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Fixed Points of Feigenbaum's Type for the Equation $f^p(\lambda x) \equiv \lambda f(x)$

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Abstract. Existence and hyperbolicity of fixed points for the map $\mathcal{N}_p : f(x) \to \lambda^{-1} f^p(\lambda x)$, with f^p p-fold iteration and $\lambda = f^p(0)$ are given for p large. These fixed points come close to being quadratic functions, and our proof consists in controlling perturbation theory about quadratic functions.

1. Introduction

The main theme of this paper is another manifestation of the observation "highly iterated maps are quadratic functions," made by Jakobson [1], Milnor [2], Guckenheimer [3], and Benedicks and Carleson [4]. We shall elaborate on this idea and use it to give a simple proof of Feigenbaum universality for certain classes of functions.

We consider maps in a class of function $\mathfrak{D}_p$ which we shall describe now informally and in more detail below. We shall say that $f \in \mathfrak{D}_p$ if $f : [-1, 1] \to [-1, 1]$, $f \in \mathscr{C}^2$ (in fact, we shall work with analytic functions below), $f(0) = 1$, $f''(0) < 0$, and, most importantly, f permutes cyclically p disjoint intervals $J_0, J_1, \ldots, J_p$ with $0 \in J_0$. The intervals are supposed to be arranged as follows and

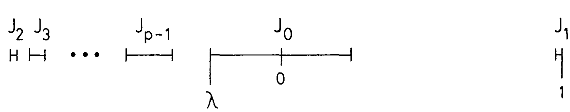

Fig. 1

the endpoints of J_0 are $f^p(0) < 0 < -f^p(0)$. Under these circumstances, setting $\lambda = f^p(0)$, one can show, see Collet, Eckmann, and Lanford [5], that $\mathcal{N}_p f(x) = \frac{1}{\lambda} f^p(\lambda x)$ is again a map of $[-1, 1]$ to itself and $(\mathcal{N}_p f)(0) = 1$. The contention is now: *If $f \in \mathfrak{D}_p$ is not too far from being a quadratic function, then the same is true for $\mathcal{N}_p f$.* Following Guckenheimer, we measure this deviation from being

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quadratic by the quantity

$$Gf(x) = \frac{f''(x)}{f'(x)} - \frac{1}{x}.$$

If f is an even quadratic function, then $Gf = 0$. Given p and $f \in \mathfrak{D}_p$ with $|Gf(x)| < 4^{-p}|x|$, then, under mild additional conditions to be given in detail below, we have $|G(\mathcal{N}_p f)(x)| < 4^{-p}|x|$. Thus, for large p, $\mathcal{N}_p f$ remains almost quadratic if f is almost quadratic.

The above observation (that $\mathcal{N}_p f$ is almost quadratic) is part of a more general phenomenon (not restricted to quadratic functions) which we describe now. Consider a function g with a quadratic critical point x_0 which is not a periodic point of g. Assume

i) some iterate of x_0 falls onto a linearly unstable periodic orbit of g,

ii) a sequence of preimages of x_0 accumulates at this unstable periodic orbit.

Under these circumstances, there are neighborhoods U of g in $\mathscr{C}^2$ and V_p of x_0 in $\mathbb{R}$ (or $\mathbb{C}$) such that the following holds: if $g_1 \in U$ satisfies $g_1^p(V_p) \in V_p$ then $g_1^p|_{V_p}$ deviates by γ^{-p} from being a quadratic function. (In this paper, $g = 1 - 2x^2$, $x_0 = 0$, the unstable periodic orbit is -1, $\gamma = 4$, $V_p = \{x \| x| \leqq \text{const}\, 4^{-p}$, with a small constant$\}$.)

We are interested in a fixed point for $\mathcal{N}_p$, and the previous discussion shows that the fixed point, if it exists, is to be sought among functions f with $|Gf(x)|$ very small. This observation can be put to work, and perturbation theory about quadratic functions is straightforward (although long) to control.

In the space of even analytic functions, we show that $\mathcal{N}_p$ has a fixed point f_p^*, for which $D\mathcal{N}_p(f_p^*)$ is hyperbolic with one eigenvalue $\delta_p > 1$ and the remainder of the spectrum inside the unit disk, and, hence, by the analysis of Collet, Eckmann, and Lanford [5], Feigenbaum universality [6] will hold in this class.

In order to orient the reader, we list here the leading behaviour of the various p dependent quantities. We shall find

$$f_p^*(x) = 1 - (2 - \mathcal{O}(4^{-p}))x^2 + \mathcal{O}(4^{-p})x^4\,,^1$$

$$\lambda_p \equiv (f_p^*)^p(0) \sim -\frac{\pi}{8} 4^{2-p} \sim -0.3927 \cdot 4^{2-p}\,,$$

$$\delta_p \sim \frac{128}{3\pi^2} 16^{2-p} \sim 4.32038 \cdot 16^{p-2}\,.$$

Furthermore, the eigendirection of δ_p is approximately x^2, while the contracting subspace is given by functions vanishing at 0, 1 (and -1, since we consider even functions, only).

Finally, let us remark that the case $p = 2$ is the one originally considered by Feigenbaum [6]. Our estimates are not good enough to yield existence of the fixed point for $p = 2$, but we hope that they shed some more light on the existing proofs for that case, in particular, the one by Campanino, Epstein, and Ruelle [7]. A way to explain the necessity of complicated proofs for the $p = 2$ case can then be seen in the observation that although $G\mathcal{N}_2 f$ becomes small, the smallness obtained is not

1 A related observation was made by Geisel and Nierwetberg [9]

sufficient, and higher order terms must be considered. This is manifest in [7] and also in Lanford's proof [8].

The remainder of this paper is organized as follows. In Sect. 2, we give precise formulations of our results and discuss the functional-analytic parts of the proofs. In Sect. 3, we study functions f which are close to the function $1-2x^2$. In Sect. 4, we construct an approximate fixed point f_p for the operator $\mathcal{N}_p$, and we study a small neighborhood of f_p in function space. In Sect. 5, we study the linear operator $D\mathcal{N}_p$.

2. Notations and Results

We denote $\mathbb{H}_p$ the space of analytic functions on $\{y||y|<p\}$, which are bounded on $|y|\leqq p$, and which are *even*, have *real coefficients* when expanded around zero, and *vanish* at $y=0$.

We shall use, for every p, a polynomial

$$E_p(x)=E_{p,1}x^2+E_{p,2}x^4\,, \qquad E_p(1)=1\,, \tag{2.1}$$

where $E_{p,1}-1=\mathcal{O}(4^{-p})$, $E_{p,2}=\mathcal{O}(4^{-p})$. This function will turn out to be an approximate eigenvector of $D\mathcal{N}_p$, for reasons which are explained towards the end of Sect. 4, and which become clear in Sect. 5. We give below an implicit definition of E_p, cf. Eq. (2.12). Once, the function E_p is given, the following construction makes sense. Every function h in $\mathbb{H}_p$ can be uniquely expanded as

$$h(x)=h_0E_p(x)+h_1(x)\,, \tag{2.2}$$

where $h_1(0)=h_1(1)$ $[=h_1(-1)]=0$. We set

$$\alpha_p=3^p\,, \tag{2.3}$$

and we equip $\mathbb{H}_p$ with the norm

$$\|h\|_p=\alpha_p|h_0|+|h_1|_p\,, \tag{2.4}$$

where

$$|h_1|_p=\sup_{|z|\leqq p}|h_1(z)|\,. \tag{2.5}$$

For convenience, we call this the decomposition $\mathbb{H}_p=\mathbb{R}\oplus\mathbb{H}'_p$.

We denote by $\mathcal{J}$ the set of functions $f\in\mathcal{C}^2$, mapping the interval $I=[-1,1]$ into itself and satisfying

$$\begin{aligned} &f(0)=1\,,\\ &f''(0)<0\,, \qquad xf'(x)<0 \quad \text{for} \quad x\in I\backslash\{0\}\,. \end{aligned} \tag{2.6}$$

The set $\mathfrak{D}_p\subset\mathcal{J}$ is the set of those functions which "exchange p disjoint intervals," $J_0, J_1, \ldots, J_{p-1}$, where

$$J_0=[f^p(0), -f^p(0)]\,, \qquad f^p(0)<0\,,$$

and

$$f(J_i)=J_{i+1}\,, \qquad i=0,1,2,\ldots,p-2\,, \qquad f(J_{p-1})\subseteqq J_0\,.$$

The intervals are supposed to be arranged as in Fig. 1. For $f \in \mathfrak{D}_p$, we define

$$(\mathcal{N}_p f)(x) = \frac{1}{\lambda} f^p(\lambda x), \tag{2.7}$$

where $\lambda = f^p(0)$. It is easy to see that $f \in \mathfrak{D}_p$ implies $\mathcal{N}_p f \in \mathcal{J}$, see e.g. Collet, Eckmann, and Lanford [5]. Below, we shall formulate results which guarantee the existence of $\mathcal{N}_p$ as a map in function space, (viz $\mathbb{H}_p + 1$).

Conventions. (i) All theorems, propositions, and lemmas are to be implicitly supplemented by the statement "There is a $p_0 < \infty$ such that for $p > p_0$ one has ... ."

(ii) All constants $K_\pi, K_\lambda, \ldots, C_1, C_2, \ldots$ are independent of p, of integers $k, j, \ldots$, and of the functions in relation to which they occur. The order symbol $\mathcal{O}$ is also independent of such quantities but $\mathcal{O}(4^{-p})x$ is a shorthand notation for $x \cdot u(x)$, $u(x) = \mathcal{O}(4^{-p})$.

(iii) The constants $K_\pi, \ldots$, retain their meaning throughout the paper, while $C_1, C_2, \ldots$ keep their meaning only through a single proof.

We now list the theorems which are needed for the proof of the existence of a fixed point for $\mathcal{N}_p$ and its hyperbolicity, as outlined in the introduction. These theorems are chosen in such a way as to allow an application of the contraction mapping principle (in a way similar to Lanford's argument [8]). The final result will be formulated in Theorem 2.4.

Theorem 2.1. *For every sufficiently large p, there is a polynomial f_p, of degree four, such that $f_p \in \mathfrak{D}_p$ and*

$$\|\mathcal{N}_p f_p - f_p\|_p \leqq K_N 16^{-p} p^8 . \tag{2.8}$$

The function f_p will be constructed in Sect. 4. It is of the form

$$f_p(x) = 1 - \mu_p x^2 + K_\alpha 4^{-p} x^4 , \tag{2.9}$$

where K_α is a universal constant, and μ_p is implicitly chosen by the condition

$$f_p(1) = \mathcal{N}_p f_p(1) , \tag{2.10}$$

i.e. $f_p(1) \cdot f_p^p(0) = f_p^{2p}(0)$.

Definition. We denote $\mathfrak{B}_p$ the set of $f \in \mathbb{H}_p + 1$, for which

$$\|f - f_p\|_p \leqq 16^{-p} p^9 . \tag{2.11}$$

Definition. The function $E_p(x)$ is given by

$$E_p(x) = \frac{f_p(0) - f_p(x) + x f_p'(x)}{f_p(0) - f_p(1) + f_p'(1)} . \tag{2.12}$$

(This is essentially the projection of the function 1 parallel to $f_p(x) - x f_p'(x)$ [= the eigenvector with eigenvalue one of $f \to f \circ \ldots \circ f$ (p times)], into $\mathbb{H}_p$.)

Theorem 2.2. *Assume $f \in \mathfrak{B}_p$. Then*

i) $f|_I \in \mathcal{J}$,

ii) *$\mathcal{N}_p f$, as given by Eq. (2.7), is defined and is in $\mathbb{H}_p + 1$.*

iii) *$\mathcal{N}_p f$ extends to an analytic and bounded function in $|y| < 2p$.*

iv) *$\mathcal{N}_p$ is infinitely differentiable from $\mathfrak{B}_p$ to $\mathbb{H}_p + 1$.*

v) *For any f in the ball $\mathfrak{B}_p$, $D\mathcal{N}_p(f)$ is a compact operator on $\mathbb{H}_p$.*

Theorem 2.1 will be proven as Theorem 4.3. Theorem 2.2i) is Remark 4.2′. Theorem 2.2ii), iii) is immediate from Corollary 4.4. Theorem 2.2iv), v) follows from ii), iii), by Montel's theorem, cf. Collet, Eckmann, and Lanford [5].

We next concentrate on the tangent map $D\mathcal{N}_p(f)$, and we show it is hyperbolic for $f \in \mathfrak{B}_p$. We identify $\mathbb{H}_p$ with $\mathbb{R} \oplus \mathbb{H}'_p$, cf. (2.2)–(2.5), and we write $D\mathcal{N}_p$ as a matrix

$$\begin{pmatrix} A_{00}(f) & A_{01}(f) \\ A_{10}(f) & A_{11}(f) \end{pmatrix},$$

where $A_{00} \in \mathbb{R}$, $A_{01} \in \mathbb{H}'^{*}_p$, $A_{10} \in \mathbb{H}'_p$, and $A_{11} \in \mathscr{L}(\mathbb{H}'_p, \mathbb{H}'_p)$. Our main estimates are summarized in

Theorem 2.3. *If $f \in \mathfrak{B}_p$, then we have*

$$0 < A_{00} = K_{00} 16^{p-2}(1 + \mathcal{O}(3^{-p})),$$

$$\|A_{10}\|_{\mathbb{H}'_p} < (\tfrac{4}{9})^p p^{20},$$

$$\|A_{01}\|_{\mathbb{H}'^{*}_p} < 12^p p^2,$$

$$\|A_{11}\|_{\mathbb{H}'_p \to \mathbb{H}'_p} < 3^{-p} p^{21},$$

where K_{00} is a universal constant.

In fact, $K_{00} = \frac{32}{3} \prod_{j=1}^{\infty} \cos^2(\pi 2^{-j-1}) = \frac{128}{3\pi^2} \approx 4.323037$. [This number should be compared with Feigenbaum's constant (case of $p=2$), $\delta = 4.6692\ldots$.]

Proof. The theorem is a consequence of the results of Sect. 5 and the definition of the $\|\ \|_p$ norm. A_{00} is estimated in Proposition 5.1. In Proposition 5.2, we show

$$|(AE_p)(z) - E_p(z)(AE_p)(1)| \leqq \mathcal{O}((\tfrac{4}{3})^p p^{19}), \quad \text{if} \quad |z| \leqq p.$$

Note that $\|E_p\|_p = \alpha_p$, so that $\|A_{10}\|_{\mathbb{H}'_p} < \alpha_p^{-1}(\tfrac{4}{3})^p p^{20} = (\tfrac{4}{9})^p p^{20}$, as asserted. In Proposition 5.3, we show that for $h \in \mathbb{H}'_p$, $|Ah(1)| < |h|_p 4^p p^2$. Therefore,

$$\|A_{01}\|_{\mathbb{H}'^{*}_p} < \alpha_p 4^p p^2 = 12^p p^2.$$

Finally, in Proposition 5.4, we bound A_{11}.

We next prove the existence of a fixed point, proceeding as in Lanford [8]. We consider the map

$$f \to \Phi_p(f) = f - U_p(\mathcal{N}_p f - f),$$

where

$$U_p = \begin{pmatrix} (K_{00}16^{p-2} - 1)^{-1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

From Theorem 2.3 one deduces easily that

$$\|D\Phi_p(f)\|_{\mathbb{H}_p \to \mathbb{H}_p} < (\tfrac{3}{4})^p p^{21} \quad \text{if} \quad f \in \mathfrak{B}_p.$$

Therefore, it will follow from the contraction mapping principle that Φ_p has a unique fixed point in $\mathfrak{B}_p$, provided

$$\frac{\|\Phi_p(f_p)-f_p\|_p}{1-(\frac{3}{4})^p p^{21}} < 16^{-p}p^9\,,$$

cf. (2.8). But this is immediate from Theorem 2.1 and the definition of U_p. In summary, the above Theorems 2.1 and 2.2 imply

Theorem 2.4. *If p is sufficiently large, the operator $\mathcal{N}_p$ has a fixed point f_p^* which is in $\mathbb{H}_p+1$ (and with $f_p^*|_I \in \mathfrak{D}_p$). The operator $D\mathcal{N}_p$ is uniformly hyperbolic on a ball of radius $16^{-p}p^9$ around f_p^*. It has exactly one eigenvalue δ_p not strictly in the unit disk. This eigenvalue is simple and $\delta_p = K_{00}\cdot 16^{p-2}(1+\mathcal{O}(3^{-p}))$. Also, $\lambda(f_p^*) \equiv f_p^{*p}(0) = -K_\lambda 4^{(2-p)}(1+\mathcal{O}(4^{-p}p^4))$.*

The assertions on the eigenvalue δ_p follow at once from Theorem 2.3. The bound on $f_p^{*p}(0)$ is a consequence of Lemma 4.2 and Lemma 4.5i). The constants K_{00} and K_λ are

$$K_{00} = \frac{32}{3}\prod_{j=1}^{\infty}\cos^2(\pi 2^{-(j+1)}) = 128/(3\pi^2) \approx 4.323037\,,$$

$$K_\lambda = \frac{1}{4}\prod_{j=1}^{\infty}\frac{1}{\cos(\pi 2^{-(j+1)})} = \pi/8 \approx 0.392699\,.$$ [2]

These constants are already rather good for $p=2$, cf. Feigenbaum [6] ($\delta = 4.66920$, $\lambda = -0.395353$).

Finally, $f_p^*|_I \in \mathfrak{D}_p$ because $f_p^* \in \mathcal{J}$, by Theorem 2.2i), and since $f_p^*(1) = \mathcal{N}_p f_p^*(1)$ implies $f_p^{*2p}(0) = f_p^{*p}(0) f_p^*(1)$, i.e. $|f_p^{*p}(\lambda(f_p^*))| < |\lambda(f_p^*)|$.

3. Orbits of Perturbations of $1-2x^2$

In this section, we first study the backward orbit of the function $\psi_2(x) = 1-2x^2$. As ψ_2 is the limit of the fixed points of $\mathcal{N}_p$ when $p\to\infty$, as we show in this paper, it, of course, plays a special role in our analysis. The fixed point f_p^* of $\mathcal{N}_p$ will lie at a distance $\sim 4^{-p}$ from ψ_2. Thus, we study here a neighborhood $\mathfrak{V}_p$ of ψ_2, defined as follows:

Definition 3.1. We say $f \in \mathfrak{V}_p$ if
 i) $f \in \mathbb{H}_p + 1$,
 ii) $\sup_{|z|\le 2} |f(z) - \psi_2(z)| < K_V 4^{-p}$,
 iii) $f|_I \in \mathfrak{D}_p$.

The following observation may be helpful in understanding the strategy of our proofs. If f is in $\mathfrak{V}_p$, then it is not only very close to ψ_2, but it shares in particular the following properties with ψ_2: It has a fixed point t_0 near -1, and the qualitative dynamics for the inverse f^{-1} on $[t_0, 0]$ is the same as that of ψ_2^{-1} on $[-1, 0]$.

2 See Lemma A.2

We shall estimate $\lambda(f)\equiv f^p(0)$ and find $0<-\lambda(f)<K_2 4^{-p}$ (Lemma 3.3), then we show $\prod_{j=2}^{p-1}(-f^j(\lambda(f)z))=K_\pi(1+\mathcal{O}(4^{-p}p^3))$ (Lemma 3.4), and finally, in Lemma 3.5, we show

$$Gf^p(z)=-K_G\cdot 4^p(1+\mathcal{O}(4^{-p}p^3))z,$$

for $|z|\leqq\mathcal{O}(4^{-p}p)$. This crucial estimate will allow us in the next section to construct an approximate fixed point for $\mathcal{N}_p$.

It is well-known that the function ψ_2 is conveniently reexpressed in the variable v, related to x by $x=\sin(\pi v/2)$. One has

$$\psi_2(x)=\sin(\pi\tilde{\psi}_2(v)/2), \quad \text{and} \quad \tilde{\psi}_2(v)=1-2|v|.$$

We shall denote by ψ_2^{-1} the negative branch of the inverse function of ψ_2, and we write ψ_2^{-k} for $\psi_2^{-1}\circ\ldots\circ\psi_2^{-1}$ (k times). It is easy to see that

$$\begin{aligned}\psi_2^{-k}(0)&=\sin(\tfrac{\pi}{2}(-1+2^{-k}))=-\cos(\pi 2^{-k-1})\\ &=-1+\frac{\pi^2}{8}4^{-k}-\mathcal{O}(16^{-k}).\end{aligned}\tag{3.1}$$

We shall use the same convention for f^{-k} as for ψ_2^{-k}.

Although the above construction is very special to the function ψ_2, we insist that only two things are really of importance:

i) The critical point is mapped to a linearly unstable periodic point (-1, in the case of ψ_2).

ii) Some preimages of the critical point accumulate at the unstable periodic point.

We now compare the backward orbits of ψ_2 and of $f\in\mathfrak{B}_p$.

Lemma 3.2. *If $f\in\mathfrak{B}_p$, then for $j=1,2,\ldots$,*

$$|f^{-j}(0)-\psi_2^{-j}(0)|\leqq K_1 4^{-p}.$$

Proof. If $f\in\mathfrak{B}_p$, then not only $f(z)$ is close to ψ_2, but we have, for $|z|\leqq 3/2$, $f''(z)=-4+\mathcal{O}(4^{-p})$ (use contour integration), and hence, $f'(z)=-(4+\mathcal{O}(4^{-p}))z$. This will be used throughout. We set $x_0=y_0=0$ and $x_j=f^{-j}(0)$, $y_j=\psi_2^{-j}(0)$. It is obvious that $|x_1-y_1|\leqq C_1 4^{-p}$. Now, if $|x_j-y_j|\leqq C_1 4^{-p}$, then

$$f(x_{j+1})=\psi_2(x_{j+1})+\mathcal{O}(4^{-p}) \quad \text{since} \quad |x_{j+1}|\leqq 3/2.$$

Thus,

$$x_{j+1}^2-y_{j+1}^2=-\tfrac{1}{2}((x_j-y_j)+\mathcal{O}(4^{-p})).$$

It follows from $|x_{j+1}+y_{j+1}|>|x_{j+1}|$ that

$$|x_{j+1}-y_{j+1}|<\frac{1}{2\cdot\left(1-\frac{\pi^2}{8}4^{-j-1}\right)}(|x_j-y_j|+\mathcal{O}(4^{-p})).$$

This clearly implies the assertion.

Our next estimate is a provisional bound on $\lambda(f)=f^p(0)$. Whenever there is no confusion possible, we shall write λ instead of $\lambda(f)$.

Lemma 3.3. *If* $f \in \mathfrak{B}_p$, *then*

$$0 < -\lambda(f) \leqq K_2 4^{-p}.$$

Proof. The inequality $\lambda < 0$ is part of the definition of $\mathfrak{D}_p$. Since $f \in \mathfrak{B}_p$, we must have $|f^p(\lambda)| \leqq |\lambda|$. Every point in J_k, $k \geqq 2$, is to the left of J_{k+1}, and the left endpoint of J_{k+1} is $f^{k+1}(0)$. Since $f \in \mathfrak{B}_p$, f is concave (contour integration again), and thus, f is increasing for $x < 0$, and hence, we have

$$f^{k+1}(0) = f^{k+1-p}(f^p(0)) < f^{k+1-p}(0), \quad \text{for} \quad k+1 < p,$$

because $f^p(0) = \lambda < 0$. Hence, by Lemma 3.2, $f^{k+1}(0) < \psi_2^{k+1-p}(0) + K_1 4^{-p}$, for $k = 2, \ldots, p-2$. Finally, since $f'(z) = -(4 + \mathcal{O}(4^{-p}))z$, as we noted in the proof of Lemma 3.2, we see, by combining the above arguments, that

$$\begin{aligned} |J_{k+1}| &> (4 + \mathcal{O}(4^{-p}))|J_k|\,|f^{k+1}(0)| \\ &> (4 + \mathcal{O}(4^{-p}))\left(1 - \frac{\pi^2}{8} 4^{k+1-p} - K_1 4^{-p}\right)|J_k|, \qquad k+1 < p. \end{aligned}$$

Finally, $|J_1| \geqq (2 + \mathcal{O}(4^{-p}))\lambda^2$. Combining these bounds, we see that $|J_{p-1}| \geqq C_1 \lambda^2 4^{p-1}$, using

$$(4 + \mathcal{O}(4^{-p}))^p = 4^p(1 + \mathcal{O}(4^{-p}p)). \tag{3.2}$$

We next note that J_{p-1} must lie to the left of $-x_0$, where $x_0 > 0$ is defined by $f(x_0) = x_0$ $[= \mathcal{O}(1)]$, since otherwise $J_0 \cap J_1 \neq \emptyset$. Thus, we find $|J_p| \geqq C_2 \lambda^2 4^p$, as before. Since we must have $J_p \subseteqq J_0 = [\lambda, -\lambda]$, the assertion follows.

We have now shown that $\lambda = \lambda(f)$ is very small. Hence, if $|z| \leqq 2p$, then λz is very close to zero. Our next lemma shows that the *whole orbit* of λz stays close to the preimages $\psi_2^{-j}(0)$ for p iterations.

Lemma 3.4. *If* $f \in \mathfrak{B}_p$ *then, for* $k \geqq 2$,

$$\prod_{j=k}^{p-1} (-f^j(\lambda(f)z)) = K_\pi(1 + \mathcal{O}(4^{k-p}p^3)), \quad \textit{for} \quad |z| \leqq 2p,$$

where

$$K_\pi = \prod_{\ell=1}^{\infty} \cos(\pi/2^{\ell+1}) = 2/\pi.$$

Proof. We shall prove below the bound, valid for $|z| \leqq 2p$, $j = 2, \ldots, p-1$,

$$|f^j(\lambda(f)z) - \psi_2^{j-p}(0)| \leqq C_3 4^{-p} p^2. \text{[3]} \tag{3.3}$$

We have already seen that $\psi_2^{j-p}(0) = \cos(\pi/2^{p-j+1})$, and hence, the assertion follows from (3.3), using again (3.2), and from

$$\prod_{\ell=1}^{k} \cos(\pi/2^{\ell+1}) = K_\pi(1 + \mathcal{O}(4^{-k})).$$

The identity $K_\pi = 2/\pi$ is derived in Lemma A.1.

3 For $j < p$, ψ_2^{j-p} should be viewed as $\psi_2^{-(p-j)}$

We now proceed to prove (3.3). We first bound $|f^j(0)-f^{j-p}(0)|$. Since $f\in\mathfrak{D}_p$, we have $|f^p(0)-0|\leqq K_2 4^{-p}$, by Lemma 3.3. Hence, by the bound $f'(\zeta) = -(4+\mathcal{O}(4^{-p}))\zeta$, for $|\zeta|\leqq 3/2$, we deduce, using Lemma 3.2, $|f^j(0)-f^{j-p}(0)| \leqq C_1 4^{-p-j}$, and thus, $|f^j(0)-\psi_2^{j-p}(0)|\leqq C_2 4^{-p}$ for $j=2,3,\dots,p-1$. Next, we observe that $f(\lambda(f)z)-f(0)=\mathcal{O}(16^{-p}p^2)$, since $f'(\lambda(f)z)=-(4+\mathcal{O}(4^{-p}))\lambda(f)z$. Since f, at ζ, $|\zeta|\leqq 1+\mathcal{O}(4^{-p})$, expands by no more than $4+\mathcal{O}(4^{-p})$, we see that

$$|f^j(\lambda(f)z)-f^j(0)|\leqq C_2 4^j 16^{-p}p^2\,. \tag{3.4}$$

Combining the above estimates, we obtain (3.3), and hence, the lemma is proved.

We next show that Gf^p is essentially a linear function, with coefficient independent of $f\in\mathfrak{V}_p$.

Lemma 3.5. *Assume* $f\in\mathfrak{V}_p$ *and* $|z|\leqq 2|\lambda(f)|p$. *Then*

$$Gf^p(z) = -K_G 4^p z(1+\mathcal{O}(4^{-p}p^3))\,,$$

$$K_G = 4/\pi^2\,.$$

Proof. The assumptions imply $(Gf)(z)=\mathcal{O}(4^{-p})z$, as is easily checked from the definition. We rewrite $Gf^p(z)-Gf(z)$, using the chain rule, and induction on p, as

$$\sum_{j=1}^{p-1}\frac{f''(f^j(z))}{f'(f^j(z))}(f^j)'(z) = zf''(0)(1+\mathcal{O}(4^{-p}))Z_f\,,$$

where

$$Z_f(z) = \sum_{j=1}^{p-1}\frac{f''(f^j(z))}{f'(f^j(z))}(f^{j-1})'(f(z))\,.$$

Using Lemma 3.2, we see that $Z_f(z)=Z\cdot(1+\mathcal{O}(4^{-p}p^3))$, where

$$Z=\sum_{j=1}^{p-1}\frac{1}{-\psi_2^{j-p}(0)}\cdot 4^{j-1}\prod_{k=1}^{j-1}(-\psi_2^{k-p}(0))$$

[we use $f'(z)=-(4+\mathcal{O}(4^{-p}))z$ and $(f''/f')(z)=(1+\mathcal{O}(4^{-p}))/z$], and we estimate $f'(f(z))\approx 4\psi^{1-p}(0)$, $f'(f^k(z))\approx -4\psi^{k-p}(0)$, if $k>1$. If we define

$$Z_{\Sigma,p}=\sum_{j=1}^{p-1}\frac{1}{-\psi_2^{j-p}(0)}4^{j-p}\prod_{k=j}^{p-1}(-\psi_2^{k-p}(0))^{-1}\,,$$

then we see that

$$Z_f = Z_{\Sigma,p}K_\pi 4^{p-1}(1+\mathcal{O}(4^{-p}p^3)) = K_G 4^{p-1}(1+\mathcal{O}(4^{-p}p^3))\,.$$

The quantities $Z_{\Sigma,p}$ and K_π are estimated in Lemma A.1, A.1′. We have $Z_\Sigma=2/\pi$, $K_\pi=2/\pi$, where $Z_\Sigma=\lim_{p\to\infty}Z_{\Sigma,p}$. The lemma is proved.

4. The Approximate Fixed Point and Its Neighborhood

In this section, we first construct an approximate fixed point f_p for $\mathcal{N}_p$; this fixed point will lie in $\mathfrak{V}_p$, provided the constant K_V in its definition is chosen sufficiently large (but, of course, independent of p). We will then collect some bounds for functions in a small neighborhood of f_p. The approximate fixed point will be constructed along the following lines. By Lemma 3.5, if $f_p(x)=1-\mu x^2+K_\alpha 4^{-p}x^4$, $\mu=2+\mathcal{O}(4^{-p})$, $K_\alpha=\mathcal{O}(1)$, then

$$Gf_p^p(x)\cong -K_G 4^p x\,.$$

We shall impose

$$f_p(1)=(\mathcal{N}_p f_p)(1)\,, \tag{4.1}$$

and this will imply, through Lemma 3.4, that

$$\lambda(f_p)\cong -K_\lambda 4^{2-p}\,,\qquad K_\lambda=1/4K_\pi\,.$$

Therefore,

$$\begin{aligned} G(\mathcal{N}_p f_p)(x) &= \lambda(f_p)(Gf_p^p)(\lambda(f_p)x)\\ &\cong -K_G K_\lambda^2 4^4 4^{-p}x\,. \end{aligned}$$

On the other hand, it is easy to see that

$$(Gf_p)(x)\cong -2K_\alpha 4^{-p}x\,.$$

Thus, if we choose $2K_\alpha=4^4K_GK_\lambda^2$, then

$$|G(\mathcal{N}_p f_p)(x)-(Gf_p)(x)|\leqq \mathcal{O}(16^{-p}p^3)\,. \tag{4.2}$$

It will be easy to see that (4.1) and (4.2) imply

$$|\mathcal{N}_p f_p(x)-f_p(x)|\leqq \mathcal{O}(16^{-p}p^3)\,,$$

so that f_p is an approximate fixed point, as asserted.

We now fill in the details of this argument. We fix

$$K_\alpha=2^7K_GK_\lambda^2 \tag{4.3}$$

(we have $K_\alpha=8$ by Lemma A.1, A.1′). We next consider the one parameter family of maps

$$g_\mu(x)=1-\mu x^2+K_\alpha 4^{-p}x^4\,. \tag{4.4}$$

For $\mu^{(0)}=2+K_\alpha 4^{-p}$, we have $g_\mu(1)=-1$, while for $\mu^{(1)}=1+K_\alpha 4^{-p}$, $g_\mu(1)=0$. This implies that $g_{\mu^{(0)}}$ and $g_{\mu^{(1)}}$ are not in $\mathfrak{D}_p$, but, by kneading theory (see e.g. [10]), there is at least one interval M of μ's, $M\subset[\mu^{(1)},\mu^{(0)}]$, so that $\mu\in M$ implies $g_\mu\in\mathfrak{D}_p$. At the boundary μ',μ'' of at least one such interval, we must have

$$g_{\mu'}^p(0)=0$$

and

$$g_{\mu''}^{2p}(0)=-g_{\mu''}^p(0)\,,$$

see the discussion in [5]. It is also shown in [5] (Proposition 3.1) that

$$\lim_{\substack{\mu\to\mu'\\ \mu\in M}}(\mathcal{N}_p g_\mu)(1)=1\,,$$

while it is obvious from the definition of $\mathcal{N}_p$ that $(\mathcal{N}_p g_{\mu''})(1) = -1$. By continuity, there exists a μ_p between μ' and μ'' for which

$$(\mathcal{N}_p g_{\mu_p})(1) = g_{\mu_p}(1) \tag{4.5}$$

(since $\mu', \mu'' < \mu^{(0)}$). We shall denote henceforth μ_p the largest such value, and we set $f_p = g_{\mu_p}$.

Lemma 4.1. *With the above definitions, we have* $|\mu_p - 2| \leqq \mathcal{O}(4^{-p})$ *and* $f_p \in \mathfrak{B}_p$.

Remark. The size of K_α and of $\mu_p - 2$ determine the choice of K_V.

Proof. By construction, we have $\mu_p \leqq 2 + K_\alpha 4^{-p}$. We now derive a lower bound for μ_p. Define v_p by $1 - v_p = \psi_\tau^{-(p-2)}(0)$, where $\tau = 2 + K_\alpha 4^{-p}$ and $\psi_\tau(x) = 1 - \tau x^2$. For $\mu \leqq 2 + K_\alpha 4^{-p}$, we have $g_\mu(x) > \psi_\tau(x)$ when $x \in I \backslash \{0\}$. Therefore, defining μ_p^* to be that (maximal) value of μ for which $g_\mu \in \mathfrak{D}_p$ and $g_\mu^p(0) = 0$, we see that

$$1 - \mu_p^* + K_\alpha 4^{-p} = g_{\mu_p^*}^2(0) = g_{\mu_p^*}^{-(p-2)}(0) < \psi_\tau^{-(p-2)}(0)\,.$$

Since $|\tau - 2| = \mathcal{O}(4^{-p})$, Lemma 3.2 and Eq. (3.1) imply $\psi_\tau^{-(p-2)}(0) < -1 + \mathcal{O}(4^{-p})$, and hence, $\mu_p^* \geqq 2 - \mathcal{O}(4^{-p})$. By the general theory of kneading, we have $\mu_p > \mu_p^*$, and thus, the first assertion of the lemma follows. The second assertion is now an immediate consequence of the definition of $\mathfrak{B}_p$.

Lemma 4.2. *With f_p as defined by* (4.1) *and* (4.3), *we have*

$$\lambda_p \equiv \lambda(f_p) = -K_\lambda 4^{2-p}(1 + \mathcal{O}(4^{-p}p^4))\,,$$

where $K_\lambda = 1/4K_\pi = \pi/8$.

Proof. By construction, we have $f_p \in \mathfrak{B}_p$ (and, a fortiori, $f_p \in \mathfrak{D}_p$). By Lemma 3.3, we have $|\lambda_p| \leqq K_2 4^{-p}$, and hence, Lemma 3.5 implies, for $|z| \leqq 2p$,

$$(G\mathcal{N}_p f_p)(z) = \lambda_p (G f_p^p)(\lambda_p z) = \mathcal{O}(4^{-p})z\,. \tag{4.6}$$

By construction, we also have

$$(G f_p)(z) = \mathcal{O}(4^{-p})z\,. \tag{4.7}$$

We next apply Lemma A.2, with $u = \mathcal{N}_p f_p$, $v = f_p$. It is easy to see from $f_p \in \mathfrak{D}_p$ that $\mathcal{N}_p f_p$ satisfies the general assumptions of Lemma A.2, and (4.6) and (4.7) imply $(Gu)(x) - (Gv)(x) = \mathcal{O}(4^{-p})x$, for $|x| \leqq p$. Hence, we conclude

$$(\mathcal{N}_p f_p)''(0) = -(4 + \mathcal{O}(4^{-p}p^4))\,. \tag{4.8}$$

By the chain rule, we have

$$(\mathcal{N}_p f_p)''(0) = f_p''(0) \cdot \prod_{j=1}^{p-1} f_p'(f_p^j(0)) \cdot \lambda_p\,. \tag{4.9}$$

Since $f_p''(0) = -4 + \mathcal{O}(4^{-p})$, we conclude from Lemma 3.4, and from $f_p'(x) = (-4 + \mathcal{O}(4^{-p}))x$, for $|x| \leqq 2$, that

$$\begin{aligned}-(4 + \mathcal{O}(4^{-p}p^4)) = (-4 + \mathcal{O}(4^{-p}))^2(+4 + \mathcal{O}(4^{-p}))^{p-2} K_\pi \\ \cdot (1 + \mathcal{O}(4^{2-p}p))\lambda_p\,,\end{aligned}$$

i.e.

$$\lambda_p = -(1+\mathcal{O}(4^{-p}p^4))/(K_\pi 4^{p-1}).$$

This proves the lemma.

Remark 4.2′. Lemma 4.2 implies $f_p(1) > -1+5^{-p}$. Indeed, assume the contrary. Then we will have $f_p^p(0) < -1+\mathcal{O}(4^p 5^{-p})$, which contradicts $|f_p^p(0)| \leqq \mathcal{O}(4^{-p})$. We now come to our first main estimate:

Theorem 4.3. *With f_p as defined by* (4.1) *and* (4.3), *we have*

$$\|\mathcal{N}_p f_p - f_p\|_p \leqq K_N 16^{-p} p^8.$$

Proof. By Lemma 4.2, we find, using Lemma 3.5,

$$\begin{aligned}(G\mathcal{N}_p f_p)(x) &= \lambda(f_p) G f_p^p(\lambda(f_p)x) \\ &= -K_\lambda^2 4^{4-2p} K_G 4^p x(1+\mathcal{O}(4^{-p}p^4)).\end{aligned}$$

By construction,

$$(Gf_p)(x) = \frac{-2\mu_p + 12K_\alpha x^2}{-2\mu_p + 4K_\alpha x^2} \cdot \frac{1}{x} - \frac{1}{x}.$$

Now Lemma 4.1 and Eq. (4.3) imply

$$\begin{aligned}(Gf_p)(x) &= -2K_\alpha x(1+\mathcal{O}(4^{-p}p^2)) \\ &= -K_\lambda^2 4^{4-2p} K_G 4^p x(1+\mathcal{O}(4^{-p}p^2)).\end{aligned}$$

Hence,

$$G\mathcal{N}_p f_p(x) - Gf_p(x) = x\mathcal{O}(16^{-p}p^4). \tag{4.10}$$

We reapply Lemma A.2 and obtain, after integrating ii),

$$\sup_{|x| \leqq p} |\mathcal{N}_p f_p(x) - f_p(x)| \leqq \mathcal{O}(16^{-p}p^8). \tag{4.11}$$

Since $\mathcal{N}_p f_p(1) - f_p(1) = 0$, this implies the assertion.

Corollary 4.4. *One has*

$$\sup_{|x| \leqq 2p} |\mathcal{N}_p f_p(x) - f_p(x)| \leqq \mathcal{O}(16^{-p}p^8),$$

and $\mathcal{N}_p f_p$ is analytic in $|x| \leqq 2p$.

Proof. This is obtained as (4.11), by an obvious modification of Lemma A.2.

Now, that we have got hold of an approximate fixed point for $\mathcal{N}_p$, namely f_p, we are in a position to define two objects: the neighborhood of f_p in which the fixed point is to be found and the approximate unstable eigendirection of $D\mathcal{N}_p$. Had we considered instead of $\mathcal{N}_p$ the operator $\mathring{\mathcal{N}}_p f = \lambda_0^{-1} \circ f^p \circ \lambda_0$, with λ_0 fixed, then this eigendirection would be approximately the function 1, however, our choice $\lambda(f) = f^p(0)$ preserves the normalization $\mathcal{N}_p f(0) = 1$, and projects 1 approximately to $1 - f_p(x) + x f_p'(x)$. We, therefore, *define*

$$E_p(x) = \frac{1 - f_p(x) + x f_p'(x)}{1 - f_p(1) + f_p'(1)}, \tag{4.12}$$

and from the definition of f_p, Eq. (4.1), we get

$$E_p(x)=x^2\cdot(1-x^2 3K_\alpha 4^{-p}/\mu_p)/(1-3K_\alpha 4^{-p}/\mu_p)\,. \tag{4.13}$$

Note that $E_p(0)=0$, $E_p(1)=1$, and that $E_p(x)$ is close [by $\mathcal{O}(4^{-p}x^4+x^2 4^{-p})$] to x^2.

We next consider the neighborhood $\mathfrak{B}_p$, defined by $f\in\mathfrak{B}_p$ if

$$f\in\mathbb{H}_p+1 \quad \text{and} \quad \|f-f_p\|_p \leqq 16^{-p}p^9\,.$$

We shall state and prove now a certain number of lemmas which will be useful in estimating the operator $D\mathcal{N}_p$ in the next section.

Lemma 4.5. *If* $f\in\mathfrak{B}_p$, *then*
i) $\lambda(f)=\lambda(f_p)(1+\mathcal{O}(3^{-p}p^{10}))$,
ii) $|f^j(\lambda(f)z)-f_p^j(\lambda(f_p)z)|\leqq 16^{-p}4^j p^{11}$, $|z|\leqq 2p$.

Note. We do not know that $f\in\mathfrak{B}_p$ implies $f\in\mathfrak{D}_p$, and hence, the estimates have to be done in a way which differs slightly from the method of Sect. 3.

Proof. If $f\in\mathfrak{B}_p$, then we have, by the definition of $\|\ \|_p$,

$$|f(1)-f_p(1)|\leqq \alpha_p^{-1}16^{-p}p^9=48^{-p}p^9\,, \tag{4.14}$$

and

$$\sup_{|z|\leqq p}|f(z)-f_p(z)-E_p(z)(f(1)-f_p(1))|\leqq 16^{-p}p^9\,,$$

which implies by (4.13),

$$\sup_{|z|\leqq p}|f(z)-f_p(z)|\leqq 2\cdot 16^{-p}p^9\,. \tag{4.15}$$

We relegate to Lemma A.3 the estimate of the orbits of f^j and f_p^j. Applying Lemma A.3 with $z=0$, we see that

$$|f^p(0)-f_p^p(0)|\leqq \mathcal{O}(48^{-p}4^p p^9)+\mathcal{O}(16^{-p}p^{10})=\mathcal{O}(12^{-p}p^{10})\,.$$

Hence, i) follows from Lemma 4.2. Write now

$$|f^j(\lambda(f)z)-f_p^j(\lambda(f_p)z)|\leqq|f^j(\lambda(f)z)-f_p^j(0)|+|f_p^j(0)-f_p^j(\lambda(f_p)z)|\,,$$

and apply Lemma A.3 to both of these terms to obtain ii).

Lemma 4.6. *If* $f\in\mathfrak{B}_p$, *then*

$$Gf_p(z)-G(\mathcal{N}_p f)(z)=\mathcal{O}(12^{-p}p^{10})z\,,\qquad |z|\leqq 2p\,. \tag{4.16}$$

Proof. By going through the proof of Lemma 3.5, we find

$$Gf^p(x)=-K_G 4^p x(1+\mathcal{O}(4^{-p}p))\,,\qquad |x|\leqq 2p|\lambda(f)|\,. \tag{4.17}$$

The assertion follows from (4.17) combined with Lemma 4.5 and the definition of f_p [cf. also (4.10)].

We now consider four quantities whose role will only become apparent in the next section, but whose estimates are a natural sequel to Lemmas 4.5 and 4.6.

The first two quantities are

$$\Delta_f(x)=(\mathcal{N}_p f)(0)-(\mathcal{N}_p f)(x)+x(\mathcal{N}_p f)'(x) \tag{4.18}$$

and

$$\Delta_f(x) - E_p(x)\Delta_f(1),$$

cf. Eq. (4.12) for the definition of E_p.

Lemma 4.7. *If $f \in \mathfrak{B}_p$, then, for $|z| \leqq p$, we have*
i) $\Delta_f(z) = -2z^2(1 + \mathcal{O}(3^{-p}p^{17}))$,
ii) $\Delta_f(z) - E_p(z)\Delta_f(1) = \mathcal{O}(12^{-p}p^{17})z^2$.

Proof. Denote $\Delta(z) = 1 - f_p(z) + zf_p'(z)$. By construction, $\Delta(z)/\Delta(1) = E_p(z)$. By Lemma 4.6, we have

$$G\mathcal{N}_p f(z) - Gf_p(z) = \mathcal{O}(12^{-p}p^{10})z. \tag{4.19}$$

Denote, momentarily, $u = \mathcal{N}_p f$, $v = f_p$. By the definition of G, (4.19) implies

$$(\log u')'(z) - (\log v')'(z) = \varepsilon z \cdot h(z), \tag{4.20}$$

with $|h| \leqq 1$, $\varepsilon = C_1 12^{-p}p^{10}$. Integrating and exponentiating we deduce from (4.20):

$$u'(z) = C_{u,v}v'(z)e^{\varepsilon H(z)}, \tag{4.21}$$

with $|H(z)| \leqq z^2$, $C_{u,v} \neq 0$. Integrating again, we get

$$u(z) = C_{u,v}v(z) + C_{u,v}K(z)\varepsilon + D_{u,v}, \tag{4.22}$$

where

$$K(z) = -\frac{1}{\varepsilon}\int_0^z v'(x)(1 - e^{\varepsilon H(x)})dx, \qquad K(0) = 0.$$

Since $v = f_p$, it is easy to see that $|K(z)| \leqq 2C_2 p^6$. Using (4.21) and (4.22), we see that

$$\frac{\Delta_u(z)}{\Delta_u(1)} = \frac{u(0) - u(z) + zu'(z)}{u(0) - u(1) + u'(1)}$$
$$= \frac{C_{u,v}(v(0) - v(z) + zv'(z) + R(z))}{C_{u,v}(v(0) - v(1) + v'(1) + R(1))},$$

where $R(z) = -K(z)\varepsilon + z(e^{\varepsilon H(z)} - 1)v'(z)$. Note now that

$$v(0) - v(z) + zv'(z) = -\mu_p z^2 + 3K_\alpha 4^{-p}z^4,$$

by the definition of $f_p = v$. Hence, $v(0) - v(1) + v'(1) = -2 + \mathcal{O}(4^{-p})$, so that

$$\frac{\Delta_u(z)}{\Delta_u(1)} = \frac{\Delta_v(z)}{\Delta_v(1)} + \mathcal{O}(12^{-p}p^{17}z^2). \tag{4.23}$$

Note that $\Delta_u(x)/x^2 \to (\mathcal{N}_p f)''(0)/2$ as $x \to 0$. Combining (4.22) with Lemma A.2 provides a bound on $(\mathcal{N}_p f)''(0)/2$, from which i) follows at once.

Using $E_p(z) = \Delta_v(z)/\Delta_v(1)$, we get ii). The proof of Lemma 4.7 is complete.

We next define, for $f \in \mathfrak{B}_p$,

$$v_j = \frac{1}{\lambda(f)}(f^{p-j-1})' \circ f \tag{4.24}$$

and

$$u_j = (v_j \cdot E_p) \circ f^j \circ \lambda(f), \tag{4.25}$$

and finally, for $h \in \mathbb{H}_p$, $h(0)=h(1)=0$,

$$\hat{u}_j = (v_j \cdot h) \circ f^j \circ \lambda(f). \tag{4.26}$$

We shall need in Sect. 5 the following bounds.

Lemma 4.8. *If* $f \in \mathfrak{B}_p$, *then, with the above definitions, for* $|z| \leqq p$,

i) $u_j(z) - u_j(0) - E_p(z)(u_j(1) - u_j(0)) = \mathcal{O}(4^{-j} p^6)$,

ii) $\hat{u}_j(z) - \hat{u}_j(0) - E_p(z)(\hat{u}_j(1) - \hat{u}_j(0)) = \mathcal{O}(4^{-p} p^8)|h|_p$.

Proof. Recall that $E_p(z) = z^2 + \mathcal{O}(4^{-p} z^2 p^2)$. Denote by g one of the functions u_j or $\hat{u}_j$. We shall first estimate

$$X = g(z) - g(0) - z^2(g(1) - g(0)). \tag{4.27}$$

In order to exhibit best the cancellations, we rewrite (4.27) as

$$X = \tilde{g}(w) - \tilde{g}(0) - w(\tilde{g}(1) - \tilde{g}(0)),$$

where $w = z^2$, $\tilde{g}(w) = g(w^{1/2})$ (since g is even, the determination of $w^{1/2}$ is irrelevant). Obviously, we must have $|X| \leqq 2p^4 \sup\limits_{|\xi| < p} |\tilde{g}''(\xi)|$. Observe now that

$$\tilde{g}'(w) = g'(w^{1/2})/2w^{1/2}$$

and

$$\tilde{g}''(w) = g''(w^{1/2})/4w - g'(w^{1/2})/4w^{3/2}.$$

Therefore, we find

$$|X| \leqq p^4 \sup_{|\xi| \leqq p} \left| \frac{g'(\xi)}{4\xi} \cdot \frac{(Gg)(\xi)}{\xi} \right|.$$

We claim $|X| \leqq \mathcal{O}(p^6 4^{-j})$ when $g = u_j$ and $\mathcal{O}(p^6 4^{-p})$, when $g = \hat{u}_j$. This claim follows by a tedious, but straightforward application of the rules of calculus of which we only indicate the steps and their leading behaviour.

We rewrite $g'(\xi) \cdot (Gg)(\xi)$ as follows: set $\tau = v_j E_p$ (respectively $v_j h$). Then

$$\begin{aligned} g' \cdot Gg &= (\tau' \circ f^j \circ \lambda) \cdot (f^j \circ \lambda)' \\ &\quad \cdot \{((G\tau) \circ f^j \circ \lambda) \cdot (f^j \circ \lambda)' + G(f^j \circ \lambda)\} \\ &= \left(\tau'' - \frac{\tau'}{x}\right) \circ (f^j \circ \lambda) \circ (f^j \circ \lambda)'^2 + (\tau' \circ f^j \circ \lambda) \cdot (f^j \circ \lambda)' G(f^j \circ \lambda). \end{aligned}$$

From Lemma A.3 and Lemma 4.5, we derive, in the case $\tau = v_j E_p$, the bounds

$$\begin{aligned} (f^j \circ \lambda)'(z) &\sim 4^j \lambda^2 z, && \text{if } |z| \leqq p, \\ \tau'(x) &\sim \frac{1}{\lambda} 4^{2(p-j)}, && \text{if } |x| \leqq 3/2, \\ \tau''(x) &\sim \frac{1}{\lambda} 4^{3(p-j)}, && \text{if } |x| \leqq 3/2, \\ G(f^j \circ \lambda)(z) &\sim 4^j \lambda^2 z, && \text{if } |z| \leqq p. \end{aligned}$$

This last bound follows as in Lemma 3.5. Combining these bounds, we get

$$|X| \leqq \mathcal{O}(p^4) \cdot \left\{ \frac{1}{\lambda} 4^{3(p-j)} 4^{2j} \lambda^4 + \frac{1}{\lambda} 4^{2(p-j)} 4^j \lambda^2 4^j \lambda^2 \right\},$$

from which the assertion follows for $g = u_j$.

When $g = \hat{u}_j$, then, since $h(\pm 1) = 0$, we have with $\tau = v_j \cdot h$,

$$\begin{aligned} \tau' \circ f^j(\lambda x) &= v_j'(f^j(\lambda x)) h(f^j(\lambda x)) + v_j(f^j(\lambda x)) h'(f^j(\lambda x)) \\ &= v_j'(f^j(\lambda x)) h(\pm 1) \\ &\quad + v_j'(f^j(\lambda x)) \{ h(f^j(\lambda x)) - h(\pm 1) \} \\ &\quad + v_j(f^j(\lambda x)) h'(f^j(\lambda x)) \\ &\sim 0 + \frac{1}{\lambda} 4^{2(p-j)} |h|_p 4^{j-p} p^2 + \frac{1}{\lambda} 4^{p-j} |h|_p , \end{aligned}$$

since, by Lemma A.3 and Eq. (3.3), $f^j(\lambda x) + 1 \sim 4^{j-p} p^2$.

Hence, $g'(x) \sim \frac{1}{\lambda} 4^{-p} |h|_p p^2$, in the case $g = \hat{u}_j$, so that the bound on X follows in all cases. Finally, we have to bound

$$Y = (E_p(x) - x^2)(g(1) - g(0)) .$$

This is achieved by writing

$$|Y| \leqq \mathcal{O}(4^{-p}(x^2 + x^4)) \sup_{|\xi| \leqq 1} |g'(\xi)| ,$$

and we obtain

$$|Y| \sim 4^{-p} p^4 \frac{1}{\lambda} 4^{-j} , \qquad \text{when} \quad g = u_j , \tag{4.28}$$

$$|Y| \sim 4^{-p} p^4 \frac{1}{\lambda} 4^{-p} |h|_p , \quad \text{when} \quad g = \hat{u}_j . \tag{4.29}$$

The lemma is proved.

Corollary 4.9. *If* $f \in \mathfrak{B}_p$, *then, with the definitions* (4.24)–(4.26), *we have, for* $|x| \leqq p$,

i) $u_j(x) - u_j(0) = \mathcal{O}(p^4 4^{p-j})$,
ii) $\hat{u}_j(x) - \hat{u}_j(0) = \mathcal{O}(p) |h|_p$,
iii) $\hat{u}_j(x) = \mathcal{O}(4^p) |h|_p$.

Proof. i) and ii) are an immediate consequence of the derivation of (4.28) and (4.29). iii) follows at once from

$$\begin{aligned} \hat{u}_j(x) &= v_j(f^j(\lambda x))(h(f^j(\lambda x)) - h(\pm 1)) \\ &\sim \frac{1}{\lambda} 4^{p-j} |h|_p 4^{j-p} . \end{aligned}$$

5. The Operator A

In this section, we consider the operator $A = A_f = D\mathcal{N}_p(f)$, when $f \in \mathfrak{B}_p$. In the decomposition $\mathbb{H}_p = \mathbb{R} + \mathbb{H}'_p$, i.e.

$$h(x) = h_0 E_p(x) + h_1(x), \quad h_1(0) = 0,$$

the operator A is "almost diagonal", with E_p "almost equal" to the unstable direction, and $\mathbb{H}'_p$ "almost equal" to the stable subspace.

We shall consider the four "matrix elements" A_{ij}, $i,j = 0, 1$, as described in Sect. 2. The explicit expression for A is given in Eq. (5.1) below, and a quick calculation shows that $A_{00} \sim 16^p$. In A_{01} we shall find a division of the leading terms by $\sim 4^p$, because A_{01} is applied to a function h which vanishes at 1. Hence, we will find $|(Ah)_0| \sim 4^p$. (This bound will use part of the compensations of Lemma 4.8.) The bounds of Lemma 4.7 will induce a cancellation of all terms $\sim 16^p$ *and* 4^p in $A_{10} = \sup|(AE_p)_1(x)|$, so that $A_{10} \sim (\frac{4}{3})^p p^{20}$, while in A_{11}, both of the above mechanisms superpose and yield $\|A_{11}\| < \mathcal{O}(3^{-p} p^{20})$.

The estimates of this section are not expressed in the norm $\|\ \|_p$, but rather in the sup (on $\mathbb{R}$) and the norm $|\ |_p$. This makes the natural orders of magnitude of the matrix elements more transparent. But in Sect. 2, we have defined $\|h\|_p = |h_0| 3^p + |h_1|_p$, and the factor 3^p "balances" the matrix and makes the neighborhood $\mathfrak{B}_p$ shorter in the E_p direction (which is very expanding) relative to the (contracting) direction $\mathbb{H}'_p$.

A straightforward calculation shows that $A_f = D\mathcal{N}_p(f)$, for $f \in \mathfrak{B}_p$, maps $\mathbb{H}_p$ to $\mathbb{H}_p$ and is given, for $h \in \mathbb{H}_p$, by

$$(A_f h)(x) = \frac{1}{\lambda(f)} \sum_{j=0}^{p-1} w_j(x) - w_j(0)(\mathcal{N}_p f(x) - x(\mathcal{N}_p f)'(x)), \tag{5.1}$$

where

$$w_j(x) = (f^{p-j-1})'(f^{j+1}(\lambda x)) h(f^j(\lambda x)). \tag{5.2}$$

Proposition 5.1. *If $f \in \mathfrak{B}_p$, then*

$$A_{00} \equiv (A_f E_p)(1) = K_{00} 16^{p-2}(1 + \mathcal{O}(3^{-p})),$$

where $K_{00} = \frac{32}{3} \prod_{j=1}^{\infty} \cos^2(\pi 2^{-(j+1)})$.

Proof. Using (5.1), (5.2), (4.24), and (4.25), we see that

$$\begin{aligned} A_{00} &= \sum_{j=0}^{p-1} u_j(1) - u_j(0) \cdot ((\mathcal{N}_p f)(1) - (\mathcal{N}_p f)'(1)) \\ &= \sum_{j=0}^{p-1} (u_j(1) - u_j(0)) + u_j(0) \Delta_f(1), \end{aligned} \tag{5.3}$$

with Δ_f as defined in (4.18). Note that $u_0(0) = 0$. We analyze now in detail $u_j(0)$, for $j = 1, 2, \ldots, p-1$. From Lemma A.3 and the standard bound $f'(x) = -(4 + \mathcal{O}(4^{-p}))x$, we derive, using the chain rule and $E_p(1) = 1$,

$$u_j(z) = \frac{-K_\pi}{K_\lambda 4^{2-p}} 4^{p-j-1}(1 + \mathcal{O}(3^{j-p})). \tag{5.4}$$

(To be honest, it has to be said that we have used quite a number of other lemmas here, namely 3.4, 4.2, and 4.5.) Summing, we get

$$\begin{aligned}\sum_{j=0}^{p-1} u_j(0) &= -\frac{K_\pi}{K_\lambda} 4^{2p-4} \sum_{j=1}^{p-1} 4^{-(j-1)}(1+\mathcal{O}(3^{j-p}))\\ &= -\frac{K_\pi}{K_\lambda} 4^{2p-4}\ \tfrac{4}{3}(1+\mathcal{O}(3^{-p}))\\ &= -K_\pi^2 4^{2p-2}\ \tfrac{1}{3}(1+\mathcal{O}(3^{-p}))\,.\end{aligned}$$

Finally, by Lemma 4.7, we have $\Delta_f(1) = -2(1+\mathcal{O}(4^{-p}))$, so that

$$\Delta_f(1) \sum_{j=1}^{p-1} u_j(0) = K_\pi^2\ \tfrac{32}{3} 16^{p-2}(1+\mathcal{O}(3^{-p}))\,. \tag{5.5}$$

It is clear that the proposition follows if we manage to show that

$$\sum_{j=0}^{p-1} u_j(1) - u_j(0)$$

is negligible with respect to (5.5), cf. (5.3). However, this is immediate from Corollary 4.9. Thus, the proof of Proposition 5.1 is complete.

We next estimate the quantity

$$A_{10}(y) = (A_f E_p)(y) - E_p(y)(A_f E_p)(1)\,.$$

Proposition 5.2. *If $f \in \mathfrak{B}_p$, then*

$$\sup_{|y| \leqq p} |A_{10}(y)| \leqq \mathcal{O}((\tfrac{4}{3})^p p^{19})\,.$$

Proof. We write A_{10} as

$$A_{10}(y) = \sum_{j=0}^{p-1} X_j\,,$$

where

$$\begin{aligned}X_j &= u_j(y) - E_p(y) u_j(1)\\ &\quad - u_j(0) \cdot \{\mathcal{N}_p f(y) - (\mathcal{N}_p f)'(y) y - E_p(y)(\mathcal{N}_p f(1) - (\mathcal{N}_p f)'(1))\}\\ &= u_j(y) - u_j(0) - E_p(y)(u_j(1) - u_j(0))\\ &\quad + u_j(0) \cdot \{\Delta_f(y) - E_p(y)\Delta_f(1)\}\,.\end{aligned}$$

These terms are bounded, respectively, in Lemmas 4.8 and 4.7, and we get

$$\begin{aligned}X_j &= \mathcal{O}(4^{-j} p^6) + \mathcal{O}\left(\frac{1}{\lambda} 4^{p-j}\right) \cdot \mathcal{O}(12^{-p} p^{19})\\ &= \mathcal{O}(4^{-j} (\tfrac{4}{3})^p p^{19})\,,\end{aligned}$$

from which the assertion follows.

We now consider functions $h \in \mathbb{H}_p'$, i.e. even functions vanishing at 0, 1 (and -1). We now have the following bounds, with $|h|_p = \sup_{|x| \leqq p} |h(x)|$:

Proposition 5.3. *If $f \in \mathfrak{B}_p$ and $h \in \mathbb{H}'_p$, then*

$$|A_f h(1)| \leqq |h|_p \mathcal{O}(4^p p).$$

Proof. Recall the definition of $\hat{u}_j$, Eq. (4.26). We have

$$A_f h(1) = \sum_{j=1}^{p-1} \hat{u}_j(1) - \hat{u}_j(0) + \sum_{j=1}^{p-1} \hat{u}_j(0) \Delta_f(1).$$

By Corollary 4.9ii), iii), and Lemma 4.7i), we get

$$\hat{u}_j(x) \sim (\mathcal{O}(p^6) + \mathcal{O}(4^p))|h|_p,$$

from which the assertion follows at once.

Finally, we estimate the "matrix element" $(A_{11}h)(x) = (A_f h)(x) - E_p(x)(A_f h)(1)$.

Proposition 5.4. *If $f \in \mathfrak{B}_p$ and $h \in \mathbb{H}'_p$, then*

$$\sup_{|x| \leqq p} |A_{11}h(x)| = \mathcal{O}(3^{-p} p^{20})|h|_p.$$

Proof. We again write a decomposition

$$(A_{11}h)(x) = \sum_{j=1}^{p-1} X_j,$$

where

$$\begin{aligned} X_j(x) = \hat{u}_j(x) - E_p(x)\hat{u}_j(1) - \hat{u}_j(0) \cdot \{1 - E_p(x)\} \\ + \hat{u}_j(0) \cdot \{\Delta_f(x) - E_p(x)\Delta_f(1)\}. \end{aligned}$$

The first line is bounded by Lemma 4.8 and the second by Lemmas 4.7 and 4.9, and we get

$$\begin{aligned} X_j(x) &= \mathcal{O}(4^{-p} p^8)|h|_p + \mathcal{O}(4^p)|h|_p \cdot \mathcal{O}(12^{-p} p^{19}) \\ &= \mathcal{O}(3^{-p} p^{19})|h|_p. \end{aligned}$$

The assertion follows.

Appendix. Some Estimates

Lemma A.1. $\prod_{j=1}^{\infty} \cos(\tau\pi/2^j) = \sin(\tau\pi)/(\tau\pi)$.

Proof (F. Leyvraz). The product

$$P_N = \prod_{j=1}^{N} \cos(\tau\pi/2^j)$$

is equal to

$$\frac{1}{2^N} \sum_{\sigma_1, \ldots, \sigma_N = \pm 1} \exp\left(i\tau\pi \sum_{j=1}^{N} \sigma_j 2^{-j} \right).$$

As N goes to infinity, P_N tends to

$$\frac{1}{2}\int_{-1}^{1} e^{i\tau\pi x}dx = \frac{1}{2}\frac{1}{\tau\pi i}(e^{i\tau\pi} - e^{-i\tau\pi}) = \frac{1}{\tau\pi}\sin(\tau\pi).$$

Lemma A.1′. $\sum_{s=1}^{\infty} 1 \Big/ \left\{\cos(\pi/2^{s+1})4^s \prod_{j=1}^{s} \cos(\pi/2^{j+1})\right\} = 2/\pi.$

Proof. We rewrite the left-hand side as

$$\sum_{s=1}^{\infty} \frac{1}{\cos(\pi/2^{s+1})} 4^{-s} \frac{\prod_{j=s+1}^{\infty} \cos(\pi/2^{j+1})}{\prod_{j=1}^{\infty} \cos(\pi/2^{j+1})},$$

which, using Lemma A.1 twice, is seen to be equal to

$$\begin{aligned}
&\sum_{s=1}^{\infty} \frac{1}{\cos(\pi/2^{s+1})} 4^{-s} \frac{\sin(\pi/2^{s+1})}{\frac{2}{\pi}\frac{\pi}{2^{s+1}}} \\
&= \sum_{s=1}^{\infty} \tan(\pi/2^{s+1})\cdot 2^{-s} \\
&= -\partial_\tau \log \prod_{s=1}^{\infty} \cos(\tau\pi/2^s)\cdot\frac{1}{\pi}\Big|_{\tau=1/2} \\
&= -\frac{1}{\pi}\partial_\tau \log(\sin(\tau\pi)/(\tau\pi))|_{\tau=1/2} = 2/\pi .
\end{aligned}$$

In the next lemma, we assume that $u'(x)x<0$, $v'(x)x<0$, when $x \neq 0$, $u(0)=v(0)=1$, $u(1)$, $v(1)<-1/2$, $u'(0)=v'(0)=0$, and $\sup_{|x|\leqq 1}|v'(x)| \leqq \mathcal{O}(1)$. The more relevant assumptions are added below.

Lemma A.2. *In addition to the above, assume, for* $|x|\leqq p$, $Gu(x)-Gv(x)=\varepsilon(x)\cdot x$, $\varepsilon \equiv \sup_{|x|\leqq p}|\varepsilon(x)|<2^{-p}$, *and* $u(1)=v(1)$. *Then*

i) $u''(0)=v''(0)(1+\varepsilon)$,

ii) $u'(x)=v'(x)(1+\varepsilon')$, $|x|\leqq p$, $|\varepsilon'|\leqq \mathcal{O}(\varepsilon p^2)$.

Proof. Recall the definition of G,

$$Gu(x)=u''(x)/u'(x)-1/x .$$

Since $u'(x)x<0$ for $x \neq 0$,

$$\frac{u'(x)}{xu''(0)} = \exp\int_0^x (Gu)(z)dz .$$

Hence,

$$\frac{u'(x)}{v'(x)} = \frac{u''(0)}{v''(0)}\exp\mathcal{O}(\varepsilon x^2), \tag{A.1}$$

where $\varepsilon = \sup|\varepsilon(x)|$. Integrating and since $u(0)=v(0)=1$, we get, for $|x|\leqq 1$,

$$u(x)-1=\frac{u''(0)}{v''(0)}(v(x)-1)+\frac{u''(0)}{v''(0)}\sup_{|y|\leqq 1}|v'(y)|(e^{\varepsilon}-1).$$

Since $u(1)=v(1)<-1/2$, we find that

$$\frac{u''(0)}{v''(0)}=1/(1+\mathcal{O}(1)\varepsilon/|u(1)-1|)=1+C_1\varepsilon.$$

This is i), and ii) follows now from (A.1).

Lemma A.3. *Assume* $f\in \mathbb{H}_p+1$ *and*

$$|f(1)-f_p(1)|\leqq\varepsilon\leqq 48^{-p}p^9, \tag{A.2}$$

and

$$\sup_{|y|\leqq p}|f(y)-f_p(y)|\leqq\varepsilon'\leqq 2\cdot 16^{-p}p^9. \tag{A.3}$$

Define $x_k=f_p^k(0)$, $y_k=f^k(z)$, $|z|<4^{-p}p^2$. *Then*

$$|x_k-y_k|<C_1\varepsilon 4^k+C_2\varepsilon' k4^{k-p}+C_3|z|^2 4^k, \qquad k\leqq p.$$

Proof. To simplify notation, set $g=f_p$. We first estimate $f^k(0)-g^k(0)$. Note that $f(0)=g(0)$ and, by assumption, $|f^2(0)-g^2(0)|\leqq\varepsilon$. We shall recursively show

$$|f^k(0)|<1+3^{-p} \tag*{$(\mathrm{A.4})_k$}$$

and

$$|f^{k+1}(0)-g^{k+1}(0)|<C_4\varepsilon\frac{(4^+)^k-1}{4^+-1}+C_5\varepsilon'(k-2)(4^+)^{k-p}, \tag*{$(\mathrm{A.5})_k$}$$

where $4^+=4(1+3\cdot 3^{-p})$. These statements are obvious for $k=1$. By the bound Eq. (3.3), we deduce $(\mathrm{A.4})_{k+1}$ from $(\mathrm{A.5})_k$. To deduce $(\mathrm{A.5})_k$ from $(\mathrm{A.4})_k$, we write

$$\begin{aligned} f^{k+1}(0)-g^{k+1}(0)=&f(f^k(0))-f(g^k(0))\\ &+\{(f-g)(g^k(0))+(f-g)(-1)\}\\ &+(f-g)(-1). \end{aligned}$$

We get, using contour integration and $(\mathrm{A.4})_k$ to bound $(f-g)'$ and Eq. (3.3) to bound $|g^k(0)|-1$,

$$\begin{aligned} |f^{k+1}(0)-g^{k+1}(0)|\leqq& 4(1+3^{-p})|f^k(0)-g^k(0)|\\ &+C_6\varepsilon'(4^+)^{k-p}+\varepsilon. \end{aligned}$$

This proves $(\mathrm{A.5})_{k+1}$.

Next we compare $f^{k+1}(z)$ to $f^{k+1}(0)$. We have

$$|f(z)-f(0)|\leqq|z|^2(2+\mathcal{O}(4^{-p})), \tag{A.6}$$

since contour integration and (A.3) imply $|f''(z)-g''(z)|\leqq\mathcal{O}(\varepsilon')$, and we already know $g''(0)=-4+\mathcal{O}(4^{-p})$, by construction. We shall show recursively that

$$|f^k(z)|<1+2\cdot 3^{-p} \tag*{$(\mathrm{A.7})_k$}$$

and

$$|f^{k+1}(z)-f^{k+1}(0)| \leqq 4^{+} |f^{k}(z)-f^{k}(0)| . \qquad (A.8)_k$$

The case $(A.8)_0$ is obvious from (A.6). Also $(A.8)_k$ implies $(A.7)_{k+1}$ as before. Finally, $(A.7)_{k+1}$ implies $(A.8)_{k+1}$ by estimating the difference as

$$|f^{k+1}(z)-f^{k+1}(0)| \leqq \sup_{|y| \leqq 1+2\cdot 3^{-p}} |f'(y)| \, |f^{k}(z)-f^{k}(0)| .$$

The lemma is proved.

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