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Classical and Quantum Mechanical Systems of Toda-Lattice Type

II. Solutions of the Classical Flows

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Abstract. Solutions to the classical periodic and non-periodic Toda lattice type Hamiltonian systems are expressed in terms of an Iwasawa-type factorization of a "large" Lie group. The scattering of these systems is determined in the non-periodic case. For the generalized periodic Toda lattices a generalization of Kostant's formula is obtained using standard representations of affine Lie groups.

Contents

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0. Introduction

This paper is the second in a proposed series of three papers on classical and quantum mechanical systems of Toda lattice type (cf. [G–W2]). The main purpose of the present paper is to study the solutions of the classical periodic and non-periodic Toda lattice type systems. The third paper (in preparation) studies the solutions of the quantized systems. (The complete integrability of both classical and quantized systems was proved in [G–W2], and the eigenfunctions for the quantized non-periodic systems were constructed in [G–W1].)

The phase spaces for these Hamiltonian systems can all be realized as coadjoint orbits for suitable finite-dimensional solvable Lie groups. The basic idea that we exploit here is that the "Lax form" of the systems immediately points to the solution in terms of an Iwasawa type factorization of a "large" Lie group. (This has been also observed by various other investigators, e.g. [Sy1, O–P, R–S1, G–S]; cf. the review article [S–T–S].) For the non-periodic Toda systems this "large" Lie group is a split finite-dimensional real semi-simple group G. Our main results in this case can be phrased in the following form: The generic Hamiltonian systems of non-periodic Toda type are linearly imbedded in the action of a vector group on the real flag manifold for G. The scattering of these systems is then naturally determined from the Bruhat decomposition of the flag manifold. We also obtain a general method for constructing new completely integrable systems in terms of the root system of G. (Special cases of this construction have been treated by Symes [Sy1, Sy2].)

In the case of the periodic Toda lattices, our results are more technical and less explicit. This time for the "large" real Lie group we must take one of the infinite-dimensional Banach Lie groups $\hat{G}$ constructed in [G–W3]. The Lie algebra of $\hat{G}$ is the (completed) affine Lie algebra associated with a finite-dimensional real semi-simple Lie algebra. The appropriate Iwasawa factorization of $\hat{G}$ was established in [G–W3]. The preliminary form of the solution to the periodic Toda systems is then given in terms of this factorization, as in the non-periodic case. We then express the solution in terms of representative functions of the "standard" (infinite-dimensional) modules for $\hat{G}$. The formula we obtain is a generalization of Kostant's formula [Ko] (which gives the solution of the non-periodic Toda lattices in terms of matrix entries of finite-dimensional representations of G) to the periodic case. To obtain explicit solutions, the next task is to calculate the representative functions defined by highest weight vectors along certain one-parameter subgroups of $\hat{G}$. We derive a non-linear system of ordinary differential equations satisfied by these functions. In the special case of $SL(2,\mathbb{R})^\frown$, we can identify these functions with Jacobi theta functions. For general groups we find the representative functions corresponding to the fixed points of the periodic Toda lattices. We expect that for general initial values these functions are given in terms of the restrictions of Riemann theta functions to an imbedding (corresponding to a specific choice of a basis of holomorphic differentials) of a hyperelliptic curve into its Jacobian variety. Evidence in this direction can be found in the papers [R–S2, A–vM].

The detailed organization of the paper should be apparent from the table of contents. The opening sections on Lax equations and Euler equations (Sects. 1.1–2.1 and 3.1) apply to both the "periodic" and "non-periodic" systems. (One of the main purposes of [G–W3] is to provide the necessary Banach-Lie group results which permit such a unified treatment.)

The middle sections (Sects. 2.2–3.6) analyze the systems of "non-periodic Toda lattice type" in terms of the Riemannian symmetric space G/K and the coadjoint orbits O of S ($G=S\cdot K$ split semi-simple as above, with K maximal compact and S solvable.) The Hamiltonians for these systems come from K-invariant functions on G/K via the Killing form of $\mathfrak{g}$, and mutually Poisson-commute on O (this is the basic "involution theorem" for Toda-type systems). One has a distinguished Hamiltonian, namely the function corresponding to the Killing form on $\mathfrak{g}$, and one looks for other functionally independent such Hamiltonians. This naturally suggests that O be considered "generic" if it has the property that we call "J-regularity": independent sets of K-invariant functions on G/K give rise to independent Hamiltonians on O. Under this condition (which we show is satisfied by the orbits associated with the generalized non-periodic Toda lattices), the scattering for the flow corresponding to the Killing form is given by a specific element of the Weyl group. When O is J-regular and has minimal dimension ($=2\,\mathrm{rank}(G/K)$), this flow is then completely integrable. We call such orbits "Toda orbits," and set up a general root-system technique for obtaining them. A related notion of Toda orbit was introduced by Symes in [Sy1]; our work corrects an error in [Sy1] concerning the appropriate form of the regularity condition. Our scattering results also yield information on the "QR" algorithm for diagonalizing a real symmetric matrix. The technical machinery used in this part of the paper consists of standard facts about the Bruhat decomposition of G, as in [He2, Wal, War], together with some root system calculations [Bo2].

The last part of the paper (Sects. 4 and 5) treats a class of systems which include the generalized periodic Toda lattices. These systems can be viewed as the geodesic flows on certain (finite-dimensional) solvable Lie groups. The "explicit" integration of the geodesic flow is then obtained via an Iwasawa factorization in a suitable infinite-dimensional group $\hat{G}$ and the representation theory of this group, as explained above. The paper [G–W3] provides the technical tools for much of this part. For this explicit solutions in terms of Jacobi elliptic functions, we use the classical work of Jacobi and his successors in the theory of elliptic functions [Han, W–W].

The principal results of this paper were the subject of lectures by the authors at the University of California, San Diego in the Spring of 1981 and at the Oberwolfach Conference on Harmonic Analysis and Representation Theory, July 1981.

1. Lie Group Factorizations and Lax Equations

1.1. Factorizations and Flows

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. (We allow $\dim\mathfrak{g}=\infty$, in which case we assume that G is a Banach-Lie group as in [Bo1].) Suppose that there are closed Lie subgroups S and K of G, with corresponding Lie algebras $\mathfrak{s}$ and $\mathfrak{k}$, such that

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{s} \quad \text{(Banach-space direct sum);} \tag{1}$$

the map $S\times K\to G$ given by $s,k\to sk$ is an analytic manifold isomorphism. (2)

Let $\pi_{\mathfrak{k}}:\mathfrak{g}\to\mathfrak{k}$ and $\pi_{\mathfrak{s}}:\mathfrak{g}\to\mathfrak{s}$ be the projections corresponding to decomposition (1), and let $\mathbf{k}:G\to K$, $\mathbf{s}:G\to S$ be the analytic maps defined implicitly by (2). Thus for $g\in G$ we have the factorization

$$g=\mathbf{s}(g)\cdot\mathbf{k}(g). \tag{3}$$

Consider the homogeneous space $S\backslash G$ with its natural right G-action. By the decomposition (3) we may identify $S\backslash G$ with K, thus making K a right G-space. Explicitly, the action of $g\in G$ on $k\in K$ is given by $k\cdot g=\mathbf{k}(kg)$. In particular, an element $X\in\mathfrak{g}$ defines a vector field $\hat{X}$ on K via the action of the one-parameter group $\exp tX$ on $K:\hat{X}f(k)=(d/dt)f(\mathbf{k}(k\exp tX))|_{t=0}$, for $f\in C^\infty(K)$, $k\in K$. We may calculate $\hat{X}$ as follows:

Lemma. *For $X\in\mathfrak{g}$, $k\in K$, one has*

$$\hat{X}_k=L(\pi_{\mathfrak{k}}(\mathrm{Ad}(k)X))_k. \tag{4}$$

Here $L(Y)$, for $Y\in\mathfrak{k}$, is the right-invariant vector field on K defined by Y: $L(Y)f(k)=(d/dt)f(\exp(tY)k)|_{t=0}$.

Proof. We can write

$$\begin{aligned} k\exp tX&=\exp[t\,\mathrm{Ad}(k)X]k\\ &=\exp[t\pi_{\mathfrak{s}}(\mathrm{Ad}(k)X)+t\pi_{\mathfrak{k}}(\mathrm{Ad}(k)X)]k.\end{aligned}$$

Hence if $f\in C^\infty(K)$ and t is near 0, then by the Campbell-Hausdorff formula, $f(\mathbf{k}(k\exp tX))=f(\exp[t\pi_{\mathfrak{k}}(\mathrm{Ad}(k)X)]k)+O(t^2)$. This implies (4). □

1.2. Solution of Lax Equations

Suppose now that in addition to the decomposition Sect. 1.1 (1), we also have a decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p} \quad \text{(Banach space direct sum),} \tag{1}$$

where $\mathfrak{p}$ is a closed subspace of $\mathfrak{g}$ such that $\mathrm{Ad}(K)\mathfrak{p}\subset\mathfrak{p}$.

Proposition. *Given X_0 and $Y_0\in\mathfrak{p}$, set $k_t=\mathbf{k}(\exp tY_0)$, $X(t)=\mathrm{Ad}(k_t)\cdot X_0$, and $Y(t)=\mathrm{Ad}(k_t)\cdot Y_0$. Then the pair $X(t)$, $Y(t)$ satisfy the "Lax equation"*

$$X'(t)=[\pi_{\mathfrak{k}}(Y(t)),X(t)]. \tag{2}$$

Proof. By Sect. 1.1, Lemma, we can write $\mathrm{Ad}(k_{t+s})=\mathrm{Ad}(\exp(sZ_t)k_t)+O(s^2)$, where $Z_t=\pi_{\mathfrak{k}}((Y(t))$. Hence

$$(d/dt)\,\mathrm{Ad}(k_t)=\mathrm{ad}(Z_t)\,\mathrm{Ad}(k_t)\,, \tag{3}$$

which yields (2). □

Corollary. *Assume that there is a non-degenerate continuous K-invariant bilinear form B on* $\mathfrak{p}$. *Suppose* $\phi\in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$ *has a gradient* $\nabla\phi$ *relative to B. Then the Lax equation*

$$dX/dt=[\pi_{\mathfrak{k}}(\nabla\phi(X)),X],\ X(0)=X_0 \tag{4}$$

on $\mathfrak{p}$ *has as solution*

$$X(t)=\mathrm{Ad}(\mathbf{k}(\exp t\nabla\phi(X_0)))\cdot X_0\,. \tag{5}$$

Remark. Here $C^\infty_{\mathbb{R}}(\mathfrak{p})^K$ denotes the real-valued smooth $\mathrm{Ad}(K)$-invariant functions on $\mathfrak{p}$. The gradient hypothesis means that there is a smooth map $\nabla\phi:\mathfrak{p}\to\mathfrak{p}$ such that $d\phi_X(Y)=B(\nabla\phi(X),Y)$, for $X,Y\in\mathfrak{p}$. This is automatic, of course, when $\dim\mathfrak{p}<\infty$, since ϕ is assumed to be smooth. The existence of B is also automatic when K is compact.

Proof of Corollary. By the K-invariance of B and ϕ we have $\nabla\phi(\mathrm{Ad}(k)\cdot X)=\mathrm{Ad}(k)\cdot\nabla\phi(X)$. Hence taking $Y_0=\nabla\phi(X_0)$ in the Proposition gives $Y(t)=\mathrm{Ad}(k_t)\cdot\nabla\phi(X_0)=\nabla\phi(\mathrm{Ad}(k_t)\cdot X_0)=\nabla\phi(X(t))$. □

2. Solution of Lax Equations on $\mathfrak{p}$

2.1. Lax Equations on Riemannian Symmetric Spaces

Let G be a finite-dimensional linear, connected semi-simple Lie group. Fix an Iwasawa decomposition $G=NAK$ ($\mathfrak{g}=\mathfrak{n}+\mathfrak{a}+\mathfrak{k}$) and a Cartan decomposition $G=\exp(\mathfrak{p})K$ ($\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$), where K is a maximal compact subgroup. Let $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ be the roots of $\mathfrak{a}$ on $\mathfrak{g}$, and Δ^+ the set of positive roots defining N. Set $S=NA$, $\mathfrak{s}=\mathfrak{n}+\mathfrak{a}$, and let B be the Killing form on $\mathfrak{p}$. The assumptions of Sects. 1.1–1.2 are satisfied here, so we can solve the Lax equation Sect. 1.2 (4) via the K-component of the one-parameter group generated by $\nabla\phi(X_0)$. Let us consider this family of one-parameter subgroups for varying ϕ and fixed $X_0\in\mathfrak{p}$.

Let $\mathfrak{a}^+$ be the open positive Weyl chamber associated with $\mathfrak{n}$. By the polar-coordinate decomposition of $\mathfrak{p}$, there exists $k_0\in K$ and H_0 in the closure of $\mathfrak{a}^+$ such that $X_0=\mathrm{Ad}(k_0)\cdot H_0$. When X is regular, the element k_0 is uniquely determined $\mathrm{mod}\,M$, where as usual M is the centralizer of A in K. If $\mathfrak{p}'$ denotes the set of regular elements of $\mathfrak{p}$, then the map $K/M\times\mathfrak{a}^+\to\mathfrak{p}'$, given by $kM,\ H\to\mathrm{Ad}(k)\cdot H$ is an analytic manifold isomorphism [He2, Chap. IX].

Suppose $\phi\in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$. If $H\in\mathfrak{a}$ then $\nabla\phi(H)\in\mathfrak{a}$ [G–W2, Lemma 8.1]. Furthermore, if H is regular, then $\mathfrak{a}=\{\nabla\phi(H):\phi\in S(\mathfrak{p})^K\}$, where $S(\mathfrak{p})$ denotes the real-valued polynomial functions on $\mathfrak{p}$. Indeed, the differentials of a set of $l=\dim\mathfrak{a}$ basic polynomial invariants are linearly independent at H [Bo2, Chap. V, Sect. 5, Proposition 5].

Suppose $X_0 = \mathrm{Ad}(k_0) \cdot H_0$ as above, and $\phi \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$. The solution $X(t)$ in Sect. 1.2, Corollary, to the Lax equation Sect. 1.2 (4) is then given as

$$X(t) = \mathrm{Ad}(\mathbf{k}(k_0 \exp t\nabla\phi(H_0))) \cdot H_0 . \tag{1}$$

To interpret this formula geometrically, observe that the right action of A on K (Sect. 1.1) gives rise to a right action, call it η, of A on K/M: $(kM) \cdot \eta(a) = \mathbf{k}(ka)M$. Hold $\xi_0 = k_0 M$ and H_0 fixed, and define $\gamma(\phi) = \xi_0 \cdot \eta(\exp \nabla\phi(H_0))$. Then $\{\gamma(\phi) : \phi \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K\} \subset \xi_0 \cdot \eta(A)$, with equality when H_0 is regular. Furthermore, $X(t) = \mathrm{Ad}(\gamma(t\phi)) \cdot H_0$. Thus it is evident that the flows on $\mathfrak{p}$ defined by the Lax equations Sect. 1.2 (4) correspond to the right action of A on K/M. We shall make this correspondence more precise at the end of Sect. 2.2.

2.2. Asymptotic Behavior of Solutions and the QR Algorithm

Continuing in the context of a Riemannian symmetric space G/K of non-compact type, we recall the Bruhat decomposition of G, in the following form [He2, Chap. IX]: Let $M' = \mathrm{Norm}_K(A)$, and set $W = M'/M$, the Weyl group of G/K. For each $w \in W$, denote by M_w the coset w viewed as a subset of M'. Let $\Delta^+_w = \{\alpha \in \Delta^+ : -w \cdot \alpha \in \Delta^+\}$, and set

$$\bar{\mathfrak{n}}_w = \sum_{\alpha \in \Delta^+_w} \mathfrak{g}_{w\cdot\alpha}, \qquad N^-_w = \exp \bar{\mathfrak{n}}_w .$$

Then the Bruhat decomposition may be written as

$$G = \bigcup_{w \in W} SN^-_w M_w \qquad \text{(disjoint union)} \tag{1}$$

[He2, Chap. IX, Sect. 1]. Since $G = SK$, we obtain from (1) a corresponding decomposition of K, in the following form:

Lemma. *For $w \in W$, define a map $\beta_w : N^-_w \times M_w \to K$ by $\bar{n}, m \mapsto \mathbf{k}(\bar{n})m$. Then β_w is a regular analytic imbedding, and*

$$K = \bigcup_{w \in W} \mathbf{k}(N^-_w) M_w \qquad \textit{(disjoint union)} . \tag{2}$$

Here the analytic manifold structure on M_w is obtained from that of M by translation.

Proof. Obviously (2) is just a restatement of (1). The map β_w is an immersion because this is true for the map $S \times N^-_w \times M_w \to G$ given by multiplication [Wal, Corollary 7.5.20]. Under the identification of K with $S\backslash G$, the set $\mathbf{k}(N^-_w)M_w$ corresponds to the orbit $SwNM$ of NM. Since there are only $|W|$ such orbits, each orbit is open in its closure, hence regularly imbedded [War, Lemma 5.2.4.1]. □

Now we combine the Bruhat decomposition (2) of K and the polar-coordinate decomposition of $\mathfrak{p}$. Letting $\mathrm{Cl}(E)$ denote the closure of a set E, we have

$$\mathfrak{p} = \mathrm{Ad}(K) \cdot \mathrm{Cl}(\mathfrak{a}^+) = \bigcup_{w \in W} \mathrm{Ad}(\mathbf{k}(N^-_w)) \cdot \mathrm{Cl}(w \cdot \mathfrak{a}^+) . \tag{3}$$

Thus $X \in \mathfrak{p}$ can be written as

$$X = \mathrm{Ad}(k) \cdot H = \mathrm{Ad}(\mathbf{k}(\bar{n}))\, w \cdot H , \tag{4}$$

where $H \in \mathrm{Cl}(\mathfrak{a}^+)$ is unique, $\bar{n} \in N_w^-$, and w is unique mod $W_H = \{r \in W : r \cdot H = H\}$. (Here $w \cdot H$ denotes the action of W on $\mathfrak{a}$.) In any event, the elements H and $w \cdot H$ are uniquely determined by X.

Theorem. *If $X \in \mathfrak{p}$ is given by* (4), *then*

$$\lim_{t \to +\infty} \mathbf{k}(\exp tX) = k_\infty \mathbf{k}(\bar{n})^{-1}, \tag{5}$$

where $k_\infty \in K$ and $\mathrm{Ad}(k_\infty) w \cdot H = w \cdot H$. *In particular,*

$$\lim_{t \to +\infty} \mathrm{Ad}(\mathbf{k}(\exp tX)) \cdot X = w \cdot H. \tag{6}$$

Proof. By (4) we have $\exp tX = \mathbf{k}(\bar{n}) \exp(tw \cdot H)\, \mathbf{k}(\bar{n})^{-1}$. But $\mathbf{k}(sgk) = \mathbf{k}(g) k$ for $s \in S$ and $k \in K$. Since $\mathbf{k}(\bar{n}) = \mathbf{s}(\bar{n})^{-1} \bar{n}$, it follows that

$$\begin{aligned} \mathbf{k}(\exp tX) &= \mathbf{k}(\bar{n} \exp tw \cdot H)\, \mathbf{k}(\bar{n})^{-1} \\ &= \mathbf{k}(\exp(-tw \cdot H) \bar{n} \exp(tw \cdot H))\, \mathbf{k}(\bar{n})^{-1}. \end{aligned}$$

The eigenvalues of $\mathrm{ad}(w \cdot H)$ on $\bar{\mathfrak{n}}_w$ are non-negative (and strictly positive if X is regular), so that

$$\lim_{t \to +\infty} \exp(-tw \cdot H) \bar{n} \exp(tw \cdot H) = \bar{n}_\infty \in G_1,$$

where $G_1 = \{g \in G : \mathrm{Ad}(g) w \cdot H = w \cdot H\}$. Note that if X is regular, then $\bar{n}_\infty = 1$.

From the root-space structure of the Lie algebra of G_1 one sees easily that $k_\infty = \mathbf{k}(\bar{n}_\infty) \in G_1$ also, which gives (5). Since $\mathrm{Ad}(\mathbf{k}(\bar{n})^{-1}) \cdot X = w \cdot H$, we obtain (6) from (5). □

Remarks. 1. If X is regular, then $k_\infty = 1$ and $\lim_{t \to +\infty} \mathbf{k}(\exp tX) = \mathbf{k}(\bar{n})^{-1}$. In this case the theorem has the following geometric interpretation: If $H \in \mathfrak{a}^+$ and ξ is in the set $\mathbf{k}(N_w^-) M_w \subset K/M$, then

$$\lim_{t \to +\infty} \xi \cdot \exp tH = w$$

(where A acts on the right on K/M as in Sect. 2.1, and we view w as a point in K/M).

2. The relation (6) is a continuous time version of the "QR algorithm" for diagonalizing a symmetric matrix [Ru, Satz 12.6]. To verify this, define Q_n, R_n, T_n by the recursive algorithm

$$T_0 = \exp X,$$

$$Q_{n+1} = \mathbf{k}(T_n), \qquad R_{n+1} = \mathbf{s}(T_n),$$

$$T_{n+1} = Q_{n+1} R_{n+1} \quad \text{(note reversal of order)}.$$

It then follows inductively that

$$T_n = Q_n \ldots Q_1 \exp(X) Q_1^{-1} \ldots Q_n^{-1}, \qquad \exp nX = R_1 \ldots R_n Q_n \ldots Q_1.$$

Hence $Q_n \ldots Q_1 = \mathbf{k}(\exp nX)$, and so by (5),

$$\lim_{n \to \infty} Q_n \ldots Q_1 = k$$

exists. Furthermore, by (6),

$$\lim_{n\to\infty} T_n = \exp \mathrm{Ad}(k) X = \exp w \cdot H . \quad \square$$

Now we introduce the following decomposition of the set of regular elements in $\mathfrak{p}$: For $w \in W$, define

$$\mathfrak{p}'(w)_+ = \mathrm{Ad}(\mathbf{k}(N_w^-))w \cdot \mathfrak{a}^+ . \tag{7}$$

By the theorem we have $\mathfrak{p}'(w)_+ = \mathrm{Ad}(\mathbf{k}(N_w^-))w \cdot \mathfrak{a}^+$. From the lemma above and the polar coordinate decomposition of $\mathfrak{p}'$, we see that $\mathfrak{p}'(w)_+$ is an imbedded analytic submanifold of $\mathfrak{p}'$ of dimension equal $\dim(\bar{\mathfrak{n}}_w) + \dim(\mathfrak{a})$. Furthermore,

$$\mathfrak{p}' = \bigcup_{w \in W} \mathfrak{p}'(w)_+ \quad \text{(disjoint union)}. \tag{8}$$

In particular, if w_0 denotes the unique element of W which sends Δ^+ to $-\Delta^+$, then $\dim \mathfrak{p}' = \dim \mathfrak{p}'(w_0)_+ > \dim \mathfrak{p}'(w)_+$, if $w \neq w_0$. Thus when we solve the Lax equation $dX/dt = [\pi_{\mathfrak{k}}(X), X]$ with "generic" initial data $X(0) \in \mathfrak{p}'(w_0)_+$, then the solution tends to the negative Weyl chamber $w_0 \cdot \mathfrak{a}^+$ as $t \to +\infty$. Thus the same behavior occurs in the discrete time QR algorithm in Remark 2 (cf. remarks after Satz 12.6 in [Ru]).

Under suitable regularity assumptions on the initial data, we can obtain the asymptotic behavior of the solutions to the general Lax equations Sect. 1.2 (4) from the theorem above, as follows:

Corollary. *Let $X \in \mathfrak{p}'$, $\phi \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$, and assume that $\nabla\phi(X) \in \mathfrak{p}'$. Write $X = \mathrm{Ad}(k) \cdot H$, with $k \in K$ and $H \in \mathfrak{a}^+$. Choose $w_1 \in W$ so that $w_1 \cdot \nabla\phi(H) \in \mathfrak{a}^+$, and choose $w_2 \in W$ so that $kw_1^{-1} \in SN_{w_2}^- M_{w_2}$. Then*

$$\lim_{t \to +\infty} \mathrm{Ad}(\mathbf{k}(\exp t \nabla\phi(X))) \cdot X = w_2 w_1 \cdot H . \tag{9}$$

Proof. There exist representatives $\bar{w}_i \in M_{w_i}$, for $i = 1, 2$, and $\bar{n}_2 \in N_{w_2}^-$ such that $k = \mathbf{k}(\bar{n}_2)\bar{w}_2\bar{w}_1$. Hence

$$\nabla\phi(X) = \mathrm{Ad}(k) \cdot \nabla\phi(H) = \mathrm{Ad}(\mathbf{k}(\bar{n}_2))w_2 \cdot H_1 ,$$

where $H_1 = w_1 \cdot \nabla\phi(H)$. Since $H_1 \in \mathfrak{a}^+$, we obtain from (5) that

$$\lim_{t \to +\infty} \mathbf{k}(\exp t \nabla\phi(X)) = \mathbf{k}(\bar{n}_2)^{-1} .$$

Thus the limit in (9) is $\mathrm{Ad}(\mathbf{k}(\bar{n}_2)^{-1}) \cdot X = w_2 w_1 \cdot H$. $\square$

Remarks on "Linearization." With the Bruhat decomposition of K/M at hand, we can be more precise about the nature of the simultaneous isospectral flows on $\mathfrak{p}$ associated with all the K-invariant polynomials on $\mathfrak{p}$. Suppose $X_0 \in \mathfrak{p}'$. Write $X_0 = \mathrm{Ad}(\mathbf{k}(\bar{n}_0)) \cdot H_0$, where $\bar{n}_0 \in N_w^-$, $H_0 \in w \cdot \mathfrak{a}^+$, and $w \in W$. As noted at the end of Sect. 2.1, the flows passing through X_0 are parametrized by the subgroup A via the formula

$$a \to \mathrm{Ad}(\mathbf{k}(a\bar{n}_0 a^{-1})) \cdot H_0 . \tag{10}$$

We may view (10) as the composite of two maps: the linear action of A on $\bar{\mathfrak{n}}_w$:

$$a \to \mathrm{Ad}(a) \cdot Z_0, \qquad Z_0 = \log \bar{n}_0, \tag{11}$$

followed by the non-linear map

$$Z \to \mathrm{Ad}(\mathbf{k}(\exp Z)) \cdot H_0, \qquad Z \in \bar{\mathfrak{n}}_w. \tag{12}$$

By the Bruhat decomposition the map (12) is injective and regular. Thus the dimension of the "isospectral leaf" through X_0 is $\dim A - \dim \mathrm{Cent}_A(Z_0)$, and the maximal leaves occur when $\mathrm{Cent}_A(Z_0) = \{1\}$. We shall study this case in detail in Sect. 3.2.

2.3. Scattering

We continue in the context of Sect. 2.2. If $X \in \mathfrak{p}$, set

$$\phi_+(X) = \lim_{t \to +\infty} \mathrm{Ad}(\mathbf{k}(\exp tX)) \cdot X. \tag{1}$$

We note that replacing X by $-X$ in (1) gives

$$-\phi_+(-X) = \lim_{t \to -\infty} \mathrm{Ad}(\mathbf{k}(\exp tX)) \cdot X. \tag{2}$$

Calling the limit on the right side of (2) $\phi_-(X)$, we thus have

$$-\phi_+(-X) = \phi_-(X). \tag{3}$$

The "scattering transformation" associated with the Lax equation $dX/dt = [\pi_{\mathfrak{k}}(X), X]$ is then the map $\phi_-(X) \to \phi_+(X)$ from $\mathfrak{a}$ to $\mathfrak{a}$. We shall calculate it for the regular trajectories of the system, i.e. when $X \in \mathfrak{p}'$.

There are elements $k_\pm \in K$ such that $\phi_\pm(X) = \mathrm{Ad}(k_\pm) \cdot X$. Thus $\phi_+(X) = \mathrm{Ad}(k_+ k_-^{-1}) \cdot \phi_-(X)$. But if two elements of $\mathfrak{a}$ are conjugate under K, then they are conjugate under W [He2, Chap. VII, Proposition 2.2]. Hence we obtain the following:

Lemma. *There exists $w = w(X) \in W$ such that $\phi_+(X) = w \cdot \phi_-(X)$.*

Remark. If $X \in \mathfrak{p}'$, then $\phi_\pm(X) \in \mathfrak{a} \cap \mathfrak{p}'$. In this case $k_+ k_-^{-1} \in M'$ so that the element w in the lemma is $k_+ k_-^{-1} M$.

To calculate the element $w(X)$, let $\mathfrak{p}'(w)_+$ be defined by Sect. 2.2 (7), and set

$$\mathfrak{p}'(w)_- = \{X \in \mathfrak{p}' : \phi_-(X) \in w w_0 \cdot \mathfrak{a}^+\}. \tag{4}$$

Taking into account the relations (3) and $w w_0 \cdot \mathfrak{a}^+ = -w \cdot \mathfrak{a}^+$, we have

$$\mathfrak{p}'(w)_- = -\mathfrak{p}'(w)_+. \tag{5}$$

Thus from Sect. 2.2 we know that $\mathfrak{p}'(w)_\pm$ are imbedded submanifolds of $\mathfrak{p}'$ of dimension equal $\dim(\mathfrak{n}_w^-) + \dim(\mathfrak{a})$. Also from Sect. 2.2 (8) and relation (5) we have

$$\mathfrak{p}' = \bigcup_{w_1, w_2 \in W} \{\mathfrak{p}'(w_1)_+ \cap \mathfrak{p}'(w_2)_-\}. \tag{6}$$

In particular, since $\dim \mathfrak{p}'(w)_{\pm} < \dim \mathfrak{p}$ if $w \neq w_0$, we see from (6) that

$$\mathfrak{p}'' := \mathfrak{p}'(w_0)_+ \cap \mathfrak{p}'(w_0)_- \tag{7}$$

is open and dense in $\mathfrak{p}$.

Theorem. L et X $\in \mathfrak{p}'(w_1)_+ \cap '(w_2)_-$. *Then* $\phi_+(X) = w_1 w_0 w_2^{-1} \cdot \phi_-(X)$. *In particular, if* $X \in \mathfrak{p}''$, *then the scattering transformation for* X *is* w_0 *("generic scattering").*

Proof. By definition, $\phi_-(X) \in w_2 w_0 \cdot \mathfrak{a}^+$ and $\phi_+(X) \in w_1 \cdot \mathfrak{a}^+$. Hence $w_1 w_0 w_2^{-1} \cdot \phi_-(X)$ and $\phi_+(X)$ are in the same Weyl chamber. Since we also know that $\phi_+(X) = w \cdot \phi_-(X)$ and $\phi_-(X)$ is regular, this implies that $w = w_1 w_0 w_2^{-1}$. $\square$

3. Integrable Hamiltonian Systems on Iwasawa Groups

3.1. Solution to Euler Equations on $\mathfrak{s}^*$

Let the notation be as in Sect. 2. We now relate the "Lax equations" on $\mathfrak{p}$ to "Euler equations" on $\mathfrak{s}^*$. This connection is by now well-known (cf. [Sy1, O–P, R–S1, Ad]). For the reader's convenience and to establish notation, we describe the result with sketches of proofs.

Let B denote the Killing form on $\mathfrak{g}$. Then $\mathfrak{k} = \mathfrak{p}^\perp$ and $\mathfrak{p} = \mathfrak{k}^\perp$ relative to B, so we have a linear isomorphism $\psi : \mathfrak{p} \to \mathfrak{s}^*$; $\psi(X)(Y) = B(X, Y)$. By the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ we also have a linear isomorphism $\pi_\mathfrak{s} : \mathfrak{p} \to \mathfrak{s}$. Since $B(X, Y) = B(\pi_\mathfrak{s}(X), Y)$ for $X, Y \in \mathfrak{p}$, it is clear that

$$\pi_\mathfrak{s}(X) = \psi^{*-1}(X^\#), \tag{1}$$

where $\psi^* : \mathfrak{s} \to \mathfrak{p}^*$ is the adjoint of ψ and $X \to X^\#$ is the map from $\mathfrak{p}$ to $\mathfrak{p}^*$ defined by the form B. One also has the map $f \to f^\flat$ from $\mathfrak{s}^*$ to $\mathfrak{s}$, such that $\psi(X)^\flat = \pi_\mathfrak{s}(X)$ for $X \in \mathfrak{p}$.

Suppose $\phi \in C^\infty_\mathbb{R}(\mathfrak{p})^K$. Then $[\nabla\phi(X), X] = 0$ for $X \in \mathfrak{p}$. To see this, take $Y \in \mathfrak{k}$ and calculate as follows: $B(Y, [\nabla\phi(X), X]) = B([X, Y], \nabla\phi(X)) = (d/dt)\phi(X - t[Y, X])|_{t=0} = (d/dt)\phi(\mathrm{Ad}(\exp - tY) \cdot X)|_{t=0} = 0$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $B|_{\mathfrak{k} \times \mathfrak{k}}$ is non-degenerate, the result follows.

One next observes that if $X, Y \in \mathfrak{p}$ and $[X, Y] = 0$, then

$$\psi([\pi_\mathfrak{k}(Y), X]) = -\pi_\mathfrak{s}(Y) \cdot \psi(X), \tag{2}$$

where the dot on the right side of (2) denotes the coadjoint action of $\mathfrak{s}$ on $\mathfrak{s}^*$. (To verify (2), note that $[X, Y] = 0$ implies $[\pi_\mathfrak{k}(Y), X] = -[\pi_\mathfrak{s}(Y), X]$ and use the invariance of the form B.) In particular, if $\phi \in C^\infty_\mathbb{R}(\mathfrak{p})^K$, then

$$\psi([\pi_\mathfrak{k}(\nabla\phi(X)), X]) = -\pi_\mathfrak{s}(\nabla\phi(X)) \cdot \psi(X). \tag{3}$$

Finally, to obtain an "Euler equation" on $\mathfrak{s}^*$ for the function $H_\phi(\xi) = \phi(\psi^{-1}(\xi))$ from the "Lax equation" for ϕ, we use (1) to calculate that $\pi_\mathfrak{s}(\nabla\phi(X)) = \psi^{*-1}(d\phi(X)) = dH_\phi(\psi(X))$. [Here $\nabla\phi(X)^\# = d\phi(X)$ by definition of the gradient, and we identify $\mathfrak{s}^{**}$ with $\mathfrak{s}$, so that dH_ϕ maps $\mathfrak{s}^*$ to $\mathfrak{s}$.] Substituting this calculation in (3) then completes the proof of the following result [Sy1, Theorem 2.2]:

Proposition. *Let $\phi \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$ and define $H_\phi(\psi(X)) = \phi(X)$ for $X \in \mathfrak{p}$. Then under the map ψ the vector field $X \to [\pi_{\mathfrak{k}}(\nabla\phi(X)), X]$ on $\mathfrak{p}$ corresponds to the vector field $\xi \to -dH_\phi(\xi)\cdot\xi$ on $\mathfrak{s}^*$.*

Corollary. *Let $\xi_0 = \psi(X_0) \in \mathfrak{s}^*$. Then the solution to the "Euler equation" $\dot{\xi} = -dH_\phi(\xi)\cdot\xi$, $\xi(0) = \xi_0$ on $\mathfrak{s}^*$ is given by*

$$\xi(t) = \mathbf{s}(\exp t\nabla\phi(X))^{-1}\cdot\xi_0\,. \tag{4}$$

Proof. By the proposition and Sect. 1.2, Corollary, one has $\xi(t) = \psi(\mathrm{Ad}(\mathbf{k}(\exp t\nabla\phi(X)))\cdot X)$. Since $[\nabla\phi(X), X] = 0$, we can exchange the K and S components of $\exp t\nabla\phi(X)$ as follows:

$$\begin{aligned}\mathrm{Ad}(\mathbf{k}(\exp t\nabla\phi(X)))\cdot X &= \mathrm{Ad}(\mathbf{s}(\exp t\nabla\phi(X))^{-1}\exp t\nabla\phi(X))\cdot X\\ &= \mathrm{Ad}(\mathbf{s}(\exp t\nabla\phi(X))^{-1})\cdot X\,.\end{aligned}$$

This implies (4) by the invariance of the form B. □

Recall that the Poisson bracket of functions F_1 and F_2 on $\mathfrak{s}^*$ is defined by $\{F_1, F_2\}(\xi) = \xi([dF_1(\xi), dF_2(\xi)])$. A basic observation in this regard is that when $F_i = H_{\phi_i}$ with $\phi_i \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$, then $\{H_{\phi_1}, H_{\phi_2}\} = 0$. There are several proofs of this "involution theorem" (cf. [Ra]). The argument which seems most suitable for both the finite and infinite-dimensional cases is due to Symes [Sy1], and goes as follows: Let $\xi = \psi(X)$. By the calculations above and the invariance of the form B, one has

$$\begin{aligned}\{H_{\phi_1}, H_{\phi_2}\}(\xi) &= B([\pi_{\mathfrak{s}}(\nabla\phi_1(X)), \pi_{\mathfrak{s}}(\nabla\phi_2(X))], X)\\ &= B(\pi_{\mathfrak{s}}(\nabla\phi_1(X)), [\pi_{\mathfrak{s}}(\nabla\phi_2(X)), X])\\ &= B([\pi_{\mathfrak{s}}(\nabla\phi_1(X)), X], \pi_{\mathfrak{k}}(\nabla\phi_2(X)))\\ &= B([X, \pi_{\mathfrak{k}}(\nabla\phi_1(X))], \pi_{\mathfrak{k}}(\nabla\phi_2(X)))\,.\end{aligned}$$

But the last expression vanishes since $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ and $\mathfrak{k} \perp \mathfrak{p}$. In particular, this proves the following result (cf. [G–W2] for a proof that also applies to the quantized systems):

Theorem. *Let $I = S(\mathfrak{p})^K$, and set $J = \{H_\phi : \phi \in I\}$. Then J is a Poisson-commutative algebra of functions on $\mathfrak{s}^*$.*

Remark. The proofs and results of this section apply equally well to the case $\dim\mathfrak{g} = \infty$, provided one assumes that $\nabla\phi$ exists in $\mathfrak{p}$ (and hence that dH_ϕ exists as a map from $\mathfrak{s}^*$ to $\mathfrak{s}$). We shall use this in Sect. 4 without further comment.

3.2. J-Regular Orbits in $\mathfrak{s}^*$

Let G be as in Sect. 2. We now make the additional assumption that $\mathfrak{g}$ is split over $\mathbb{R}$. In this case we have the triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}$, and $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Set $l = \dim\mathfrak{a}$.

Given $\xi \in \mathfrak{s}^*$, set

$$\mathfrak{s}_\xi = \{Y \in \mathfrak{s} : Y\cdot\xi = 0\}, \qquad S_\xi = \{s \in S : s\cdot\xi = \xi\}\,.$$

Since S is an exponential-solvable Lie group, the coadjoint orbit $O=S\cdot\xi$ is an imbedded manifold in $\mathfrak{s}^*$, analytically isomorphic to S/S_ξ [Be, Chap. I]. It is a symplectic manifold relative to the canonical Kirillov-Kostant form ω^O.

If $\xi\in\mathfrak{n}^*$ we extend ξ to $\mathfrak{s}$ by $\xi(\mathfrak{a})=0$; similarly, if $\beta\in\mathfrak{a}^*$, we extend β to $\mathfrak{s}$ by $\beta(\mathfrak{n})=0$. With these conventions we have

$$S\cdot(\beta+\xi)=\beta+S\cdot\xi \tag{1}$$

for $\beta\in\mathfrak{a}^*$ and $\xi\in\mathfrak{n}^*$, since $\beta([\mathfrak{s},\mathfrak{s}])=\beta(\mathfrak{n})=0$. Thus for calculating S orbits in $\mathfrak{s}^*$ it is enough to consider orbits of elements on $\mathfrak{n}^*$.

In this section we study the isotropic foliations of these orbits associated with the Poisson-commutative algebra J in Sect. 3.1, Theorem. (Since G is split, the functions in J in fact come from G-invariant polynomials on $\mathfrak{g}$; cf. Sect. 4.1, Lemma 1.)

Given a function $F\in C^\infty(\mathfrak{s}^*)$, we have the differential $dF:\mathfrak{s}^*\to\mathfrak{s}$. The Hamiltonian vector field X^F on the symplectic manifold (O,ω) corresponding to F is $(X^F)_\xi=-dF(\xi)\cdot\xi$, [G–W2, Sect. 7]. Consider the distribution of tangent spaces $\xi\to L_\xi=\{(X^F)_\xi:F\in J\}$ in the tangent bundle of $\mathfrak{s}^*$. Since J is Poisson-commutative, the subspace L_ξ is isotropic for $(\omega^O)_\xi$. In particular, one has

$$\dim O\geqq 2\max_{\xi\in O}\{\dim L_\xi\}.$$

Furthermore, if $\phi_1,\ldots,\phi_l$ are homogeneous polynomials which generate $S(\mathfrak{p})^K$ (cf. [He1, Chap. X]), and $F_i=H_{\phi_i}$, then the vector fields $\{X^{F_i}:1\leqq i\leqq l\}$ span L at every point. In particular, $\dim L_\xi\leqq l$ for all $\xi\in\mathfrak{s}^*$.

Lemma 1. *Let $X\in\mathfrak{p}$ and set $\xi=\psi(X)$. Then $\dim L_\xi=l$ iff X is regular and satisfies the transversality condition*

$$\mathfrak{g}^X\cap\mathfrak{a}=\{0\}. \tag{T}$$

(Here $\mathfrak{g}^X$ is the centralizer of X in $\mathfrak{g}$.)

Proof. Let $\mathfrak{u}=\{\nabla\phi(X):\phi\in S(\mathfrak{p})^K\}$. We observed in Sect. 2.1 that $\dim\mathfrak{u}\leqq l$, with equality iff X is regular. The linear map $\nabla\phi(X)\to dH_\phi(\xi)\cdot\xi$ from $\mathfrak{u}$ to L_ξ is surjective, by definition. Thus we may assume that X is regular. Now the centralizer of X in $\mathfrak{k}$ is trivial, since $\mathfrak{g}$ is split over $\mathbb{R}$. Thus $\mathfrak{g}^X=\mathfrak{u}\subset\mathfrak{p}$, as noted in Sect. 2.1.

The kernel of the map above is characterized by the equation

$$[\pi_{\mathfrak{k}}(\nabla\phi(X)),X]=0, \tag{2}$$

by Sect. 3.1, Proposition. Thus the solutions to (2) satisfy $\nabla\phi(X)\in\mathfrak{p}\cap\mathfrak{s}=\mathfrak{a}$. Hence $\dim L_\xi<l$ implies that condition (T) does not hold.

Conversely, suppose that (T) fails. Hence there is some ϕ with $0\neq\nabla\phi(X)\in\mathfrak{a}$. Then ϕ satisfies (1), which implies that $\dim L_\xi<l$. $\square$

Let $O\subset\mathfrak{s}^*$ be an S orbit. Define $O_{\text{reg}}=\{\xi\in O:\dim L_\xi=l\}$. Since O_{reg} is the subset of O on which the analytic vector fields X^{F_i}, $1\leqq i\leqq l$, are linearly independent, it is clear that O_{reg} is open, and is either empty or dense in O. We shall say that O is *J-regular* when O_{reg} is nonempty. To state a criterion for J-regularity, we need some additional notation.

For $\alpha \in \Delta^+$ choose $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, and let $\{X_\alpha^* : \alpha \in \Delta^+\}$ be the basis for $\mathfrak{n}^*$ dual to $\{X_\alpha\}$. Given $\xi \in \mathfrak{s}^*$, define $\Phi_\xi = \{\alpha \in \Delta^+ : \xi(X_\alpha) \neq 0\}$. Writing $\alpha \in \Delta^+$ in terms of the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ as $\alpha = n_1\alpha_1 + \ldots + n_l\alpha_l$, we define the *support* of α to be $\operatorname{Supp}(\alpha) = \{\alpha_i : n_i \neq 0\}$. We then associate with ξ the following subset of Π:

$$\operatorname{Supp}(\Phi_\xi) = \bigcup_{\alpha \in \Phi_\xi} \operatorname{Supp}(\alpha).$$

Note that Φ_ξ measures the extent to which ξ is "non-diagonal;" e.g. Φ_ξ is empty if $\xi \in \mathfrak{a}^*$. Our criterion for J-regularity is then the following root system condition:

Theorem 1. *The set O_{reg} is nonempty* iff *there exists $\xi \in O$ such that* $\operatorname{Supp}(\Phi_\xi) = \Pi$.

Proof. The necessity of the criterion in Theorem 1 is easily established. Indeed, suppose that there exists a simple root α such that for all $\xi \in O$, $\alpha \notin \operatorname{Supp}(\Phi_\xi)$. Choose $H \in \mathfrak{a}$ with $\alpha(H) = 1$ and $\beta(H) = 0$ for $\beta \in \Pi \setminus \{\alpha\}$. Then we have $H \perp \Phi_\xi$, and hence $H \cdot \xi = 0$, for all $\xi \in O$. But $H \cdot \psi(X) = \psi([H, X])$, for $X \in \mathfrak{p}$. It follows that no element of $\psi^{-1}(O)$ satisfies condition (T) in Lemma 1, and hence O_{reg} is empty.

The proof of the sufficiency of the criterion in Theorem 1 requires some preparation.

Let $\bar{\mathfrak{s}} = \mathfrak{a} + \bar{\mathfrak{n}}$, and use the form B to define a linear isomorphism $\gamma : \bar{\mathfrak{s}} \to \mathfrak{s}^*$. Denote by $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{p}'$ the regular elements in $\mathfrak{a}$. We shall first prove

$$\begin{aligned}&\text{there exists } X \in \mathfrak{p}' \text{ with } \psi(X) \in O \text{ iff there exists}\\ &H \in \mathfrak{a}' \text{ and } Y \in \bar{\mathfrak{n}} \text{ with } \gamma(H + Y) \in O.\end{aligned} \tag{A}$$

Indeed, given $H \in \mathfrak{a}'$ and $Y \in \bar{\mathfrak{n}}$, a well-known result of Harish-Chandra [He2, Chap. IX, Lemma 1.5] asserts that there exists $\bar{n} \in N^-$ such that $\operatorname{Ad}(\bar{n})H = H + Y$. Set $X = \operatorname{Ad}(\mathbf{k}(\bar{n}))H \in \mathfrak{p}'$. Then

$$\psi(X) = \mathbf{s}(\bar{n})^{-1} \cdot \gamma(H + Y), \tag{3}$$

by the invariance of the form B. Hence if $\gamma(H + Y) \in O$, then $\psi(X) \in O$ also. Conversely, if $X \in \mathfrak{p}'$, then there exists $k \in K$ and $H \in \mathfrak{a}'$ such that $\operatorname{Ad}(k)H = X$. Applying Sect. 2.2, Lemma, to write $k = s\bar{n}m$, with $s \in S$, $\bar{n} \in N_w^-$, $m \in M_w$, one has $\psi(X) = s \cdot \gamma(\operatorname{Ad}(\bar{n}m)H) = s \cdot \gamma(w \cdot H + Y)$ for some $Y \in \bar{\mathfrak{n}}_w$. Since $w \cdot H \in \mathfrak{a}'$, this completes the proof of (A).

Next, let $\alpha \in \Pi$. We claim that

$$X_\alpha \perp O \text{ iff} \quad \text{for all} \quad \xi \in O, \alpha \notin \operatorname{Supp}(\Phi_\xi). \tag{B}$$

To verify this, suppose that $X_\alpha \perp O$ and $\xi \in O$. Then $U(\mathfrak{n}) \cdot \xi$ vanishes on X_α. But by Lemma 1 of the appendix, for every $\gamma \in \Delta^+$ such that $\alpha \in \operatorname{Supp}(\gamma)$, there are positive roots $\beta_1, \ldots, \beta_r$ such that $\operatorname{ad} X_{\beta_r} \ldots \operatorname{ad} X_{\beta_1}(X_\alpha) = cX_\gamma$, with $c \neq 0$. Hence $\xi(X_\gamma) = 0$, so that $\gamma \notin \Phi_\xi$. This shows that $\alpha \notin \operatorname{Supp}(\Phi_\xi)$. Conversely, if there exists some $\xi \in O$ with $\xi(X_\alpha) \neq 0$, then trivially $\alpha \in \operatorname{Supp}(\Phi_\xi)$. This proves (B).

We can now complete the proof of the sufficiency of the criterion in Theorem 1. Assume that there exists $\xi_0 \in O$ with $\operatorname{Supp}\Phi_{\xi_0} = \Pi$. For each $\alpha \in \Pi$, define an analytic function r_α on O by

$$r_\alpha(\xi) = \xi(X_\alpha), \qquad \xi \in O. \tag{4}$$

From (B) we have $r_\alpha \neq 0$. Since $r_\alpha(a \cdot \xi) = a^{-\alpha} r_\alpha(\xi)$ for $a \in A$, it follows that r_α is nonconstant, for every $\alpha \in \Pi$. Since O is connected, there thus exists $\xi_1 \in O$ such that for all $\alpha \in \Pi$, $r_\alpha(\xi_1) \neq 0$. Observing that $X_\alpha \cdot X_\beta^*|_{\mathfrak{a}} = \delta_{\alpha\beta}\beta$ for $\alpha, \beta \in \Delta^+$, we then have $\mathfrak{s} \cdot \xi_1|_{\mathfrak{a}} = \mathfrak{a}^*$. It follows that the projection map from O to $\mathfrak{a}^*(\xi \to \xi|_{\mathfrak{a}})$ is a submersion at ξ_1. Hence the image of O under this map contains a nonempty open set, by the implicit function theorem. We conclude from (A) that $O \cap \psi(\mathfrak{p}')$ is nonempty. Finally, since this set is open in O, there exists $\xi_2 \in O \cap \psi(\mathfrak{p}')$ such that $r_\alpha(\xi_2) \neq 0$ for all $\alpha \in \Pi$. Then $X = \psi^{-1}(\xi_2)$ is easily seen to satisfy condition (T). Hence $\xi_2 \in O_{\mathrm{reg}}$ by Lemma 1. □

Remark. From the proof just given one obtains the following alternate necessary and sufficient geometric condition for O_{reg} to be nonempty:

There exists $\xi \in O$ such that the projection of O into $\mathfrak{a}^*$ is a submersion at ξ.

We can also express the criterion for O_{reg} to be nonempty in terms of Iwasawa subgroups of S. For this, we introduce the following notation: Given $\Pi_1 \subset \Pi$, denote by $\Delta^+(\Pi_1)$ the positive roots in the span of Π_1. Define

$$\mathfrak{n}(\Pi_1) = \sum_{\alpha \in \Delta^+(\Pi_1)} \mathfrak{g}_\alpha .$$

Clearly $\mathfrak{n}(\Pi_1)$ is a subalgebra of $\mathfrak{n}$ stable under $\mathrm{Ad}(A)$. Let

$$\mathfrak{a}(\Pi_1) = \mathrm{span}\{H_\alpha : \alpha \in \Pi_1\},$$

where H_α is the coroot to α. [So $B(H_\alpha, H) = 2\alpha(H)/(\alpha, \alpha)$.] Define

$$\mathfrak{s}(\Pi_1) = \mathfrak{a}(\Pi_1) + \mathfrak{n}(\Pi_1).$$

The corresponding connected subgroup $S(\Pi_1) \subset S$ is an Iwasawa group for the split semisimple group $G(\Pi_1) \subset G$ with Lie algebra $\mathfrak{g}(\Pi_1) = \mathfrak{n}(\Pi_1) \oplus \mathfrak{a}(\Pi_1) \oplus \mathfrak{n}(\Pi_1)^-$.

Via the root space decomposition, we identify $\mathfrak{n}(\Pi_1)^*$ with the subspace of $\mathfrak{n}^*$ spanned by $\{X_\alpha^* : \alpha \in \Delta^+(\Pi_1)\}$. We identify $\mathfrak{a}(\Pi_1)^*$ with the span of Π_1. Then $\mathfrak{n}(\Pi_1)^*$ consists of all $\xi \in \mathfrak{n}^*$ such that $\Phi_\xi \subset \Delta^+(\Pi_1)$, and we have $\mathfrak{s}(\Pi_1)^* \subset \mathfrak{s}^*$. These identifications are consistent with the coadjoint representation:

Lemma 2. *Let $\xi_1 \in \mathfrak{n}(\Pi_1)^*$. If $X \in \mathfrak{s}(\Pi_1)$, then $\mathrm{ad}_{\mathfrak{s}(\Pi_1)}(X)^* \xi_1 = X \cdot \xi_1$ (where the dot denotes the coadjoint action of $\mathfrak{s}$ on $\mathfrak{s}^*$). Furthermore, $S(\Pi_1) \cdot \xi_1 = S \cdot \xi_1$.*

Proof. Let $\alpha, \beta, \gamma \in \Delta^+$. Then $(X_\alpha \cdot X_\beta^*)(X_\gamma) = 0$ if $\beta \neq \alpha + \gamma$. On the other hand, if $\alpha, \beta \in \Delta^+(\Pi_1)$ and $\beta = \alpha + \gamma$, then $\gamma \in \Delta^+(\Pi_1)$. Furthermore, $X_\alpha \cdot X_\beta^*|_{\mathfrak{a}} = \delta_{\alpha\beta}\alpha$, $H \cdot X_\beta^* = -\beta(H) X_\beta^*$, for $H \in \mathfrak{a}$. From the definition of the embedding of $\mathfrak{s}(\Pi_1)^*$ into $\mathfrak{s}^*$ and these calculations it is clear that the first statement of the lemma holds. The same calculations show that $X_\beta \cdot \xi_1 = 0$ and $H \cdot \xi_1 = 0$ if $\beta \notin \Delta^+(\Pi_1)$ and $H \in \Pi_1^\perp$. Since $\mathfrak{a} = \mathfrak{a}(\Pi_1) \oplus \Pi_1^\perp$, this gives the second statement. □

Combining Theorem 1 [and statement (A) in its proof] with Lemma 2, we obtain the following description of coadjoint orbits:

Theorem 2. *Let $O \subset \mathfrak{s}^*$ be a coadjoint S orbit. Define $\Pi_O = \{\alpha \in \Pi : \langle X_\alpha, O \rangle \neq 0\}$. Then there exists $\delta \in \mathfrak{a}^*$ and $\xi \in \mathfrak{n}(\Pi_O)^*$ such that $O = \delta + S(\Pi_O) \cdot \xi$. Furthermore, the following are equivalent:*

(i) *O is J-regular;*

(ii) *$\Pi_O = \Pi$;*

(iii) *there exists $\xi \in O$ such that* $\mathrm{Supp}\, \Phi_\xi = \Pi$.

3.3. Orbits of Toda Type

Let $O \subset \mathfrak{s}^*$ be a coadjoint S-orbit. We shall call O a *Toda orbit* if $\dim O = 2l$ and O is J-regular, in the sense of Sect. 3.2. (Here $l = \dim \mathfrak{a}$.)

These orbits are of interest because of the "involution theorem" of Sect. 3.1. If O is a Toda orbit and $\phi \in C^\infty_{\mathbb{R}}(\mathfrak{p})^K$, then the Hamiltonian $H = H_\phi|_O$ is completely integrable in the classical sense: Take $\phi_1, \dots, \phi_l$ as generators for $S(\mathfrak{p})^K$, and define a map $\Lambda: O \to \mathbb{R}^l$ by $\Lambda(\xi) = (\phi_1(\xi), \dots, \phi_l(\xi))$. If $\Lambda(\xi) = c$, then the level surface ("isospectral leaf") $\Lambda^{-1}(c)$ through ξ is smooth and has tangent space L_ξ, with $\dim L_\xi = \frac{1}{2} \dim O$, for all ξ in a dense open subset of O. These Lagrangian submanifolds give a foliation of O (possibly with singularities). The flow generated by H (cf. Sect. 3.1, Corollary) follows the leaves of this foliation.

We observe that the Toda orbits are those of minimal dimension, if we exclude orbits belonging to proper Iwasawa subgroups (Sect. 3.2, Theorem 2). We also have from Sect. 3.2, Theorem 1, the following criterion for Toda orbits:

Proposition. *Let $O \subset \mathfrak{s}^*$ be an orbit of dimension $2l$. Then O is a Toda orbit* iff *there exists $\xi \in O$ with* $\operatorname{Supp} \Phi_\xi = \Pi$.

Example (Jacobi Matrices). The best-known example of a Toda orbit is the following: Choose $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_l)$ with each $\varepsilon_i = \pm 1$, and set

$$J_{\boldsymbol{\varepsilon}} = \left\{ H + \sum_{i=1}^{l} c_i (X_{\alpha_i} - \theta X_{\alpha_i}) \right\} \subset \mathfrak{p},$$

where $H \in \mathfrak{a}$, $c_i \varepsilon_i > 0$, and θ is the Cartan involution. Then $O = \psi(J_{\boldsymbol{\varepsilon}})$ is the S-orbit of the element

$$\xi = \sum_{i=1}^{l} \varepsilon_i X^*_{\alpha_i}.$$

The structure of O has been studied by several people, especially Kostant [Ko] (cf. [G–W2, Sy1] for further citations). It is obvious that $\operatorname{Supp} \Phi_\xi = \Pi$ and $\dim O = 2l$, so O is a Toda orbit by the proposition.

One of Kostant's results is that $O = O_{\mathrm{reg}}$ in this case, so that the foliation of O by isospectral leaves has no singularities. We now show that this result follows easily from the results of Sect. 3.2. Let $X \in J_{\boldsymbol{\varepsilon}}$. Clearly X satisfies the transversality condition (T) of Sect. 3.2, Lemma 1, so we only need to prove that X is regular. Write $X = \operatorname{Ad}(\mathbf{k}(\bar{n})) \cdot H$ as in Sect. 2.2 (4), where $w \in W$, $H \in \operatorname{Cl}(w \cdot \mathfrak{a}^+)$ and $\bar{n} \in N_w^-$. Then $\operatorname{Ad}(\mathbf{s}(\bar{n}))X = \operatorname{Ad}(\bar{n})H = H + Y$, where $Y \in \bar{\mathfrak{n}}_w$. It follows from Sect. 3.2 (3) that $\gamma(H+Y) \in O$. This forces $w = w_0$, since $\gamma(Y)(X_\alpha) \neq 0$ for all $\alpha \in \Pi$. Furthermore, writing $\bar{n} = \exp Z$, with $Z \in \bar{\mathfrak{n}}$, we see that $Y = [Z_{-1}, H]$, where $Z = Z_{-1} + Z_{-2} + \dots$ is the principal gradation of Z (cf. Sect. 4.2). Hence $\alpha(H) \neq 0$ for $\alpha \in \Pi$. But we already have $\alpha(H) \leq 0$ for $\alpha \in \Pi$, since $H \in \operatorname{Cl}(w_0 \cdot \mathfrak{a}^+)$. Thus H is regular, and so is X. $\square$

3.4. Construction of Toda Orbits (Basic Examples)

In this section we construct Toda orbits under the assumption that $\mathfrak{g}$ is split and simple, i.e. the set Π of simple roots defines a connected Dynkin diagram. The orbits will be of the form $S \cdot X_\alpha^*$ for suitable $\alpha \in \Delta^+$. We first observe that the calculation of the dimension of such an orbit can be done via the root system, as follows:

Lemma 1. *For $\alpha \in \Delta^+$, set $\Gamma_\alpha = \{\beta \in \Delta^+ : \alpha - \beta \in \Delta^+\}$. Then*

$$\dim S \cdot X_\alpha^* = \operatorname{Card}(\Gamma_\alpha) + 2. \tag{1}$$

Proof. Let $\mathfrak{s}_0 = \{Z \in \mathfrak{s} : Z \cdot X_\alpha^* = 0\}$ be the isotropy algebra of X_α^*. We observe that $\mathfrak{s}_0$ is stable under $\mathrm{Ad}(A)$, and calculate that $X_\beta \in \mathfrak{s}_0$ iff $\beta \notin \{\alpha\} \cup \Gamma_\alpha$. Thus

$$\mathfrak{s}_0 = \operatorname{Ker}(\alpha|_{\mathfrak{a}}) + \sum \mathfrak{g}_\beta, \tag{2}$$

where the sum is over $\beta \in \Delta^+ \backslash (\{\alpha\} \cup \Gamma_\alpha)$. Since $\dim S \cdot X_\alpha^* = \dim \mathfrak{s} - \dim \mathfrak{s}_0$, we obtain (1) from (2). (Recall that $\dim \mathfrak{g}_\beta = 1$, since $\mathfrak{g}$ is split.) $\square$

Theorem. *Suppose that either $\alpha = \alpha_1 + \ldots + \alpha_l$ or else $\alpha = (H_1 + \ldots + H_l)^\vee$, where H_i is the coroot to α_i, and $^\vee$ is the "root$\leftrightarrow$coroot" operation. Then $S \cdot X_\alpha^*$ is a Toda orbit.*

Proof. By Lemma 1 of this section and Lemmas 2 and 4 of the appendix, we see that $\dim S \cdot X_\alpha^* = 2l$. Since $\Phi_\xi = \{\alpha\}$ when $\xi = X_\alpha^*$, it is clear that $\operatorname{Supp} \Phi_\xi = \Pi$. The theorem then follows by Sect. 3.3, Proposition. $\square$

Remarks. 1. In the simply-laced case (Δ of type A, D, or E), the two choices of α in the theorem coincide, giving rise to two such Toda orbits for each Dynkin diagram (we could have taken $-X_\alpha^*$ instead of X_α^* in the choice of basis). In the multiply-laced cases (Δ of type B, C, F, or G), the two choices of α are distinct (the first is a short root; the second is long). Hence we obtain four such Toda orbits for each of these Dynkin diagrams.

2. For Δ of type A_{n-1} ($\mathfrak{g} = sl(n, \mathbb{R})$), these Toda orbits were found by Symes [Sy2, Sect. 10].

We turn now to a more detailed description of the orbit $O = S \cdot X_\alpha^*$ when $\alpha = (H_1 + \ldots + H_l)^\vee$. Thus α is a long root. Similar results can be obtained in the multiply-laced case for the short root $\alpha = \alpha_1 + \ldots + \alpha_l$. Instead of using Lemmas 3 and 4 from the appendix, however, one must do a number of root calculations on a case-by-case basis. We omit the details.

We first construct a $2l$-dimensional subgroup of S which acts simply-transitively on O. Let $\{\beta_1, \ldots, \beta_{l-1}\} \cup \{\gamma_1, \ldots, \gamma_{l-1}\}$ be the polarization of Γ_α described in Lemma 4 and Table 1 of the appendix. Set

$$X_i = X_{\beta_i}, \qquad Y_i = X_{\gamma_i}, \qquad Z = X_\alpha. \tag{3}$$

By Lemma 3 of the appendix, we have the following commutation relations (after an appropriate rescaling of Z):

$$[X_i, X_j] = [Y_i, Y_j] = 0,$$

$$[X_i, Z] = [Y_i, Z] = 0, \qquad [X_i, Y_j] = \delta_{ij} Z,$$

$$[H_\alpha, X_i] = X_i, \qquad [H_\alpha, Y_i] = Y_i, \qquad [H_\alpha, Z] = 2Z.$$

Here H_α is the coroot to α. Set $\mathfrak{u} = \operatorname{span}\{X_i, Y_i, Z : 1 \leqq i \leqq l-1\}$. Then $\mathfrak{u}$ is either abelian (if $l = 1$), or else a $2l-1$ dimensional Heisenberg algebra with center spanned by Z. Let $U=\exp\mathfrak{u}$ be the corresponding Heisenberg group. Clearly A normalizes U. Also, $\operatorname{ad} H_\alpha$ generates the usual group of dilations of U, since $\beta_i(H_\alpha)=\gamma_i(H_\alpha)=1$ (cf. appendix, Lemma 3). Let $R=\exp(\mathbb{R}H_\alpha)U$ be the semi-direct product of U with the dilation group. The Lie algebra of R is

$$\mathfrak{r}=\mathbb{R}H_\alpha+\mathfrak{u}. \tag{4}$$

Let $\mathfrak{s}_0$ be the isotropy algebra of X_α^*. By (2) and (4) it is clear that

$$\mathfrak{s}=\mathfrak{r}\oplus\mathfrak{s}_0 \quad \text{(vector space direct sum)}. \tag{5}$$

Since S is exponential-solvable, we know that the isotropy group S_0 of X_α^* is connected. It follows from (5) that $S_0\cap R=\{1\}$ and $O=R\cdot X_\alpha^*$. Thus R acts simply-transitively on O. To obtain an explicit parametrization of O in terms of R, we make the following calculation, where $\{X_i^*, Y_i^*, Z^*\}$ are dual to $\{X_i, Y_i, Z\}$:

Lemma 2. *Suppose $f\in U\cdot Z^*$. Write $f=\exp X\exp(Y+\zeta Z)\cdot Z^*$, where $X\in\operatorname{span}\{X_i\}$ and $Y\in\operatorname{span}\{Y_i\}$. Define $\xi_i=X_i^*(X)$ and $\eta_i=Y_i^*(Y)$. Then the projection of f onto $\mathfrak{u}^*$ is*

$$Z^*+\sum_{i=1}^{l-1}\eta_i X_i^*-\xi_i Y_i^*.$$

The projection of f onto $\mathfrak{a}^*$ is

$$\zeta\alpha+\sum_{i=1}^{l-1}(\xi_i\eta_i)\beta_i.$$

Proof. We first observe that

$$\mathfrak{s}\cdot\mathfrak{a}^*=0, \tag{6}$$

$$Y_i\cdot Z^*=X_i^*. \tag{7}$$

Next, we claim that

$$Y_i\cdot X_j^*=0 \tag{8}$$

for all i, j. Indeed, the left side of (8) has weight $\gamma_i-\beta_j$ relative to the coadjoint action of $\mathfrak{a}$. But by Lemma 4(i) of the appendix, we know that $\gamma_i>\beta_j$, while all weights of $\mathfrak{a}$ on $\mathfrak{s}^*$ are negative, relative to the order on Δ defined by Δ^+. This proves (8). Obviously $Z\cdot Z^*=\alpha$, so combining (6), (7), and (8) gives

$$\exp(\zeta Z+Y)\cdot Z^*=\zeta\alpha+Z^*+\sum_{i=1}^{l-1}\eta_i X_i^*. \tag{9}$$

Now consider the action of $\exp X$ on (9). We have

$$X_j\cdot Z^*=-Y_j^*. \tag{10}$$

The higher order terms $X_{i_1}\dots X_{i_m}\cdot Y_j^*\in\mathfrak{u}^\perp$, $X_{i_1}\dots X_{i_m}\cdot X_j^*\in\mathfrak{u}^\perp$, by Lemma 4(ii) of the appendix, if $m\geqq 1$. It follows from (9) and (10) that the projection of f onto $\mathfrak{u}^*$ is as claimed. Since $X_i\cdot X_i^*=\beta_i$, we also obtain the projection of f onto $\mathfrak{a}^*$ from (6) and (9) in the form stated. $\square$

We can now give a set of global canonical coordinates on O.

Proposition. *Let $O = S \cdot X_\alpha^*$, where $\alpha = (H_1 + \ldots + H_l)^\vee$. Let X_i, Y_i, Z be as in (3). Then $f(Z) > 0$ for $f \in O$. The functions $p_i(f) = f(X_i)/f(Z)^{1/2}$, $q_i(f) = f(Y_i)/f(Z)^{1/2}$ for $1 \leqq i \leqq l-1$ and $p_l(f) = f(H_\alpha)$, $q_l(f) = \frac{1}{2}\log f(Z)$ are global canonical symplectic coordinates on O ($\{p_i, q_i\} = 1$ and all other Poisson brackets are zero). The map $f \to (p_1(f), \ldots, p_l(f), q_1(f), \ldots, q_l(f))$ is an analytic manifold isomorphism from O onto $\mathbb{R}^{2l}$.*

Proof. Let $f \in O$. Then $f = \exp(tH_\alpha)\exp(X)\exp(Y + \zeta Z) \cdot Z^*$, where

$$X = \sum_{i=1}^{l-1} \xi_i X_i \quad \text{and} \quad Y = \sum_{i=1}^{l-1} \eta_i Y_i .$$

Since R acts simply-transitively on O, it is clear that $\{t, \xi_1, \ldots, \xi_{l-1}, \eta_1, \ldots, \eta_{l-1}, \zeta\}$ is a global coordinate system on O. By Lemma 2 and the fact that $\beta_i(H_\alpha) = \gamma_i(H_\alpha) = 1$, we find that $f(Z) = e^{2t}$, and for $1 \leqq i \leqq l-1$, $p_i(f) = \eta_i$, $q_i(f) = -\xi_i$. Also $q_l(f) = -t$ and

$$p_l(f) = 2\zeta + \sum_{i=1}^{l-1} \xi_i \eta_i .$$

This shows that the p's and q's give global analytic coordinates on O. From the commutation relations after Eq. (3), it is easily checked that the only non-zero Poisson bracket among the p_i and q_j is $\{p_i, q_i\} = 1$ (use the same argument as on p. 380 of [G–W2]). □

Example. Let $G = \mathrm{SL}(n, \mathbb{R})$, $S =$ upper triangular unimodular matrices. Identify $\mathfrak{s}^*$ with the lower triangular trace-zero matrices via the trace form. Let O be the orbit of the elementary matrix E_{n1}. If $n = 2$, then $S = R$ is the "$ax+b$" group and $\dim O = 2$. The parametrization in the proposition above is

$$(p, q) \to \begin{pmatrix} p & 0 \\ e^q & -p \end{pmatrix} .$$

When $n = 3$, we still have $N = U$, but $\dim S/R = 1$. Now $\dim O = 4$, and in terms of the canonical coordinates in the Proposition, O consists of the matrices

$$\begin{bmatrix} p_2 & 0 & 0 \\ -p_1 e^{-q_2} & p_1 q_1 & 0 \\ e^{-2q_2} & q_1 e^{-q_2} & -p_2 - p_1 q_1 \end{bmatrix} .$$

When $n \geqq 4$, then $N \neq U$ [$\dim N = n(n-1)/2$ while $\dim U = 2n-3$]. For $n = 4$, one has $\dim N/U = 1$. An explicit matrix calculation of O in canonical coordinates slightly different than those used above may be found in [Sy2, Sect. 10].

3.5. Construction of Toda Orbits (Amalgamation)

In this section we develop inductive procedures for obtaining Toda orbits of $S = S(\Pi)$ from Toda orbits for smaller Iwasawa groups $S(\Pi_1)$, where $\Pi_1 \subset \Pi$. We do not require that the Dynkin diagram for Π be connected.

Recall from Sect. 3.2 that if O is a coadjoint S-orbit, then there is a unique subset $\Pi_O = \Pi_1 \subset \Pi$, and a $\xi \in \mathfrak{s}^*$ with $\mathrm{Supp}\,\Phi_\xi = \Pi_1$, such that $O = S(\Pi_1) \cdot \xi$.

Writing $\alpha = \xi|_{\mathfrak{a}}$ and $\xi_1 = \xi|_{\mathfrak{n}(\Pi_1)}$, we have $\operatorname{Supp}\Phi_{\xi_1} = \Pi_1$ and $O = \alpha + O_1$. Here $O_1 = S(\Pi_1)\cdot\xi_1$ is a coadjoint $S(\Pi_1)$-orbit, by Sect. 3.2, Lemma 2. Conversely, every such orbit O_1 may be viewed as an S-orbit. In this connection, we will say that O_1 is J_1-*regular* when it satisfies the regularity condition of Sect. 3.2 relative to the group $S(\Pi_1)$.

Proposition 1. *Let $\Pi = \Pi_1 \cup \Pi_2$, with Π_1 and Π_2 disjoint. Assume that O_i is a coadjoint $S(\Pi_i)$-orbit for $i = 1, 2$ and set $O = O_1 + O_2$ (vector sum). Then*

(a) *O is a coadjoint S orbit;*

(b) *$\Pi_O = \Pi_{O_1} \cup \Pi_{O_2}$;*

(c) $\dim O = \dim O_1 + \dim O_2$.

In particular, if O_i are J_i-regular (respectively of Toda type) relative to $S(\Pi_i)$ for $i = 1, 2$, then O is J-regular (respectively of Toda type) relative to S.

Proof. Pick $\xi_i \in O_i$ with $\operatorname{Supp}\Phi_{\xi_i} = \Pi_{O_i}$, and set $\xi = \xi_1 + \xi_2$. Since $S(\Pi_i)$ fixes $\mathfrak{s}(\Pi_j)^*$ for $i \neq j$, one has $(s_1 s_2)\cdot\xi = s_1\cdot\xi_1 + s_2\cdot\xi_2 = (s_2 s_1)\cdot\xi$, when $s_i \in S(\Pi_i)$. The group S is generated by $S(\Pi_1)$ and $S(\Pi_2)$, so $S\cdot\xi = O_1 + O_2$, proving (a).

By definition, $\Pi_O = \{\alpha \in \Pi : X_\alpha \notin O^\perp\}$. Since $\mathfrak{n}(\Pi_i) \perp \mathfrak{n}(\Pi_j)^*$ for $i \neq j$, it is thus clear that (b) holds. We also have $\mathfrak{s}(\Pi_1)^* \cap \mathfrak{s}(\Pi_2)^* = \{0\}$, so (c) is obvious.

When O_i is J_i-regular for $S(\Pi_i)$, then $\Pi_{O_i} = \Pi_i$, so that by (b) we have $\Pi_O = \Pi$. Thus O is J-regular, by Sect. 3.2, Theorem 2. If O_i is a Toda orbit relative to $S(\Pi_i)$, then $\Pi_{O_i} = \Pi_i$ and $\dim O_i = 2\operatorname{Card}(\Pi_i)$. Hence O is J-regular and $\dim O = 2\operatorname{Card}(\Pi)$ by (b) and (c). □

Corollary. *Let $\Pi = \Pi_1 \cup \ldots \cup \Pi_r$ be a disjoint union, and suppose that $O_i \subset \mathfrak{s}(\Pi_i)^*$ are Toda orbits, for $1 \leqq i \leqq r$. Set $O = O_1 + \ldots + O_r$ (vector sum). Then O is a Toda orbit for S.*

Examples. 1. Take $\Pi_i = \{\alpha_i\}$, $O_i = S(\Pi_i)\cdot X^*_{\alpha_i}$, for $1 \leqq i \leqq l$. Then each O_i is a two-dimensional orbit associated with a non-periodic Toda lattice of one degree of freedom. Forming $O = O_1 + \ldots + O_l$, we obtain the Toda orbit for the non-periodic generalized Toda lattice associated with Π (cf. Sect. 3.3, Example).

2. Let Π be of type $A_{2k}(G = \mathrm{SL}(2k+1, \mathbb{R}))$. Take $\Pi_i = \{\alpha_{2i-1}, \alpha_{2i}\}$, for $1 \leqq i \leqq k$. Then each subgroup $G(\Pi_i)$ is a copy of $\mathrm{SL}(3, \mathbb{R})$, embedded in block diagonal form in G. Take O_i to be the four-dimensional Toda orbit for the corresponding Iwasawa group S_i described in Sect. 3.4, Example, and set $O = O_1 + \ldots + O_k$. Then the canonical coordinates $p_i, p_{i+1}, q_i, q_{i+1}$ on each orbit O_i jointly give a canonical parametrization of O. When $k = 2$, then O consists of the 5×5 lower triangular matrices of the form

$$\begin{bmatrix} p_2 & 0 & 0 & 0 & 0 \\ p_1 e^{-q_2} & p_1 p_2 & 0 & 0 & 0 \\ e^{-2q_2} & q_1 e^{-q_2} & p_3 - p_2 - p_1 q_1 & 0 & 0 \\ 0 & 0 & -p_3 e^{-q_4} & p_3 p_4 & 0 \\ 0 & 0 & e^{-2q_4} & q_3 e^{-q_4} & -p_4 - p_3 q_3 \end{bmatrix}$$

The boxes in dashed lines indicate the orbits O_1 and O_2, with overlap in the middle of the matrix.

3. *(Amalgamation with overlap).* Let $\Pi = \Pi_1 \cup \Pi_2$, as before, but now allow an overlap of one root:

$$\Pi_1 \cap \Pi_2 = \{\delta\}. \tag{1}$$

Take $\beta_i \in \Delta^+(\Pi_i)$ such that $\mathrm{Supp}(\beta_i) = \Pi_i$ and $\dim S(\Pi_i) \cdot X^*_{\beta_i} = 2\,\mathrm{Card}\,\Pi_i$, for $i = 1, 2$ (cf. Sect. 3.4, theorem, for examples of such roots β_i). With this choice, the orbit of $X^*_{\beta_i}$ under $S(\Pi_i)$ is a Toda orbit relative to $\mathfrak{s}(\Pi_i)^*$. Set $\xi = X^*_{\beta_1} + X^*_{\beta_2}$. We want to determine whether $S \cdot f$ is a Toda orbit relative to $\mathfrak{s}^*$. Since $\Gamma_{\beta_i} \subset \Delta^+(\Pi_i)$ and $\Delta^+(\Pi_1) \cap \Delta^+(\Pi_2) = \{\delta\}$ by condition (1), the subsets Γ_{β_i}, $i = 1, 2$, are either disjoint or else satisfy

$$\Gamma_{\beta_i} \cap \Gamma_{\beta_2} = \{\delta\} \cdot \tag{2}$$

Proposition 2. *Assume* (1) *and* (2) *hold, with* β_i, ξ *as above. Then* $S \cdot \xi$ *is a Toda orbit in* $\mathfrak{s}^*$.

Proof. Clearly $\mathrm{Supp}(\Phi_\xi) = \Pi_1 \cup \Pi_2 = \Pi$, so by Sect. 3.3, Proposition, we only need to check that $\dim S \cdot \xi = 2l$, i.e. that $\dim \mathfrak{s}_\xi = \dim \mathfrak{s} - 2l$.

To obtain a set of equations defining $\mathfrak{s}_\xi$, we note that if $X = H + \sum a_\alpha X_\alpha$ is in $\mathfrak{s}$, with $H \in \mathfrak{a}$, then

$$X \cdot X^*_{\beta_i} = -a_{\beta_i}\beta_i - \beta_i(H) X^*_{\beta_i} + \sum_{\alpha \in \Gamma_{\beta_i}} a_\alpha N_{\alpha, \beta_i - \alpha} X^*_\alpha\,. \tag{3}$$

Here $N_{\alpha,\beta}$ are the structure constants defined by $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$. From (3) we see that $X \in \mathfrak{s}_\xi$ iff

$$a_{\beta_1-\delta} N_{\delta,\beta_1-\delta} + a_{\beta_2-\delta} N_{\delta,\beta_2-\delta} = 0\,, \tag{4}$$

$$a_\alpha = 0, \quad \text{for} \quad \alpha \in \Gamma_{\beta_1} \cup \Gamma_{\beta_2} \setminus \{\beta_1 - \delta, \beta_2 - \delta\}\,, \tag{5}$$

$$a_{\beta_1} = a_{\beta_2} = \beta_1(H) = \beta_2(H) = 0\,. \tag{6}$$

Counting equations, we find that $\dim S \cdot \xi = 3 + \mathrm{Card}(\Gamma_{\beta_1} \cup \Gamma_{\beta_2}) = 2 + \mathrm{Card}(\Gamma_{\beta_1}) + \mathrm{Card}(\Gamma_{\beta_2})$, by condition (2). But we know by Sect. 3.4, Lemma 1, that $\mathrm{Card}(\Gamma_{\beta_i}) = 2\,\mathrm{Card}(\Pi_i) - 2$. It follows that $\dim S \cdot \xi = 2\,\mathrm{Card}(\Pi_1) + 2\,\mathrm{Card}(\Pi_2) - 2 = 2l$. $\square$

3.6. Scattering on J-Regular Orbits

Let $O \subset \mathfrak{s}^*$ be a J-regular S orbit, in the sense of Sect. 3.2. Consider the asymptotics of the Hamiltonian flow on O coming from the Killing form on $\mathfrak{p}$. We saw in Sect. 2.3 that this flow, viewed as a flow on $\mathfrak{p}$, has for "generic" scattering transformation the longest element w_0 of the Weyl group ("generic" in this case meaning on the dense open subset $\mathfrak{p}''$). Now we shall sharpen this result by showing that for almost all points of O, the scattering transformation is still given by w_0.

Theorem. *Let O be a J-regular S orbit in $\mathfrak{s}^*$. Let $\mathfrak{p}'' \subset \mathfrak{p}$ be defined by Sect. 2.3 (7). Then $\psi(\mathfrak{p}'') \cap O$ has complement of measure zero in O (relative to the canonical* *measure on O). Thus for almost all choices of initial data in O, the Hamiltonian system $\dot{\xi} = -\xi^{\flat} \cdot \xi$ on O has scattering transformation $\xi(+\infty) = w_0 \cdot \xi(-\infty)$.*

Proof. (A) For $\alpha \in \Pi$, let r_α be the analytic function on O defined in Sect. 3.2 (4). Then r_α is non-constant, by Sect. 3.2, Theorem 1.

(B) Define $O' = \{\xi \in O : \xi(H_\beta) \neq 0, \forall \beta \in \Delta^+\}$. Then O' is open in O. Define $\tau : O' \to O$ as follows: Given $\xi \in O'$, write $\xi = \gamma(H + Y)$, where $H \in \mathfrak{a}'$ and $Y \in \bar{\mathfrak{n}}$. Define $\bar{n} \in N^-$ implicitly as a function of H and Y by the equation $\mathrm{Ad}(\bar{n})H = H + Y$, and set $\tau(\xi) = \mathbf{s}(\bar{n})^{-1} \cdot \xi$ [cf. Sect. 3.2, Theorem 1, proof of statement (A)]. The map τ is analytic on O'.

(C) For $\alpha \in \Pi$, set $O'_\alpha = \{\xi \in O' : \xi(X_\alpha) = 0\}$. Then O'_α has measure zero in O by (A). Hence $\tau(O'_\alpha)$ also has measure zero in O, by (B). But if $w \in W$, and $w \cdot \alpha \in \Delta^+$, then we claim that

$$\psi(\mathfrak{p}'(w)_+) \cap O \subset \tau(O'_\alpha). \tag{1}$$

Indeed, if $X \in \mathfrak{p}'(w)_+$ then by Sect. 2.2 (7) and the proof of Sect. 3.2, Theorem 1, we can write $\psi(X) = \tau(\gamma(H + Y))$, with $H \in w \cdot \mathfrak{a}^+$ and $Y \in \bar{\mathfrak{n}}_w$. Since $X_\alpha \perp \bar{\mathfrak{n}}_w$, we have $\gamma(H + Y) \in O'_\alpha$, proving (1). In particular, if $w \neq w_0$, then there exists $\alpha \in \Pi$ such that $w \cdot \alpha \in \Delta^+$. Hence by (1) we conclude that $\psi(\mathfrak{p}'(w)_+) \cap O$ has measure zero in O in this case.

(D) Since $O_{\mathrm{reg}} \subset \psi(\mathfrak{p}')$, we may use the decomposition Sect. 2.2 (8) of $\mathfrak{p}'$ to write

$$O_{\mathrm{reg}} = \bigcup_{w \in W} \{\psi(\mathfrak{p}'(w)_+) \cap O_{\mathrm{reg}}\}.$$

By (C) all terms on the right have measure zero in O except for the term with $w = w_0$. Since the same argument applies to the J-regular orbit $-O$, we conclude from Sect. 2.3 (5) that $\psi(\mathfrak{p}'') \cap O$ has complement of measure zero in O. Now apply Sect. 2.3, theorem. □

Remarks. 1. For the examples of Toda orbits in Sect. 3.4, Proposition, one can show by some detailed calculation that the sets O'_α in part (C) of the proof just given are empty, when the root system is of type B, C, F, or G (multiply-laced). Thus $O_{\mathrm{reg}} \subset \psi(\mathfrak{p}'')$ in these cases, and every element of O_{reg} has scattering transformation w_0. For the simply-laced root systems (ADE-type), the sets O'_α can be non-empty for certain α.

2. In connection with the QR algorithm (cf. Sect. 2.2), it was known that for a "generic" symmetric matrix, the diagonal entries produced by the algorithm appear in monotone order [Ru, Satz 12.6, Remarks]. The stronger assertion made by the theorem just proved is that this behavior is still "generic" among the matrices restricted to lie on $\psi^{-1}(O)$, where O is any J-regular orbit.

3. In the case of the Toda orbit O of Jacobi matrices described in Sect. 3.3, we already noted Kostant's result that $O = O_{\mathrm{reg}}$. It is obvious that O'_α is empty for every $\alpha \in \Pi$, by the explicit parametrization of the orbit. Hence by parts (C) and (D) of the proof just given, we have $O \subset \psi(\mathfrak{p}'')$ in this case. This proves the following generalization of J. Moser's scattering results for the original non-periodic Toda lattice (cf. [Ko, Chap. 7]):

Corollary. *The scattering transformation for the generalized non-periodic Toda lattices is always given by the longest element of the Weyl group.*

4. Hamiltonian Systems Associated with Affine Root Systems

4.1. Lax equations on loop groups

In this chapter we study a class of (finite-dimensional) Hamiltonian systems which are obtained from affine root systems. To give a unified treatment of all these systems within the framework of Chap. 1, we need to introduce some infinite-dimensional Lie groups associated with affine root systems, and a suitable Poisson-commutative algebra of functions. We first recall some well-known structural properties of semi-simple Lie groups [He2], and the analogous properties of the associated "loop groups" [G–W3].

Let $G_{\mathbb{C}}$ be a simply-connected complex Lie group, whose Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is simple. Let $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ be a normal real form, and let $G \subset G_{\mathbb{C}}$ be the corresponding connected real Lie group. Denote by σ the involutions of $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ defined by this real form. Fix Iwasawa and Cartan decompositions

$$G = KAN, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n},$$

$$G = KP, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

Then $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is a compact form of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u} + \mathfrak{e}$ is a Cartan decomposition, where $\mathfrak{e} = i\mathfrak{u}$. Furthermore, $\mathfrak{a}$ is maximal abelian in $\mathfrak{e}$ as well as in $\mathfrak{p}$. Let $U \subset G_{\mathbb{C}}$ be the connected group with Lie algebra $\mathfrak{u}$. Denote by τ the involution (respectively conjugation) of $G_{\mathbb{C}}$ (respectively $\mathfrak{g}_{\mathbb{C}}$) whose fixed-point set is U (respectively $\mathfrak{u}$). The following result is an immediate consequence of the Chevalley restriction theorem [He1, Chap. X, Theorem 6.10]:

Lemma 1. *The restriction map from $S(\mathfrak{e})^U$ to $S(\mathfrak{p})^K$, $f \to f|_{\mathfrak{p}}$, is bijective. Denote the inverse map by $\phi \to \check{\phi}$. (Here we identify $S(\mathfrak{e})$ with the polynomial functions on $\mathfrak{e}$ via the Killing form, as usual.)*

Let $\tilde{G}_{\mathbb{C}} = C^\infty(\mathbb{T}, G_{\mathbb{C}})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, be the smooth "loop group" (or "current group") associated with $G_{\mathbb{C}}$. With the C^∞ topology, it is a Fréchet Lie group, with Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}} = C^\infty(\mathbb{T}, \mathfrak{g}_{\mathbb{C}})$. We extend the conjugation σ of $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ to a conjugation on the loop group and algebra by setting $(\sigma f)(z) = \sigma(f(\bar{z}))$, ($\bar{z}$ = complex conjugate of z). We denote by $\tilde{G}$ and $\tilde{\mathfrak{g}}$ the fixed-point sets of the extended σ. Then $\tilde{G}$ is a real form of $\tilde{G}_{\mathbb{C}}$, with Lie algebra $\tilde{\mathfrak{g}}$. (In terms of Fourier-series expansions on $\mathbb{T}$, $\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}_{\mathbb{C}}$ consists of the elements whose Fourier coefficients are in $\mathfrak{g}$.) We extend the involution τ to $\tilde{G}_{\mathbb{C}}$ and $\tilde{\mathfrak{g}}_{\mathbb{C}}$ by $(\tau f)(z) = \tau(f(z))$, for $z \in \mathbb{T}$. [This formula for the extended involution τ can be viewed as follows: If f has a finite Fourier series, for example, and is extended holomorphically to $\mathbb{C}^\times$, then $(\tau f)(\zeta) = \tau(f(\bar{\zeta}^{-1}))$, where $\zeta \in \mathbb{C}^\times$. Note that $\zeta \to \bar{\zeta}^{-1}$ is the involution of $\mathbb{C}^\times$ whose fixed-point set is the compact real form $\mathbb{T}$.]

The extended involutions τ and σ commute, so $\tilde{G}$ and $\tilde{\mathfrak{g}}$ are invariant under τ. Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}}$ be the decomposition of $\tilde{\mathfrak{g}}$ into $+1$ and -1 eigenspaces for τ. Let $\tilde{K}$ be the fixed-point set of τ in $\tilde{G}$. Then $G = \tilde{K} \cdot \tilde{P}$, where $\tilde{P} = \exp(\tilde{\mathfrak{p}})$ (cf. [G–W3, Chap. 6]). Observe that if $f \in \tilde{\mathfrak{g}}$, then $f \in \tilde{\mathfrak{k}}$ (respectively $\tilde{\mathfrak{p}}$) iff for all $z \in \mathbb{T}$, $f(z) \in \mathfrak{u}$ (respectively $\mathfrak{e}$).

From the Killing form B on $\mathfrak{g}$, we obtain a bilinear form $\tilde{B}$ on $\tilde{\mathfrak{g}}$ by integration over $\mathbb{T}$:

$$\tilde{B}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} B(x(e^{i\theta}), y(e^{i\theta}))\, d\theta .$$

Clearly $\tilde{B}$ is positive-definite on $\tilde{\mathfrak{p}}$, by the corresponding property of B on $\mathfrak{p}$. Likewise, one has $\tilde{\mathfrak{p}} \perp \tilde{\mathfrak{k}}$ relative to $\tilde{B}$. Let d_0 be the degree-derivation of $\tilde{\mathfrak{g}}$: $d_0 x(e^{i\theta}) = -i(d/d\theta)\, x(e^{i\theta})$. Denote by $\tilde{\mathfrak{g}}^e$ the semi-direct product of $\tilde{\mathfrak{g}}$ with $\mathbb{R}d_0$, and set $\mathfrak{a}^e = \mathfrak{a} \oplus \mathbb{R}d_0$. Integrating by parts and using the $\mathfrak{g}$-invariance of B shows that $\mathrm{ad}(x)|_{\tilde{\mathfrak{g}}}$ is skew-symmetric relative to $\tilde{B}$ for any $x \in \tilde{\mathfrak{g}}^e$.

Given $\phi \in S(\mathfrak{p})^K$, we may similarly define a function $\tilde{\phi}$ on $\tilde{\mathfrak{p}}$ by integration over $\mathbb{T}$ (taking into account Lemma 1 and the remarks above):

$$\tilde{\phi}(x) = \frac{1}{2\pi} \int_0^{2\pi} \check{\phi}(x(e^{i\theta}))\, d\theta .$$

Lemma 2. *If ϕ is a K-invariant polynomial on $\mathfrak{p}$, then $\tilde{\phi}$ is invariant under the adjoint action of $\tilde{K}$ on $\tilde{\mathfrak{p}}$. Furthermore, $\tilde{\phi}$ is differentiable, and $d\tilde{\phi}(x)(y) = \tilde{B}(\nabla\tilde{\phi}(x), y)$, for $x, y \in \tilde{\mathfrak{p}}$, where $\nabla\tilde{\phi}(x)(t) = (\nabla\check{\phi})(x(t))$ for $t \in \mathbb{T}$. Thus $\tilde{\phi}$ has a gradient, relative to the form $\tilde{B}$.*

Proof. The $\tilde{K}$ invariance of $\tilde{\phi}$ is obvious, as is the differentiability. The formula for the gradient of $\tilde{\phi}$, as a map from $\tilde{\mathfrak{p}}$ to $\tilde{\mathfrak{p}}_{\mathbb{C}}$ follows from the integral formula. Note that $\check{\phi}(z) = \check{\phi}(\sigma z)^-$ for $z \in \mathfrak{p}_{\mathbb{C}}$, which implies that $(\nabla\check{\phi})(\sigma z) = \sigma(\nabla\check{\phi}(z))$. Hence $\nabla\tilde{\phi}(x) \in \tilde{\mathfrak{p}}$ if $x \in \tilde{\mathfrak{p}}$. □

Let $\delta \in (\mathfrak{a}^e)^*$ be defined by $\delta(\mathfrak{a}) = 0$ and $\delta(d_0) = 1$. With the notation as in Sect. 3.2, let $\tilde{\alpha} = \sum_{1 \leqq i \leqq l} n_i \alpha_i$ be the largest positive root, and set $\alpha_0 = \delta - \tilde{\alpha}$. Then the roots of $\mathfrak{a}^e$ on $\tilde{\mathfrak{g}}^e$ are integral combinations, with all coefficients of the same sign, of the roots $\alpha_0, \alpha_1, \ldots, \alpha_l$. Take $H_0 \in \mathfrak{a}$ satisfying $\alpha_i(H_0) = 1$, for $1 \leqq i \leqq l$. Set

$$h = 1 + \tilde{\alpha}(H_0) = 1 + \sum n_i$$

(Coxeter number of the root system of $\mathfrak{g}$). Let $H_0^e = hd_0 + H_0$, and define the *principal derivation* of $\tilde{\mathfrak{g}}^e$ to be $\mathrm{ad}(H_0^e)$. Note that this operator is skew-symmetric relative to the form $\tilde{B}$. Since $\alpha_i(H_0^e) = 1$ for $0 \leqq i \leqq l$, one has the *principal gradation*

$$\tilde{\mathfrak{g}}^e = \mathfrak{a}^e + \sum_{n \neq 0} \tilde{\mathfrak{g}}_n ,$$

where $\tilde{\mathfrak{g}}_n$ is the eigenspace for $\mathrm{ad}(H_0^e)$ with eigenvalue n.

We recall from [G–W3, Sect. 6.8] the following properties of the Banach-Lie group $\tilde{G}_w \subset \tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}_w \subset \tilde{\mathfrak{g}}$. Here w is a *weight function* on $\mathbb{Z}$, i.e. w is a positive function on the integers such that $w(k+m) \leqq w(k)\, w(m)$, and $w(k) = w(-k)$. We shall assume that w is rapidly increasing:

$$\lim_{n \to \infty} w(n) n^{-s} = \infty \tag{1}$$

for all $s > 0$. We shall also assume that w is of non-analytic type:

$$\lim_{n \to \infty} w(n)^{1/n} = 1 . \tag{2}$$

Take any faithful, finite-dimensional representation of G. Then $\tilde{G}_w$ (respectively $\tilde{\mathfrak{g}}_w$) consists of the elements x in $\tilde{G}$ (respectively $\tilde{\mathfrak{g}}$) whose matrix-valued Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ is absolutely convergent relative to the weight w:

$$\|x\|_w = \sum_{n\in\mathbb{Z}} \|a_n\| \, w(n) < \infty . \tag{3}$$

Note that by condition (1), the convergence of (3) implies that x is a C^∞ function. Using (2) and results on the inner-outer factorization of matrix functions on the circle, one obtains that $\tilde{G}_w$ has Cartan and Iwasawa decompositions

$$\tilde{G}_w = \tilde{K}_w \cdot \tilde{P}_w, \qquad \tilde{\mathfrak{g}}_w = \tilde{\mathfrak{k}}_w + \tilde{\mathfrak{p}}_w \tag{4}$$

$$\tilde{G}_w = \tilde{K}_w \cdot A \cdot \tilde{N}_w, \qquad \tilde{\mathfrak{g}}_w = \tilde{\mathfrak{k}}_w + \mathfrak{a} + \tilde{\mathfrak{n}}_w . \tag{5}$$

Here $\tilde{K}_w = \tilde{K} \cap \tilde{G}_w$ is a Banach-Lie subgroup of $\tilde{G}_w$ with Lie algebra $\tilde{\mathfrak{k}}_w = \tilde{\mathfrak{g}}_w \cap \tilde{\mathfrak{k}}$, and $\tilde{P}_w = \exp(\tilde{\mathfrak{p}}_w)$. One has $\tilde{N}_w = \exp(\tilde{\mathfrak{n}}_w)$, with $\tilde{\mathfrak{n}}_w$ the closed span, in the w-norm, of $\tilde{\mathfrak{g}}_n$, $n>0$.

We set $\tilde{S}_w = A\tilde{N}_w$, $\tilde{\mathfrak{s}}_w = \mathfrak{a} + \tilde{\mathfrak{n}}_w$, and denote by $\mathbf{k} : \tilde{G}_w \to \tilde{K}_w$, $\mathbf{s} : \tilde{G}_w \to \tilde{S}_w$, the analytic maps defined by the factorization $g = \mathbf{s}(g)\,\mathbf{k}(g)$. We denote by $\pi_{\mathfrak{k}} : \tilde{\mathfrak{g}}_w \to \tilde{\mathfrak{k}}_w$ the projection corresponding to the decomposition (5), and denote by $\psi : \tilde{\mathfrak{p}}_w \to (\tilde{\mathfrak{s}}_w)^*$ the map defined by the form $\tilde{B}$. It is easy to check that Lemma 2 is valid with $\tilde{K}$ and $\tilde{\mathfrak{p}}$ replaced by $\tilde{K}_w$ and $\tilde{\mathfrak{p}}_w$. Also, if we take $\phi \in S(\mathfrak{p})^K$ and define a function H_ϕ on $(\tilde{\mathfrak{s}}_w)^*$ via the map ψ, then H_ϕ is differentiable, and $dH_\phi : (\tilde{\mathfrak{s}}_w)^* \to \tilde{\mathfrak{s}}_w$ is given by $dH_\phi(\psi(x)) = \psi(\nabla\tilde{\phi}(x))$.

By virtue of Lemma 2 and the properties just recalled, the results of Sects. 1.1–1.2 and 3.1 can now be applied in the present context, replacing G by $\tilde{G}_w$, K by $\tilde{K}_w$, P by $\tilde{P}_w$, S by $\tilde{S}_w$, etc. We summarize the results as follows:

Theorem. *Let $\phi \in S(\mathfrak{p})^K$. Then the Lax equation*

$$\dot{X} = [\pi_{\mathfrak{k}}(\nabla\tilde{\phi}(X)), X], \qquad X(0) = x_0 \in \tilde{\mathfrak{p}}_w , \tag{6}$$

has as solution the curve in $\tilde{\mathfrak{p}}_w$

$$X(t) = \mathrm{Ad}(k(t)) \cdot x_0 , \tag{7}$$

where $k(t) = \mathbf{k}(\exp(\nabla\tilde{\phi}(x_0))$. If $F \in S(\mathfrak{p})^K$, then $\tilde{F}$ is constant on the curve (7). Furthermore, the solution to the Euler equation

$$\dot{\xi} = -dH_\phi(\xi) \cdot \xi, \qquad \xi(0) = \xi_0 = \psi(x_0) \tag{8}$$

on $(\tilde{\mathfrak{s}}_w)^*$ (where $\cdot$ denotes the coadjoint action) is given by

$$\xi(t) = s(t)^{-1} \cdot \xi_0 , \tag{9}$$

where $s(t) = \mathbf{s}(\exp t\nabla\tilde{\phi}(x_0))$.

Remarks. 1. Take a faithful matrix representation of $G_{\mathbb{C}}$ so that $\tau(g)^{-1} = g^*$ is the usual conjugate-transpose map. The solution (9) can be calculated from the "inner-outer" factorization of the positive-definite matrix valued function $\theta \to \exp ty(e^{i\theta})$ on $\mathbb{T}$, where $y = \nabla\tilde{\phi}(x_0) \in \tilde{\mathfrak{p}}_w$, and the variable t now plays the role of a parameter. To see this, combine the Iwasawa factorization $\exp ty = \mathbf{s}(\exp ty)\,\mathbf{k}(\exp ty)$ and the equation $\exp 2ty = (\exp ty)(\exp ty)^*$ to write

$$\exp 2ty = \mathfrak{s}(\exp ty)\,\mathfrak{s}(\exp ty)^* . \tag{10}$$

Thus $\mathfrak{s}(\exp ty)$ is the (suitably normalized) "inner factor" of $\exp 2ty$.

2. The theorem is also valid if the weight function w only satisfies (2), but is not necessarily rapidly increasing, e.g. $w=1$. In this case, $\tilde{G}_w$ is a subgroup of the continuous loop group on G. Conditions of the form (1) will be used when we calculate in Sect. 5 the solution $X(t)$, for special choices of x_0, using representation theory.

4.2. Finite-Dimensional Subquotients of $\tilde{\mathfrak{g}}$

To apply the results of the previous section to finite-dimensional Hamiltonian systems, we consider in more detail the principal gradation of $\tilde{\mathfrak{g}}^e$ (cf. [A–vM]). As in Sect. 4.1 we take $H_0^e = hd_0 + H_0$, where h is the Coxeter number of the root system of $\mathfrak{g}$. Then $\mathfrak{g}$ has the principal gradation

$$\mathfrak{g} = \sum_{-h<n<h} \mathfrak{g}_n ,$$

where $\mathfrak{g}_n$ is the eigenspace for $\mathrm{ad}(H_0)$ with eigenvalue n. Since $\tau(H_0) = -H_0$, one has $\tau(\mathfrak{g}_n) = \mathfrak{g}_{-n}$. Let $\tilde{\mathfrak{g}}_n$ be the eigenspace for $\mathrm{ad}(H_0^e)$ with eigenvalue n. For each n, $\tilde{\mathfrak{g}}_n$ is finite-dimensional, and is spanned by elements $xe^{ik\theta}$, where $x \in \mathfrak{g}_r$ and $r + kh = n$. Thus if $n > 0$ and $1 \leqq r < h$, then

$$\tilde{\mathfrak{g}}_{nh+r} = \mathfrak{g}_r e^{in\theta} + \mathfrak{g}_{r-h} e^{i(n+1)\theta} = \tilde{\mathfrak{g}}_r e^{in\theta} . \tag{1}$$

Since $\tau(\tilde{\mathfrak{g}}_k) = \tilde{\mathfrak{g}}_{-k}$, we have

$$\tilde{\mathfrak{p}} = \mathfrak{a} + \sum_{k>0} \tilde{\mathfrak{p}}_k ,$$

where $\tilde{\mathfrak{p}}_k = \{x + \tau(x) \mid x \in \tilde{\mathfrak{g}}_k\}$.

Now consider the subalgebras $\tilde{\mathfrak{n}} = \sum_{k>0} \tilde{\mathfrak{g}}_k$ (topological direct sum in $\tilde{\mathfrak{g}}$) and $\tilde{\mathfrak{s}} = \mathfrak{a} + \tilde{\mathfrak{n}}$. From the above description of the root spaces, it is clear that $\tilde{\mathfrak{n}}$ is generated by $\tilde{\mathfrak{g}}_1$, and that $\tilde{\mathfrak{n}}^k = \sum_{r \geqq k} \tilde{\mathfrak{g}}_r$. Thus the quotient algebra $\mathfrak{b}_k = \tilde{\mathfrak{s}}/\tilde{\mathfrak{n}}^{k+1}$ is a finite-dimensional, exponential-solvable Lie algebra, with nilradical $\mathfrak{u}_k = \tilde{\mathfrak{n}}/\tilde{\mathfrak{n}}^{k+1}$. As an $\mathfrak{a}^e$ module,

$$\mathfrak{u}_k = \sum_{1 \leqq r \leqq k} \tilde{\mathfrak{g}}_r . \tag{2}$$

Examples. 1. Consider $\mathfrak{b}_1$. The space $\mathfrak{g}_{1-h} = \mathfrak{g}_{-\tilde{\alpha}}$ is one-dimensional, and by (1) we have $\mathfrak{b}_1 = \mathfrak{a} \oplus \mathfrak{u}$, where $\mathfrak{u} = \mathfrak{u}_1$ is an $l+1$ dimensional abelian ideal. Under the adjoint action of $\mathfrak{a}$, $\mathfrak{u}$ is the sum of one-dimensional weight spaces with weights $\{-\tilde{\alpha}\} \cup \Pi$. These algebras were studied in [G–W2]. Note that if we form the algebra $\mathfrak{b}_1^e$ by adjoining the derivation d_0, then the weights of $\mathfrak{a}^e$ on $\mathfrak{u}$ are $\alpha_0, \alpha_1, \ldots, \alpha_l$.

2. If $k \geqq h-1$, then we see from (1) that the Iwasawa algebra $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$ for $\mathfrak{g}$ can be viewed as a subalgebra of $\mathfrak{b}_k$. Relative to the derivation d_0,

$$\mathfrak{b}_k = \mathfrak{s} + \mathfrak{v}_k ,$$

where $\mathfrak{v}_k$ is an ideal (the sum of the positive eigenspaces of d_0). We may identify $\mathfrak{s}^*$ with the subspace $\mathfrak{v}_k^\perp \subset \mathfrak{b}_k^*$. Since $\mathfrak{v}_k$ acts trivially on $\mathfrak{v}_k^\perp$, the coadjoint actions of $\mathfrak{b}_k$ and $\mathfrak{s}$ coincide under this identification. For example, fix $k=h-1$ and write $\mathfrak{v}=\mathfrak{v}_k$. Then one finds that $\mathfrak{b}_k=\mathfrak{s}+\mathfrak{v}$, where $\mathfrak{v}$ is an abelian ideal in $\mathfrak{b}_k$. As an $\mathfrak{a}^e$ module, $\mathfrak{v} \approx e^{i\theta}\bar{\mathfrak{n}}$, where $\bar{\mathfrak{n}}$ is the span of the negative root spaces in $\mathfrak{g}$. Relative to the adjoint action of $\mathfrak{n}$ on $\mathfrak{v}$, the commutation relations are $\mathrm{ad}_{\mathfrak{b}}(x_n)(x_{-m}e^{i\theta})=[x_n, x_{-m}]_-\, e^{i\theta}$, where $\mathfrak{b}=\mathfrak{b}_k$, $x_i \in \mathfrak{g}_i$, $m>0$, $n>0$, and $[\cdot,\cdot]_-$ denotes the projection onto $\bar{\mathfrak{n}}$ along $\mathfrak{s}$ of the bracket in $\mathfrak{g}$. In particular, the center of the nilradical $\mathfrak{u}_k=\mathfrak{n}+\mathfrak{v}$ of $\mathfrak{b}_k$ is $l+1$ dimensional, and isomorphic to $\mathfrak{g}_k \oplus \mathfrak{g}_{-1}e^{i\theta}$ as an $\mathfrak{a}^e$ module.

Denote by B_k the connected and simply-connected Lie group with Lie algebra $\mathfrak{b}_k$. It is clear from (2) that $B_k=AU_k$, where $U_k=\exp\mathfrak{u}_k$, and that the map $h, u \to \exp h \exp u$ from $\mathfrak{a} \times \mathfrak{u}_k$ to B_k is an analytic manifold isomorphism. Consider the coadjoint orbits of B_k, with their canonical symplectic structure. Given $f \in \mathfrak{b}_k^*$, and a weight function w as in Sect. 4.1, we may naturally view f as an element of $(\tilde{\mathfrak{s}}_w)^*$ which vanishes on $\tilde{\mathfrak{n}}_w^{k+1}$. (Obviously replacing $\tilde{\mathfrak{s}}$ by $\tilde{\mathfrak{s}}_w$ makes no difference in the definition of $\mathfrak{b}_k$.) The Lie group B_k may be identified with $\tilde{S}_w/V_k$, where V_k is the closed normal Lie subgroup of $\tilde{S}_w$ with Lie algebra $\tilde{\mathfrak{n}}_w^{k+1}$. The coadjoint B_k-orbit O of f is the same as the orbit of f under the action of $\tilde{S}_w$, and the functions H_ϕ, $\phi \in S(\mathfrak{p})^K$, restrict to analytic functions on O.

Remark. When $k \geqq h-1$, we may view S as a subgroup of B_k, and identify $\mathfrak{s}^*$ with a subspace of $\mathfrak{b}_k^*$ as above. Clearly the S and B_k orbits of elements in $\mathfrak{s}^*$ coincide, and the functions H_ϕ have the same restriction to these orbits as in Sect. 3.

Since $\tilde{B}(\tilde{\mathfrak{g}}_i, \tilde{\mathfrak{g}}_j)=0$ if $i+j \neq 0$, and $\tilde{\mathfrak{g}}_i$ is non-singularly paired with $\tilde{\mathfrak{g}}_{-i}$, it is clear that via the form $\tilde{B}$, we have a linear isomorphism $\nu_k : \tilde{\mathfrak{p}}^k \to \mathfrak{b}_k^*$, where

$$\tilde{\mathfrak{p}}^k=\mathfrak{a}+\sum_{1 \leqq j \leqq k} \tilde{\mathfrak{p}}_j. \tag{3}$$

Use this isomorphism to define an inner product $(\cdot,\cdot)$ on $\mathfrak{b}_k^*$ from the form $\tilde{B}$ on $\tilde{\mathfrak{p}}^k$. The theorem of Sect. 4.1 when applied in this case then yields the following result:

Theorem. *Let $O \subset \mathfrak{b}_k^*$ be a coadjoint B_k orbit, and let $\phi \in S(\mathfrak{p})^K$. The Hamiltonian flow on O generated by H_ϕ has the trajectories*

$$t \to s(t)^{-1} \cdot f, \qquad f \in O, \tag{4}$$

where $s(t)=\mathbf{s}(\exp t\nabla\tilde{\phi}(x))$ and $x=\nu_k^{-1}(f) \in \tilde{\mathfrak{p}}^k$. In particular, the flow generated by the Hamiltonian $H(f)=\frac{1}{2}(f,f)$ is

$$t \to \mathbf{s}(\exp tx)^{-1} \cdot f, \tag{5}$$

and the functions H_ϕ, $\phi \in S(\mathfrak{p})^K$, are constants of motion.

4.3. Geodesic Flow on B_k

As we have seen in Sect. 4.2, the form $\tilde{B}$ on $\tilde{\mathfrak{p}}$ gives rise to an inner product on $\mathfrak{b}_k^*$, $k=1,2,\ldots$. This in turn induces a left-invariant Riemannian structure on the group B_k. Since the inner product is not $\mathrm{Ad}(B_k)$ invariant, however, the geodesics for this metric are not one-parameter subgroups of B_k. In this section we show how the geodesics can be calculated from the flow 4.2 (5).

Recall that the cotangent bundle T^*B_k can be canonically trivialized as $B_k \times \mathfrak{b}_k^*$, with the left B_k-invariant functions on T^*B_k being identified with the functions on $\mathfrak{b}_k^*$ [G–W2, Sect. 7]. Let $\flat: \mathfrak{b}_k^* \to \mathfrak{b}_k$ and $\#: \mathfrak{b}_k \to \mathfrak{b}_k^*$ be the maps induced by the inner product on $\mathfrak{b}_k^*$. Define a function H on $B_k \times \mathfrak{b}_k^*$ by $H(b,f) = \frac{1}{2}(f,f)$. Then the integral curve through (b,f) for the Hamiltonian vector field generated by H on the symplectic manifold T^*B_k is

$$t \to (b\gamma(t), (dL(\gamma(t)^{-1})_{\gamma(t)}\dot{\gamma}(t))^{\#}). \tag{1}$$

Here γ is the geodesic through 1 with tangent vector $\dot{\gamma}(0) = f^{\flat}$, and $L(b)$ is left translation by $b \in B_k$. (This is the "geodesic flow" on T^*B_k; cf. [A–M, Chap. 3, Sect. 3.7].)

Theorem. *Let $x \in \tilde{\mathfrak{p}}^k$, $f = \psi(x) \in \mathfrak{b}_k^*$, and let $Q_k: \tilde{S}_w \to B_k$ be the quotient map. Then the integral curve of the geodesic flow on T^*B_k passing through $(1,f)$ is*

$$t \to (Q_k(\mathbf{s}(\exp tx)), \mathbf{s}(\exp tx)^{-1} \cdot f). \tag{2}$$

In particular, the geodesic through 1 *with tangent vector $f^{\flat}$ is the curve $t \to Q_k(\mathbf{s}(\exp tx))$.*

Proof. Set $s(t) = \mathbf{s}(\exp tx)$ and $s_k(t) = Q_k(s(t))$. We first calculate that

$$(s(t)^{-1} \cdot f)^{\flat} = dL(s_k(t)^{-1})_{s_k(t)} \dot{s}_k(t). \tag{3}$$

For this, it simplifies the notation to take a faithful matrix representation of $G_{\mathbb{C}}$, so that the elements of $\tilde{G}$ and $\tilde{\mathfrak{g}}$ are matrix-valued functions. Then $dL(s(t)^{-1})_{s(t)} \dot{s}(t) = s(t)^{-1} \dot{s}(t)$ (pointwise matrix multiplication). Write $s(t) = \exp(tx)\, k(t)^{-1}$, where $k(t) = \mathbf{k}(\exp(tx))$. Differentiating gives the equation $s(t)^{-1} \dot{s}(t) = k(t)\, x k(t)^{-1} - \dot{k}(t)\, k(t)^{-1}$ in $\tilde{\mathfrak{g}}$. It follows from the orthogonality of $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{p}}$ that

$$(s(t)^{-1} \dot{s}(t))^{\#} = \psi(k(t) \cdot x) = s(t)^{-1} \cdot f \tag{4}$$

(cf. proof of Sect. 3.1, Corollary). Projecting this equation onto B_k, we obtain (3).

Now let $t \to \gamma(t)$ be the geodesic through 1 with tangent vector $f^{\flat}$. The projections onto $\mathfrak{b}_k^*$ of the geodesic flow are the integral curves for the Euler field $f \to -f^{\flat} \cdot f$ (cf. [G–W2, Sect. 7]). Applying the theorem of Sect. 4.2 and (3) above, we conclude that

$$dL(\gamma(t)^{-1})_{\gamma(t)} \dot{\gamma}(t) = dL(s_k(t)^{-1})_{s_k(t)} \dot{s}_k(t) \tag{5}$$

for all t. From (5) and the formula for the differential of the exponential map [He2, Chap. II, Sect. 4], it is a straightforward induction, whose details we leave to the reader, to show that $\gamma^{(n)}(0) = s_k^{(n)}(0)$ for $n = 1, 2, \ldots$. Since $\gamma(0) = s_k(0) = 1$, it follows by the analyticity of the curves that $\gamma(t) = s_k(t)$ for all t. □

Corollary. *Define curves $h(t)$ in $\mathfrak{a}$ and $u(t)$ in $\mathfrak{u}_k$ by the factorization $Q_k\mathbf{s}(\exp tx) = \exp u(t) \exp h(t)$. Then the tangent vector field along the geodesic $\gamma(t)$, when translated back to* 1, *is given by*

$$dL(\gamma(t)^{-1})_{\gamma(t)} \dot{\gamma}(t) = \dot{h}(t) + e^{-\operatorname{ad} h(t)} \left\{ \frac{1 - e^{-\operatorname{ad} u(t)}}{\operatorname{ad} u(t)} \right\} \dot{u}(t). \tag{6}$$

Proof. With the notation as in the proof above, the left side of (6) is the projection onto $\mathfrak{b}_k$ of $s(t)^{-1}\dot{s}(t)$, under the quotient map from $\tilde{\mathfrak{s}}_w$ to $\mathfrak{b}_k$ [cf. (4) and (5)]. By the Iwasawa decomposition Sect. 4.1 (5), we have $s(t)=\exp\tilde{u}(t)\exp h(t)$, where $\tilde{u}(t)\in\tilde{\mathfrak{n}}_w$ projects onto $u(t)$. By the formula for the differential of the exponential map, it follows that

$$\dot{s}(t)=e^{\tilde{u}(t)}\left\{\frac{1-e^{-\operatorname{ad}\tilde{u}(t)}}{\operatorname{ad}\tilde{u}(t)}\right\}\dot{\tilde{u}}(t)e^{h(t)}+s(t)\,\dot{h}(t).$$

Multiplying on the left by $s(t)$ and projecting onto $\mathfrak{b}_k$ then yields (6). □

4.4. Solution of Periodic Toda Lattices

We now specialize the results of the previous section to the group B_1. In this case, the nilradical $\mathfrak{u}=\mathfrak{u}_1$ is abelian, and hence formula Sect. 4.3 (6) simplifies. As a result, we can calculate the solution to the "generalized periodic Toda lattice" Hamiltonian system from the A-component in the Iwasawa factorization of $\exp tx$, $x\in\tilde{\mathfrak{p}}^1$, as follows:

Theorem. *Let $f_0\in\mathfrak{b}_1^*$, $x=\nu_1^{-1}(f_0)\in\tilde{\mathfrak{p}}^1$, and let $s(\exp tx)=\exp u(t)\exp h(t)$ as in Sect. 4.3, Corollary. Then the integral curve with initial datum f_0 for the system with Hamiltonian $\frac{1}{2}(f,f)$ is given by*

$$f(t)=\dot{h}(t)^{\#}+\sum_{i=0}^{l}f_0(X_i)e^{\alpha_i(h(t))}X_i^*. \tag{1}$$

Here $\{X_i;\ 0\leqq i\leqq l\}$ is a basis for $\mathfrak{u}$ with $X_i\in\mathfrak{u}_{\alpha_i}$, and $\{X_i^*\}$ is the dual basis.

Proof. By equation Sect. 4.3 (6), we have

$$f(t)^{\flat}=\dot{h}(t)+e^{-\operatorname{ad}h(t)}\dot{u}(t). \tag{2}$$

Take $X\in\mathfrak{u}_\alpha$, write $f(t)=f_t$, and consider the function $q(t)=f_t(X)$. From the Hamiltonian equations for the flow and (2), we calculate that

$$\dot{q}(t)=-(f_t^{\flat}\cdot f_t)(X)=f_t([f_t^{\flat},X])=\alpha(\dot{h}(t))\,q(t).$$

Since $h(0)=0$, it follows that

$$q(t)=f_0(X)e^{\alpha(h(t))}. \tag{3}$$

Expanding the $\mathfrak{u}^*$ component of $f(t)$ according to the basis $\{X_i\}$ and dual basis $\{X_i^*\}$ and using (2) and (3), we obtain (1). □

Assume that f_0 is generic, in the sense that $c_i=f_0(X_i)\neq 0$ for $0\leqq i\leqq l$. The orbit $O=B_1\cdot f_0$ then has dimension $2l$, and we can write the solution (1) in terms of canonical symplectic coordinates $q_1,\dots,q_l,\ p_1,\dots,p_l$ on O as follows: As in [G–W2, Sect. 7], we parametrize points of O as

$$f=\sum_{i=1}^{l}p_i\alpha_i+\sum_{j=0}^{l}\varepsilon_j e^{-q_j}X_j^*, \tag{4}$$

where $\varepsilon_j = \operatorname{sgn}(c_j)$ and

$$q_0 = \gamma - \sum_{i=1}^{l} n_i q_i . \tag{5}$$

$\left(\text{Recall that } \alpha_0 = -\sum_{1 \leqq i \leqq l} n_i \alpha_i \text{ on } \mathfrak{a}.\right)$ Here γ is a constant on the orbit, with $e^\gamma = |c_0 c_1^{n_1} \dots c_l^{n_l}|$ the value of the $\mathrm{Ad}^*(B_1)$-invariant function $|\Xi|$ [G–W2, Eq. (9.1)]. Note that if

$$f_0|_{\mathfrak{a}} = \sum_{i=1}^{l} a_i \alpha_i ,$$

then f_0 has coordinates $p_i = a_i$, $q_i = -\log|c_i|$. If we take X_i to be a unit vector relative to the inner product on $\mathfrak{b}_1$, then the Hamiltonian $H(f) = \frac{1}{2}(f, f)$ in these coordinates becomes

$$H = 1/2 \sum_{i,j=1}^{l} (\alpha_i, \alpha_j) p_i p_j + 1/2 \sum_{j=0}^{l} e^{-2q_j} . \tag{6}$$

Comparing (1) and (4), we see that along the solution curve,

$$q_j(t) = -\log|c_j| - \alpha_j(h(t)), \quad \text{for} \quad 0 \leqq j \leqq l . \tag{7}$$

Since $\dot{p}_j = \{H, p_j\} = \partial H/\partial q_j = n_j e^{-2q_0} - e^{-2q_j}$, we can calculate $p_j(t)$ by a quadrature from (7). Or we can use the equation $\dot{q}_k = \{H, q_k\}$ to obtain p_j by inverting the linear system

$$\sum_{j=1}^{l} (\alpha_k, \alpha_j) p_j = -\dot{q}_k , \qquad 1 \leqq k \leqq l . \tag{8}$$

Remark. From the calculations above, it is easy to see that on each orbit O, the flow has exactly one fixed point, characterized by the equations

$$p_j = 0 , \qquad q_j = q_0 - \tfrac{1}{2} \log n_j , \qquad 1 \leqq j \leqq l . \tag{9}$$

To prove this, it suffices to show that Eq. (9) determines q_j uniquely, when q_0 given by (5). But the coefficient matrix is $I + v^{\mathrm{T}} w$, where $v = [1\ 1 \dots 1]$ and $w = [n_1 n_2 \dots n_l]$. Any vector in the null space of this matrix must be a multiple of v. Since $w^{\mathrm{T}} v > 0$, v is not in this null space. Hence the matrix is invertible.

5. Periodic Toda Lattices and Representations of Affine Groups

5.1. Standard Representations

In this chapter we show how the solution to the (generalized) periodic Toda lattice systems in Sect. 4.4 can be calculated in terms of representative functions on a Banach-Lie group $\hat{G}_w$, which is a central extension of the loop group $\tilde{G}_w$. The structure and representation theory of these groups was worked out in [G–W3]. We summarize now the results relevant for the present application.

Let w be a weight function as in Sect. 4.1. Assume that w satisfies the non-analyticity condition Sect. 4.1 (2) and the following stronger version of the rapidly increasing condition Sect. 4.1 (1):

$$\exists \sigma, 1<\sigma<2, \quad \text{such that} \quad \lim_{n\to\infty} |n|^{-1/\sigma} \log w(n) = \infty\,. \tag{1}$$

[For an example of a weight function satisfying both these conditions, take $\frac{1}{2}<s<1/\sigma$ and set $w(n)=\exp(|n|^s)$.] Then there is a complex Banach-Lie group $[\hat{G}_{\mathbb{C}}]_w$ which is a central extension of $[\tilde{G}_{\mathbb{C}}]_w$ by $\mathbb{C}^\times$. The Lie algebra $[\hat{\mathfrak{g}}_{\mathbb{C}}]_w$ of this group is the corresponding one-dimensional central extension of $[\tilde{\mathfrak{g}}_{\mathbb{C}}]_w$, and is just the completion in the w-norm Sect. 4.1 (3) of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_{\mathbb{C}}$ associated with $\mathfrak{g}_{\mathbb{C}}$. We shall denote the corresponding completed "normal real forms" by $\hat{G}_w$ and $\hat{\mathfrak{g}}_w$. Thus $\hat{\mathfrak{g}}_w$ is a central extension of the Lie algebra $\tilde{\mathfrak{g}}_w$ in Sect. 4.1 by $\mathbb{R}$. There is a Cartan decomposition $\hat{G}_w=\hat{K}_w\cdot\hat{P}_w$, $\hat{P}_w=\exp\hat{\mathfrak{p}}_w$, and an Iwasawa decomposition $\hat{G}_w=\hat{N}_w\cdot\hat{A}\cdot\hat{K}_w$, obtained by lifting the corresponding decompositions of $\tilde{G}_w$. Here $\hat{A}=A\cdot\exp\mathbb{R}c$, with c a basis for the center of $\hat{\mathfrak{g}}$, and $\hat{N}_w\approx\tilde{N}_w$. There is a projection

$$\hat{\mathfrak{p}}_w\to\tilde{\mathfrak{p}}_w \tag{2}$$

with kernel $\mathbb{R}c$. For $k\geqq 0$, define the finite-dimensional subspace $\hat{\mathfrak{p}}^k$ of $\hat{\mathfrak{p}}_w$ to be the inverse image of $\tilde{\mathfrak{p}}^k$ under (2), and let $\Psi:\hat{\mathfrak{p}}^k\to\mathfrak{b}_k^*$ be the composition of the map (2), restricted to $\hat{\mathfrak{p}}^k$, with the map v_k in Sect. 4.2. Thus Ψ is surjective, with kernel $\mathbb{R}c$.

The algebra $\hat{\mathfrak{g}}_{\mathbb{C}}$ admits a family of irreducible "standard modules" V^λ, parametrized by the dominant integral functionals λ on $\hat{\mathfrak{a}}$, that are completely analogous to the irreducible finite-dimensional representations of $\mathfrak{g}_{\mathbb{C}}$. These modules carry a positive-definite Hermitian form $\langle\cdot|\cdot\rangle$ which is contravariant relative to the involution τ of Sect. 4.1: $\langle X\cdot u|v\rangle=-\langle u|\tau(X)\cdot v\rangle$ for $X\in\hat{\mathfrak{g}}_{\mathbb{C}}$ and $u,v\in V^\lambda$. Let H^λ be the completion of V^λ in the norm defined by this inner product. If σ and the weight w are related by (1), then there is a Fréchet space S_σ^λ, of "Gevrey vectors of order σ", with $V^\lambda\subset S_\sigma^\lambda\subset H^\lambda$. The representation of $\hat{\mathfrak{g}}_{\mathbb{C}}$ on V^λ extends by continuity to a continuous representation of $[\hat{\mathfrak{g}}_{\mathbb{C}}]_w$ on S_σ^λ. Furthermore, this representation can be integrated to a holomorphic representation π^λ of the group $[\hat{G}_{\mathbb{C}}]_w$ on S_σ^λ.

For any pair of vectors u, $v\in S_\sigma^\lambda$, one thus has a holomorphic function $g\to\langle\pi^\lambda(g)u|v\rangle$ on $[\hat{G}_{\mathbb{C}}]_w$. In particular, let v_λ be a normalized highest weight vector for V^λ, and define

$$\psi_\lambda(g)=\langle\pi^\lambda(g)v_\lambda|v_\lambda\rangle \quad \text{for} \quad g\in[\hat{G}_{\mathbb{C}}]_w\,. \tag{3}$$

If $g=\exp X$, with $\tau(X)=-X$, then $\psi_\lambda(g)=\psi_\lambda(g^{-1})>0$. (For further properties of the functions ψ_λ, cf. [G–W3, Chap. 6].) When λ is one of the "fundamental weights" $\hat{\omega}_i$, $0\leqq i\leqq l$, then π^λ is called a "fundamental representation." We shall write π^i for π^λ, V^i for V^λ, v_i for v^λ and ψ_i for ψ^λ in this case. We note from [G–W3, Sect. 6.2] that if $\lambda=\sum m_i\hat{\omega}_i$, where $\{m_i\}$ are non-negative integers, then

$$\psi_\lambda(g)=\prod_{i=0}^{l}\psi_i(g)^{m_i}\,. \tag{4}$$

Lemma. *Let* $x\in\tilde{\mathfrak{p}}_w$, $t\in\mathbb{R}$, *and define* $h(t)\in\mathfrak{a}$ *by the Iwasawa factorization* $\exp tx=n\cdot\exp h(t)\cdot k$, *where* $n\in\tilde{N}_w$ *and* $k\in\tilde{K}_w$. *Take* $X\in\hat{\mathfrak{p}}_w$ *which projects onto* x *in* (2).

Then

$$h(t)=\sum_{i=0}^{l} c_i(t)H_i\,, \tag{5}$$

where $c_i(t)=-\frac{1}{2}\log\psi_i(\exp -2tX)$. *Here* H_i *is the coroot to* α_i *for* $1\leqq i\leqq l$, *and* H_0 *is the coroot to* $-\tilde{\alpha}$. *(Recall that the coroot* $H_\alpha\in\mathfrak{a}$ *to* $\alpha\in\mathfrak{a}^*$ *is defined by* $(H_\alpha, H)=2\alpha(H)/(\alpha,\alpha)$, *for* $H\in\mathfrak{a}$.*)*

Proof. This follows from [G–W3; formulas 6.5 (2) and 6.6 (1)]. □

5.2. Solution of Periodic Toda Lattices via Representative Functions

Continuing with the notation of the previous section, we recall that the extended Cartan matrix $[A_{ij}]_{0\leqq i,j\leqq l}$ of the root system of $\mathfrak{g}$ is defined by $A_{ij}=2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j)$, where $\alpha_0=-\tilde{\alpha}$. Let $\{e_i, f_i, h_i: 0\leqq i\leqq l\}$ be a set of canonical generators of the affine algebra $\hat{\mathfrak{g}}$. The commutation relations are

$$[e_i,f_j]=\delta_{ij}h_i\,,\qquad [h_i,h_j]=0\,,$$
$$[h_j,e_i]=A_{ij}e_i\,,\qquad [h_j,f_i]=-A_{ij}f_i\,.$$

Note that $\hat{\mathfrak{p}}^1$ has basis $\{h_i, e_i+f_i: 0\leqq i\leqq l\}$.

Theorem. *The solution to the generalized periodic Toda lattice system, with Hamiltonian Sect. 4.4* (6) *and initial data* $p_i(0)$, $q_i(0)$, $1\leqq i\leqq l$, *is given in terms of the fundamental representative functions as follows:*

$$q_i(t)=q_i(0)+1/2\sum_{j=0}^{l} A_{ij}\log\psi_j(\exp -tX)\,, \tag{1}$$

and p_i *is obtained either by quadrature from*

$$\dot{p}_j=n_je^{-2q_0}-e^{-2q_j}\,, \tag{2}$$

or by inverting the linear system

$$\sum_{j=1}^{l}(\alpha_i,\alpha_j)p_j=1/2\sum_{j=0}^{l}A_{ij}\frac{\langle\pi^j(\exp -tX)X\cdot v_j|v_j\rangle}{\psi_j(\exp -tX)} \tag{3}$$

for $1\leqq i\leqq l$. *Here* $X\in\hat{\mathfrak{p}}^1$ *is defined by*

$$X=\sum_{i=1}^{l}(\alpha_i,\alpha_i)\,p_i(0)h_i+\sum_{j=0}^{l}(\alpha_j,\alpha_j)^{1/2}e^{-q_i(0)}(e_j+f_j)\,, \tag{4}$$

and q_0 *is defined by Sect.* 4.4 (5).

Remark. Equation (1) also holds for $q_0(t)$, as is easily checked.

Proof. It is a straightforward calculation, using the invariant form and the commutation relations $[h_i,e_i]=2e_i$, $[e_i,f_i]=h_i$, to verify that we may take the set $\{\mu_i\Psi(e_i+f_i)\}$ as the orthonormal basis $\{X_i^*\}$ in Theorem 4.4, where $\mu_i^2=(\alpha_i,\alpha_i)/4$. With X defined by (4), we then have

$$\Psi(X)=\sum_{i=1}^{l}2p_i(0)\alpha_i+\sum_{j=0}^{l}2e^{-q_i(0)}X_i^*\,. \tag{5}$$

Now apply Sect. 4.4, Theorem and Sect. 5.1, Lemma to the solution with initial data f_0 given by $\Psi(\frac{1}{2}X)$. □

Corollary. *The functions $e^{-2q_i(t)}$ extend meromorphically in t, and are the ratio of two entire functions of exponential order of growth $\leqq 2$.*

Proof. By [G–W3, Theorem 6.1], we know that $\phi_i(t)=\psi_i(\exp tX)$ is an entire function of t, and satisfies the growth estimate $|\phi_i(t)|\leqq A\exp B|t|^{2+\varepsilon}$ for all $\varepsilon>0$, with constants A, B depending on ε and X. (Since X is in the finite-dimensional space $\tilde{\mathfrak{p}}^1$, $\|X\|_w<\infty$ for any admissible weight function w.) The result now follows from formula (1). □

Remark. If we let $\gamma\to+\infty$ in the defining relation Sect. 4.4 (5) for q_0 [i.e. set the coefficient of X_0^* in (5) to zero], then the element X in (4) lies in the finite-dimensional algebra $\mathfrak{g}$. In this case

$$\psi_\lambda(\exp tX)=\sum_{\mu\in\Sigma} c_\mu e^{\mu t},$$

where Σ is the spectrum of the self-adjoint operator $\varrho_\Lambda(X)$, $c_\mu\geqq 0$, and $\sum c_\mu=1$. Here ϱ_Λ is the irreducible finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $\Lambda=\lambda|_{\mathfrak{a}_{\mathbb{C}}}$. In this case formulas (1) and (3) become Kostant's formulas for the solution of the generalized non-periodic Toda lattices ([Ko, Theorem 7.5]; see also [Sy1]).

Example. Take $G=\mathrm{SL}(n,\mathbb{R})$, $n\geqq 3$. In this case Sect. 4.4 (6) is the periodic Toda lattice Hamiltonian, in a particular choice of canonical coordinates. The extended Cartan matrix $A_{ij}=-1$ if $i-j=\pm 1(\mathrm{mod}\,n)$, $A_{ii}=2$, and all other entries are zero. If we define $y_i=\log[\phi_i/\phi_{i-1}]$, where $\phi_i(t)=\psi_i(\exp -tX)$ and the subscripts are read $\mathrm{mod}(n)$, then we can write (1) as

$$q_i(t)=q_i(0)+[y_i(t)-y_{i+1}(t)]/2. \tag{6}$$

We then obtain $p_i(t)$ by quadrature from

$$\dot{p}_i=c_0^2\frac{\phi_i\phi_1}{\phi_0^2}-c_i^2\frac{\phi_{i-1}\phi_{i+1}}{\phi_i^2}, \tag{7}$$

where $\log c_i=-q_i(0)$.

For the case $\mathrm{SL}(2,\mathbb{R})$ (the periodic Toda lattice with one degree of freedom), the extended Cartan matrix has $A_{10}=-2$, and the formulas above become

$$q_1(t)=q_1(0)+\log[\phi_1(t)/\phi_0(t)], \tag{8}$$

$$\dot{p}_1=c_0^2(\phi_1/\phi_0)^2-c_1^2(\phi_0/\phi_1)^2. \tag{9}$$

5.3. Differential Equations for Representative Functions

Using representation theory, we now obtain a system of non-linear differential equations satisfied by the basic representative functions ψ_i along certain one-parameter subgroups $\exp tX$. Assume that

$$X=\sum_{i=0}^{l} c_i(e_i+f_i) \tag{1}$$

is in $\hat{\mathfrak{p}}^1$, where $c_i \in \mathbb{R}$ [e.g. in Sect. 5.2 (4), take initial data $p_i(0)=0$]. For λ a dominant integral functional on $\hat{\mathfrak{a}}$, set $\phi_\lambda(t)=\psi_\lambda(\exp -tX)$, with X given by (1). When $\lambda=\hat{\omega}_i$ is a fundamental weight, write $\phi_\lambda=\phi_i$. Recall (Sect. 5.2, Corollary) that ϕ_i extends holomorphically to an entire function of t and $\{c_i\}$, of exponential order $\leqq 2$ in t.

Proposition. *For* $0 \leqq i \leqq l$, *one has*

$$\frac{d^2}{dt^2}\log\phi_i(t)=c_i^2\prod_{j=0}^{l}\phi_j(t)^{-A_{ij}}, \tag{2}$$

with initial conditions $\phi_i(0)=1$ *and* $\phi_i'(0)=0$. *In particular,* ϕ_i *is an even function of* t *and* $\{c_j\}$.

Proof. We first recall that the action of the canonical generators e_i, f_i, and h_i on the highest weight vector v_j is given by

$$e_i\cdot v_j=0, \qquad h_i\cdot v_j=\delta_{ij}v_j, \tag{3}$$

and if $i \neq j$, then

$$f_i\cdot v_j=0. \tag{4}$$

From the commutation relations for the canonical generators, one calculates that

$$h_j f_i\cdot v_i=(\delta_{ij}-A_{ij})\,f_i\cdot v_i, \qquad e_j f_i\cdot v_i=\delta_{ij}\,h_i\cdot v_i. \tag{5}$$

Now fix i, write $e_i=e$, $f_i=f$, $h_i=h$, $v_i=v$, and consider the vector

$$\xi=2^{-1/2}\{v\otimes f\cdot v-f\cdot v\otimes v\}$$

in $V^i\otimes V^i$. From (3) and (5), it follows that ξ is a highest weight vector in the tensor product representation, with weight

$$\lambda=\sum_{j=0}^{l}(2\delta_{ij}-A_{ij})\hat{\omega}_j.$$

Furthermore, $\langle f\cdot v|v\rangle=0$ and

$$\|f\cdot v\|^2=\langle ef\cdot v|v\rangle=\langle h\cdot v|v\rangle=1,$$

so that ξ is a unit vector, relative to the canonical inner product on $V^i\otimes V^i$. Thus we may calculate the representative function ψ_λ using the vector ξ and the representation $\pi^i\otimes\pi^i$:

$$\begin{aligned}\psi_\lambda(g)&=\langle(\pi^i(g)\otimes\pi^i(g))\,\xi|\xi\rangle\\ &=\langle\pi^i(g)f\cdot v|f\cdot v\rangle\,\psi_i(g)-\langle\pi^i(g)f\cdot v|v\rangle\,\langle\pi^i(g)v|f\cdot v\rangle,\end{aligned} \tag{6}$$

for $g\in[\hat{G}_{\mathbb{C}}]_w$. When $g=\exp tX$, we can calculate the right side of (6) in terms of derivatives of ϕ_i, using (3), (4), and (5):

$$(d/dt)\,\phi_i(t)=-\langle\pi^i(\exp -tX)X\cdot v|v\rangle=-c_i\langle\pi^i(\exp -tX)f\cdot v|v\rangle \tag{7}$$

$$(d/dt)^2\,\phi_i(t)=c_i\langle\pi^i(\exp -tX)f\cdot v|X\cdot v\rangle=c_i^2\langle\pi^i(\exp -tX)f\cdot v|f\cdot v\rangle \tag{8}$$

(In the last equation we have used the self-adjointness of X.) Thus (6) implies that

$$c_i^2\phi_\lambda(t)=\phi_i''(t)\,\phi_i(t)-[\phi_i'(t)]^2. \tag{9}$$

On the other hand, we can also calculate ϕ_λ using Sect. 5.1 (4), which gives

$$\phi_\lambda = \prod_{j \neq i} \phi_j(t)^{-A_{ij}}.$$

Substituting this in (9) and dividing by ϕ_i^2, we obtain (2), since $A_{ii}=2$. The initial condition $\phi_i'(0)=0$ follows from (7). The uniqueness of solutions to the system (2) implies the symmetry under changes of sign of t and c_i. □

Example. Take $G=\mathrm{SL}(n,\mathbb{R})$. Then Eq. (2) reads

$$\frac{d^2}{dt^2}\log\phi_i(t) = \frac{c_i^2\phi_{i-1}(t)\,\phi_{i+1}(t)}{\phi_i(t)^2}, \tag{10}$$

where the subscripts are read $\bmod(n)$.

5.4. Matrix Entries Associated with Fixed Points

We assume for simplicity that the root system of $\mathfrak{g}$ is simply-laced, and we normalize the inner product so that $(\alpha_i, \alpha_i)=2$, for $1 \leqq i \leqq l$. Take

$$X = e_0 + f_0 + \sum_{i=1}^{l} c_i(e_i + f_i),$$

where the coefficients c_i satisfy

$$\sum_{i=1}^{l} c_i^2 \alpha_i = \tilde{\alpha}.$$

(There are 2^l choices of sign for these coefficients. Let $n_i = c_i^2$ as usual.) This choice of X corresponds to the fixed-points of the corresponding periodic Toda lattice (cf. Sect. 4.3 and Sect. 5.2, Theorem). We write

$$u = e_0 + \sum_{i=1}^{l} c_i e_i, \qquad v = f_0 + \sum_{i=1}^{l} c_i f_i.$$

Then from the commutation relations among the canonical generators (Sect. 5.2), we have $[u,v]=c$, where

$$c = h_0 + \sum_{i=1}^{l} n_i h_i$$

spans the center of $\hat{\mathfrak{g}}$ (cf. [G–W3, Sect. 1.3] for details). Hence

$$\begin{aligned}\exp tv \exp tu &= \exp t(u+v)\exp \tfrac{1}{2}t^2[v,u]\\ &= \exp tX \exp -\tfrac{1}{2}t^2 c.\end{aligned}$$

Thus $\pi^\lambda(\exp tX)v_\lambda = \exp[\tfrac{1}{2}t^2\lambda(c)]\,\pi^\lambda(\exp tv)\cdot v_\lambda$. Now $\langle \pi^\lambda(\exp tv)v_\lambda | v_\lambda\rangle = 1$. Hence we can calculate the representative functions ψ_λ along the subgroup generated by X:

$$\psi_\lambda(\exp tX) = e^{t^2\lambda(c)/2}. \tag{1}$$

Taking $\lambda = \hat{\omega}_i$ to be a fundamental weight, we have $\lambda(c)=1$ for $i=0$, while $\lambda(c)=n_i$ for $1 \leqq i \leqq l$ [G–W3, Sect. 1.3]. Hence the functions $\phi_i(t) = \psi_\lambda(\exp tX)$ are as follows:

$$\phi_i(t) = \exp[\tfrac{1}{2} n_i t^2], \quad 0 \leqq i \leqq l, \tag{2}$$

where $n_0 = 1$.

5.5. *Explicit Solutions for* $\mathrm{SL}(2,\mathbb{R})^\wedge$

We conclude by calculating the basic representative functions ϕ_i of Sect. 5.3 in terms of theta functions and elementary functions in the case of the group $\mathrm{SL}(2,\mathbb{R})^\wedge$. For this, we will use the connection between representative functions and the periodic Toda lattice [Sect. 5.2, Eqs. (8) and (9)], together with the differential equations derived in Sect. 5.3.

Let X be defined by 5.3 (1), with $l=1$ and $c_0 = \frac{1}{2}(1+k)$, $c_1 = \frac{1}{2}(1-k)$. Here k is a parameter which we take in the range $-1 < k < 1$. Define the functions $\phi_0(t)$ and $\phi_1(t)$ as in Sect. 5.3 in terms of X. Let q_1, p_1 be the canonical coordinates for the periodic Toda lattice with one degree of freedom, and set $q_0 = \gamma - q_1$, as before, with γ depending on the associated coadjoint orbit. Relative to the dual of the Killing form for $\mathfrak{sl}(2,\mathbb{R})$, one has $(\alpha_1, \alpha_1) = \frac{1}{2}$. Hence by Sect. 4.4 (6), along the trajectories of the system one has $\dot{q}_1 = -\partial H/\partial p_1 = -\frac{1}{2}p_1$, while $\dot{q}_0 = -\dot{q}_1$ and $\dot{p}_1 = e^{-2q_0} - e^{-2q_1}$, as derived in Sect. 4.4. From the equation for $\dot{p}_1$, it is natural to define

$$x = 2^{-1/2}(e^{-q_0} - e^{-q_1}), \quad y = \tfrac{1}{2}p_1, \quad z = 2^{-1/2}(e^{-q_0} + e^{-q_1}).$$

Then along the trajectories, x, y, and z satisfy the system of bilinear differential equations

$$\dot{x} = -yz, \quad \dot{y} = xz, \quad \dot{z} = -xy. \tag{1}$$

Now choose the initial data and coadjoint orbit so that

$$p_0(0) = 0, \quad q_0(0) = -\log[2^{1/2}(1+k)], \quad q_1(0) = -\log[2^{1/2}(1-k)].$$

Then $x(0) = k$, $y(0) = 0$, and $z(0) = 1$. Hence it follows from (1) that x, y, z are given in terms of the Jacobi elliptic functions as

$$x = k\,\mathrm{cn}(t,k), \quad y = k\,\mathrm{sn}(t,k), \quad z = \mathrm{dn}(t,k) \tag{2}$$

[W–W, p. 493]. Returning to the canonical coordinates, we thus have the solution to the periodic Toda lattice for this choice of initial data:

$$q_1(t) = q_1(0) + \log\left\{\frac{\mathrm{dn}(t,k) + k\,\mathrm{cn}(t,k)}{1+k}\right\}, \tag{3}$$

$$p_1(t) = 2k\,\mathrm{sn}(t,k). \tag{4}$$

Comparing (3) with Sect. 5.2 (8), we see that

$$\frac{\phi_1(t)}{\phi_0(t)} = \frac{\mathrm{dn}(t,k) + k\,\mathrm{cn}(t,k)}{1+k}. \tag{5}$$

Using (5) in Sect. 5.3 (10), together with the basic identities

$$\frac{\mathrm{dn}(t,k) + k\,\mathrm{cn}(t,k)}{\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)} = \frac{1+k}{1-k},$$

$$k^2\,\mathrm{cn}(t,k)^2 = \mathrm{dn}(t,k)^2 + k^2 - 1,$$

we find that ϕ_0 and ϕ_1 satisfy the equations

$$2(\log\phi_0)'' = \mathrm{dn}^2 + k\,\mathrm{dn}\,\mathrm{cn} + 2(k^2-1)\,, \tag{6}$$

$$2(\log\phi_1)'' = \mathrm{dn}^2 - k\,\mathrm{dn}\,\mathrm{cn} + 2(k^2-1)\,. \tag{7}$$

These equations can be integrated as follows: Following the standard notation in the theory of elliptic functions, as in [W–W], we let

$$K = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{-1/2}\,d\theta, \qquad E = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\,d\theta$$

be the complete elliptic integrals of the first and second kind with modulus k. Let the number q, $0 \leqq q < 1$, be defined implicitly in terms of k by the equation [W–W, p. 481]

$$1-k^2 = \prod_{n=1}^{\infty} \left\{\frac{1-q^{2n-1}}{1+q^{2n-1}}\right\}^8 .$$

Take Jacobi's original theta function $\Theta(t) = \Theta_4(\pi t/2K, q)$. Then $(d/dt)^2 \log\Theta(t) = \mathrm{dn}^2(t,k) - E/K$ [W–W, Sect. 22.73]. Furthermore,

$$(d/dt)^2 \log \frac{\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)}{1-k} = k\,\mathrm{dn}(t,k)\,\mathrm{cn}(t,k)\,.$$

[W–W, p. 516]. Recalling from Proposition 5.3 that $\phi_i'(0) = 0$, and $\phi_i(0) = 1$, we calculate from (6) and the cited formulas that

$$\phi_0(t)^2 = \frac{\Theta(t)}{\Theta(0)} \cdot \frac{\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)}{1-k} \cdot e^{2\nu t^2}, \tag{8}$$

where $\nu = (k^2-1)/8 + E/4K$. Similarly, starting with (7), we find that $\phi_1(t)^2$ is given by the right side of (8), with k replaced by $-k$.

As noted in Sect. 5.2, we know that the functions ϕ_i are entire functions of t. Hence the right side of (8) must be the square of an entire function. To calculate this function explicitly, we use the infinite product expansions [Hancock, p. 255 (1)]:

$$\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k) = \prod_{n=0}^{\infty} \frac{(1-a_n)(1-b_n)}{(1+a_n)(1+b_n)},$$

$$\Theta(t) = G \prod_{n=0}^{\infty} (1-a_n^2)(1-b_n^2)\,,$$

where $a_n = q^{n+1/2}e^{iu}$, $b_n = q^{n+1/2}e^{-iu}$, and $u = \pi t/2K$. Here $G = G(k)$ is independent of t. Using these factorizations in (8), we see that the zeros of Θ indeed cancel the poles of $\mathrm{dn} - k\,\mathrm{cn}$, and we obtain the factorization

$$\phi_0(t) = G_0 e^{\nu t^2} \prod_{n=0}^{\infty} (1-a_n)(1-b_n)\,, \tag{9}$$

with a_n, b_n, ν as above, and G_0 a constant (depending only on k), determined by the initial condition $\phi_0(0) = 1$. Similarly, we have

$$\phi_1(t) = G_1 e^{\nu t^2} \prod_{n=0}^{\infty} (1+a_n)(1+b_n)\,. \tag{10}$$

From Jacobi's infinite product expansions of the theta functions [W–W, Sect. 21.3], we may also write these formulas as

$$\phi_0(t) = e^{vt^2}\Theta_4(\tfrac{1}{2}u, q^{1/2})/\Theta_4(0, q^{1/2}), \tag{9$'$}$$

$$\phi_1(t) = e^{vt^2}\Theta_3(\tfrac{1}{2}u, q^{1/2})/\Theta_3(0, q^{1/2}), \tag{10$'$}$$

with u and v as above.

From their representation-theoretic definition, we know that the functions $\phi_i(t)$ are positive for real t, and positive-definite for purely imaginary t [G–W3, Sect. 6]. From the Fourier series for Θ_4 and formula (9)$'$, we calculate that

$$\phi_0(it) = C_0 e^{-bt^2} \sum_{n\in\mathbb{Z}} (-1)^n e^{-\varepsilon(n+ct)^2}, \tag{11}$$

where C_0, $\varepsilon = -\frac{1}{2}\log q$, $b = v + 1/(\varepsilon K^2)$, and $c = \pi/(4\varepsilon K)$ are positive constants depending on k. There is a similar formula for ϕ_1. It would be interesting to have a representation-theoretic interpretation (or derivation) of these formulas, as well as a "physical" interpretation via the periodic Toda lattice.

Remark. In the case of the "twisted" affine Lie algebra A_2^2 (cf. [G–W3, Sect. 6.9] and [R–S1]), the solutions to the corresponding Toda-type system can be expressed in terms of the Weierstrass σ-function.

Appendix. Some Root System Results

Let Δ be a reduced root system, Δ^+ a set of positive roots, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of simple roots in Δ^+. Define, for $1 \leqq i \leqq l$,

$$\Delta_i^+ = \left\{\gamma \in \Delta^+ : \gamma = \sum_{j=1}^{l} n_j\alpha_j, \quad n_i > 0\right\}$$

(the set of positive roots containing α_i). For γ as above, set $|\gamma| = \sum n_i$.

Lemma 1. *Let $\gamma \in \Delta_i^+$ and suppose that $|\gamma| \geqq 2$. Then there exist $\beta_1, \dots, \beta_r \in \Delta^+$ such that*

(a) $\alpha_i + \beta_1 + \dots + \beta_k \in \Delta^+$ *for* $1 \leqq k \leqq r$;
(b) $\alpha_i + \beta_1 + \dots + \beta_r = \gamma$.

Proof. By induction on $|\gamma|$. If $|\gamma| = 2$, then $\gamma = \alpha_i + \alpha_j$. Thus we may take $r = 1$, $\beta_1 = \alpha_j$ in this case. Assume now that the lemma holds for roots of length $< m$, and take γ with $|\gamma| = m$. Then $\gamma \notin \Pi$, so there exist $\alpha, \beta \in \Delta^+$ such that $\gamma = \alpha + \beta$. Since $\gamma \in \Delta_i^+$, we may assume that $\alpha \in \Delta_i^+$. By induction, we can find a sequence $\beta_1, \dots, \beta_r$ for α. Adjoin $\beta_{r+1} = \beta$ to get a sequence which works for γ. $\square$

Assume now that Δ is irreducible. Given $\alpha \in \Delta^+$, set $\Gamma_\alpha = \{\beta \in \Delta^+ : \alpha - \beta \in \Delta^+\}$.

Lemma 2. *Let $\alpha = \alpha_1 + \dots + \alpha_l$. Then* $\mathrm{Card}(\Gamma_\alpha) = 2l - 2$.

Proof. By [Bo2, Chap. VI, Sect. 1, Corollary 3 to Proposition 19], Γ_α consists of all roots of the form

$$\sum_{i\in Y} \alpha_i,$$

where Y and its complement are non-empty connected subsets of the Dynkin diagram for Π. Using the classification of Dynkin diagrams, it is easily verified that there are $2l-2$ such subsets Y (cf. Lemma 3 and Table 1). For a proof without classification, we could also invoke the following combinatorial result, whose proof we leave to the reader (cf. [Bo2, Chap. 4, Annexe, Proposition 2]):

Scholium. *Let Γ be a tree with l vertices. Then there are exactly $2l$ connected subsets of Γ whose complements are also connected. (Here we allow the empty set as a connected subset.)*

Recall that Δ has elements of at most two lengths (which we call short and long; in the case of only one root length, all roots will be called long).

Lemma 3. *Suppose that $\alpha \in \Delta^+$ is long.*
(i) *If $\beta, \gamma \in \Gamma_\alpha$ and $\beta+\gamma \in \Delta$, then $\beta+\gamma=\alpha$;*
(ii) *If $\beta \in \Gamma_\alpha$, then $\alpha+\beta \notin \Delta$;*
(iii) $\mathrm{Card}(\Gamma_\alpha)$ *is even.*

Proof. (This argument was suggested by [Jo, Sect. 2].) We first claim that if $\beta \in \Gamma_\alpha$, then

$$2(\alpha, \beta)/(\alpha, \alpha)=1\,. \tag{1}$$

Indeed, we have $\|\alpha-\beta\|^2=\|\alpha\|^2-2(\alpha,\beta)+\|\beta\|^2$, so the assertion follows immediately once we know that β and $\alpha-\beta$ have the same length. But the case β short, $\alpha-\beta$ long (or vice versa) cannot occur, since it would imply $2(\alpha,\beta) =\|\beta\|^2<\|\alpha\|^2$, contradicting the root system axiom that $2(\alpha,\beta)/(\alpha,\alpha)$ be an integer.

With (1) established, now let $\beta, \gamma \in \Gamma_\alpha$ and assume that $\beta+\gamma \in \Delta$. Then $(\beta+\gamma, \alpha) =(\alpha,\alpha)$ by (1), while $\|\beta+\gamma\| \leqq \|\alpha\|$ since $\beta+\gamma$ is a root. Hence the Cauchy-Schwarz inequality forces $\beta+\gamma=\alpha$. Similarly, $\|\alpha+\beta\|^2=\|\alpha\|^2+2(\alpha,\beta)+\|\beta\|^2=2\|\alpha\|^2 +\|\beta\|^2>\|\alpha\|^2$, so $\alpha+\beta \notin \Delta$, since α is long. This proves (i) and (ii). As for (iii), we observe that the map sending β to $\alpha-\beta$ has no fixed points on Γ_α, since $2\beta \notin \Delta$. Since this map is an involution, we obtain (iii). □

Definition. Let $\alpha \in \Delta^+$ be long. A polarization of Γ_α is a partition of Γ_α into complementary subsets $\{\beta_1, \dots, \beta_r\}$ and $\{\gamma_1, \dots, \gamma_r\}$, such that $\beta_i+\gamma_i=\alpha$ for $1 \leqq i \leqq r$ (where $2r=\mathrm{Card}\,\Gamma_\alpha$).

By Lemma 3 it is clear that polarizations exist, and have the property that

$$\beta_i+\beta_j \notin \Delta^+, \qquad \gamma_i+\gamma_j \notin \Delta^+\,. \tag{2}$$

We now fix

$$\alpha=(H_1+\ldots+H_l)^\vee, \tag{3}$$

where H_i is the coroot to α_i, and $^\vee$ is the operation of passing from root to coroot. Thus when all roots have the same length, then

$$\alpha=\alpha_1+\ldots+\alpha_l\,. \tag*{$(3)_{\mathrm{ADE}}$}$$

When the ratio of squared root lengths is 2:1, then

$$\alpha=2\sum_{\text{short}}\alpha_i+\sum_{\text{long}}\alpha_i\,. \tag*{$(3)_{\mathrm{BCF}}$}$$

Finally, when this ratio is $3:1$ (G_2 root system), then

$$\alpha = 3\alpha_1 + \alpha_2\,, \tag{3}_G$$

where α_1 is short. In all cases α is long (see Table 1).

Lemma 4. *Let α be as in $(3)_{A-G}$. Then the set Γ_α has $2l-2$ elements, and admits a polarization $\Gamma_\alpha = \{\beta_1, \ldots, \beta_{l-1}\} \cup \{\gamma_1, \ldots, \gamma_{l-1}\}$ with the following properties:*

(i) *There is a monotone ordering $\beta_1 < \beta_2 < \ldots < \beta_{l-1} < \gamma_i < \alpha$ for all i, relative to the lexicographic order on Δ^+ associated with the set Π;*

(ii) *For any choice of indices $i_1, \ldots, i_n$ with $n \geqq 2$, one has*

$$\gamma_{i_1} - \beta_{i_2} - \ldots - \beta_{i_n} \notin \Gamma_\alpha, \qquad \beta_{i_1} - \beta_{i_2} - \ldots - \beta_{i_n} \notin \Gamma_\alpha.$$

Proof. See Table 1 for the existence of a polarization having property (i). In the calculation of Table 1 we make frequent use of the property that the sum of the simple roots in any connected subset of the Dynkin diagram for Π is a root (cf. Lemma 2). Property (ii) follows immediately from Lemma 3 (i). $\square$

Table 1. Polarizations of Γ_α, $\alpha = (H_{\alpha_1} + \ldots + H_{\alpha_l})^\vee$

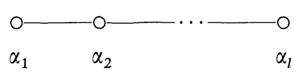
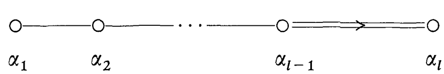
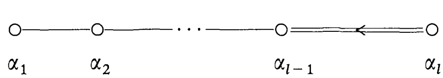
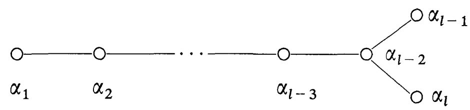
$\mathbf{A}_l$	Diagram:	
		$\alpha = \alpha_1 + \ldots + \alpha_l$
	Polarization:	$\beta_i = \alpha_{i+1} + \ldots + \alpha_l$,
		$\gamma_i = \alpha_1 + \ldots + \alpha_i \quad (1 \leqq i \leqq l-1)$.
$\mathbf{B}_l$	Diagram:	
		$\alpha = \alpha_1 + \ldots + \alpha_{l-1} + 2\alpha_l$
	Polarization:	$\beta_i = \alpha_{i+1} + \ldots + \alpha_{l-1} + 2\alpha_l$,
		$\gamma_i = \alpha_1 + \ldots + \alpha_i \quad (1 \leqq i \leqq l-2)$,
		$\beta_{l-1} = \alpha_l, \quad \gamma_{l-1} = \alpha_1 + \ldots + \alpha_l$.
$\mathbf{C}_l$	Diagram:	
		$\alpha = 2\alpha_1 + \ldots + 2\alpha_{l-1} + \alpha_l$
	Polarization:	$\beta_i = \alpha_1 + \ldots + \alpha_{l-i}$,
		$\gamma_i = \alpha_1 + \ldots + \alpha_{l-i} + 2\alpha_{l-i+1} + \ldots + 2\alpha_{l-1} + \alpha_l \quad (1 \leqq i \leqq l-1)$,
		with
		$\gamma_{l-1} = \alpha_1 + \ldots + \alpha_l$.

Table 1 (continued)

D_l Diagram:



$\alpha=\alpha_1+\ldots+\alpha_l$

Polarization:

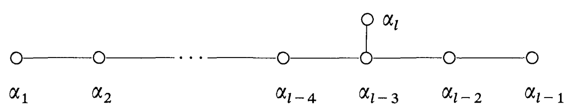
$\beta_i=\alpha_{i+1}+\ldots+\alpha_l,$

$\gamma_i=\alpha_1+\ldots+\alpha_i \quad (1\leqq i\leqq l-3),$

$\beta_{l-2}=\alpha_{l-1}, \quad \gamma_{l-2}=\alpha_1+\ldots+\alpha_{l-2}+\alpha_l,$

$\beta_{l-1}=\alpha_l, \quad \gamma_{l-1}=\alpha_1+\ldots+\alpha_{l-1}.$

E_l Diagram:



$\alpha=\alpha_1+\ldots+\alpha_l$

Polarization:

$\beta_i=\alpha_{i+1}+\ldots+\alpha_l,$

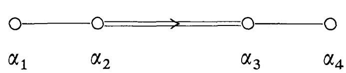
$\gamma_i=\alpha_1+\ldots+\alpha_i \quad (1\leqq i\leqq l-4),$

$\beta_{l-3}=\alpha_{l-2}+\alpha_{l-1}, \quad \gamma_{l-3}=\alpha_1+\ldots+\alpha_{l-3}+\alpha_l,$

$\beta_{l-2}=\alpha_{l-1}, \quad \gamma_{l-2}=\alpha_1+\ldots+\alpha_{l-2}+\alpha_l,$

$\beta_{l-1}=\alpha_l, \quad \gamma_{l-1}=\alpha_1+\ldots+\alpha_{l-1}.$

F_4 Diagram:



$\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$

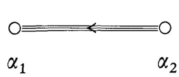
Polarization:

$\beta_1=\alpha_2+2\alpha_3+2\alpha_4, \quad \gamma_1=\alpha_1$

$\beta_2=\alpha_3+\alpha_4, \quad \gamma_2=\alpha_1+\alpha_2+\alpha_3+\alpha_4,$

$\beta_3=\alpha_4, \quad \gamma_3=\alpha_1+\alpha_2+2\alpha_3+\alpha_4.$

G_2 Diagram:



$\alpha=3\alpha_1+\alpha_2$

Polarization:

$\beta_1=\alpha_1, \quad \gamma_1=2\alpha_1+\alpha_2.$

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