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Skeleton Inequalities and the Asymptotic Nature of Perturbation Theory for Φ^4-Theories in Two and Three Dimensions

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Abstract. We use the polymer representation of ϕ^4-quantum field theories to prove an infinite family of correlation inequalities, called "skeleton inequalities", for the $2n$-point Green's functions. As an application, we show that they imply that Feynman perturbation theory is asymptotic in less than four dimensions.

I. Introduction

Recently there has been a revival of interest in Symanzik's polymer representation [1] of quantum field theories. The probably most important results of this development so far have been the proofs of the triviality of the continuum limits of the Ising- and ϕ^4-models in dimensions larger then four, due to Aizenman [2] and Fröhlich [3, 4].

In recent papers, Brydges, Fröhlich and Sokal [5, 6] have shown, however, that the polymer representation may also be useful to study the theory in lower dimensions. Their main result was a new, very simple proof of the existence and nontriviality of the continuum limit of ϕ^4 in two and three dimensions. The new proof of Brydges et al. rests on new correlation inequalities, called "skeleton inequalities", which may be described as follows.

Call a full skeleton amplitude a Feynman diagram without selfenergy insertions where all the lines stand for full propagators. The "skeleton series" is the power series in $(-\lambda)$ with coefficients given by the full skeleton amplitudes associated with the skeletons of perturbation theory. Then, the partial skeleton series to even (odd) order are rigorous upper (lower) bounds for the corresponding Green's functions.

In [5] this conjecture has been proven up to order $n=2$ in $(-\lambda)$. In the present paper we give a complete proof to all orders in $(-\lambda)$. As an application we will use these bounds to obtain a new proof that perturbation theory gives asymptotic expansions for the continuum Green's functions for the one and two-component ϕ^4 theory in dimensions less than four. Again, this is a known result [8, 9], but our proof is considerably simpler and possibly more transparent than the original ones.

Despite the advantage of enormous simplicity, we would like to mention a few shortcomings of the present approach. First, it has not been possible so far to establish Euclidean invariance of the continuum limit within this framework. Second, the method only works for one and two-component theories, basically because one needs to know Griffith's inequalities. An exception is the "zero-component model" (known as the Edwards model [10]), for which we have recently proven results analogous to those presented here [11]. In fact, in many respects the proofs for the Edwards model are even simpler than for the ϕ^4-theory, and in particular the proof of the skeleton inequalities there is recommended as a warm-up for the more complicated one given in the present paper.

The remainder of this paper is organized as follows. In Sect. II we collect a number of definitions and facts regarding the polymer representation that we will need later on. In Sect. III we present our proof of the skeleton inequalities to all orders. Section IV contains the proof of the asymptoticity of continuum perturbation theory in two and three dimensions. In Sect. V we draw our conclusions.

II. The Polymer Representation

This section is intended to provide some basic facts about the polymer representation of ϕ^4-theories which we will need later. Since everything in this section is well-known, we do not give proofs or derivations. For detailed reviews see, e.g. [5, 6, 12].

Let $\boldsymbol{\phi}(x)$ be an N-component real scalar field on a lattice $(a\mathbb{Z})^d$, $\{t_x\}$ a field of real, positive "local times." We define a probability measure $d\mu^{(\lambda)}(\boldsymbol{\phi})[t]$ on $\boldsymbol{\phi}$ depending on the local times t_x by

$$d\mu^{(\lambda)}(\boldsymbol{\phi})[t]=\frac{1}{Z(t)}\prod_{x\in(a\mathbb{Z})^d} d\phi(x)f(|\boldsymbol{\phi}(x)|^2+2t_x)e^{\frac{1}{2}(\boldsymbol{\phi},\tilde{\Delta}\boldsymbol{\phi})}. \tag{2.1}$$

where $\tilde{\Delta}$ is the off-diagonal part of the Laplacian operator,

$$(\boldsymbol{\phi},\tilde{\Delta}\boldsymbol{\phi})\equiv\int_a dxdy\boldsymbol{\phi}(x)\cdot\tilde{\Delta}_{xy}\boldsymbol{\phi}(y)=\int_a dx\sum_{\mu=1}^{d}\frac{\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x+ae_\mu)}{a^2}, \tag{2.2}$$

e_μ is a unit vector in the μ^{th} direction, and

$$f(u)\equiv e^{-a^d\frac{\lambda}{4}u^2-\frac{1}{2}a^d\left(\frac{2d}{a^2}+m^2\right)u}. \tag{2.3}$$

We adopt the convention to write

$$\int_a dx(\cdot)\equiv a^d\sum_{x\in(a\mathbb{Z})^d}(\cdot). \tag{2.4}$$

$Z(t)$ is the partition function defined by demanding the measure (2.1) to be normalized. $d\mu^{(\lambda)}(\phi)[0]\equiv d\mu^{(\lambda)}(\phi)$ is the ordinary measure of the lattice $\frac{\lambda}{4}\phi_d^4$-theory. Notice that the local times t_x could alternatively be viewed as introducing a space-dependent mass

$$m_x^2=m^2+\lambda t_x.$$

The following identity gives rise to the polymer representation for the propagator. We have [1, 12]

$$G^{(\lambda)}(x,y)(t) \equiv \langle \phi^i(x)\phi^i(y)\rangle(t) \equiv \int \phi^i(x)\phi^i(y)d\mu^{(\lambda)}(\phi)[t]$$
$$= \sum_{\omega:x\to y} \int dv_\omega(t')Z(t+t')/Z(t)\,. \tag{2.5}$$

Here the sum is over all random walks ω going from x to y. The ω-dependent measure $dv_\omega(t)$ is given by

$$dv_\omega(t) = (a^{d-2})^{|\omega|} \prod_{x\in(a\mathbb{Z})^d} dv_{n_x(\omega)}(t_x)\,, \tag{2.6}$$

with

$$dv_n(t) = \begin{cases} \theta(t)dt t^{n-1}/(n-1)! & \text{for} \quad n>0 \\ \delta(t)dt & \text{for} \quad n=0 \end{cases}. \tag{2.7}$$

$n_x(\omega)$ is the number of times ω visits the site x, and $|\omega|$ denotes the length of ω.

In particular, (2.5) with $t=0$ allows us to express the propagator of a ϕ^4-theory as a sum over random walks,

$$G^{(\lambda)}(x,y) = \sum_{\omega:x\to y} dv_\omega(t)z(t) \tag{2.8}$$

with $z(t) \equiv Z(t)/Z(0)$.

Corresponding expressions hold for general Green's functions. Let

$$F_n^{(\lambda)}(x_1,y_1;\ldots;x_n,y_n) \equiv \sum_{\substack{\omega_\iota:x_\iota\to y_\iota\\ i=1,\ldots,n}} \int \prod_{i=1}^n dv_{\omega_\iota}(t^i) z\left(\sum_{i=1}^n t^i\right). \tag{2.9}$$

Then

$$\langle \phi^{i_1}(x_1^1)\ldots\phi^{i_1}(x_{2n_1}^1)\ldots\phi^{i_k}(x_1^k)\ldots\phi^{i_k}(x_{2n_k}^k)\rangle$$
$$= \sum_{p_1\ldots p_k} F^{(\lambda)}_{\sum_{\ell=1}^k n_\ell}(x^1_{p_1(1)}, x^1_{p_1(2)};\ldots;x^1_{p_1(2n_1-1)}, x^1_{p_1(2n_1)};\ldots$$
$$\ldots; x^k_{p_k(1)}, x^k_{p_k(2)};\ldots;x^k_{p_k(2n_k-1)}, x^k_{p_k(2n_k)})\,. \tag{2.10}$$

Here the sum is over all pairings p_i of $2n_i$ objects, respectively. Finally, let us define the four-point Ursell-function $u_4^{(\lambda)}$ by

$$u_4^{(\lambda)}(x_1,\ldots,x_4) \equiv \langle \phi^i(x_1)\ldots\phi^i(x_4)\rangle$$
$$-\sum_p \langle \phi^i(x_{p(1)})\phi^i(x_{p(2)})\rangle\langle \phi^i(x_{p(3)})\phi^i(x_{p(4)})\rangle\,. \tag{2.11}$$

From the above equations it is obvious that

$$u_4^{(\lambda)}(x_1,\ldots,x_4) = \sum_p \sum_{\substack{\omega_1:x_{p(1)}\to x_{p(2)}\\ \omega_2:x_{p(3)}\to x_{p(4)}}} \int dv_{\omega_1}(t^1)dv_{\omega_2}(t^2)[z(t^1+t^2)-z(t^1)z(t^2)]\,, \tag{2.12}$$

and further

$$u_4^{(\lambda)}(x_1,\ldots,x_4) = \sum_p \sum_{\omega:x_{p(1)}\to x_{p(2)}} \int dv_\omega(t)z(t)[G^{(\lambda)}(x_{p(3)},x_{p(4)})(t) - G^{(\lambda)}(x_{p(3)},x_{p(4)})(0)] \tag{2.13}$$

We conclude this section by stating an important lemma on the splitting of paths. For a proof see [5].

Lemma 2.1 (Brydges, Fröhlich, Sokal) [5]. *Let $g(t)$ be some function of the local times t_x. Then*

$$\sum_{\omega:x\to y} \int dv_\omega(t) t_{x_1} \dots t_{x_n} g(t)$$

$$= \sum_{\pi\in\sigma_n} \sum_{\substack{\omega_0:x\to x_{\pi(1)} \\ \omega_1:x_{\pi(1)}\to x_{\pi(2)} \\ \dots \\ \omega_{n-1}:x_{\pi(n-1)}\to x_{\pi(n)} \\ \omega_n:x_{\pi(n)}\to y}} \int \prod_{i=0}^{n} dv_{\omega_i}(t^i) g\left(\sum_{i=0}^{n} t^i\right). \tag{2.14}$$

Here σ_n denotes the group of permutations of n objects.

III. The Skeleton Inequalities

In this section we prove the skeleton inequalities for $2p$-point Green's functions, that have recently been conjectured by Brydges et al. [5]. Before stating the precise form of our main theorem, we introduce some convenient notation.

Let $\mathscr{G}_n$ be some graph arising in the perturbation series for a p-point function. We call $\mathscr{G}_n$ a "*skeleton graph*", if by cutting no more than two lines no subset of inner vertices can be disconnected from the external points of the graph; that is to say, if the graph contains no self-energy insertions [13]. With a graph $\mathscr{G}_n$ we associate an amplitude $A^{(\lambda)}(\mathscr{G}_n)$ defined as

$$A^{(\lambda)}(\mathscr{G}_n)(y_1, \dots, y_n) = \int \prod_{\substack{a \text{ lines } \ell \\ \text{in } \mathscr{G}_n}} G^{(\lambda)}(x_{\ell_1}, x_{\ell_2}) \prod_{\substack{\text{vertices} \\ \ell_j \text{ in } \mathscr{G}_n}} dx_{\ell_j}. \tag{3.1}$$

Here x_{ℓ_1}, x_{ℓ_2} are the endpoints of the line ℓ, and ℓ_j is the vertex associated with the spatial point x_{ℓ_j}. This amplitude is a function of the p points $y_1, \dots y_n$ which are associated with the external points of $\mathscr{G}_n$. If λ is put equal to zero in (3.1), we call $A^{(0)}(\mathscr{G}_n)$ a "*bare*" amplitude, otherwise a "*full*" amplitude. If $\mathscr{G}_n$ is a skeleton graph, we call $A(\mathscr{G}_n)$ a (full or bare) "*skeleton amplitude*". Mostly, in this section, we will be concerned with full skeleton amplitudes, and skeleton amplitude will mean "full skeleton amplitude," if not otherwise indicated.

In the course of the proof we will frequently have to deal with skeleton graphs with some of their external points pairwise coinciding. We will use the special name of "*preskeleton graphs*" for them, and call the coinciding external points "*splitting points*", for reasons to become clear later. A typical preskeleton graph is depicted in Fig. 1.

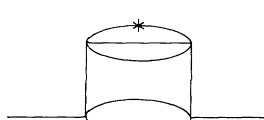


Fig. 1. A "preskeleton graph"

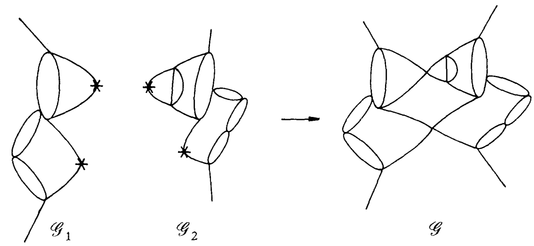


Fig. 2. Composition of two preskeleton graphs

The following lemma states an important property of preskeleton amplitudes.

Lemma 3.1. *Let $\mathcal{G}_1$, $\mathcal{G}_2$ be preskeleton graphs. Then*

$$\begin{aligned}\int_a \prod_{i=1}^{r} dy_{\ell+i} A^{(\lambda)}(\mathcal{G}_1)(y_1, \ldots, y_\ell, y_{\ell+1}, y_{\ell+1}, \ldots, y_{\ell+r}, y_{\ell+r}) \\ \cdot A^{(\lambda)}(\mathcal{G}_2)(y_1', \ldots, y_{\ell'}', y_{\ell+1}, y_{\ell+1}, \ldots, y_{\ell+r}, y_{\ell+r}) \\ = A^{(\lambda)}(\mathcal{G})(y_1, \ldots, y_\ell, y_1', \ldots, y_{\ell'}')\end{aligned} \tag{3.2}$$

and $\mathcal{G}$ is a preskeleton graph.

Figure 2 depicts how $\mathcal{G}$ is obtained from $\mathcal{G}_1$ and $\mathcal{G}_2$. The only thing one has to show to prove Lemma 3.1 is that $\mathcal{G}$ has no self-energy insertions. But this follows since $\mathcal{G}_1$ and $\mathcal{G}_2$ are preskeleton graphs, and since it already needs two cuts to remove, say, $y_{\ell+1}$ from the external points of $\mathcal{G}_1$, and then it is still connected to those of $\mathcal{G}_2$. □

Now we introduce a very convenient relation, denoted "$\lessgtr$", between functions and power series.

Definition 3.1. Let $f(\lambda)$ be a function of λ, $\{f_n(\lambda)\}_{n=0}^{\infty}$ be the coefficients of a power series in $(-\lambda)$. We write

$$f(\lambda) \lessgtr \sum^{N} (-\lambda)^n f_n(\lambda), \tag{3.3}$$

iff, for all $\lambda \geqq 0$,

$$f(\lambda) \leqq \sum_{n=0}^{2k} (-\lambda)^n f_n(\lambda), \quad \text{if } 2k \leqq N,$$

and

$$f(\lambda) \geqq \sum_{n=0}^{2k+1} (-\lambda)^n f_n(\lambda), \quad \text{if } 2k+1 \leqq N.$$

We write $f(\lambda) \lessgtr \sum (-\lambda)^n f_n(\lambda)$ if (3.3) holds for all N. The following two lemmas state simple, but important properties of the relation $\lessgtr$.

Lemma 3.2. *Let $f(\lambda) \lessgtr \sum^{N} (-\lambda)^n f_n(\lambda)$, $g(\lambda) \lessgtr \sum^{N} (-\lambda)^n g_n(\lambda)$, and $f(\lambda) \geqq 0$, $g(\lambda) \geqq 0$. Then $f(\lambda) g(\lambda) \lessgtr \sum^{N} (-\lambda)^n h_n(\lambda)$, where $\{h_n(\lambda)\}$ is the series obtained by formally multiplying the two power series $\{f_n(\lambda)\}$ and $\{g_n(\lambda)\}$.*

Lemma 3.3. *Let* $f(\lambda) \leqq \sum^{N} (-\lambda)^n f_n(\lambda)$, $f_n(\lambda) \leqq \sum^{N-n} (-\lambda)^m f_{nm}(\lambda)$. *Then* $f(\lambda) \leqq \sum^{N} (-\lambda)^n h_n(\lambda)$, *where* $\{h_n(\lambda)\}$ *is the series obtained by formally replacing the coefficients* $\{f_n(\lambda)\}$ *by the power series* $\{f_{nm}(\lambda)\}$.

The simple proofs of these lemmas are given in an appendix. We are now in the position to state our main theorem.

Theorem 3.1. *For one- and two-component* $\frac{\lambda}{4}\phi^4$*-theories the* $2p$*-point Green's functions satisfy:*

$$\langle \phi^i(x_1)\dots\phi^i(x_{2n_1})\phi^j(x_{2n_1+1})\dots\phi^j(x_{2n})\rangle \leqq \sum (-\lambda)^k s^{(\lambda)}(x_1,\dots,x_{2n_1};x_{2n_1+1},\dots,x_{2n}) \tag{3.4}$$

with the coefficients $s^{(\lambda)}(x_1,\dots x_{2p})$ *given by full skeleton amplitudes.*

Theorem 3.1 immediately implies a stronger statement about the coefficients $s^{(\lambda)}(x_1,\dots,x_{2p})$. We have the

Corollary 3.1. *The series* $\sum(-\lambda)^n s_n^{(\lambda)}(x_1,\dots,x_{2n})$ *of Theorem 3.1 is obtained by removing all but the bare skeleton amplitudes from the perturbation expansion of* $\langle \phi^i(x_1)\dots\phi^i(x_{2n_1})\phi^j(x_{2n_1+1})\dots\phi^j(x_{2n})\rangle$ *and replacing the bare propagators in the latter by full propagators.*

Proof of Corollary 3.1. Consider the theory on the lattice with finite spacing. Then, perturbation theory yields asymptotic expansions, since all Feynman diagrams converge (and the remainders of the partial series can be bounded by Feynman diagrams). Inserting the perturbation expansion for the propagator into Eq. (3.4), we get thus an asymptotic expansion for the n-point function, which, by uniqueness of asymptotic expansions, must coincide with the ordinary perturbation theory expansion. In particular, it must contain the same skeleton graphs, with the same coefficients. But each bare skeleton graph in the series produced from (3.4) arises from just replacing the full propagator by the lowest term in the expansion, the bare propagator. This proves the corollary. □

We now turn to the proof of Theorem 3.1. The following lemma provides the basic identity we will need.

Lemma 3.4 [5]. *For the N-component* $\frac{\lambda}{4}|\phi|^4$ *theory, we have*

$$\begin{aligned}\langle \phi^1(x)\phi^1(y)\rangle(t) = \langle \phi^1(x)\phi^1(y)\rangle(0) - \lambda \int_0^1 d\alpha \int_a dj t_j \\ \cdot \{2\langle \phi^1(x)\phi^1(j)\rangle(\alpha t)\langle \phi^1(j)\phi^1(y)\rangle(\alpha t) \\ + u_4(x,y,j,j)(\alpha t) + \sum_{p=2}^{N} \langle \phi^1(x)\phi^1(y);\phi^p(j)\phi^p(j)\rangle\}.\end{aligned} \tag{3.5}$$

Proof. The lemma is proven by writing the identity

$$\begin{aligned}\langle \phi^1(x)\phi^1(y)\rangle(t) - \langle \phi^1(x)\phi^1(y)\rangle(0) &= \int_0^1 d\alpha \frac{d}{d\alpha}\langle \phi^1(x)\phi^1(y)\rangle(\alpha t) \\ &= -\lambda \int_a dj t_j \langle \phi^1(x)\phi^1(y);|\phi(j)|^2\rangle.\end{aligned} \tag{3.6}$$

The semicolon denotes truncation: $\langle A;B\rangle=\langle AB\rangle-\langle A\rangle\langle B\rangle$. Using the definition of u_4, we get (3.5). □

Now, from Eq. (3.6) and Griffith's second inequality [14], we get

Lemma 3.5. *For one- and two-component* $\frac{\lambda}{4}\phi^4$ *theories,*

$$\langle\phi^1(x)\phi^1(y)\rangle(t)\leqq\langle\phi^1(x)\phi^1(y)\rangle(0). \tag{3.7}$$

The restriction to one- and two-component models arises here since Griffith's inequality is only known in these cases. *If* (3.7) holds for general N-component models, then all our results carry over for these cases. Unfortunately, we do not know how to prove (3.7) without using Griffith's inequality.

For the rest of the proof we will, for notational convenience, consider the one-component case. Then (3.5) reads

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle(t)=\langle\phi(x)\phi(y)\rangle(0)-\lambda\int_0^1 d\alpha\int_a djt_j\\ \cdot\{\langle\phi(x)\phi(j)\rangle(\alpha t)\langle\phi(j)\phi(y)\rangle(\alpha t)+u_4(x,y,j,j)(\alpha t)\}.\end{aligned} \tag{3.8}$$

From Eqs. (2.9) and (2.5) we obtain further

$$F_p^{(\lambda)}(x_1,y_1;\ldots;x_p,y_p)=\sum_{\substack{\omega_i:x_i\to y_i\\ i=1,\ldots,p-1}}\int\prod_{i=1}^{p-1}dv_{\omega_i}(t^i)z\left(\sum_{i=1}^{p-1}t^i\right)\langle\phi(x_p)\phi(y_p)\rangle\left(\sum_{i=1}^{p-1}t^i\right). \tag{3.9}$$

Equation (3.9) together with (3.8) provide a machinery that, with (3.7) as input, allows us to produce the skeleton inequalities order by order. To start, one inserts (3.7) into (3.9) with $p=2$. Using the definition of u_4 and Eq. (2.10) we get

$$u_4(x_1,\ldots,x_4)\leqq0, \tag{3.10}$$

which is the Lebowitz inequality [15]. Using this and again (3.7) in Eq. (3.8) gives the improved bound

$$\langle\phi(x)\phi(y)\rangle(t)\geqq-\lambda\int_a djt_j\langle\phi(x)\phi(j)\rangle\langle\phi(j)\phi(y)\rangle+\langle\phi(x)\phi(y)\rangle. \tag{3.11}$$

Inserting this in (3.9) and using the splitting Lemma 2.1 gives

$$\begin{aligned}u_4(x_1,\ldots,x_4)\geqq-3\lambda\int_a dj\langle\phi(x_1)\phi(j)\rangle\\ \cdot\langle\phi(x_2)\phi(j)\rangle\langle\phi(x_3)\phi(j)\rangle\langle\phi(x_4)\phi(j)\rangle.\end{aligned} \tag{3.12}$$

This process could be continued *ad infinitum*. However, going to higher order in λ would become extremely tedious and impracticable. To prove Theorem 3.1, we can instead devise a rather simple inductive proof, and the actual bounds can then easily be calculated from Corollary 3.1.

We will prove inductively, for all N and p, the following set of relations:

$$u_4^{(\lambda)}(x_1, \ldots, x_4) \lessgtr \sum^{N} (-\lambda)^k S_k^{(\lambda)}(x_1, \ldots, x_4), \tag{3.13}$$

$$\langle \phi(x)\phi(y)\rangle(t) \lessgtr \sum^{N} (-\lambda)^k g_k^{(\lambda)}(x, y)(t), \tag{3.14}$$

and

$$F_p^{(\lambda)}(x_1, y_1; \ldots; x_p, y_p) \lessgtr \sum^{N} (-\lambda)^k S_k^{(\lambda)}(x_1, y_1; \ldots; x_p, y_p), \tag{3.15}$$

with

$$S_k^{(\lambda)}(x_1, \ldots, x_4) = \sum_{\mathscr{S}_k} c(\mathscr{S}_k) A(\mathscr{S}_k)(x_1, \ldots, x_4), \tag{3.16}$$

$$S_k^{(\lambda)}(x_1, y_1; \ldots; x_p, y_p) = \sum_{\mathscr{S}_k} c(\mathscr{S}_k) A(\mathscr{S}_k)(x_1, y_1, \ldots, x_p, y_p), \tag{3.17}$$

where the $\mathscr{S}_k$ are skeleton graphs with k inner vertices, and external points $x_1, \ldots, x_4$ and $x_1, y_1, \ldots, x_p, y_p$, respectively. The $c(\mathscr{S}_k)$ are constant coefficients depending on $\mathscr{S}_k$, but not on the external points. We will not keep track of their values.

Furthermore

$$g_k^{(\lambda)}(x, y)(t) = \sum_{\mathscr{S}_k} c(\mathscr{S}_k) \int \prod_{i=1}^{\ell} dx_i t_{x_i} A(\mathscr{S}_k)(x, y, x_1, x_1, x_2, x_2, \ldots, x_\ell, x_\ell), \tag{3.18}$$

where in the terminology introduced above, $\mathscr{S}_k$ is a "pre-skeleton graph" with splitting points $x_1, \ldots, x_\ell$. (Note that for each splitting point there is a factor of t_{x_i}, which will, by the splitting lemma, allow us to split the path ω at these points.) In particular, $g_0^{(\lambda)}(x, y)(t) = \langle \phi(x)\phi(y)\rangle(0)$.

To start our inductive proof, we need to verify (3.13) and (3.14) for $N=0$, and (3.15) for $N=0$ and all p, and for $p=1$ and all N. But for $N=0$, (3.13) follows from (3.10), and (3.14) from (3.7). Equation (3.7) and successive use of (3.9) yields, for all p, the Gaussian inequality [16, 17]

$$F_p^{(\lambda)}(x_1, y_1; \ldots; x_p, y_p) \leqq \langle \phi(x_1, y_1)\rangle \langle \phi(x_2, y_2)\rangle \ldots \langle \phi(x_p)\phi(y_p)\rangle, \tag{3.19}$$

which gives (3.15) for $N=0$ and all p.

Finally, for $p=1$,

$$F^{(\lambda)}(x, y) = \langle \phi(x)\phi(y)\rangle \lessgtr \sum (-\lambda)^n S_n^{(\lambda)}(x, y), \tag{3.20}$$

with $S_0^{(\lambda)}(x, y) = \langle \phi(x)\phi(y)\rangle$, $S_k^{(\lambda)}(x, y) = 0$ for $k > 0$, establishes (3.15) for $p=1$ and all N.

We now assume that (3.13)–(3.15) hold for all p and $N = M-1$. We will show that then (3.13)–(3.15) hold for $N = M$. To do this, consider Eq. (3.8). Our aim is to replace the terms on the right-hand side by the series (3.13)–(3.15) and to show that the series we obtain satisfies (3.14) for $N = M$. We have to deal with the two terms $\langle \phi(x)\phi(j)\rangle(\alpha t)\langle \phi(j)\phi(y)\rangle(\alpha t)$ and $u_4(x, y, j. j)(\alpha t)$ separately. First we get

$$\int_a dj t_j \langle \phi(x)\phi(j)\rangle(\alpha t)\langle \phi(j)\phi(y)\rangle(\alpha t) \lessgtr \sum^{M-1} (-\lambda)^k \tilde{g}_k(x, y)(t), \tag{3.21}$$

with

$$\tilde{g}_k(x,y)(t) = \sum_{\substack{\mathscr{S}'_{k'},\mathscr{S}''_{k''}\\ k'+k''=k}} c(\mathscr{S}'_{k'})c(\mathscr{S}''_{k''})\alpha^{\ell'+\ell''}$$

$$\cdot \int_a djt_j \prod_{i'=1}^{\ell'} dx'_{i'}t_{x'_{i'}} \prod_{i''=1}^{\ell''} dx''_{i''}t_{x''_{i''}}t_j$$

$$\cdot A(\mathscr{S}'_{k'})(x,j,x'_1,x'_1,\ldots,x'_{\ell'},x'_{\ell'})$$

$$\cdot A(\mathscr{S}''_{k''})(j,y,x''_1,x''_1,\ldots,x''_{\ell''},x''_{\ell''})$$

$$= \sum_{\mathscr{S}_k} c(\mathscr{S}_k)\alpha^{\ell} \int_a djt_j \prod_{i=1}^{\ell} dx_it_{x_i}$$

$$\cdot A(\mathscr{S}_k)(x,y,j,j,x_1,x_1,\ldots,x_\ell,x_\ell)\,. \tag{3.22}$$

In the last equality we made use of Lemma 3.1. The $\mathscr{S}_k$ are preskeletons obtained by composing $\mathscr{S}'_{k'}$ and $\mathscr{S}''_{k''}$. We see from (3.22) that the $\tilde{g}_k(x,y)(t)$ have the same form as demanded for the $g_k(x,y)(t)$ in Eq. (3.18).

Next, consider the $u_4(x,y,j,j)(\alpha t)$. We can use the expression (3.13) to write

$$\int_a djt_ju_4(x,y,j,j)(\alpha t) \lessgtr \sum^{M-1} (-\lambda)^k \int djt_js_k(x,y,j,j)(\alpha t)$$

$$= \sum^{M-1} (-\lambda)^k \sum_{\mathscr{S}_k} c(\mathscr{S}_k) \int djt_jA(\mathscr{S}_k)(x,y,j,j)(\alpha t)\,. \tag{3.23}$$

Here $A(\mathscr{S}_k)(x,y,j,j)(\alpha t)$[1] stands for the amplitude with propagators $\langle\phi(x_1)\phi(x_2)\rangle(\alpha t)$ corresponding to the lines of $\mathscr{S}_k$. Thus, the coefficients of the series (3.23) are products of t-dependent propagators. To bring them in the desired form, we have to again replace them by the series for $\langle\phi(x)\phi(y)\rangle(t)$. Lemmas 3.2 and 3.3 guarantee that doing this, we will produce a series of alternating bounds to order $M-1$. Furthermore, by replacing the lines in the preskeletons $\mathscr{S}_k$ by preskeletons with two external points we produce only preskeletons. Thus we get

$$\int_a djt_ju_4(x,y,j,j)(\alpha t) \lessgtr \sum^{M-1} (-\lambda)^k \sum_{\mathscr{S}_k} c(\mathscr{S}_k)\alpha^{\ell} \int_a djt_j \prod_{i=1}^{\ell} dx_it_{x_i}$$

$$\cdot\, A(\mathscr{S}_k)(x,y,j,j,x_1,x_1,\ldots,x_\ell,x_\ell) = \sum^{M-1} (-\lambda)^k \tilde{\tilde{g}}_k(x,y)(t)\,, \tag{3.24}$$

with $\tilde{\tilde{g}}_k(x,y)(t)$ again having the form demanded for $g_k(x,y)(t)$. Plugging these results into Eq. (3.8), we obtain

$$-[\langle\phi(x)\phi(y)\rangle(t) - \langle\phi(x)\phi(y)\rangle(0)]$$

$$\lessgtr \lambda \sum (-\lambda)^k \int_0^1 d\alpha[\tilde{g}_k(x,y)(t) + \tilde{\tilde{g}}_k(x,y)(t)]\,, \tag{3.25}$$

---

1 Since the t_x may be considered as a space dependent mass, it is trivial that (3.13) generalizes to $u_4(x_1,\ldots,x_4)(t)$ in the manner exploited here

and thus

$$\langle\phi(x)\phi(y)\rangle(t) \lessgtr \sum^{M}(-\lambda)^k g_k(x,y)(t)\,, \tag{3.26}$$

with

$$g_0(x,y)(t) = \langle\phi(x)\phi(y)\rangle(0)\,, \tag{3.27}$$

and

$$g_k(x,y)(t) = \int_0^1 d\alpha[\tilde{g}_{k-1}(x,y)(t) + \tilde{\tilde{g}}_{k-1}(x,y)(t)]\,. \tag{3.28}$$

Note that the α-integration can be trivially performed, contributing to the coefficients $c(\mathscr{S}_k)$ only.

By construction, $g_k(x,y)(t)$ is of the form (3.18). This proves (3.14) for $N=M$.

The next step is to prove (3.15) for $N=M$, all p, by induction on p. Assume (3.15) holds for $p \leqq q-1$. Then we write (3.9) for $p=q$, and use (3.14) for $N=M$. This gives

$$\begin{aligned} F(x_1,y_1;\ldots;x_q,y_q) \lessgtr \sum^{M}(-\lambda)^k & \sum_{\substack{\omega_i:x_i\to y_i\\ i=1,\ldots,q-1}} \int \prod_{i=1}^{q-1} dv_{\omega_i}(t^i) \\ & \cdot z\left(\sum_{i=1}^{q-1} t^i\right) g_k(x_q,y_q)\left(\sum_{i=1}^{q-1} t^i\right). \end{aligned} \tag{3.29}$$

Now the term with $k=0$ in this sum is simply given by

$$F(x_1,y_1;\ldots;x_{q-1},y_{q-1})\langle\phi(x_q)\phi(y_q)\rangle\,. \tag{3.30}$$

By hypothesis, the skeleton expansion for $F(x_1,\ldots,y_{q-1})$ produces bounds up to order M which can be inserted into the $k=0$ term (3.30). The coefficients for $k>0$ are of the form

$$\begin{aligned} & \sum_{\mathscr{S}_k} \sum_{\substack{\omega_i:x_i\to y_i\\ i=1,\ldots,q-1}} \int \prod_{i=1}^{q-1} dv_{\omega_i}(t^i) z\left(\sum_{i=1}^{q-1} t^i\right) \\ & \cdot \int_a \prod_{j=1}^{\ell} dz_j \left(\sum_{i=1}^{q-1} t^i_{z_j}\right) c(\mathscr{S}_k)A(\mathscr{S}_k)(x_q,y_q,z_1,z_1,\ldots,z_\ell,z_\ell)\,. \end{aligned} \tag{3.31}$$

We now carry out the product $\prod_{j=1}^{\ell}\left(\sum_{i=1}^{q-1} t^i_{z_j}\right) = \sum_{\{i_j\}} \prod_{j=1}^{\ell} t^{i_j}_{z_j}$, where the $\sum_{\{i_j\}}$ is over all the ℓ-tuples $\{i_j\}$ with $1 \leqq i_j \leqq q-1$. We will take the $\sum_{\{i_j\}}$ out of the sum over walks and consider a typical term. Using the splitting Lemma 2.1, we can write it as

$$\int \prod_{j=1}^{\ell} dz_j c(\mathscr{S}_k) F(u_1,u_2;\ldots;u_{m-1},u_m) A(\mathscr{S}_k)(x_q,y_q,z_1,z_1,\ldots,z_\ell,z_\ell)\,. \tag{3.32}$$

Here $(u_1,u_2,\ldots,u_{m-1},u_m)$, $m=2(q-1+\ell)$, stands for some permutation of $(x_1,y_1,\ldots,x_{q-1},y_{q-1},z_1,z_1,\ldots,z_\ell,z_\ell)$.

We can now use the series (3.15) for $N=M-k$, $p=q-1+\ell$ to get

$$(3.23) \leqq \sum^{M-k} (-\lambda)^{k'} \sum_{\mathcal{S}_{k'}} c(\mathcal{S}_k)c(\mathcal{S}_{k'}) \cdot \int \prod_{j=1}^{\ell} dz_j A(\mathcal{S}_{k'})(u_1, \ldots, u_m) A(\mathcal{S}_k)(x_q, \ldots, z_\ell)$$

$$= \sum^{M-k} (-\lambda)^{k'} \sum_{\mathcal{S}_{k+k'}} c(\mathcal{S}_{k+k'}) A(\mathcal{S}_{k+k'})(x_1, y_1, \ldots, x_q, y_q). \tag{3.33}$$

By Lemma 3.1, $\mathcal{S}_{k+k'}$ is a skeleton graph. By Lemma 3.3, adding all the terms (3.33) up and inserting them into (3.29) yields

$$F(x_1, y_1; \ldots; x_q, y_q) \leqq \sum^{M} (-\lambda)^k s_k(x_1, y_1; \ldots; x_q, y_q), \tag{3.34}$$

with the $s_k(x_1, \ldots, y_q)$ having the form (3.17). This proves (3.15) for $N=M$ and $p=q$. By induction it thus holds for all p.

As a trivial corollary, it follows now that (3.13) also holds for $N=M$, and the inductive step in N is completed. Thus (3.15) holds for all N and all p which, by (2.10) proves the theorem. □

A few remarks are in order:

(1) The above proof mimics the actual constructive process one could go through to derive explicitly the skeleton inequalities. Of course, in order to derive the bounds for finite p and to some finite order N, the process is finite, since there is no need to complete the induction over p each time we proceed to higher order in N. In fact, to construct the bounds for, say, u_4 up to order N, we need the bounds for the $2q$-point functions only up to order $N-q+2$.

(2) In this proof, we show explicitly that all the bounds are in terms of skeleton amplitudes. In principle, this can also be shown by an abstract argument similar to the one used to prove Corollary 3.1.

(3) In [11] we have also proven the skeleton inequalities for the zero-component model (Edwards model). In that case the proof is simpler, since no induction over N was necessary. The reason is that we have the relations

$$z(t^1+t^2) = z(t^1)z(t^2)\exp\left(-2\lambda \int_a t_x^1 t_x^2 dx\right),$$

and

$$e^{-\lambda x} \leqq \sum (-\lambda)^k \frac{x^k}{k!},$$

which replaces (3.13).

IV. Asymptotic of Perturbation Theory

We will now use the skeleton inequalities of the preceding section to give a new, simpler proof of asymptoticity of the Feynman perturbation expansion of the Euclidean Green's functions to all orders, for the weakly coupled $\lambda\phi^4$ model in two and three dimensions.

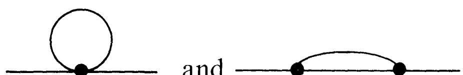
As in [6] an essential rôle will be played by the *Schwinger-Dyson equation*,

$$G^{(\lambda)}(x,y)=G^{(0)}(x,y)-\int_a dz G^{(0)}(x,z)\langle(\lambda\phi^2(z)+\delta m^2)\phi(z)\phi(y)\rangle\,. \tag{4.1}$$

Before we state our results, let us first recall some facts about renormalized perturbation theory. One is interested in constructing the continuum limit $a\to 0$ of the lattice Green's functions $\langle\phi(x_1)\ldots\phi(x_{2p})\rangle$. To do so one has to let diverge the mass counterterm $\delta m^2(\lambda,a)$, with

$$\begin{aligned}\delta m^2(\lambda,a)&=-3\lambda G^{(0)}(0,0)\,, &&\text{for } d=2\,,\\ \delta m^2(\lambda,a)&=-3\lambda G^{(0)}(0,0)+6\lambda^2\int_a dx[G^{(0)}(0,x)]^3\,, &&\text{for } d=3\,.\end{aligned} \tag{4.2}$$

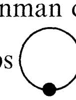
Existence of the continuum limit for small λ and $m_0>0$ (and a suitable sequence of lattice spacings a) can be proven by using the skeleton inequalities to lowest order (see [6]). We will see that the mass insertions (4.2) produce in perturbation theory Feynman diagrams that cancel the ultraviolet divergences of the self-energies:

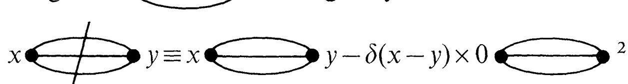
 (4.3)

(We use the graphical notation of [6], i.e. x  $y\equiv G^{(\lambda)}(x,y)$, x  y $\equiv G^{(0)}(x,y)$.)

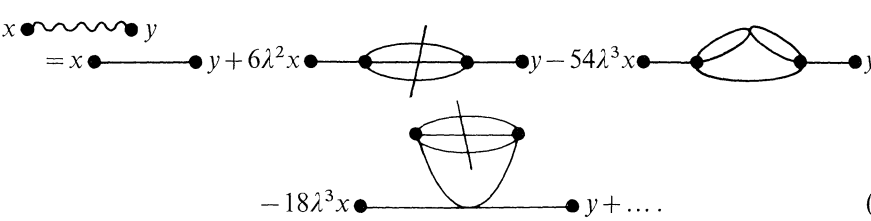
In fact we will see that in the formal perturbation expansion of the Green's functions

$$\langle\phi(x_1)\ldots\phi(x_{2p})\rangle=\sum_{n=0}^{\infty}\lambda^n f_n(x_1,\ldots,x_{2p})\,, \tag{4.4}$$

the coefficients f_n can be written as a linear combination of Feynman diagrams with all divergencies removed by counterterms, i.e. with no loops  present and with subdiagrams  coming only in the combination


x … $y\equiv x$ … $y-\delta(x-y)\times 0$ … [2], (4.5)

which is ultraviolet finite. We call these diagrams "properly renormalized". For example, the lowest order terms in the expansion of the propagator can be expressed as


x … y $=x$ … $y+6\lambda^2x$ … $y-54\lambda^3x$ … y $-18\lambda^3x$ … $y+\ldots$. (4.6)

Power counting shows that amplitudes of properly renormalized diagrams converge to well-defined distributions in the continuum limit. This shows that the coefficients of the perturbation expansion are finite.

2 We adopt the convention of integration over unlabeled points

Asymptoticity up to order 2 for the propagator was proven in [6]:

Theorem 4.1 (Brydges, Fröhlich, Sokal). *Let $G^{(\lambda)}(x, y)$ be the lattice two-point function. Then*

$$\|G^{(\lambda)}(0,\cdot)-G^{(0)}(0,\cdot)\|_1+\|G^{(\lambda)}(0,\cdot)-G^{(0)}(0,\cdot)\|_\infty \leqq c\lambda^2\,, \tag{4.7}$$

where c is a constant, independent of the lattice spacing a.

This theorem can be proven using the skeleton inequalities to lowest order together with the Schwinger-Dyson equation (4.1).

Theorem 4.2. *Let $\langle\phi(x_1)\ldots\phi(x_{2p})\rangle$ be the lattice $2p$-point function, and let*

$$\langle\phi(x_1)\ldots\phi(x_{2p})\rangle \sim \sum_{n=0}^{\infty} \lambda^n f_n(x_1, \ldots, x_{2p}) \tag{4.8}$$

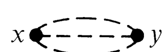
be its formal Feynman perturbation expansion. Then for any test function $\zeta\in\mathscr{S}(\mathbb{R}^d)$,

$$\left|\int_a \left[\langle\phi(x_1)\ldots\phi(x_{2p})\rangle - \sum_{k=0}^{n} \lambda^k f_k(x_1, \ldots, x_{2p})\right] \prod_{i=1}^{2p} \zeta(x_i)dx_i\right| \leqq c_n(\zeta)\lambda^{n+1}, \tag{4.9}$$

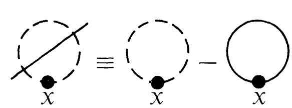
where $c_n(\zeta)$ is independent of a.

Note that independency of a implies that the bounds (4.7), (4.9) carry over to the continuum limit.

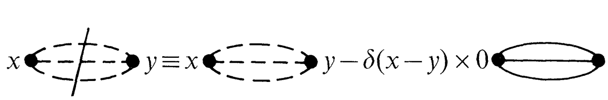
Proof of Theorem 4.2. First we prove the theorem for the propagator. To do this we insert the skeleton inequalities for the four-point function into the Schwinger-Dyson equation, producing this way Feynman diagrams which contain free as well as full (interacting) propagators. Let us thus extend the definition of a properly renormalized diagram to this case: We say that a diagram is *properly renormalized*, if all divergent subdiagrams come together with the corresponding counterterms, i.e. the insertions

 and  (4.10)

(x•- - - -•y denotes either x•———•y or x•∿∿∿∿•y) come only in the combination



and

 (4.11)

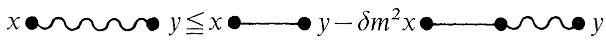
In order to prove the theorem, we will prove inductively bounds of the form

$$G^{(\lambda)}(x, y) \leqq \sum_{k=0}^{m} \lambda^k f_k(x, y) + \sum_{k=m+1}^{m+3} \lambda^k f_k^+(x, y)\,, \tag{4.12}$$

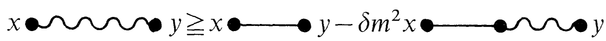
$$G^{(\lambda)}(x, y) \geqq \sum_{k=0}^{m} \lambda^k f_k(x, y) + \sum_{k=m+1}^{m+3} \lambda^k f_k^-(x, y)\,, \tag{4.13}$$

$m=0,1,\ldots$, where the $f_k(x,y)$ are given in terms of properly renormalized Feynman diagrams with free propagators (we will see later that they are really the coefficients of formal perturbation theory), and the $f_k^{\pm}(x,y)$ are also given by properly renormalized diagrams, but may contain full propagators as well.

To prove (4.12), (4.13) we replace the four-point function in the Schwinger-Dyson equation (4.1) by the skeleton inequalities of order m and $m+1$:



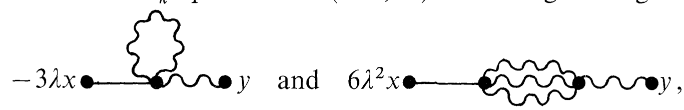
$$+\sum_{k=1}^{m+1}(-\lambda)^k \int_a dz G^{(0)}(x,z)s_{k-1}(z,z,z,y)\,, \tag{4.14}$$



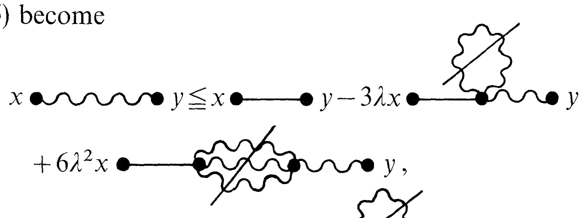
$$+\sum_{k=1}^{m+2}(-\lambda)^k \int_a dz G^{(0)}(x,z)s_{k-1}(z,z,z,y)\,. \tag{4.15}$$

(We assumed m to be odd. If m is even the inequalities are reversed.)

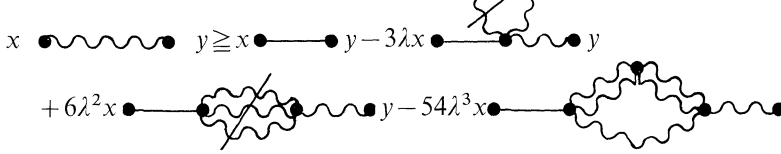
The skeletons do not contain any self-energy insertions. However, identifying three arguments of the s_k's produces in (4.14, 15) two divergent diagrams, namely



but for these diagrams we have the counterterm  [see (4.2)] and (4.14, 15) are bounds in terms of properly renormalized diagrams. To lowest order (4.14, 15) become



(4.16)



(4.17)

which proves (4.12, 13) for $m=0$.

Now assume that (4.12, 13) hold for $m=0, 1, \ldots, n-1$. We want to use these bounds to eliminate successively the full propagators up to order n in (4.14, 15) with $m=n$. This will give (4.12, 13) for $m=n$, thus proving the induction step. Consider a full propagator which appears in a diagram of order k, $1\leqq k\leqq n$, with ℓ full propagators, on the right-hand side of (4.14) and (4.15). Replace this propagator by the bounds (4.12, 13) with $m=n-k$. Expressions (4.14, 15) become

$$\leqq \sum_{k=0}^{n}\lambda^k \bar{f}_k(x,y)+\sum_{k=n+1}^{n+3}\lambda^k \bar{f}_k^{+}(x,y)\,, \tag{4.18}$$

$$\geqq \sum_{k=0}^{n}\lambda^k \bar{f}_k(x,y)+\sum_{k=n+1}^{n+3}\lambda^k \bar{f}_k^{-}(x,y)\,. \tag{4.19}$$

All $\bar{f}_k$, $\bar{f}_k^{\pm}$ are given in terms of properly renormalized diagrams, by induction hypothesis, and the new diagrams we produced are either of order $k > n$, and are thus included in $\bar{f}_k^{\pm}$ or of order $k \leqq n$ and have $\ell - 1$ full propagators. Then we repeat this procedure for all full propagators which are still present in the $\bar{f}_k$'s, $1 \leqq k \leqq n$, until we are left with bare propagators only, up to order n. Thus the proof of (4.12) and (4.13) is complete.

The generalization of (4.12, 13) to higher $2p$-point functions

$$\langle \phi(x_1) \dots \phi(x_{2p}) \rangle \leqq \sum_{k=0}^{m} \lambda^k f_k(x_1, \dots, x_{2p}) + \sum_{k=m+1}^{m+3} \lambda^k f_k^+(x_1, \dots, x_{2p}), \quad (4.20)$$

$$\langle \phi(x_1) \dots \phi(x_{2p}) \rangle \geqq \sum_{k=0}^{m} \lambda^k f_k(x_1, \dots, x_{2p}) + \sum_{k=m+1}^{m+3} \lambda^k f_k^-(x_1, \dots, x_{2p}), \quad (4.21)$$

is easily proven by replacing the full propagators of the skeleton inequalities for $2p$-point functions (3.4) by the bounds (4.12, 13). To conclude the proof of the theorem we have to estimate the remainders $f_k^{\pm}(x_1, \dots, x_{2p})$ which, as we have seen, are linear combinations of amplitudes of properly renormalized diagrams with bare as well as full propagators. Writing for these full propagators $G^{(\lambda)}(x, y) = G^{(0)}(x, y) + E(x, y)$, with $E(x, \cdot) \in L^1 \cap L^\infty$ with norm bounded uniformly in a, by Theorem 4.1, we see that $f_k^{\pm}(x_1, \dots, x_{2p})$ can be expressed as a sum over amplitudes of properly renormalized diagrams with, as propagators $G^{(0)}(x, y)$ and $E(x, y)$. We know from power counting that properly renormalized diagrams with $G^{(0)}$ propagators converge (when smeared out with test functions of $\mathscr{S}$) in the continuum limit, and so do *a fortiori* diagrams with E propagators, which have no small-distance singularities.

Thus one has the bound

$$\left| \int_a f_k^{\pm}(x_1, \dots, x_{2p}) \zeta(x_1) \dots \zeta(x_{2p}) dx_1 \dots dx_{2p} \right| \leqq c_k(\zeta),$$

with $c_k(\zeta)$ independent of a. As in Corollary 3.1, it follows from the uniqueness of the asymptotic expansions on the lattice that the coefficients f_k are really the ones given by perturbation theory, and the proof is complete. □

V. Conclusions

In this paper we have pushed the new "random walk"-approach to the construction of superrenormalizable ϕ^4-models of [6] a little further, showing that the asymptoticity of perturbation theory can be obtained easily in this context from correlation inequalities that follow from the random walk expansion.

Still, however, the results of this new, simple approach do not match the ones previously obtained. Probably the most embarassing shortcoming of this method is its restriction to one- and two-component ϕ^4-models, and its failure to allow for a proof of Euclidean invariance in the continuum limit.

Both these shortcomings are consequences of the dependence of this method on the Griffith's inequalities as the basic input in the proof of the skeleton inequalities. This restricts us to one- and two-component ϕ^4-models, and forces us to use the non-rotational invariant lattice cut-off.

As an exception, the zero-component model (Edwards model) is also tractable, without having to use any a priori inequalities. There, the weights on the local times $z(t)$ are known explicitly, not only as ratios of partion functions. In [11], all results of [6] and of this paper were derived for the Edwards model. Furthermore, in this model it is possible to prove also Euclidean invariance of the limiting theory. To do this, one defines the model directly in the continuum, as a measure on the space of paths

$$d\mu^{(\varepsilon)}_{x,y,T}(\omega)=\exp\left[-\lambda\int_0^T\int_0^T dtds\delta_\varepsilon(x_t(\omega)-x_s(\omega))-m^2T\right]dW_{x,y,T}(\omega),$$

where $dW_{x,y,T}(\omega)$ is the conditional Wiener measure on paths which reach y from x in time T, and $x_t(\omega)$ is the position of the path ω at time t, $0\leqq t\leqq T$. Here δ_ε is a rotational invariant, non-negative regularization of the δ function, e.g.

$$\delta_\varepsilon(x)=(2\pi\varepsilon^2)^{-d/2}e^{-\frac{x^2}{2\varepsilon^2}}.$$

One can again prove a Schwinger-Dyson equation and the skeleton inequalities, and hence, by the same methods used in [5] and here, show the existence of the $\varepsilon\to 0$ limit, which will be manifestly Euclidean invariant.

A similar procedure could be adopted in the field theory case only if Griffith's inequalities were known for a rotationally invariant cut-off, which is not the case.

Appendix

In this appendix we give the proofs of Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. By hypothesis we have

$$g(\lambda)f(\lambda)\lessgtr\sum^N(-\lambda)^n f_n(\lambda)g(\lambda).$$

Now since $f_n(\lambda)$ is non-negative by assumption,

$$g(\lambda)f(\lambda)\leqq\sum_{n=0}^{2k}(-\lambda)^n f_n(\lambda)\sum_{n'=0}^{2k-n}(-\lambda)^{n'}g_{n'}(\lambda)\leqq\sum_{n=0}^{2k}(-\lambda)^n\sum_{\substack{k',k''\\k'+k''=n}}f_{k'}(\lambda)g_{k''}(\lambda),$$

and similarly

$$g(\lambda)f(\lambda)\geqq\sum_{n=0}^{2k+1}(-\lambda)^n\sum_{k',k'':k'+k''=n}f_{k'}(\lambda)g_{k''}(\lambda)$$

for $2k$, $2k+1\leqq N$, which proves Lemma 3.2. □

Proof of Lemma 3.3. We have by hypothesis

$$\begin{aligned}f(\lambda)&\leqq\sum_{n=0}^{2k}(-\lambda)^n f_n(\lambda)\leqq\sum_{n=0}^{2k}(-\lambda)^n\sum_{n'=0}^{2k-n}(-\lambda)^{n'}f_{nn'}(\lambda)\\&\leqq\sum_{n=0}^{2k}(-\lambda)^n\sum_{\substack{k',k''\\k'+k''=n}}f_{k'k''}(\lambda),\end{aligned}$$

and similarly

$$f(\lambda) \geqq \sum_{n=0}^{2k+1} (-\lambda)^n \sum_{\substack{k',k'' \\ k'+k''=n}} f_{k'k''}(\lambda), \qquad \text{for} \quad 2k, 2k+1 \leqq N.$$

This proves Lemma 3.3. □

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