Commun. Math. Phys. 93, 367–378 (1984)

Communications in
Mathematical
Physics

Integrality of the Monopole Number in SU(2) Yang-Mills-Higgs Theory on $\mathbb{R}^3$ ⋆

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Abstract. We prove that in classical SU(2) Yang-Mills-Higgs theories on $\mathbb{R}^3$ with a Higgs field in the adjoint representation, an integer-valued monopole number (magnetic charge) is canonically defined for any finite-action $L^2_{1,\mathrm{loc}}$ configuration. In particular the result is true for smooth configurations. The monopole number is shown to decompose the configuration space into path components.

Introduction

In classical Yang-Mills-Higgs theories over a Riemannian manifold M, one fixes a principal bundle

$$\begin{array}{c} P \\ \downarrow G \\ M \end{array}$$

and studies the *action functional*

$$\mathscr{a}(A, \Phi) = \tfrac{1}{2} \int_M (|F_A|^2 + |d_A\Phi|^2). \tag{1}$$

This functional is defined on the *configuration space*

$$\mathscr{C} = \{(A, \Phi) \in \mathscr{A} \times \mathscr{E} \,|\, \mathscr{a}(A, \Phi) < \infty\}, \tag{2}$$

where $\mathscr{A}$ is the space of $L^2_{1,\mathrm{loc}}$ connections on P, and $\mathscr{E}$ is the space of $L^2_{1,\mathrm{loc}}$ sections of some (fixed) associated vector bundle E. (This is the most general configuration space. Often one considers only *smooth* connections A and sections Φ, but for many applications this restriction is inconvenient. If a potential term is included in the action, the configuration space again is smaller.) The symbols F_A, $d_A\Phi$ denote the curvature of $A \in \mathscr{A}$ and the covariant derivative of $\Phi \in \mathscr{E}$, respectively; norms

⋆ Research supported in part by NSF Grants 8120790 and PHY-03669

are defined by fixing a metric on M, an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$, and a metric on E. Multiplication theorems for $L^2_{1,\text{loc}}$ (see [1, Chap. 9]) imply that F_A and $d_A\Phi$, defined by their usual coordinate expressions, are locally square integrable. Finite action ensures global square integrability.

When $M=\mathbb{R}^3$ and E is the adjoint bundle, one considers the flux expression

$$N_1(A,\Phi)=\lim_{R\to\infty}\frac{1}{4\pi}\int_{|x|=R}|\Phi|^{-1}(\Phi,F_A). \tag{3}$$

If $G=\mathrm{SU}(2)$, a sufficient condition that the "magnetic charge" or "monopole number" $N_1(A,\Phi)$ be well-defined by the formula (3), and be integer-valued, is that $(A,\Phi)\in\mathscr{C}$ be a critical point of $\mathscr{a}$ (and that Φ not vanish identically). It has been conjectured that finite action alone is sufficient to define $N_1(A,\Phi)\in\mathbb{Z}$ in a natural way.

This conjecture is true.

Theorem. *Let $P\to\mathbb{R}^3$ be a principal* SU(2)*-bundle and let E be the bundle of Lie algebras associated to P by the adjoint representation. Let $\mathbb{R}^3$ be given the standard metric and let a metric on E be induced by minus one-half the Killing form of $\mathfrak{su}(2)$. Let $\mathscr{C}$ be the space of finite-action $L^2_{1,\text{loc}}$ configurations given by* (1), (2). *If $(A,\Phi)\in\mathscr{C}$, there is a unique number M such that $M-|\Phi|\in L^6(\mathbb{R}^3)$, and if $M\neq 0$ then*

$$N(A,\Phi)=\frac{1}{4\pi M}\int d_A\Phi\wedge F_A$$

is an integer. This integer agrees with (3) *(*i.e. $N=N_1$*) if*

$$\lim_{R\to\infty}\sup_{|x|\geq R}|M-|\Phi||=0, \tag{4}$$

provided A and Φ are continuously differentiable. There is a natural topology on $\mathscr{C}$ (see Definition 2) such that N is continuous on $\dot{\mathscr{C}}=\{(A,\Phi)\in\mathscr{C}\,|\,M\neq 0\}$, and $N:\mathscr{C}\to\mathbb{Z}$ decomposes $\dot{\mathscr{C}}$ into path components.

Remarks. 1. By "$\wedge$" in "$d_A\Phi\wedge F_A$", we mean inner product on E and wedge product on forms. Similarly, the inner product (Φ,F_A) in (3) is inner product only on E.

2. The set $B=\mathscr{C}-\dot{\mathscr{C}}$ is that subset of $\mathscr{C}$ for which $|\Phi|\in L^6(\mathbb{R}^3)$ (equivalently, for which $M=0$). Configurations in B are in any case never of physical interest; the action (1) has no critical points in B except $(A\equiv 0,\Phi\equiv 0)$ and its gauge translates. Any symmetry-breaking potential added to the action density – e.g. $(1-|\Phi|^2)^2$ – leads to a configuration space $\mathscr{C}'\subset\mathscr{C}$ disjoint from B.

3. Our proof will show that, when (4) holds with $M\neq 0$, $N(A,\Phi)$ is given by (3) not only if $(A,\Phi)\in C^1$, but more generally if $A\in L^2_{1,\text{loc}}$ and $\Phi\in L^p_{1,\text{loc}}$ for some $p>3$. In the general case one must take some care interpreting the integral in (3), which is defined only for almost all large R. We deal with this problem and state the stronger version of the theorem in Sect. 2.

4. Uhlenbeck [2] has proven an integrality theorem for pure Yang-Mills fields in even dimensions ≥ 4, valid for any compact structure group G: if $A\in L^n_{1,\text{loc}}(\mathbb{R}^{2n})$ and $F\in L^n(\mathbb{R}^{2n})$, then the Chern number of any vector bundle

associated by a special unitary representation is an integer. The question in [2] is essentially whether a connection on a bundle over $\mathbb{R}^{2n}$ *extends* to the compactification S^{2n}. By contrast, here we essentially study how a connection on $\mathbb{R}^3$ *restricts* to ever-larger two-spheres, and the methods we employ are quite different from Uhlenbeck's.

The theorem above is more or less a corollary of the following proposition, proven in Sect. 2.

Proposition. *Let* $(A, \Phi) \in \mathscr{C}$ *be smooth and suppose that*

$$\lim_{R\to\infty} \sup_{|x|\geqq R} |1-|\Phi|| = 0 .$$

Then $N_1(A, \Phi)$, *as given by* (3), *is well-defined, integer-valued, and*

$$N_1(A,\Phi) = \frac{1}{4\pi}\int_{\mathbb{R}^3} d_A\Phi \wedge F_A = \lim_{R\to\infty}\frac{1}{4\pi}\int_{|x|=R}(\Phi, F_A).$$

The theorem is proven in two steps. The first step, the subject of Sect. 1, shows that it suffices to consider continuous Φ (more precisely, $\Phi \in L^p_{1,\mathrm{loc}}$ for some $p>3$) which satisfy an asymptotic condition of the form (4). The second step (Sect. 2) shows that for such Φ the monopole number – which we redefine for convenience as a volume integral rather than the surface integral (3) – is an integer. (Whenever the limit in (3) exists, the two definitions are equivalent.)

1. Reducing to the Case in which $|\Phi|$ tends Uniformly to a Constant

Several arguments below make use of a "cutoff function" which we now define.

Definition 1. Fix a smooth function $\beta: \mathbb{R}^3 \to \mathbb{R}$ such that $0 \leqq \beta \leqq 1$, $\beta(x) \equiv 0$ for $|x| \geqq 1$, and $\beta(x) \equiv 1$ for $|x| \leqq 1/2$. For $R>0$ define $\beta_R(x) = \beta(x/R)$.

Notation. Henceforth, for $\Phi \in \mathscr{E}$, let $\hat{\Phi} = \Phi/|\Phi|$ wherever $\Phi \neq 0$. Also, let d^*, d_A^* denote the formal L^2 adjoints of the exterior derivatives d, d_A respectively; thus d^*d and $d_A^* d_A$ are the nonnegative Laplacians. The L^2 inner product will be denoted by $\langle\,,\rangle$. The operator $\nabla_A : \Gamma(E\otimes(T^*\mathbb{R}^3)^p) \to \Gamma(E\otimes(T^*\mathbb{R}^3)^{p+1})$, the *full* covariant derivative, is the tensor product connection determined by d_A and the Levi-Civita connection ∇ on $\mathbb{R}^3$.

The following fact is well-known (cf. [3, Lemma 4.12]).

Lemma 1. *Let V be the space of real valued functions $f \in L^2_{1,\mathrm{loc}}(\mathbb{R}^3)$ for which $df \in L^2(\mathbb{R}^3)$. Let H be the completion of $C_0^\infty(\mathbb{R}^3)$ (the compactly supported smooth functions) in the norm $\|f\|_H = \|df\|_2$. Then for each $f \in V$ there is a unique number $M(f) \in \mathbb{R}$ such that $f - M(f) \in H$. In other words, there is a canonical isomorphism $V \equiv H \oplus \mathbb{R}$.*

Proof. Let $f \in V$ and define $Q: H \to \mathbb{R}$ by $Q(g) = \langle d(f-g), d(f-g)\rangle$. The functional Q is weakly lower semicontinuous on H and coercive: $Q(g) \geqq \frac{1}{4}\|g\|_H^2 - \|f\|_H^2$. Hence it achieves a minimum (see [4, Chap. 6], e.g.), say at g, which is unique since Q is strictly convex. Let $h = f - g$; then $d^*dh = 0$ weakly. Note that $H \hookrightarrow L^2_{1,\mathrm{loc}}(\mathbb{R}^3)$, since $H \hookrightarrow L^6$ by a Sobolev inequality and $L^6 \hookrightarrow L^6_{\mathrm{loc}} \hookrightarrow L^2_{\mathrm{loc}}$. Hence $h \in L^2_{1,\mathrm{loc}}$. But any weakly harmonic $L^2_{1,\mathrm{loc}}$ function on $\mathbb{R}^n$ is C^∞ and (strongly) harmonic (see [5, Chap. 8]), and so are all its derivatives. Let $h_i = \partial h/\partial x^i$, $i=1,2,3$. Then for each i and any $R>0$,

$$0 = \langle \beta_R h_i, d^* dh_i \rangle = \langle \beta_R dh_i, dh_i \rangle + \tfrac{1}{2}\langle d^* d\beta_R, h_i^2 \rangle .$$

As $R \to \infty$, the first term approaches $\|dh_i\|_2^2$ while the second, bounded by

$$\mathrm{const} \cdot R^{-2} \|h_i\|_2^2 \leqq \mathrm{const} \cdot R^{-2} \|d(f-g)\|_2^2 < \infty ,$$

approaches zero. Hence $\|dh_i\|_2 = 0$ and h_i is constant. But $h_i = \partial_i(f-g) \in L^2$, so $h_i \equiv 0$ for each i. Therefore $f-g=h=\mathrm{const}=M(f)$, and $f-M(f)\in H$. Uniqueness is clear. □

Kato's inequality $\|d|\Phi|\,\|_2 \leqq \|d_A\Phi\|_2$ (see [6], proof of Lemma 2.1a) implies that $M(|\Phi|)$ is defined by Lemma 1 whenever $d_A\Phi \in L^2$. This enables the following definitions.

Definition 2. Let $\mathscr{C} = \{(A,\Phi) \in L^2_{1,\mathrm{loc}} | \mathscr{a}(A,\Phi) < \infty\}$. Endow $\mathscr{C}$ with the weakest topology that renders continuous the functions

$$F_A : \mathscr{C} \to L^2 ,$$
$$d_A\Phi : \mathscr{C} \to L^2 ,$$
$$M(|\Phi|) : \mathscr{C} \to \mathbb{R} .$$

Define $B = \{(A,\Phi) \in \mathscr{C} | M(|\Phi|) = 0\}$ and let $\dot{\mathscr{C}} = \mathscr{C} - B$.

Remark. An alternate description of the topology above is as follows. Let V be as in Lemma 1, with the topology induced by the isomorphism $V \cong H \oplus \mathbb{R}$, and give $\Omega^k(\mathrm{Ad}P)$ the L^2 topology for $k=1,2$. Define $\iota : \mathscr{C} \to \Omega^2(\mathrm{Ad}P) \times \Omega^1(\mathrm{Ad}P) \times V$ by $\iota(A,\Phi) = (F_A, d_A\Phi, |\Phi|)$. The topology on $\mathscr{C}$ is the one induced by the inclusion ι.

Definition 3. Define $N' : \mathscr{C} \to \mathbb{R}$ by

$$N'(A,\Phi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d_A\Phi \wedge F_A .$$

Define the *monopole number* $N : \mathscr{C} \to \mathbb{R}$ by

$$N(A,\Phi) = \begin{cases} M(|\Phi|)^{-1} N'(A,\Phi) , & (A,\Phi) \in \dot{\mathscr{C}} \\ N'(A,\Phi) , & (A,\Phi) \in B . \end{cases}$$

(We will soon see that in fact $N' \equiv 0$ on B.)

Remark. The path components of $\mathscr{C}$ are, in fact, labeled precisely by the values of N. We defer the proof of this fact to the appendix, since it uses ideas developed in Sect. 2.

Lemma 2. *For $A \in \mathscr{A}$ let H_A be the completion of the space of compactly supported sections of E in the norm $\|\varphi\|_{H_A} = \|d_A\varphi\|_2$.*

(a) *If (A,Φ_1) and (A,Φ_2) are in $\mathscr{C}$ and $\Phi_2 - \Phi_1 \in H_A$, then $N'(A,\Phi_1) = N'(A,\Phi_2)$.*

(b) *Given $(A,\Phi) \in \mathscr{C}$ there exists a unique Φ' such that $\Phi' - \Phi \in H_A$ and $d_A^* d_A \Phi' = 0$ (weakly).*

Proof. (a) Let $\varphi=\Phi_1-\Phi_2$. Since $d_A\varphi$ and F_A are in L^2,

$$|\int d_A\varphi\wedge F_A|=\left|\lim_{R\to\infty}\int\beta_R d_A\varphi\wedge F_A\right|=\lim_{R\to\infty}|\int(d\beta_R)\varphi F_A|,$$

by Stokes' theorem and the Bianchi identity ($d_AF_A=0$). Since $d\beta_R\equiv 0$ for $|x|\leqq R/2$, the last integral is bounded by

$$\|d\beta_R\|_3\|F_A\|_2\left(\int_{|x|\geqq R/2}|\varphi|^6\right)^{1/6}$$

(Hölder's inequality). Using Kato's and Sobolev's inequalities, $\|\varphi\|_6\leqq\text{const}\cdot\|d|\varphi|\|_2\leqq\text{const}\cdot\|d_A\varphi\|_2<\infty$, so $\varphi\in L^6$. Since $\|d\beta_R\|_3$ is independent of R, the expression above tends to zero as $R\to\infty$.

(b) The functional $Q(\varphi)=\mathscr{a}(A,\Phi+\varphi)$ is strictly convex, weakly lower semi-continuous, and coercive on H_A. Hence it achieves a unique absolute minimum, say at φ. Then $\Phi'=\Phi+\varphi$ solves $d_A^*d_A\Phi'=0$ weakly. □

Lemma 3. *Suppose* $(A,\Phi)\in L^2_{1,\text{loc}}$, $d_A\Phi\in L^2$, *and* $d_A^*d_A\Phi=0$ *weakly. Then* Φ *is continuous (in fact* $\Phi\in L^p_{1,\text{loc}}$ *and* $d_A\Phi\in L^p(\mathbb{R}^3)$ *for* $2\leqq p\leqq 6$*) and*

$$\lim_{R\to\infty}\sup_{|x|\geqq R}|M-|\Phi||=0,$$

where $M=M(|\Phi|)$ *is given by Lemma 1.*

Proof. The local integrability conditions on (A,Φ) together with $d_A^*d_A\Phi=0$ imply that $\Phi\in L^p_{1,\text{loc}}$ for $2\leqq p\leqq 6$ (see [7, Theorem 5.5.3]). By the Sobolev embedding theorem Φ is continuous (in fact $C^{1/2}$). Hence

$$|\Phi||d_A\hat{\Phi}|^2=|\Phi|^{-1}(|d_A\Phi|^2-(\hat{\Phi},d_A\Phi)^2)$$

is a well-defined distribution. By Lemma 1, $g=M(|\Phi|)-|\Phi|\in H\hookrightarrow L^2_{1,\text{loc}}$, and we compute $d^*dg=|\Phi||d_A\hat{\Phi}|^2\geqq 0$ (weakly). By the weak maximum principle (see [4, Proposition VI.3.2]) $g\geqq 0$ everywhere, so $0\leqq|\Phi|\leqq M(|\Phi|)$ and $\Phi\in L^\infty$. The inequality

$$\|\nabla_A\nabla_A\Phi\|_2\leqq\text{const}(\|d_A^*d_A\Phi\|_2+\|\Phi\|_\infty\|F_A\|_2+\|d_A\Phi\|_2\|F_A\|_2^2),$$

which is formally a result of integration by parts and a Sobolev inequality (for a proof see [6, Appendix B]), therefore implies $\nabla_A\nabla_A\Phi\in L^2$. Since $\nabla_A\Phi\equiv d_A\Phi\in L^2$, Sobolev's and Kato's inequalities imply $\|g\|_6\leqq\|\Phi\|_\infty\|d_A\Phi\|_2<\infty$ and

$$\|dg\|_6\leqq\|\Phi\|_\infty\|d_A\Phi\|_6\leqq\text{const}\cdot\|\Phi\|_\infty(\|d_A\Phi\|_2+\|\nabla_A\nabla_A\Phi\|_2)<\infty.$$

Therefore $g\in L^6_1$. But functions in $L^p_1(\mathbb{R}^n)$ for $p>n$ have uniform decay (see [4, Proposition III.7.5]), so $\lim_{R\to\infty}\sup_{|x|\geqq R}|g|=0$. □

The proof above also yields the following corollary.

Corollary. $N'\equiv 0$ *on* B.

Proof. Let $(A,\Phi)\in B$ and let Φ' be given by Lemma 2b. By Lemma 2a, $N'(A,\Phi')=N'(A,\Phi)$. The weak maximum principle implies $|\Phi'|\leqq M(|\Phi'|)$, and since $M(|\Phi'|)=0$ we conclude $\Phi'\equiv 0$. Hence $N'(A,\Phi')=0$. □

Lemmas 2 and 3 imply that to prove integrality of N we need only consider configurations for which $|\Phi|$ is continuous and approaches 1 uniformly at spatial infinity.

2. Integrality of $N(A, \Phi)$ when $|\Phi|$ tends Uniformly to a Constant

To stress the main ideas, we first prove that N is an integer for *smooth* (A, Φ) satisfying the asymptotic condition (4) with $M=1$. Later we relax the smoothness constraint.

Proposition. *Let* $(A, \Phi) \in \mathscr{C}$ *be smooth and suppose that*

$$\lim_{R\to\infty} \sup_{|x|\geqq R} |1-|\Phi|| = 0 .$$

Then $N(A, \Phi) \in \mathbb{Z}$, *and*

$$\lim_{R\to\infty} \frac{1}{4\pi} \int_{|x|=R} (\hat{\Phi}, F_A) = \lim_{R\to\infty} \frac{1}{4\pi} \int_{|x|=R} (\Phi, F_A) = N(A, \Phi) .$$

Moreover, restricting Φ *to large two-spheres determines a homotopy class of maps from* S^2 *to* S^2, *and* $N(A, \Phi)$ *is the Brouwer degree of this class.*

Proof. Let R_0 be large enough that $|\Phi(x)| \geqq \frac{1}{2}$ if $|x| \geqq R_0$. On the set $U = \{|x| \geqq R_0\}$ there is an orthogonal splitting $E = E^L \oplus E^T$, where the "longitudinal" line bundle E^L is the stabilizer of Φ and the "transverse" 2-plane bundle E^T is the orthogonal complement of E^L. On U let $\hat{\Phi} = \Phi/|\Phi|$, and let $\hat{\Phi}_R = \hat{\Phi}|_{|x|=R}$. The endomorphism

$$J = \mathrm{ad}\,\hat{\Phi} = [\hat{\Phi}, \cdot\,] ,$$

restricted to

$$E_R^T = E^T|_{|x|=R} ,$$

where $R \geqq R_0$, satisfies $J^2 = -1$. Therefore J defines a complex line bundle

$$L_R \subset E_R^T \bigotimes_{\mathbb{R}} \mathbb{C} ,$$

for each $R \geqq R_0$, on which J acts as multiplication by i. Fix a global trivialization of P momentarily so that Φ may be regarded as a map from $\mathbb{R}^3$ to $\mathfrak{su}(2)$. Topologically, $L_R \cong \hat{\Phi} = {}^*K$, where K is the canonical bundle of $\mathbb{C}P^1 \cong$(unit sphere in g). Thus the Chern classes are related by

$$c_1(L_R) = \deg(\hat{\Phi}_R) c_1(K) = -2\deg(\hat{\Phi}_R) , \tag{5}$$

where the orientations are redefined, if necessary, to make the sign in (5) correct. The value of $\deg(\hat{\Phi}_R)$ is independent of $R \geqq R_0$ and of the choice of trivialization [since $\pi_2(\mathrm{SU}(2)) = 0$].

By orthogonal projection the connection A on $E|_U$ induces a connection $\tilde{A}$ on $E^T|_U$, defined by setting

$$d_{\tilde{A}} s = (d_A s)^T = d_A s - (d_A s, \hat{\Phi})\hat{\Phi}$$

for $s \in \Gamma(E^T)$. The curvature $\tilde{F}_{\tilde{A}}$, viewed as an End(E_T)-valued two-form, is given by

$$\tilde{F}_{\tilde{A}}(s) = d_{\tilde{A}} d_{\tilde{A}} s = ([F_A, s] - (d_A \hat{\Phi}, s) \wedge d_A \hat{\Phi})^T .$$

Using $[E^T, E^T] \subset E^L$ and $[E^L, E^T] \subset E^T$, one finds that $\tilde{F}_{\tilde{A}}(s) = f_{\tilde{A}}[\hat{\Phi}, s]$, where

$$f_{\tilde{A}} = (\hat{\Phi}, F_A - \tfrac{1}{2}[d_A \hat{\Phi}, d_A \hat{\Phi}]) .$$

Complexifying induces a connection on the line bundle L_R with curvature form

$$\omega = i f_{\tilde{A}} .$$

Since

$$c_1(L_R) = \frac{i}{2\pi} \int_{|x|=R} \omega ,$$

(5) implies that

$$c_1(L_R) = -\frac{1}{2\pi} \int_{|x|=R} (\hat{\Phi}, F_A - \tfrac{1}{2}[d_A \hat{\Phi}, d_A \hat{\Phi}]) . \tag{6}$$

The covering homotopy theorem implies that the left-hand side of (6) is independent of $R \geqq R_0$. We examine the limit as $R \to \infty$ of the terms on the right arising from F_A and $[d_A \hat{\Phi}, d_A \hat{\Phi}]$ separately. For $R \geqq 2R_0$,

$$\begin{aligned} \int_{|x|=R} (\hat{\Phi}, F_A) &= \int_{|x|=R} (1 - \beta_{2R_0})(\hat{\Phi}, F_A) \\ &= -\int_{|x| \leqq R} d\beta_{2R_0}(\hat{\Phi}, F_A) + \int_{|x| \leqq R} (1 - \beta_{2R_0}) F_A \wedge d_A \hat{\Phi} . \end{aligned} \tag{7}$$

The $d\beta$ integral is independent of $R \geqq 2R_0$. Also, for such R,

$$|d_A \Phi|^2 \geqq |\Phi|^2 |d_A \hat{\Phi}|^2 \geqq \tfrac{1}{4} |d_A \hat{\Phi}|^2 ,$$

so $|d_A \hat{\Phi}| \leqq 2 |d_A \Phi|$, and the last integral in (7) is absolutely convergent as $R \to \infty$. Hence

$$\lim_{R \to \infty} \int_{|x|=R} (\hat{\Phi}, F_A)$$

exists, and since the left-hand side of (6) is independent of (large) R we conclude that

$$\lim_{R \to \infty} \int_{|x|=R} (\hat{\Phi}, [d_A \hat{\Phi}, d_A \hat{\Phi}])$$

exists. This limit must be zero for otherwise $\|d_A \Phi\|_2$ would be infinite. Therefore, by (5) and (6),

$$\lim_{R \to \infty} \frac{1}{4\pi} \int_{|x|=R} (\hat{\Phi}, F_A) = \deg(\hat{\Phi}_{R_0}) \in \mathbb{Z} . \tag{8}$$

To complete the proof we must replace $(\hat{\Phi}, F_A)$ in (8) by (Φ, F_A). But

$$\begin{aligned} \left| \int_{|x|=R} (\Phi - \hat{\Phi}, F_A) \right| &= \left| \int_{|x|=R} (1 - \beta_R)(\Phi - \hat{\Phi}, F_A) \right| \\ &\leqq \int_{|x| \leqq R} |d\beta_R| \, |1 - |\Phi|| \, |F_A| \\ &\quad + \int_{|x| \geqq R/2} |d_A \Phi - d_A \hat{\Phi}| \, |F_A| . \end{aligned} \tag{9}$$

For $|x| \geqq R_0$, $|d_A\Phi - d_A\hat{\Phi}| \leqq |d_A\Phi|$, so the last integral above $\to 0$ as $R \to \infty$. The first integral on the right in (9) is

$$\leqq \|d\beta_R\|_3 \|F_A\|_2 \left(\int_{|x| \geqq R/2} (1-|\Phi|)^6 \right)^{1/6}, \tag{10}$$

since $d\beta_R = 0$ for $|x| \leqq R/2$. Arguing as in Lemma 2a, the expression (10) $\to 0$ as $R \to \infty$. Hence $\hat{\Phi}$ may be replaced by Φ in (8) as desired. □

The crux of the argument above is that the right-hand side of (6) is automatically an integer because it is the Chern number. If we only assume $A \in L^2_{1,\mathrm{loc}}$ we cannot make this statement directly because the induced connection form on L_R need not be continuously differentiable. However, the statement is still true in a measure-theoretic sense.

Lemma 4. *Let* $(A, \Phi) \in \mathscr{C}$. *Suppose that* $\Phi \in L^p_{1,\mathrm{loc}}$ *for some* $p > 3$ *and that* $|\Phi| \geqq \frac{1}{2}$ *for* $r \geqq R_0$. (Since $L^p_{1,\mathrm{loc}} \hookrightarrow C^0$, $|\Phi|$ is continuous.) *Then*

$$f(R) = -\frac{1}{2\pi} \int_{|x|=R} (\hat{\Phi}, F_A - \tfrac{1}{2}[d_A\hat{\Phi}, d_A\hat{\Phi}]) \tag{11}$$

is a measurable function of $R \geqq R_0$, *and* $f(R) = c_1(L_R)$ *for almost every* $R \geqq R_0$.

Proof. For $|x| > R_0$, the 1-form $d_A\hat{\Phi} = |\Phi|^{-1}(d_A\Phi - (\hat{\Phi}, d_A\Phi)\hat{\Phi})$ is locally square integrable since $d_A\Phi \in L^2_{\mathrm{loc}}$ and $\Phi \in C^0$. Hence

$$|(\hat{\Phi}, [d_A\hat{\Phi}, d_A\hat{\Phi}])| \in L^1_{\mathrm{loc}}(\mathbb{R}^3),$$

and $|(\hat{\Phi}, F_A)| \in L^2_{\mathrm{loc}} \hookrightarrow L^1_{\mathrm{loc}}$ as well. Fix $R_1 > R_0$ and let J be the annulus $R_0 \leqq |x| \leqq R_1$. If I is the interval $[R_0, R_1]$, then $J \cong I \times S^2$, so Fubini's theorem implies that $f(R)$ is measurable on I and defined almost everywhere.

Let $\{(A_i, \Phi_i)\}$ be a sequence of smooth configurations such that $A_i \to A$ strongly in $L^2_1(J)$ and $\Phi_i \to \Phi$ strongly in $L^p_1(J)$. Since $L^p_{1,\mathrm{loc}} \hookrightarrow C^0$, $\Phi_i \to \Phi$ uniformly, and we may assume without loss of generality that each Φ_i is nonvanishing on J. Hence the bundles $(L_R)_i$ arising from Φ_i are all isomorphic to the L_R arising from Φ. If $f_i(R)$ is the integral (11) with (A, Φ) replaced by (A_i, Φ_i), then $f_i(R)$ equals the Chern number n of the bundle L_R, independent of $R \in I$ and i. Thus for all such R, i,

$$(R - R_0)n = \int_{R_0}^{R} f_i(r)dr = \int_{R_0 \leqq |x| \leqq R} dr \wedge (\hat{\Phi}_i, F_{A_i} - \tfrac{1}{2}[d_{A_i}\hat{\Phi}_i, d_{A_i}\hat{\Phi}_i]). \tag{12}$$

The last integrand converges strongly in $L^1(J)$, so as $i \to \infty$ we may erase all the i's in (12). Hence

$$\int_{R_0}^{R} (f(r) - n)dr = 0$$

for all $R \in I$. It follows that $f(r) - n$ integrates to zero over every measurable subset of I, and hence $f(r) = n$ almost everywhere in I. As R_1 was arbitrary, we are done. □

To further mimic the proof of the proposition, we need one last definition and lemma.

Definition 4. A measurable function $f:[0,\infty)\to\mathbb{R}$ has an *essential limit* at infinity $\left(\text{written ess}\lim_{r\to\infty} f(r)\right)$ if there exists a set $U\subset[0,\infty)$ of measure zero such that $\lim_{n\to\infty} f(r_n)$ exists for every sequence $\{r_n\}\to\infty$ which lies entirely in the complement of U. (An essential limit is necessarily unique.)

Lemma 5. *Let (A,Φ) satisfy the hypothesis of Lemma 4. Then*

$$\operatorname{ess}\lim_{R\to\infty}\int_{|x|=R}(\hat{\Phi},F_A),\qquad \operatorname{ess}\lim_{R\to\infty}\int_{|x|=R}(\Phi,F_A)$$

exist and are equal to $\int_{\mathbb{R}^3} d_A\Phi\wedge F_A$.

Proof. Let R_0, R_1, J, I be as in Lemma 4, but now assume $R_1>2R_0$. Let $(A_i,\Phi_i)\to(A,\Phi)$ just as in Lemma 4. Equation (7) is valid for (A_i,Φ_i), so for $2R_0\leqq R\leqq R_1$,

$$\int_{2R_0}^{R} dr\int_{|x|=r}(\hat{\Phi}_i,F_{A_i})=\int_{2R_0}^{R} dr\left\{-\int_{|x|\leqq r}(\hat{\Phi}_i,F_{A_i})d\beta_{2R_0}+\int_{|x|\leqq r}(1-\beta_{2R_0})F_{A_i}\wedge d_{A_i}\hat{\Phi}_i\right\}.$$

Taking limits as $i\to\infty$ and arguing just as in Lemma 4, we find that

$$g(R)=\int_{|x|=R}(\hat{\Phi},F_A)=-\int_{|x|\leqq R}(\hat{\Phi},F_A)d\beta_{2R_0}+\int_{|x|\leqq R}(1-\beta_{2R_0})F_A\wedge d_A\hat{\Phi}\tag{13}$$

for almost every R. Again the first integral is independent of R and the second has a limit as $R\to\infty$, so $\operatorname{ess}\lim_{R\to\infty} g(R)$ exists. The arguments that this limit is the same with $\hat{\Phi}$ replaced by Φ and that

$$\operatorname{ess}\lim_{R\to\infty}\int(\Phi,F_A)=\int_{\mathbb{R}^3} d_A\Phi\wedge F_A$$

are similar modifications of those given in proving the proposition. We omit the details. □

We may now prove the theorem stated in the introduction. For convenience we rephrase the statement.

Theorem. *Let $\mathscr{C}$ be the space of $L^2_{1,\mathrm{loc}}$ configurations given by Definition 2. Let $\dot{\mathscr{C}}$, B, and $N:\mathscr{C}\to\mathbb{R}$ be given by Definitions 2 and 3. Then N is integer-valued. Moreover N is continuous on $\dot{\mathscr{C}}$, and for $(A,\Phi)\in\dot{\mathscr{C}}$,*

$$\operatorname{ess}\lim_{R\to\infty}(4\pi)^{-1}\int_{|x|=R}(\hat{\Phi},F_A)=\operatorname{ess}\lim_{R\to\infty}(4\pi M)^{-1}\int_{|x|=R}(\Phi,F_A)=N(A,\Phi)\tag{14}$$

exist, where $M=M(|\Phi|)$ is given by Lemma 1. The configuration (A,Φ) determines a homotopy class of maps from S^2 to S^2 and a complex line bundle over S^2, and $N(A,\Phi)$ coincides with the degree of these maps and with the Chern number of the line bundle.

Proof. Let $(A,\Phi)\in\mathscr{C}$. By Lemma 2 there exists Φ' such that $(A,\Phi')\in\mathscr{C}$, $N(A,\Phi)=N(A,\Phi')$, and $d_A^*d_A\Phi'=0$ weakly. By Lemma 3, $\Phi'\in L^p_{1,\mathrm{loc}}$ for some $p>3$, and $|\Phi'|\to M$ uniformly at spatial infinity. If $M=0$, then $N=0$, so assume $M\neq 0$.

Replacing Φ' by Φ'/M, we may assume $M=1$, Lemma 4 then applies, and the function $f(R)$ of (11) is the constant $c_1(L_{R_0})\in\mathbb{Z}$ for almost every $R\geqq R_0$. Together with Lemma 5 this implies that

$$\operatorname{ess}\lim_{R\to\infty}\int_{|x|=R}(\hat{\Phi}',[d_A\hat{\Phi}',d_A\hat{\Phi}])$$

exists. This limit must be zero since $d_A\hat{\Phi}'\in L^2(\{|x|\geqq R_0\})$. Equation (14) then follows from Lemmas 4 and 5. Now Φ' determines the line bundle L_{R_0} and the homotopy class in $\mathrm{Maps}(S^2,S^2)$ discussed in proving the proposition, and (A,Φ) determines Φ'. Hence (A,Φ) itself determines the line bundle and homotopy class.

Continuity of N on $\dot{\mathscr{C}}$ follows immediately from Definitions 2 and 3. (A proof that the $\{N=\text{constant}\}$ subsets of $\dot{\mathscr{C}}$ are connected is given in the appendix.) □

Appendix. The Path Components of $\dot{\mathscr{C}}$

For $k\in\mathbb{Z}$, let $\mathscr{C}_k=\{(A,\Phi)\in\mathscr{C}\mid N(A,\Phi)=k\}$ and let $\dot{\mathscr{C}}_k=\mathscr{C}_k\cap\dot{\mathscr{C}}$. We will show that the path components of $\dot{\mathscr{C}}$ are exactly the $\dot{\mathscr{C}}_k$. Since N is clearly continuous on $\dot{\mathscr{C}}$, it suffices to show that each $\dot{\mathscr{C}}_k$ is path-connected. Essentially, this was shown by Taubes in the appendix of [3], but since he only considered smooth configurations we paraphrase his argument.

Theorem. *$\dot{\mathscr{C}}_k$ is path-connected for each $k\in\mathbb{Z}$.*

Proof. Below, $r=|x|$, $\hat{x}=x/|x|$, $S_r^2\subset\mathbb{R}^3$ is the two-sphere of radius r, and $S_{\mathfrak{g}}^2$ is the unit sphere in $\mathfrak{su}(2)$.

Fix k, and let $e:S_1^2\to S_{\mathfrak{g}}^2$ be any smooth map of degree k. Fix a global trivialization of P (henceforth "reference gauge") and define $\Phi_0\in\Gamma(\mathrm{Ad}P)$ by $\Phi_0(x)=(1-\beta(x))e(\hat{x})$ in this gauge. Define a connection form by $A_0=-[\Phi_0,d\Phi_0]$ in reference gauge. Then $d_{A_0}\Phi_0\equiv 0$ for $r\geqq 1$ and $|F_{A_0}|\leqq\text{const}\cdot r^{-2}$ for $r\geqq 1$. Lemma 1 shows that $M(|\Phi|)=1$, so $(A_0,\Phi_0)\in\dot{\mathscr{C}}$. We will show that any configuration in $\dot{\mathscr{C}}_k$ can be joined to (A_0,Φ_0) by a continuous arc lying in $\dot{\mathscr{C}}$.

Let $(A,\Phi)\in\dot{\mathscr{C}}_k$, let $M=M(|\Phi|)$, and let Φ' be as in Lemma 2b. Since $t\mapsto(A,(1-t)\Phi+t\Phi')$ and $t\mapsto(A,M^{-t}\Phi')$ are continuous: $[0,1]\to\dot{\mathscr{C}}$, there exists a curve $c_1(t)$ in $\dot{\mathscr{C}}$ connecting (A,Φ) to $(A,M^{-1}\Phi')=(A,\Phi_1)$. By Lemma 3, Φ_1 is in $L^6_{1,\mathrm{loc}}\subset C^0$, $|1-|\Phi_1||$ has uniform decay, and $d_A\Phi_1\in L^2\cap L^4$. Fix a positive $\varepsilon<1/4$ and an arbitrary smooth section $\Phi_2^{(1)}$ of $\mathrm{Ad}P$ over the closed unit ball. There exists a smooth extension of $\Phi_2^{(1)}$ to some Φ_2, defined globally, such that for $n=2,3,4,\ldots$,

$$\text{(a)}\qquad |\Phi_2(x)-\Phi_1(x)|\leqq\begin{cases}\varepsilon/2^n, & 2n-2\leqq r\leqq 2n+1\\ \varepsilon/2^{n+1}, & 2n\leqq r\leqq 2n+1\,,\end{cases}$$

and

$$\text{(b)}\qquad \text{for } p=2 \text{ and } p=4,\quad \int_{2n-1\leqq r\leqq 2n+1}|d_A\Phi_2-d_A\Phi_1|^p\leqq\varepsilon/2^n\,.$$

Hence $|\Phi_2-\Phi_1|$ decays uniformly and $d_A\Phi_2\in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$.

For $t\in[0,1]$, $(A,(1-t)\Phi_1+t\Phi_2)=(A,\Phi_{2,t})=c_2(t)\in\mathscr{C}$, and $|\Phi_{2,t}|\to 1$ uniformly as $r\to\infty$. Thus $M(|\Phi_{2,t}|)=1$ (Lemma 1), so $c_2:[0,1]\to\dot{\mathscr{C}}$ is continuous, and it joins (A, Φ_1) to (A, Φ_2). Similarly, let $R \geqq 2$ be such that $r \geqq R/2$ implies $|\Phi_2(x)| \geqq 1/2$; then there is a curve $c_3(t)$ connecting (A, ϕ_2) to $(A, (1-\beta_R)\Phi_2) = (A, \Phi_3)$.

Next, let $A_{4,t} = A - t[\Phi_3, d_A\Phi_3]$, $t \in [0,1]$, and define $c_4(t) = (A_{4,t}, \Phi_3)$. From the formula $F_{A+\eta} = F_A + d_A\eta + \frac{1}{2}[\eta, \eta]$, the fact that $d_A\Phi_3 \in L^4$ (since $d_A\Phi_2 \in L^4$), and the fact that $|\Phi_3|$ is bounded, it follows that $t \mapsto F_{A_{4,t}}$ is continuous: $[0,1] \to L^2$. Hence $c_4(t)$ is a curve in $\dot{\mathscr{C}}$ from (A, Φ_3) to

$$(A_4, \Phi_4) = (A_3 - [\Phi_3, d_A\Phi_3], \Phi_3).$$

Let $U = \{r \geqq R\}$ and note that $d_{A_4}\Phi_4 \equiv 0$ on U. Let $\operatorname{Aut} P = P \times_{\operatorname{Ad}} \mathrm{SU}(2)$ be the bundle whose sections are gauge transformations, and let $\mathscr{S}$ be the unit sphere bundle of $\operatorname{Ad} P$. In reference gauge, define Φ_5' by $\Phi_5'(x) = \Phi_0(x/R)$. If $g \in \operatorname{Aut} P$ is in the fiber over $x \in U$, set $\pi(g) = \operatorname{Ad} g(\Phi_5'(x))$; this defines a fibering of $(\operatorname{Aut} P)|_U$ over $\mathscr{S}|_U$. The reference gauge gives a bundle isomorphism

$$(\operatorname{Aut} P|_U \to \mathscr{S}|_U) \cong (\operatorname{Aut} P|_{S_R^2} \to \mathscr{S}|_{S_R^2}) \times [R, \infty).$$

For $r \geqq R$, $\Phi_4|_{S_r^2}$ is homotopic (as a map from S^2 to S^2) to $\Phi_4|_{S_R^2} = \Phi_2|_{S_R^2}$, hence to $\Phi_1|_{S_R^2}$. The Theorem of Sect. 2 (or rather its proof) shows that $\deg \Phi_1|_{S_R^2} = k$, so $\Phi_1|_{S_R^2}$ and $\Phi_5'|_{S_R^2}$ are homotopic. The homotopy lifting property of fibrations (see I.11.7 of [8], e.g.) implies that there exists a smooth section $\tilde{g}$ of $\operatorname{Aut} P|_U$ which covers Φ_4; i.e. satisfies $\Phi_4 = \operatorname{Ad} \tilde{g}(\Phi_5')$. This $\tilde{g}$ can be extended smoothly to all of $\mathbb{R}^3$. Since Maps $(\mathbb{R}^3, \mathrm{SU}(2))$ is connected, there exists a smooth $h: \mathbb{R}^3 \times [0,1] \to \mathrm{SU}(2)$ such that $h_t = h(\cdot, t)$ satisfies $h_0 \equiv$ identity and $h_1 \equiv \tilde{g}$. Then $c_5(t) = (\operatorname{Ad} h_t^{-1}(A_4) + h_t^{-1} dh_t^{-1}, \operatorname{Ad} h_t^{-1}(\Phi_4))$ defines a curve in $\dot{\mathscr{C}}$ from (A_4, Φ_4) to some (A_5, Φ_5) (with $\Phi_5 \equiv \Phi_5'$ on U).

On $\{r \geqq R/2\}$ write $A_5 = A_5^L + A_5^T$, with $A^L = (\Phi_5, A_5)\Phi_5$. Since (A_5, Φ_5) is a gauge transform of (A_3, Φ_3), we know that $d_{A_5}\Phi_5$ vanishes on U, and it follows that $A_5^T = -[\Phi_5, d\Phi_5]$ for $r \geqq R$. By construction Φ_5 is radially constant on U, so $|A_5^T| \leqq \mathrm{const} \cdot r^{-1}$, $|dA_5^T| \leqq \mathrm{const} \cdot r^{-2}$, and hence $F_{A_5^T} \in L^2(U)$. But $F_{A_5} = F_{A_5^T} + d_{A_5^T} A_5^L \in L^2(U)$, so $dA_5^T A_5^L \in L^2(U)$. It follows that $c_6(t) = (A_5 - t(1-\beta_R)A_5^L, \Phi_5)$ is continuous: $[0,1] \to \dot{\mathscr{C}}$. Write $c_6(1) = (A_6, \Phi_6)$; a continuous rescaling of the $\mathbb{R}^3$ variable connects $(A_{6,j}(x)dx^j, \Phi_6(x))$ (in reference gauge) to $(R^{-1}A_{6,j}(x/R)dx^j, \Phi_6(x/R)) = (A_7, \Phi_7)$. Then for $r \geqq 1$, $\Phi_7 \equiv \Phi_0$, $d_{A_7}\Phi_7 \equiv 0$, and $A_7 \perp \Phi_7$ pointwise, whence $A_7 \equiv A_0$ for $r \geqq 1$. Finally, $c_8(t) = (1-t)(A_7, \Phi_7) + t(A_0, \Phi_0)$ connects (A_7, Φ_7) to (A_0, Φ_0). □

Acknowledgements. The author is grateful to A. Jaffe for asking him the question answered by the proposition, and to C. Taubes for pointing out why the general theorem follows from the proposition and for the idea behind Lemma 1. He also thanks T. Parker for several suggestions.

References

1. Palais, R.S.: Foundations of global nonlinear analysis. New York: Benjamin 1968
2. Uhlenbeck, K.K.: The Chern classes of Sobolev connections. Commun. Math. Phys. (to appear)
3. Taubes, C.H.: The existence of a non-minimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$. Part I. Commun. Math. Phys. **86**, 257 (1982)

4. Jaffe, A.M., Taubes, C.H.: Vortices and monopoles: structure of static gauge theories. Boston: Birkhäuser 1980
5. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1977
6. Groisser, D.: SU(2) Yang-Mills-Higgs theory on $\mathbb{R}^3$. Harvard University Preprint (1983)
7. Morrey, C.B.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
8. Steenrod, N.: The topology of fibre bundles. Princeton, NJ: Princeton University Press 1951

Communicated by A. Jaffe

Received October 26, 1983; in revised form December 19, 1983