of horizontal tangency of $\mathscr{C}$ (see Fig. 2).

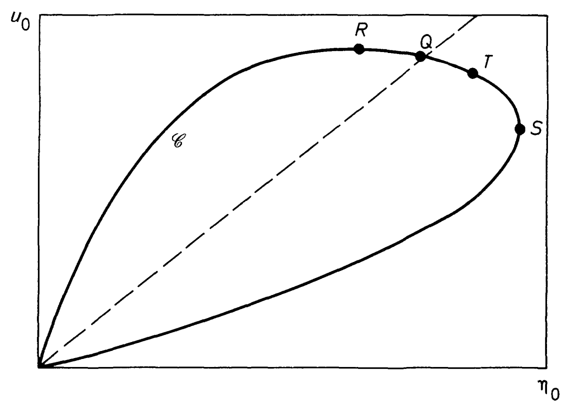

Fig. 2. —— denotes the line $u_0 = 2c\eta_0/(c^2+1)$. So $(u(0), \eta(0))$ must lie to the right of Q on Γ. If T denotes the point, where the slope of $\mathscr{C}$ is $-c^{-1}$, then $(u(0), \eta(0))$ lies on Γ to the left of T (see Lemmas 2.4 and 2.5)

Lemma 2.5. *If* (u, η) *satisfies* (2.1)–(2.5), *then*

$$u(0) < \left(\frac{2c}{c^2+1}\right)\eta(0).$$

In particular,

$$u(0) < \frac{3c^4 - 2c^2 - 1}{2c(c^2+1)} \quad \textit{and} \quad \eta(0) > \frac{3c^4 - 2c^2 - 1}{4c^2}.$$

Proof. According to Lemma 2.3,

$$u(0) \leqq \frac{1}{c}\left\{1 + \frac{\eta(0)}{2+\eta(0)}\right\}\eta(0),$$

which, on account of (2.10), yields the estimate

$$u(0) \leqq \frac{1}{c}\left\{1 + \frac{c^2-1}{c^2+1}\right\}\eta(0) = \left(\frac{2c}{c^2+1}\right)\eta(0).$$

Hence $(u(0), \eta(0))$ lies on $\mathscr{C}$ to the right of the point, where the line $u_0 = 2c\eta_0/(c^2+1)$ intersects $\mathscr{C}$. A calculation now yields this point to be

$$(u, \eta) = Q = \frac{(3c^4 - 2c^2 - 1)}{2c}\left(\frac{1}{c^2+1}, \frac{1}{2c}\right). \quad \text{q.e.d.}$$

We will return to these estimates of the location of $(u(0), \eta(0))$ in Sect. IV. At present we turn our attention to the question of uniqueness.

III. Uniqueness near Bifurcation

The proof is by *reductio ad absurdum*. Suppose that there are two solutions (u_1, η_1) and (u_2, η_2) of the problem (2.1)–(2.5) for some fixed $c>1$, and without loss of generality suppose that $\eta_1(0)>\eta_2(0)$. [If $\eta_1(0)=\eta_2(0)$, then $u_1(0)=u_2(0)$ because of (2.7) and the observation that $(u_1(0), \eta_1(0))$ and $(u_2(0), \eta_2(0))$ must lie on Γ.] From Lemma 2.4 and the remark following it we know that

$$0<u_2(0)-u_1(0)<\frac{1}{c}(\eta_1(0)-\eta_2(0)). \tag{3.1}$$

Let

$$Z=\sup\{z\in[0,\infty): u_2(x)\geqq u_1(x),\ x\in[0,z]\}. \tag{3.2}$$

Clearly, $Z>0$.

Lemma 3.1. *If* $3u_1(0)/2c\leqq 1$, *then* $Z=\infty$.

Proof. Let W be defined on $[0,Z]$ by

$$W(x)=cu_1(x)+2\eta_1(x)-cu_2(x)-2\eta_2(x). \tag{3.3}$$

Then W satisfies the identities

$$\begin{aligned} W-\tfrac{1}{3}W'' &= \frac{3}{c}(u_1+u_1\eta_1-u_2-u_2\eta_2)+\tfrac{1}{2}(u_1^2-u_2^2) \\ &= (u_1-u_2)\left\{\frac{3}{c}+\tfrac{1}{2}(u_1+u_2)\right\}+\frac{3}{c}(u_1\eta_1-u_2\eta_2). \end{aligned} \tag{3.4}$$

Now

$$u_1\eta_1-u_2\eta_2=\frac{u_1}{2}W+(u_1-u_2)\left(\eta_2-\frac{cu_1}{2}\right),$$

and so (3.4) becomes

$$(1-3u_1/2c)W-\tfrac{1}{3}W''=(u_1-u_2)\left\{\frac{3}{c}+\tfrac{1}{2}(u_1+u_2)+\frac{3}{c}\left(\eta_2-\frac{cu_1}{2}\right)\right\}.$$

Since by definition $u_2(x)\geqq u_1(x)$, $x\in[0,Z]$, and since for all $x\in\mathbb{R}$, $2\eta_2(x)\geqq cu_2(x)$, it follows that $\eta_2-cu_1/2\geqq 0$ on $[0,Z]$, and so

$$\left(1-\frac{3u_1}{2c}\right)W-\tfrac{1}{3}W''<0 \quad \text{on} \quad [0,Z].$$

Now observe that $W(0)>\eta_1(0)-\eta_2(0)>0$, by (3.1). Therefore, $W''(0)>0$, since by assumption $3u_1(0)/2c\leqq 1$.

Since $3u_1(x)/2c\leqq 1$, $x\in[0,Z]$, and $W'(0)=0$, it follows that

$$W(x)>0, \quad W'(x)>0, \quad W''(x)>0, \quad x\in[0,Z]. \tag{3.5}$$

Now the ratio u_1/u_2 satisfies

$$\frac{u_1(x)}{u_2(x)} \leqq 1, \quad x \in [0, Z], \tag{3.6}$$

$$\left(\frac{u_1}{u_2}\right)'(0) = 0. \tag{3.7}$$

and

$$\begin{aligned}\left(\frac{u_1}{u_2}\right)'' + \left(\frac{2u_2'}{u_2}\right)\left(\frac{u_1}{u_2}\right)' &= \left(\frac{u_1''}{u_1} - \frac{u_2''}{u_2}\right)\left(\frac{u_1}{u_2}\right) \\ &= -\frac{3}{c}\left\{\frac{\eta_1}{c} + \frac{u_1}{2} - \frac{\eta_2}{c} - \frac{u_2}{2}\right\} = -\frac{3}{2c^2}W \\ &< 0 \quad \text{for all} \quad x \in [0, Z]. \end{aligned}\tag{3.8}$$

However, since (3.7) holds, the inequality (3.8) yields that

$$\left(\frac{u_1}{u_2}\right)''(0) < 0,$$

and it follows that

$$\left(\frac{u_1}{u_2}\right)'(x) < 0, \quad x \in (0, \varepsilon), \tag{3.9}$$

for some $\varepsilon > 0$. However, (3.8) now yields that

$$\left(\frac{u_1}{u_2}\right)'' < -\left(\frac{2u_2'}{u_2}\right)\left(\frac{u_1}{u_2}\right)' < 0 \quad \text{on} \quad (0, \varepsilon),$$

since $u_2' < 0$ on $(0, \infty)$ and (3.9) holds. It follows at once that

$$\left(\frac{u_1}{u_2}\right)' < 0 \quad \text{and} \quad \left(\frac{u_1}{u_2}\right)'' < 0 \quad \text{on} \quad [0, Z]. \tag{3.10}$$

If $Z < \infty$, then $u_2(Z) = u_1(Z)$, by definition. However, $u_1(0)/u_2(0) < 1$ and (3.10) implies that $u_1(Z)/u_2(Z) < 1$. This contradiction implies that $Z = \infty$. q.e.d.

The main result of this section is then the following.

Theorem 3.2. *If* $3u_1(0)/2c \leqq 1$, *then* (u_1, η_1) *is the unique solution of* (2.1)–(2.5).

Proof. Suppose, as before, that there are two solutions (u_1, η_1) and (u_2, η_2). Since $Z = \infty$, we find from (3.10) that

$$\left(\frac{u_1}{u_2}\right) > 0, \quad \left(\frac{u_1}{u_2}\right)' < 0, \quad \left(\frac{u_1}{u_2}\right)'' < 0 \quad \text{on} \quad (0, \infty).$$

Clearly, no function can satisfy such inequalities, and this contradiction proves the result. q.e.d.

Remark. The conclusion implies that if there is a solution (u, η) of (2.1)–(2.5) for some $c > 1$ which satisfies the condition $3u(0)/2c \leqq 1$, then it is the unique solution for that value of c. Since the solutions (u, η) bifurcate from $(0,0) \in L_\infty(\mathbb{R}) \times L_\infty(\mathbb{R})$ at $c=1$, this result indicates that in a substantial neighbourhood of the bifurcation point the solution set comprises a continuous curve of solutions, since already we know it contains a connected set of solutions (see [3]).

From the estimate in Lemma 2.5 we know that for any solution

$$\frac{3u(0)}{2c} < \frac{3}{4}\left(\frac{3c^4-2c^2-1}{c^2(c^2+1)}\right).$$

Hence if $c<1.504$, there is a unique solution. Using the fact that for every solution $u(0)<2(c-1)$, we obtain

$$\frac{3u(0)}{2c} \leqq \frac{3u(0)}{2+u(0)},$$

and so far any $c>1$ there is at most one solution with $u(0) \leqq 1$.

Clearly, as it stands our understanding is incomplete, because for every $c>1$

$$u(0)>(c-1/c)$$

(see Fig. 2), and for $c>\sqrt{3}$ the hypotheses of Theorem 3.2 are violated.

However, in Sect. V we will see that for large c there is exactly one solution, using a different method.

IV. Asymptotic Behaviour as $c \to \infty$

4.1. Further Estimates for $(u(0), \eta(0))$

To recap what we know so far:

$$u(0) < \frac{3c}{2} - \frac{5c^2+1}{2(c^3+c)}, \tag{4.1}$$

$$\eta(0) > \frac{3c^2}{4} - \frac{2c^2+1}{4c^2} \quad \text{(Lemma 2.5)}. \tag{4.2}$$

The next result gives the opposite bounds for $u(0), \eta(0)$, which are effective for large c.

Lemma 4.1. *If* (u, η) *solves* (2.1)–(2.5), *then*

$$\eta(0) \leqq \frac{cu(0)}{2} + \frac{u(0)}{2c-u(0)}, \tag{4.3}$$

whence

$$\frac{cu(0)}{2} \leqq \eta(0) \leqq \frac{cu(0)}{2} + 3. \tag{4.4}$$

Proof. By Lemma 2.4 and inequality (2.17)

$$\frac{cu(0)-\eta(0)-\frac{1}{2}u(0)^2}{u(0)+u(0)\eta(0)-c\eta(0)} \geqq -\frac{1}{c}.$$

Therefore, since the denominator of the left-hand side is $-c\eta''(0)/3>0$, it follows that

$$\eta(0) \leqq \frac{cu(0)}{2} + \frac{u(0)}{2c-u(0)}. \tag{4.5}$$

Therefore, by (4.1)

$$\eta(0) \leqq \frac{cu(0)}{2} + \frac{3c}{2\left(\frac{1}{2}c + \frac{5c^2+1}{2(c^3+1)}\right)}.$$

The other part of (4.4) has been proved in Lemma 2.3. q.e.d.

Remark. This shows that the estimate given in Lemma 2.3 is sharp for large values of c.

Corollary 4.2. *If* $(u(0), \eta(0))$ *solves* (2.1)–(2.5), *then*

$$\eta(0) \leqq \frac{3c^2}{4} + 3,$$

and

$$u(0) \geqq \frac{3c}{2} - \frac{2c^2+1}{2c^3} - \frac{4}{c^2}. \tag{4.7}$$

Proof. From (4.1) and (4.4) it follows that

$$\eta(0) \leqq \frac{3c^2}{4} + 3.$$

Hence by (4.2) and (4.4)

$$u(0) \geqq \frac{3c}{2} - \frac{2c^2+1}{2c^3} - \frac{6}{c}. \quad \text{q.e.d.}$$

Remark. These estimates give precise asymptotic values for $(u(0), \eta(0))$ as $c \to \infty$, viz

$$(u(0), \eta(0)) \sim \left(\frac{3c}{2} + O\left(\frac{1}{c}\right), \frac{3c^2}{4} + O(1)\right). \tag{4.8}$$

This observation is central in the next section.

4.2. *Limiting Equation as* $c \to \infty$ *is* K.dV

If (u, η) is a solution of (2.1)–(2.5) for some $c>1$, let $cU(x) = u(x)$, $c^2V(x) = \eta(x)$, and $\lambda = 1/c^2$. Then U, V satisfy the equations

$$U - \tfrac{1}{3}U'' = V - \tfrac{1}{3}V'' + \tfrac{1}{2}U^2, \tag{4.9}$$

$$V - \tfrac{1}{3}V'' = \lambda U + UV, \tag{4.10}$$

and the boundary conditions (2.3)–(2.5). Here values of λ close to zero correspond to large values of c, and so (4.8) implies that if (U, V) satisfies (4.9), (4.10), then

$$(U(0), V(0)) = (\tfrac{3}{2}, \tfrac{3}{4}) + O(\lambda) \quad \text{as} \quad \lambda \to 0. \tag{4.11}$$

Theorem 4.3. *When* $\lambda = 0$ *Eqs.* (4.9), (4.10) *with the boundary condition* (2.3)–(2.5) *have a unique solution. This solution is* $U = 2\phi$, $V = \phi$, *where* ϕ *is the solitary wave solution of the* Kd.V *equation travelling wave equation*

$$-\tfrac{1}{3}\phi'' = -\phi + 2\phi^2, \tag{4.12}$$

namely,

$$\phi(x) = \tfrac{3}{4}\operatorname{sech}^2(\sqrt{\tfrac{3}{4}}\,x). \tag{4.13}$$

Proof. It is sufficient to show that if (U, V) is any solution of (4.9), (4.10) with $\lambda = 0$ and the boundary conditions (2.3)–(2.5), then $U = 2V$. Then a substitution reduces both (4.9) and (4.10) to (4.12) and the unique (non-zero) solitary wave solution is that given in (4.13).

If (U, V) solves (4.9) and (2.5), it follows by the Maximum Principle that $U > V$ on $\mathbb{R}$. Moreover, from (4.9) and (4.10) with $\lambda = 0$ we obtain

$$(U - 2V) - \tfrac{1}{3}(U - 2V)'' = \tfrac{1}{2}U^2 - UV = \tfrac{1}{2}U(U - 2V).$$

Hence

$$[1 - \tfrac{1}{2}U](U - 2V) - \tfrac{1}{3}(U - 2V)'' = 0.$$

However, $\tfrac{1}{2}U(0) \leq 1$ because $U'(0) = V'(0) = 0$ and $\tfrac{1}{2}((\lambda + V)U^2 + V^2) - UV - \tfrac{1}{6}V'^2 + \tfrac{1}{3}U'V' \equiv 0$ [from (2.6)] implies that $U(0) < 2$. Hence the maximum principle implies that $U - 2V$ has neither a positive local maximum, nor a negative local minimum. Hence $U \equiv 2V$. q.e.d.

Corollary 4.4. *Suppose that* (U_n, V_n) *satisfies* (4.9), (4.10) *and the boundary conditions* (2.3)–(2.5) *for some* λ_n. *If* $\lambda_n \to 0$ *then* $(U_n, V_n) \to (2\phi, \phi)$ *in* $H^1(\mathbb{R})$ *as* $n \to \infty$. *Hence* $(U_n, V_n) \to (2\phi, \phi)$ *uniformly on* $\mathbb{R}$ *as* $n \to \infty$.

Proof. From the fact that U_n and V_n are bounded independently of n [see (4.11)] it follows from (4.9) and (4.10) that U_n, V_n, U_n', V_n' are uniformly bounded and equicontinuous on $\mathbb{R}$. Hence there exists a subsequence (U_{n_k}, V_{n_k}) which converges in $C^2(I)$ for every compact interval I of $\mathbb{R}$ to a solution of (4.9), (4.10) with $\lambda = 0$. On account of (4.11), this solution is non-trivial and must therefore be $U = 2\phi$, $V = \phi$, where ϕ is given by (4.13). Now because $U_{n_k} \to 2\phi$, $V_{n_k} \to \phi$ uniformly on I for every compact interval I, and because U_{n_k}, V_{n_k} and ϕ are even, monotonic non-increasing on $(0, \infty)$ and converge to zero as $x \to \infty$, the uniform convergence of (U_{n_k}, V_{n_k}) to $(2\phi, \phi)$ follows. We have shown that *every* sequence $\{(U_n, V_n)\}$ of solutions of (4.9), (4.10) with $\lambda_n \to 0$ has a subsequence which converges uniformly to $(2\phi, \phi)$. It is, therefore, immediate that $(U_n, V_n) \to (2\phi, \phi)$ uniformly on $\mathbb{R}$ as $n \to \infty$. Since (U_n, V_n) satisfy the differential equations in (4.9), (4.10), the convergence in $H^1(\mathbb{R})$ follows at once. q.e.d.

In the next section we use the implicit function theorem to show that for large values of c the solution of (4.9), (4.10) is unique.

V. Uniqueness for Large c

If we write a solution of (4.9), (4.10) as $U=2\phi+h$, $V=\phi+k$, then we obtain equations which h and k must satisfy:

$$h-\tfrac{1}{3}h''=k-\tfrac{1}{3}k''+2\phi h+\tfrac{1}{2}h^2\,, \tag{5.1}$$

$$k-\tfrac{1}{3}k''=2\lambda\phi+\lambda h+\phi h+2\phi k+hk\,. \tag{5.2}$$

In addition we require that

$$h,\ k \text{ are even on } \mathbb{R}\,, \tag{5.3}$$

$$\lim_{x\to\infty} h(x)=\lim_{x\to\infty} k(x)=0\,. \tag{5.4}$$

If G denotes the Green's function for the differential operator L given by

$$Lu=u-\tfrac{1}{3}u'' \quad \text{on } \mathbb{R}$$

subject to zero boundary conditions at $x=\pm\infty$, then (5.1), (5.2) can be rewritten

$$h-k-2A(\phi h)-\tfrac{1}{2}A(h^2)=0\,, \tag{5.5}$$

$$k-A(\phi h)-2A(\phi k)-A(hk)-\lambda A(h)=2\lambda A(\phi)\,, \tag{5.6}$$

where the operator A is defined for continuous functions u which converge to zero at $x=\pm\infty$ by

$$A(u)(x)=\int_{\mathbb{R}} G(x,y)u(y)dy\,.$$

Here

$$G(x,y)=\sqrt{\tfrac{3}{2}}\exp(-\sqrt{3}|x-y|)\,, \qquad x,y\in\mathbb{R}\,.$$

Let $H_e^1(\mathbb{R})$ denote the Hilbert space of even functions in $H^1(\mathbb{R})$. The following properties of A are now easily verified:

$$A: H_e^1(\mathbb{R})\to H_e^1(\mathbb{R})\,,$$

$$|A(u)|_{H_e^1(\mathbb{R})}\leqq(\text{const})|u|_{L_2(\mathbb{R})}\leqq(\text{const})|u|_{H_e^1(\mathbb{R})}\,,$$

$$|A(uv)|_{H_e^1(\mathbb{R})}\leqq|u|_{L^\infty(\mathbb{R})}|v|_{H_e^1(\mathbb{R})}\leqq|u|_{H_e^1(\mathbb{R})}|v|_{H_e^1(\mathbb{R})}\,.$$

[Note the last property makes sense since $H_e^1(\mathbb{R})$ is an algebra.] These properties can be seen most simply by noting that

$$\int_{\mathbb{R}} u(x)^2dx=\int_{\mathbb{R}}\hat{u}(k)^2dk \quad \text{(Parseval's identity)},$$

$$|u|^2_{H_e^1(\mathbb{R})}=\frac{1}{2\pi}\int_{\mathbb{R}}(1+k^2)\hat{u}(k)^2dk\,, \qquad u\in H_e^1(\mathbb{R})\,,$$

and

$$(A(u))\hat{}\,(k)=\frac{3\hat{u}(k)}{3+k^2}\,,$$

where ^ denotes Fourier transform. Let $X=H_e^1(\mathbb{R})\times H_e^1(\mathbb{R})\times\mathbb{R}$.

Now define the operator $\mathscr{A}: X \to X$ by

$$\mathscr{A}\begin{pmatrix} h \\ k \\ \lambda \end{pmatrix} = \begin{pmatrix} h-k-2A(\phi h)-\frac{1}{2}A(h^2) \\ k-A(\phi h)-2A(\phi k)-A(hk)-\lambda A(h) \\ \lambda \end{pmatrix}.$$

Clearly, $\mathscr{A}: X \to X$ is a smooth mapping, and we want to investigate its invertability in a neighbourhood of $(0,0,0) \in X$ using the Inverse Function Theorem. The linearization of $\mathscr{A}$ about $(0,0,0) \in X$ is

$$\mathscr{L}\begin{pmatrix} h \\ k \\ \lambda \end{pmatrix} = \begin{pmatrix} h-k-2A(\phi h) \\ k-A(\phi h)-2A(\phi k) \\ \lambda \end{pmatrix}.$$

Lemma 5.1. *The operator $\mathscr{L}: X \to X$ has a continuous inverse $\mathscr{L}^{-1}: X \to X$.*

Proof. Since $\mathscr{L}: X \to X$ is a bounded linear operator between Banach spaces it suffices, by the Open Mapping Theorem, to show that $\mathscr{L}$ is invertable. Let $(\xi, \varrho, \mu) \in X$. We need to show that the equation

$$\mathscr{L}\begin{pmatrix} h \\ k \\ \lambda \end{pmatrix} = \begin{pmatrix} \xi \\ \varrho \\ \mu \end{pmatrix} \tag{5.7}$$

has a unique solution $(h, k, \lambda) \in X$ for every right-hand side. This is so if and only if the equations

$$h-3A(\phi h)-2A(\phi k)=\xi+\varrho \quad (=\gamma \text{ say}), \tag{5.8}$$

$$k-A(\phi h)-2A(\phi k)=\varrho, \tag{5.9}$$

$$\lambda=\mu, \tag{5.10}$$

having a unique solution for every right-hand side $(\gamma, \varrho, \mu) \in X$. Clearly, $\lambda=\mu$. Now since $\phi(x)$ decays to zero at $x=\pm\infty$, it follows immediately that the mapping

$$\begin{pmatrix} h \\ k \end{pmatrix} \mapsto \begin{pmatrix} 3A(\phi h)+2A(\phi k) \\ A(\phi h)+2A(\phi k) \end{pmatrix}$$

maps $H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$ compactly into itself. Hence by the Fredholm Alternative, (5.8), (5.9) are uniquely solvable for every right-hand side if and only if the system

$$h-3A(\phi h)-2A(\phi k)=0, \tag{5.11}$$

$$k-A(\phi h)-2A(\phi k)=0 \tag{5.12}$$

has a unique solution in $H_e^1(\mathbb{R}) \times H_e^1(\mathbb{R})$, namely, $h=k=0$. Now if (h,k) satisfies (5.11) and (5.12) then clearly $h, k \in C^2(\mathbb{R})$ and are even,

$$h-\tfrac{1}{3}h''=3\phi h+2\phi k, \tag{5.13}$$

$$k-\tfrac{1}{3}k''=\phi h+2\phi k, \tag{5.14}$$

and

$$\lim_{x\to\infty} h(x)=\lim_{x\to\infty} k(x)=0. \tag{5.15}$$

Now multiplying (5.14) by 2 and subtracting from (5.13) gives

$$(1-\phi)(h-2k)-\tfrac{1}{3}(h-2k)''=0\,.$$

By (4.13), $(1-\phi)\geqq 1/4$ on $\mathbb{R}$, and so by the Maximum Principle $h-2k$ can neither have a positive maximum nor a negative minimum on $\mathbb{R}$. Hence, $h=2k$ if (5.13) and (5.14) hold. It suffices, therefore, to show that in $H^1(\mathbb{R})$ the only *even* solution of the problem

$$h-\tfrac{1}{3}h''=4\phi h\,, \tag{5.16}$$

$$h(x)\to 0 \quad \text{as} \quad |x|\to\infty \tag{5.17}$$

is $h\equiv 0$ on $\mathbb{R}$. Here ϕ is given explicitly by (4.13). Note that ϕ' satisfies (5.16) [this is clear from differentiating (4.12)]; however, ϕ' is an odd function and so does not lie in $H^1_e(\mathbb{R})$. But it is easy to see that ϕ' is the only (normalized) solution of (5.16) which decays to zero at $x=\pm\infty$. [To see this, suppose that ϕ' and h are such solutions. Then $(\phi'h'-\phi''h)'=0$, whence $\phi'h'-\phi''h=0$ on account of the asymptotic behaviour, and $\phi'/h=\text{const.}$] This completes the proof of the lemma. q.e.d.

Theorem 5.2. *For all c sufficiently large there is a unique solution of* (2.1)–(2.5). *Moreover, the solution (u,η) is a smooth function of $c^{-2}\in\mathbb{R}$.*

Proof. By the Inverse Mapping Theorem the smooth function $\mathscr{A}:X\to X$ is injective in a neighbourhood of zero in X, and its inverse is a smooth function. Now this means that for each $\mu>0$ sufficiently small there exists a unique solution of the equation

$$\mathscr{A}\begin{pmatrix} h\\ k\\ \lambda\end{pmatrix}=\begin{pmatrix} h-k-2A(\phi h)-\tfrac{1}{2}A(h^2)\\ k-A(\phi h)-2A(\phi k)-A(hk)-\lambda A(h)\\ \lambda\end{pmatrix}=\begin{pmatrix} 0\\ 2\mu A(\phi)\\ \mu\end{pmatrix}.$$

Clearly, $\lambda=\mu$ and (h,k,λ) is a solution of (5.5), (5.6). But since (5.5), (5.6) is another version of (5.1), (5.2), we have established the existence of a unique smooth curve of solutions for $\lambda>0$ but close to $(0,0,0)$ of (4.9), (4.10). Since for $\lambda>0$ sufficiently small all solutions are close to $(0,0,0)$ by Corollary 4.4 this completes the proof of the theorem. q.e.d.

Remark 1. It is conspicuous that in applying the Inverse Function Theorem here we have restricted attention to values of $\lambda>0$, even though it applies equally well for $\lambda<0$. Recall that in (4.9), (4.10), $\lambda=1/c^2>0$. The case of $\lambda<0$ corresponds to considering imaginary values of c, the significance of which is unclear for travelling wave solutions of the system under consideration here. Such solutions, however, do exist, as we have shown for imaginary ic with c sufficiently large.

Remark 2. Using the analytic expression for G it is possible to obtain an estimate for the size of c, where uniqueness is guaranteed, using the contraction mapping principle. We have not attempted this.

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