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Small $\hbar$ Asymptotics for Quantum Partition Functions Associated to Particles in External Yang-Mills Potentials

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Abstract. To a gauge field on a principal G-bundle $P\to M$ is associated a sequence of quantum mechanical Hamiltonians, as Planck's constant $\hbar\to 0$ and a sequence of representations π_n of G is taken. This paper studies the associated quantum partition functions, trace $\exp(-tH_n)$, and produces a complete asymptotic expansion, as $\hbar\to 0$, $\hbar=1/n$, of which the principal term, proportional to the classical partition function, is the familiar classical limit.

1. Introduction

In this paper we study the limit as $\hbar\to 0$ of the (non-relativistic) quantum partition function associated with the Hamiltonian for motion in a Yang-Mills field. More specifically, let M be a compact Riemannian manifold, and let $P\to M$ be a principal G-bundle, G a compact connected Lie group. We suppose a connection is given on P; this determines a gauge field. We can regard the connection as a $\mathfrak{g}$-valued one-form θ. We have an associated covariant derivative on any associated vector bundle $E=P\times_\pi V$, where π is a representation of G on a vector space V. With respect to a local frame, this is given by

$$\nabla_X^\pi u = X\cdot u + \pi(\theta(X))u\,, \tag{1.1}$$

where X is a tangent vector to M, $u\in C^\infty(M,E)$. Here $X\cdot u$ represents the action of X componentwise on u, and $\theta(X)$ is the element of $\mathfrak{g}$ defined by the connection 1-form θ. In local coordinates, on a coordinate patch $\mathcal{O}\subset M$, with $X=\partial/\partial x_j=\partial_j$ and

$$\theta=\sum A_j(x)dx_j;\qquad A_j\in C^\infty(\mathcal{O},\mathfrak{g})\,, \tag{1.2}$$

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we can write (1.1) as

$$\nabla_j^\pi = \partial_j + \pi(A_j(x)). \tag{1.3}$$

The quantum mechanical Hamiltonian we consider is of the form

$$H_{\hbar,\pi} = -\hbar^2 g^{-1/2} \nabla_j^\pi g^{jk} g^{1/2} \nabla_k^\pi - i\hbar\pi(A_0(x)) + V(x). \tag{1.4}$$

We use the summation convention. Here $A_0(x)$ is a smooth section of $P \times_{\mathrm{Ad}} \mathfrak{g}$ and $V(x)$ is given as a smooth real valued function on M. g_{jk} is the metric tensor and g^{jk} the associated metric on cotangent vectors.

There are two main contexts in which to study the behavior of a quantum system and relate it to the behavior of a classical system, when $\hbar$, the Planck constant, tends to zero. Historically, both were discussed already in early stages of the development of (non-relativistic) quantum mechanics. The first compares classical and quantum mechanical observables as they appear for example in the description of a particle moving in a potential. Ehrenfest [4] was the first to relate these two types of observables. It is remarkable that only fairly recently it was realized by Hepp [8] that the Ehrenfest relations are compatible with the time evolution of the classical and the quantum mechanical system (see also the discussion in [26]). Recently, Hepp's treatment has been extended to cover the case of a particle moving in an external (static) metric field [10] and in an external Yang-Mills field [11]. In the first case the classical equations of motion are of course the geodesic equations, whereas in the second case one obtains generalized versions of the Lorentz equations for a classical particle described by a position, a momentum, and a classical isospin [in the case $G = \mathrm{SU}(2)$]. These equations are sometimes called the Wong equations [30]; see also [20] and [7].

The second main context in which one may discuss the $\hbar \to 0$ limit and with which we will be concerned here deals with the Gibbs canonical partition functions as obtained from the Hamiltonians H_{cl} and H_{qm} which describe the (one-particle) system. In the quantum mechanical case one looks at $Z_{\mathrm{qm}}(\beta) = \mathrm{trace}\, e^{-\beta H_{\mathrm{qm}}}$, with $\beta = (kT)^{-1}$, T = temperature, k = Boltzmann constant, where H_{qm} contains $\hbar$, and compares it with the classical expression $Z_{\mathrm{cl}}(\beta) = \int e^{-\beta H_{\mathrm{cl}}} d\,\mathrm{vol}$, where integration is over the classical phase space of the system.

Here we will obtain a complete asymptotic expansion as $\hbar \to 0$ of the trace of $\exp(-tH_{\hbar,\pi})$, with fixed t ($t = \beta$). For reasons that have been developed elsewhere (see [11]) we must vary the representation π as $\hbar \to 0$. In fact, if we picture the irreducible representations of G as indexed by a lattice in a Weyl chamber, we can pick some point λ_1 in this lattice, corresponding to a representation π_1 of G, and let π_n be the representation of G corresponding to the point $n\lambda_1$. We then look at the sequence of Hamiltonians

$$H_n = H_{\hbar,\pi_n}, \qquad \hbar = 1/n. \tag{1.5}$$

The main result of this paper is the following, which extends previous results along these lines.

Theorem A. *Let d_n be the dimension of the representation space of π_n. Fix $t > 0$. Then there is a complete asymptotic expansion as $\hbar \to 0$ ($\hbar = 1/n$) of the form*

$$d_n^{-1}\, \mathrm{trace}\, e^{-tH_n} \sim \hbar^{-\nu}(a_0(t) + a_1(t)\hbar + a_2(t)\hbar^2 + \ldots), \tag{1.6}$$

where

$$\nu = \dim M . \tag{1.7}$$

Here $a_0(t)$ *is given (up to a factor* $\|\lambda_1\|^\nu$*) by the integral formula* (4.88), *or equally, by* (5.90).

The fact that $\hbar^\nu d_n^{-1}\,\mathrm{trace}\, e^{-tH_n}$ tends in the limit to $a_0(t)$ has been proved before, first in the case of Abelian gauge fields ($G=S^1$) in [2, 18, and 19], and recently, for general gauge fields [in the case of a product bundle over $M=R^\nu$, and with appropriate conditions on $V(x)$ as $|x|\to\infty$ in this non-compact situation] in [11]. Theorem A refines these results insofar as it produces a complete asymptotic expansion.

In the Abelian case $G=S^1$, we have $\pi_n(A_j(x))$ equal to $n\pi_1(A_j(x))$, so in this case we can write H_n as

$$H_n = -g^{-1/2}(\hbar\partial_j + iA_j)g^{jk}g^{1/2}(\hbar\partial_k + iA_k) + V \quad (\hbar = 1/n) \tag{1.8}$$

[replacing $V(x) - i\pi_1(A_0(x))$, which is real valued, by V, and denoting $\pi_1(A_j)$ by iA_j]. In Sect. 2 we will analyze a class of singular perturbation problems. A special case will include the qualitative analysis of e^{-tH_n} in case (1.8). The singular perturbation problems treated in Sect. 2 are not restricted to scalar problems. These singular perturbation problems have some points in common with the work on first order hyperbolic systems with a small viscosity term in [22], but the analysis is very much simpler in the present case.

In the special case when the potentials A_j are absent from (1.8), Uhlenbeck and Gropper [27] and Wigner [29] were the first to derive recursion relations for the quantum partition function in terms of powers of $\hbar$ (see [13] for a lucid discussion). Considerable effort has been put into trying to prove that their expansions are asymptotic [3]; to our knowledge the proof we present seems to be the first one, even in this case. We remark that three ingredients go into the proof of the validity of the expansion produced in Sect. 2. There is a qualitative analysis of the amplitudes obtained by solving certain transport equations (Lemma 2.1), then an interpretation of this analysis in terms of symbol estimates (Lemma 2.2) and use of the pseudodifferential operator calculus, and finally an appeal to certain energy estimates (Lemma 2.3), which follow from Gårding's inequality.

It is a remarkable fact, first observed by Lieb [14] (see also [9]) that in some situations it is possible to take the classical limit only for a subset of dynamical variables, while retaining the quantum mechanical properties of the remaining dynamical variables. The analysis in Sect. 2 in case A_j are general gauge potentials (but one does not replace $n\pi_1$ by π_n) provides another example of this phenomenon by retaining the quantum mechanical property of the isospin and letting position and momentum become classical observables (see the end of Sect. 2).

To treat the quantum partition function associated with (1.8), for general gauge fields, we will use two approaches, both involving the study of harmonic analysis on compact Lie groups. One approach involves the method of "coherent states", which has also been applied in the study of quantum partition functions in [18, 11], following other applications given in [14, 5, 6, 12, 16]. In Sect. 5 we amalgamate the singular perturbation analysis of Sect. 2 with the use of projections onto coherent states to prove Theorem A. The other method we use involves fitting all the representations of G (including the sequence π_n) into the regular representation. Proving Theorem A becomes a task in microlocal analysis on P, which we tackle with pseudodifferential operators. We briefly describe some aspects of representation theory and harmonic analysis on a compact Lie group in Sect. 3, including some results on harmonic analysis using pseudodifferential operators developed in Chap. XII, Sect. 6, of the book [21].

We use the following notational conventions for pseudodifferential operators. We have

$$p(x, D)u = \int p(x, \xi) e^{ix\cdot\xi} \hat{u}(\xi) d\xi, \tag{1.9}$$

where $\hat{u}(\xi)$ is the Fourier transform of u. We say

$$p(x, \xi) \in S^m_{\varrho,\delta} \quad \text{and} \quad p(x, D) \in OPS^m_{\varrho,\delta}, \tag{1.10}$$

provided

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \leqq C_{\alpha\beta}(1 + |\xi|)^{m - \varrho|\alpha| + \delta|\beta|}. \tag{1.11}$$

We will use partitions of unity and work on coordinate patches in M, in a standard fashion. The symbol class $S^m \subset S^m_{1,0}$ consists of functions with an asymptotic expansion

$$p(x, \xi) \sim \sum_{j \geqq 0} p_j(x, \xi), \tag{1.12}$$

where $p_j(x, \xi)$ is homogeneous in ξ of degree $m - j$, for $|\xi|$ large.

2. Uniform Parametrix for a Singular Perturbation Problem

In this section we shall construct a uniform parametrix for solutions to initial value problems

$$\partial u/\partial t = -H_\varepsilon u, \qquad u(0) = f, \tag{2.1}$$

with H_ε a family of second order differential operators on a compact Riemannian manifold M of the form

$$-H_\varepsilon = \varepsilon^2 L + \varepsilon X + V_1. \tag{2.2}$$

We suppose

L is a negative definite strongly elliptic scalar second order differential operator on M, (2.3)

X is a first order differential operator on M (scalar), (2.4)

and

V_1 is a smooth (scalar) function on M. (2.5)

A special case of this arose in Sect. 1, namely, in local coordinates, and with the summation convention,

$$H_\varepsilon = -g^{-1/2}(\varepsilon\partial_j + iA_j) g^{jk} g^{1/2}(\varepsilon\partial_k + iA_k) + V. \tag{2.6}$$

In this case we have

$$L=\Delta\,,\quad X=2iA_j g^{jk}\partial_k+ig^{-1/2}(\partial_j g^{1/2}g^{jk}A_k)\,,\quad V_1=-V-g^{jk}A_jA_k\,. \tag{2.7}$$

In the more general situation we take, in local coordinates,

$$L=g^{jk}(x)\partial_j\partial_k+b^j(x)\partial_j+c(x)\,,\quad X=B^j(x)\partial_j\,. \tag{2.8}$$

We will use ε rather than $\hbar$ in this section as the small parameter to emphasize that we are dealing with a more general class of singular perturbation problems than arise from Sect. 1.

In local coordinates on M, the uniform parametrix will be of the form

$$U(t,\varepsilon)f=\int a(t,\varepsilon,x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi\,, \tag{2.9}$$

where the amplitude $a(t,\varepsilon,x,\xi)$ will be an asymptotic sum

$$a(t,\varepsilon,x,\xi)\sim\sum_{j\geqq 0}a_j(t,\varepsilon,x,\xi)\,, \tag{2.10}$$

the terms $a_j(t,\varepsilon,x,\xi)$ being determined by transport equations, which we proceed to derive. This derivation will have some points in common with the work [22] on hyperbolic systems with a small viscosity term, though it will be somewhat simpler. The transport equations are determined by applying $\partial/\partial t+H_\varepsilon$ to (2.9). If we set $\psi(x,\xi)=x\cdot\xi$, a straightforward computation gives

$$L(ae^{i\psi})e^{-i\psi}=-\|\xi\|^2a+2i\langle\xi,\nabla a\rangle+i(B\psi)a+La\,. \tag{2.11}$$

Here we set

$$\|\xi\|^2=g^{jk}(x)\xi_j\xi_k\,,\quad \langle\xi,v\rangle=g^{jk}(x)\xi_j v_k\,, \tag{2.12}$$

and

$$B\psi=b^j(x)\partial_j\psi\,. \tag{2.13}$$

Similarly, we have

$$X(ae^{i\psi})e^{-i\psi}=i(X^{\#}\psi)a+Xa\,,\quad X^{\#}=2iA_jg^{jk}\partial_k\,. \tag{2.14}$$

Consequently, we require of the amplitude $a=a(t,\varepsilon,x,\xi)$ that, in an appropriate sense,

$$-\partial a/\partial t-\varepsilon^2\|\xi\|^2a+2i\varepsilon^2\langle\xi,\nabla a\rangle+i\varepsilon^2(B\psi)a+\varepsilon^2La+i\varepsilon(X^{\#}\psi)a+\varepsilon Xa+V_1a\sim 0\,. \tag{2.15}$$

It will be convenient to group terms together by weight, where weights are assigned as follows:

$$-\partial a_j/\partial t-\varepsilon^2\|\xi\|^2a_j+i\varepsilon(X^{\#}\psi)a_j+V_1a_j \text{ has weight } -j\,, \tag{2.16}$$

and

$$2i\varepsilon^2\langle\xi,\nabla a_j\rangle+\varepsilon^2La_j+i\varepsilon^2(B\psi)a_j+\varepsilon Xa_j \text{ has weight } -j-1\,. \tag{2.17}$$

Our iterative procedure will consist of requiring the sums of all terms of weight $0, -1, -2$, etc., to vanish. Requiring the terms of weight 0 to sum to zero leads to the "first transport equation":

$$\partial a_0/\partial t = (-\varepsilon^2 \|\xi\|^2 + i\varepsilon(X^{\#}\psi) + V_1)a_0 . \tag{2.18}$$

Since $U(0, \varepsilon)$ is to be the identity operator, the appropriate initial condition is

$$a_0 = I \quad \text{as} \quad t = 0 . \tag{2.19}$$

More precisely, we should set $a_0 = \varphi_j(x)$ at $t = 0$, where φ_j is an element of a partition of unity, but we can safely ignore this point. We have

$$a_0(t, \varepsilon, x, \xi) = e^{t(-\varepsilon^2\|\xi\|^2 + i\varepsilon(X^{\#}\psi) + V_1)} . \tag{2.20}$$

Let us denote the exponent in (2.20) by $-t\Gamma$:

$$\Gamma(\varepsilon, x, \xi) = \varepsilon^2 \|\xi\|^2 - i\varepsilon(X^{\#}\psi) - V_1 = \tilde{\Gamma}(x, \varepsilon\xi) . \tag{2.21}$$

In other words, we have

$$a_0 = e^{-t\tilde{\Gamma}(x, \varepsilon\xi)} . \tag{2.22}$$

For $j \geqq 1$, the transport equation for a_j becomes

$$\partial a_j/\partial t = -\Gamma a_j + \Omega_j , \tag{2.23}$$

where

$$\Omega_j(t, \varepsilon, x, \xi) = 2i\varepsilon^2 \langle \xi, \nabla a_{j-1} \rangle + \varepsilon^2 L a_{j-1} + i\varepsilon^2 (B\psi) a_{j-1} + \varepsilon X a_{j-1} . \tag{2.24}$$

In this case, the appropriate initial condition is

$$a_j(0, \varepsilon, x, \xi) = 0 , \quad j \geqq 1 , \tag{2.23}$$

so the solution to (2.23) is

$$a_j = \int_0^t e^{-(t-s)\Gamma} \Omega_j(s, \varepsilon, x, \xi) ds . \tag{2.26}$$

The following gives important qualitative information on the amplitudes a_j.

Lemma 2.1. *For each j, there is a smooth function* $\tilde{a}_j(t, \varepsilon, x, \zeta)$, *such that*

$$a_j(t, \varepsilon, x, \xi) = t^j \varepsilon^j \tilde{a}_j(t, \varepsilon, x, \varepsilon\xi) e^{-t\tilde{\Gamma}(x, \varepsilon\xi)} . \tag{2.27}$$

More precisely,

$$\tilde{a}_j(t, \varepsilon, x, \varepsilon\xi) = a_j^{\#}(t, \varepsilon, x, \zeta, \omega, \sigma) , \tag{2.28}$$

with

$$\zeta = \varepsilon\xi , \quad \omega = t^{1/2}\varepsilon\xi , \quad \sigma = t\varepsilon\xi , \tag{2.29}$$

where $a_j^{\#}$ *is smooth in all its arguments, and a polynomial in* ε, ω, ζ, *and* σ. *It contains only even powers of* ω. *Its order in* ζ *is not greater than j.*

Proof. Write

$$t\tilde{\Gamma}(x, \varepsilon\xi) = \Gamma^{\#}(x, \omega, \sigma) = \|\omega\|^2 - V_1 + iX^{\#}\psi_1 , \quad \psi_1 = x \cdot \sigma .$$

We use induction on j. For $j=0$, the result follows from (2.22). Suppose (2.27) and (2.28) hold. We will verify the analogous formulas for a_{j+1}. First of all, by (2.24) with j replaced by $j+1$, we see that

$$\Omega_{j+1}=t^j\varepsilon^{j+1}\Omega_j^{\#}(t,\varepsilon,x,\zeta,\omega,\sigma)e^{-\Gamma^{\#}(x,\omega,\sigma)},$$

where $\Omega_j^{\#}$ is a polynomial in ε, ζ, ω, σ, even in ω, whose order in ζ can exceed that of $a_j^{\#}$ by at most 1. Now (2.26), with j replaced by $j+1$, gives

$$a_{j+1}(t,\varepsilon,x,\xi)=t^{j+1}\varepsilon^{j+1}\left(t^{-j-1}\int_0^t \Omega_{j+1}^{\#}(s,\varepsilon,x,\zeta,s^{1/2}\zeta,s\zeta)s^j ds\right)e^{-\Gamma^{\#}(x,\omega,\sigma)},$$

so the degree of $a_{j+1}^{\#}(t,\varepsilon,x,\zeta,\omega,\sigma)$ in ζ exceeds that of $a_j^{\#}$ by at most 1. This completes the proof.

Recall from (2.21) that

$$\tilde{\Gamma}(x,\zeta)=\|\zeta\|^2-iX^{\#}\psi_2-V_1 \quad (\psi_2=x\cdot\zeta) \tag{2.30}$$

is a second order polynomial in ζ, with real part satisfying

$$\operatorname{Re}\tilde{\Gamma}(x,\zeta)\geqq C|\zeta|^2-C'.$$

This enables us to prove a result on uniform boundedness of the amplitudes a_j in appropriate symbol classes. Recall that a smooth function $p(x,\xi)$ is said to be in the symbol class $S^m_{1,0}$ provided

$$|D_x^\beta D_\xi^\alpha p(x,\xi)|\leqq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}. \tag{2.31}$$

In that case we say $p(x,D)$ belongs to $OPS^m_{1,0}$, where

$$p(x,D)f=\int p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi. \tag{2.32}$$

A bounded subset of $S^m_{1,0}$ is a set of functions satisfying the estimates (2.31) with uniform bounds $C_{\alpha\beta}$, and they give rise to a bounded family of operators in $OPS^m_{1,0}$.

Lemma 2.2. *Fix positive T and E. For $0\leqq t\leqq T$, $0\leqq\varepsilon\leqq E$, we have*

$$a_j(t,\varepsilon,\cdot,\cdot)\ \textit{bounded on}\ S^{-j}_{1,0}, \tag{2.33}$$

and, for $0\leqq\ell\leqq j$,

$$\varepsilon^{-\ell}a_j(t,\varepsilon,\cdot,\cdot)\ \textit{bounded in}\ S^{-(j-\ell)}_{1,0}. \tag{2.34}$$

Proof. Since $t^{j/2}|\zeta|^je^{-t\tilde{\Gamma}/4}=t^{j/2}\varepsilon^j|\xi|^je^{-t\tilde{\Gamma}/4}\leqq K_j$, we deduce from (2.27)–(2.28) that

$$|a_j|\leqq C_jt^j\varepsilon^j|\zeta|^je^{-t\tilde{\Gamma}/2}\leqq C'_jt^{j/2}\varepsilon^je^{-t\tilde{\Gamma}/4}\leqq C''_j(1+|\xi|)^{-j}.$$

Derivatives of a_j have similar estimates, so (2.33) and (2.34) follow. Note that taking k ε-derivatives raises the order by $2k$, as does taking k t-derivatives.

Let us now consider a partial sum of the expansion (2.10):

$$A_\ell(t,\varepsilon,x,\xi)=\sum_{j=0}^{\ell}a_j(t,\varepsilon,x,\xi). \tag{2.35}$$

Form

$$W_\ell(t)f(x)=\int A_\ell(t,\varepsilon,x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi\,. \tag{2.36}$$

More precisely, using a partition of unity subordinate to a coordinate chart, paste together operators of the form (2.36) to form $W_\ell(t)$. We have

$$W_\ell(0)=I\,, \tag{2.31}$$

and

$$(\partial/\partial t+H_\varepsilon)W_\ell(t)f(x)=\int B_\ell(t,\varepsilon,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi\,, \tag{2.38}$$

where B_ℓ is an expression of the form (2.17) with $j=\ell$, i.e., $B_\ell=\Omega_\ell$. We conclude from Lemma 2.2 that

$$\begin{aligned}|D_x^\beta D_\xi^\alpha B_\ell(t,\varepsilon,x,\xi)|&\leqq C_{\ell\alpha\beta}(t^{1/2}\varepsilon)^{\ell-2}(1+|\xi|)^{-|\alpha|}e^{-t\tilde{r}/4}\\&\leqq C'_{\ell\alpha\beta}(1+|\xi|)^{-(\ell-2)-|\alpha|}\quad\text{if}\quad \ell\geqq 2\,.\end{aligned} \tag{2.39}$$

In order to compare $W_\ell(t)$ and e^{-tH_ε}, we need the following energy estimate:

Lemma 2.3. *Let $v(t,\varepsilon,x)$ satisfy*

$$(\partial/\partial t+H_\varepsilon)v=g(t,\varepsilon,x)\,,\qquad v(0,\varepsilon,x)=h(\varepsilon,x)\,. \tag{2.40}$$

Then, for $0\leqq t\leqq T$, $0\leqq\varepsilon\leqq E$, we have

$$\sup_t\|v(t,\varepsilon,\cdot)\|_{H^s(M)}\leqq C_1\|h(\varepsilon,\cdot)\|_{H^s}+C_2\sup_t\|g(t,\varepsilon,\cdot)\|_{H^s}\,, \tag{2.41}$$

where C_1 and C_2 are independent of $t\in[0,T]$ and $\varepsilon\in[0,E]$. One has similar estimates on

$$\left\|\sup_t D_t^\mu D_\varepsilon^\nu v(t,\varepsilon,\cdot)\right\|_{H^{s-2(\mu+\nu)}}\,. \tag{2.42}$$

Proof. Let $(\,,)_s$ denote a Hilbert space inner product on the Sobolev space $H^s(M)$. If we apply Gårding's inequality to $(-Lu,u)_s$, we get the estimate

$$\begin{aligned}\operatorname{Re}(H_\varepsilon u,u)_s&=\varepsilon^2\operatorname{Re}(-Lu,u)_s+\varepsilon\operatorname{Re}(-Xu,u)_s+(-V_1u,u)_s\\&\geqq C_0\varepsilon^2\|u\|_{s+1}^2-K\varepsilon\|u\|_{s+1}\|u\|_s-C_1\|u\|_s^2\\&\geqq C_0\varepsilon^2\|u\|_{s+1}^2-[\tfrac{1}{2}C_0\varepsilon^2\|u\|_{s+1}^2+2K^2C_0^{-1}\|u\|_s^2]-C_1\|u\|_s^2\\&\geqq\tfrac{1}{2}C_0\varepsilon^2\|u\|_{s+1}^2-(C_1+2K^2C_0^{-1})\|u\|_s^2\\&\geqq -C\|u\|_s^2\,.\end{aligned} \tag{2.43}$$

Here, C_0, C_1, K, and C are all positive and independent of $\varepsilon\in[0,E]$. It follows that

$$e^{-tH_\varepsilon}:H^s(M)\to H^s(M) \tag{2.44}$$

with operator norm

$$\|e^{-tH_\varepsilon}\|_{\mathscr{L}(H^s)}\leqq C(s)e^{tK(s)} \tag{2.45}$$

bounded independently of $\varepsilon \in [0, E]$. Now Duhamel's principle gives for the solution to (2.40):

$$v(t, \varepsilon, x) = e^{-tH_\varepsilon} h + \int_0^t e^{-(t-\tau)H_\varepsilon} g(\tau, \varepsilon, x) d\tau, \tag{2.46}$$

from which (2.41) follows. Differentiating (2.40) with respect to ε gives for $v_1 = \partial v/\partial \varepsilon$,

$$(\partial/\partial t + H_\varepsilon) v_1 = \partial g/\partial \varepsilon + (2\varepsilon L v + X v), \quad v_1|_{t=0} = \partial h/\partial \varepsilon. \tag{2.47}$$

The analysis just given yields bounds on v_1, and inductively, one analyses $v_j = \partial^j v/\partial \varepsilon^j$; t derivatives are bounded similarly, and the proof of Lemma 2.3 is complete.

Now we can asymptotically sum:

$$U(t, \varepsilon) \sim \sum_{j \geqq 0} U_j(t, \varepsilon), \tag{2.48}$$

where

$$U_j(t, \varepsilon) f(x) = \int a_j(t, \varepsilon, x, \xi) e^{ix \cdot \xi} d\xi, \tag{2.49}$$

and conclude that, for any $f \in \mathscr{D}'(M)$,

$$e^{-tH_\varepsilon} f - U(t, \varepsilon) f = h(\varepsilon, t, x) \tag{2.50}$$

is a smooth function of $(\varepsilon, t, x) \in [0, E] \times [0, T] \times M$, which is rapidly decreasing as $\varepsilon \to 0$. In particular, for any $t > 0$,

$$\operatorname{trace} e^{-tH_\varepsilon} - \operatorname{trace} U(t, \varepsilon) \tag{2.51}$$

is rapidly decreasing as $\varepsilon \to 0$. We proceed to produce an asymptotic expansion for $\operatorname{trace} U(t, \varepsilon)$ as $\varepsilon \to 0$, with $t > 0$ fixed. We analyze $\operatorname{trace} U_j(t, \varepsilon)$, where $U_j(t, \varepsilon)$ is given by (2.49). If we write, in local coordinates,

$$U_j(t, \varepsilon) f(x) = \int W_j(t, \varepsilon, x, y) f(y) dy, \tag{2.52}$$

we have

$$W_j(t, \varepsilon, x, y) = \int a_j(t, \varepsilon, x, \xi) e^{i(x-y)\cdot \xi} d\xi. \tag{2.53}$$

In particular

$$W_j(t, \varepsilon, x, x) = \int a_j(t, \varepsilon, x, \xi) d\xi, \tag{2.54}$$

and hence,

$$\operatorname{trace} U_j(t, \varepsilon) = \int_M W_j(t, \varepsilon, x, x) d\operatorname{vol}(x) = \iint a_j(t, \varepsilon, x, \xi) d\xi d\operatorname{vol}(x). \tag{2.55}$$

Now formula (2.27) gives

$$\operatorname{trace} U_j(t, \varepsilon) = t^j \varepsilon^j \iint \tilde{a}_j(t, \varepsilon, x, \varepsilon\xi) e^{-t\tilde{\Gamma}(x, \varepsilon\xi)} d\xi d\operatorname{vol}(x). \tag{2.56}$$

A change of variable gives

$$\int \tilde{a}_j(t, \varepsilon, x, \varepsilon\xi) e^{-t\tilde{\Gamma}(x, \varepsilon\xi)} d\xi = \varepsilon^{-v} A_j(t, \varepsilon, x) \quad (v = \dim M), \tag{2.57}$$

and hence,

$$\operatorname{trace} U_j(t, \varepsilon) = \varepsilon^{-\nu+j} \int_M B_j(t, \varepsilon, x) d\,\mathrm{vol}(x), \tag{2.58}$$

where $B_j(t, \varepsilon, x) = t^j A_j(t, \varepsilon, x)$ is smooth on $(0, \infty) \times [0, E] \times M$. In light of the rapid decrease of (2.51), we deduce the following.

Proposition 2.4. *With* $\nu = \dim M$, $t > 0$ *fixed, we have an asymptotic expansion as* $\varepsilon \to 0$:

$$\operatorname{trace} e^{-tH_\varepsilon} \sim \varepsilon^{-\nu}(B_0(t) + \varepsilon B_1(t) + \varepsilon^2 B_2(t) + \ldots). \tag{2.59}$$

The leading coefficient is

$$B_0(t) = \iint e^{-t\tilde{\Gamma}(x, \zeta)} d\zeta d\,\mathrm{vol}(x). \tag{2.60}$$

In case H_ε is given by (2.6), we have

$$B_0(t) = \iint \exp[-t(\xi_j - A_j(x)) g^{jk}(\xi_k - A_k(x)) - tV] d\xi d\,\mathrm{vol}(x), \tag{2.61}$$

and we can change variables to get

$$B_0(t) = \iint \exp[-t\|\xi\|^2 - tV] d\xi d\,\mathrm{vol}(x). \tag{2.62}$$

This special case of Proposition 2.4 implies Theorem A in the scalar case.

We now turn to the modification of the analysis above that is required if one is to generalize (2.2), allowing X and V_1 to be $K \times K$ matrices, rather than merely scalars. In fact, we will stick to the construction of the amplitude, via (2.22)–(2.26). In this more general situation, each a_j is a $K \times K$ matrix valued function. Now behind the proof of Lemma 2.1 is the identity

$$(\partial/\partial x_j) e^{-\Gamma^\#(x, \omega, \sigma)} = -(\partial \Gamma^\# / \partial x_j) e^{-\Gamma^\#}, \tag{2.63}$$

valid when $\Gamma^\#$ is scalar, but not valid for general matrix valued $\Gamma^\#$. To treat the more general case, we need a replacement for (2.63). To phrase more precisely what we want, note that

$$\Gamma^\# = t\tilde{\Gamma} = t\|\varepsilon\xi\|^2 - 2A_j(x) g^{jk}(x) \sigma_k - tV_1(x),$$

so

$$e^{-\Gamma^\#} = e^{-t\|\varepsilon\xi\|^2} \exp[2A_j(x) g^{jk} \sigma_k + tV_1]. \tag{2.64}$$

The first factor on the right side of (2.64) is scalar, so we need to understand how to differentiate the last factor. Let us set

$$L(A_1, \ldots, A_\nu, V_1; x, \sigma, t) = 2A_j g^{jk}(x) \sigma_k + tV_1, \tag{2.65}$$

so

$$e^{-\Gamma^\#} = \exp L(A_1, \ldots, A_\nu, V_1; x, \sigma, t) e^{-t\|\varepsilon\xi\|^2}. \tag{2.66}$$

Now we want to understand derivatives with respect to x of $\exp L(x)$ for a general smooth $K \times K$ matrix function $L(x)$; we will adopt the hypothesis that $L(x)$ is self-adjoint for each x, so we assume

$$A_1, \ldots, A_\nu \text{ and } V_1 \text{ are self-adjoint}. \tag{2.67}$$

More generally, we have a formula for $(\partial/\partial x_j)f(L(x))$, where $L(x)$ is self-adjoint and f is a smooth function on R, so $f(L(x))$ is defined by the spectral theorem, given as follows:

$$(\partial/\partial x_j)f(L(x)) = g(\partial L/\partial x_j, L), \tag{2.68}$$

where g is a smooth function on R^2 defined by

$$g(s, \tau) = sf'(\tau). \tag{2.69}$$

The right side of (2.68), a function of two (generally noncommuting) self adjoint matrices, is defined by the Weyl calculus. More generally, if $(T_1, \ldots, T_k)$ is a k-tuple of bounded self-adjoint operators on some Hilbert space, and $f \in \mathcal{S}(R^k)$, one defines the Weyl calculus by

$$f(T) = (2\pi)^{-k/2} \int \hat{f}(\xi) \exp[i\xi_1 T_1 + \ldots + i\xi_k T_k] d\xi. \tag{2.70}$$

It is easy to verify, using the Paley-Wiener theorem, that for a given $(T_1, \ldots, T_k)$, with $\sum \|T_j\|^2 \leqq M^2$, $f(T)$ depends only on the restriction of f to the closed ball B_M of radius M, and the norm of (2.70) satisfies an estimate:

$$\sum \|T_j\|^2 \leqq M^2 \Rightarrow \|f(T)\| \leqq C_{k,M} \|f\|_{C^k(B_M)}. \tag{2.71}$$

See [25]. Then we have a natural extension of $f(T)$ from $f \in \mathcal{S}(\mathbb{R}^k)$ to $f \in C^\infty(\mathbb{R}^k)$, and if, for example, $f(\tau_1, \ldots, \tau_k) = \varphi(\alpha_1 \tau_1 + \ldots + \alpha_k \tau_k)$, $\alpha_j \in R$, then $f(T)$ defined by such a Weyl calculus is equal to $\varphi(\alpha_1 T_1 + \ldots + \alpha_k T_k)$, defined by the spectral theorem. The identity (2.68) and (2.69) is proved in [25], and is a simple consequence of the elementary identities

$$\begin{aligned}(\partial/\partial x_j)L(x)^k = (\partial L/\partial x_j)L(x) \ldots L(x) + L(x)(\partial L/\partial x_j)L(x) \ldots L(x) + \ldots \\ + L(x) \ldots L(x)(\partial L/\partial x_j),\end{aligned} \tag{2.72}$$

where each term on the right in (2.72) contains $k-1$ factors of $L(x)$, and there are k terms. More generally, we have

$$(\partial/\partial x_j)f(L_1(x), \ldots, L_k(x)) = \sum_{\ell=1}^{k} g_\ell(\partial L_\ell/\partial x_j, L_1(x), \ldots, L_k(x)), \tag{2.73}$$

with

$$g_\ell(s, \tau_1, \ldots, \tau_k) = s\partial f/\partial \tau_\ell,$$

where both sides of (2.73) are evaluated via the Weyl calculus.

We will use these results on the Weyl calculus to prove the following analogue of Lemma 2.1.

Lemma 2.5. *Under the hypothesis* (2.67), *we have*

$$a_j(t, \varepsilon, x, \xi) = \varepsilon^j t^j b_j(t, \varepsilon, x, \varepsilon\xi) e^{-t\|\varepsilon\xi\|^2} \tag{2.75}$$

with a polynomial representation

$$b_j(t, \varepsilon, x, \varepsilon\xi) = \sum_{\substack{|\alpha| \leqq j \\ |\beta| \text{ even}}} E_{j\alpha\beta}(t, \varepsilon, x, \zeta)\omega^\beta \zeta^\alpha, \tag{2.76}$$

where

$$\zeta=\varepsilon\xi, \qquad \omega=t^{1/2}\varepsilon\xi, \tag{2.77}$$

and, for some $B=B_{j\sigma\gamma}<\infty$, *the* $K\times K$ *matrix function* $E_{j\alpha\beta}$ *satisfies*

$$|D_x^\gamma D_\xi^\sigma E_{j\alpha\beta}(t,\varepsilon,x,\varepsilon\xi)|\leqq C_{j\alpha\beta\gamma\sigma}(\varepsilon t)^{|\sigma|}e^{Bt|\zeta|}. \tag{2.78}$$

Proof. The proof will proceed by induction on j. For $j=0$ we have

$$b_0(t,\varepsilon,x,\varepsilon\xi)=\exp L(A_1(x),\ldots,A_\nu(x),V_1(x);x,\sigma,t)=b_0^\#(t,x,\sigma), \tag{2.79}$$

where L is given by (2.65), $\sigma=t\varepsilon\xi$, and the right side of (2.79) can be taken to be defined by the Weyl calculus, as mentioned in the discussion above. We have $E_0(t,\varepsilon,x,\zeta)=b_0(t,\varepsilon,x,\zeta)$, and we want to establish (2.78). Note that the absolute value on the left side of (2.78) refers to the operator norm on $\mathbb{C}^K$, since $E_{j\alpha\beta}$ is a $K\times K$ matrix. Now we have

$$\partial b_0/\partial x_j=F_2(\partial L/\partial x_j, L), \tag{2.80}$$

where

$$F_2(s,\tau)=se^\tau, \tag{2.81}$$

and

$$\partial b_0/\partial\xi_j=t\varepsilon F_2(\partial L/\partial\sigma_j, L). \tag{2.82}$$

Using the estimate (2.71), we see that (2.78) holds for E_0 if $|\gamma|+|\sigma|=1$. The required estimates on $D_x^\gamma D_\xi^\sigma E_0$ for general γ and σ follow similarly, and the result is established for $j=0$. Now we want to show that, if (2.75)–(2.78) hold for a_j, the analogous results hold for a_{j+1}. Recall that

$$a_{j+1}(t,\varepsilon,x,\xi)=\int_0^t e^{-(t-s)\tilde{\Gamma}}\Omega_{j+1}(s,\varepsilon,x,\xi)ds, \tag{2.83}$$

where Ω_{j+1} is given by (2.24), i.e.,

$$\Omega_{j+1}(t,\varepsilon,x,\xi)=2i\varepsilon\langle\zeta,\nabla_x a_j\rangle+\varepsilon^2La_j+\beta(x,\zeta)a_j+\varepsilon Xa_j, \tag{2.84}$$

where $\beta(x,\zeta)$ is linear in ζ. It follows that, if (2.25)–(2.78) hold for a_j, then

$$\Omega_{j+1}(t,\varepsilon,x,\zeta)=\varepsilon^{j+1}t^j\sum_{\substack{|\alpha|\leqq j+1\\|\beta|\,\text{even}}}F_{j\alpha\beta}(t,\varepsilon,x,\zeta)\omega^\beta\zeta^\alpha e^{-t\|\varepsilon\xi\|^2}, \tag{2.85}$$

where $F_{j\alpha\beta}$ satisfies the estimates (2.78). Consequently, a_{j+1} is of the form (2.75)–(2.76), with

$$E_{j+1,\alpha\beta}(t,\varepsilon,x,\zeta)=t^{-(j+1+\frac{1}{2}|\beta|)}\int_0^t b_0^\#(t-s,x,(t-s)\zeta)F_{j\alpha\beta}(s,\varepsilon,x,\zeta)s^{j+\frac{1}{2}|\beta|}ds. \tag{2.86}$$

Since (2.78) holds for $b_0^\#$ and for $F_{j\alpha\beta}$, such an estimate also holds for $E_{j+1,\alpha\beta}$. The proof of Lemma 2.5 is complete.

From Lemma 2.5 one concludes that

$$\begin{aligned}|a_j| &\leqq C(\varepsilon t)^j \langle\omega\rangle^J \langle\zeta\rangle^j e^{Bt|\zeta|} e^{-t\|\zeta\|^2} \quad (J=J_j)\\ &\leqq C'(t^{1/2}\varepsilon)^j e^{-t\|\zeta\|^2/4}\\ &\leqq C''(1+|\xi|)^{-j},\end{aligned} \tag{2.87}$$

with similar estimates on the derivatives, so Lemma 2.2 continues to hold. Lemma 2.3 also holds, with no change, and hence (2.51) is still rapidly decreasing as $\varepsilon\to 0$, so for any $t>0$, as $\varepsilon\to 0$,

$$\operatorname{trace} e^{-tH_\varepsilon} \sim \sum_{j\geqq 0} \operatorname{trace} U_j(t,\varepsilon). \tag{2.88}$$

In this case, replacing (2.56)–(2.58), we have

$$\begin{aligned}\operatorname{trace} U_j(t,\varepsilon) &= t^j \varepsilon^j \iint \operatorname{tr} b_j(t,\varepsilon,x,\varepsilon\xi) e^{-t\|\varepsilon\xi\|^2} d\xi\, d\operatorname{vol}(x)\\ &= t^j \varepsilon^{-\nu+j} \iint \operatorname{tr} b_j(t,\varepsilon,x,\zeta) e^{-t\|\zeta\|^2} d\zeta\, d\operatorname{vol}(x).\end{aligned} \tag{2.89}$$

Above, we take the trace of the $K\times K$ matrix valued integrand. Thus Proposition 2.4 continues to hold in this more general case, with the formula for the leading term modified to

$$B_0(t) = \iint \operatorname{tr} e^{-t\tilde{r}(x,\zeta)} d\zeta\, d\operatorname{vol}(x). \tag{2.90}$$

In particular, if H_ε is given by (2.6), with $A_j(x)$ [and possibly $V(x)$] self-adjoint matrix valued, we have

$$B_0(t) = \iint \operatorname{tr} \exp[-t(\xi_j - A_j(x)) g^{jk}(x)(\xi_k - A_k(x)) - tV] d\xi\, d\operatorname{vol}(x). \tag{2.91}$$

As opposed to the scalar case (2.61)–(2.62), (2.91) need not be independent of the matrix terms $A_j(x)$.

As a final comment in this section, we note that even the self-adjointness hypothesis (2.67) could be dropped. We would need to exploit the complex analyticity of the functions being applied to matrices above, and replace (2.70) by the Dunford calculus. The identity (2.68)–(2.69) is also well-known in this context. In fact, we could use the following more elementary derivation of $(\partial/\partial x_j)e^{L(x)}$. If we let $u(t)=e^{tL(x)}f$, then $v=\partial u/\partial x_j$ solves

$$\partial v/\partial t = L(x)v + (\partial L/\partial x_j)u, \qquad v(0)=0,$$

so Duhamel's principle gives $v=\int_0^t e^{(t-s)L(x)}(\partial L/\partial x_j)u(s)ds$, so

$$(\partial/\partial x_j)e^{tL(x)} = \int_0^t e^{(t-s)L(x)}(\partial L/\partial x_j)e^{sL(x)}ds,$$

and setting $t=1$ gives

$$(\partial/\partial x_j)e^{L(x)} = \int_0^1 e^{(1-s)L(x)}(\partial L/\partial x_j)e^{sL(x)}ds. \tag{2.92}$$

This provides a perfectly adequate replacement for (2.80) and (2.81) in the proof of Lemma 2.5.

3. Harmonic Analysis on a Compact Lie Group

The purpose of this section is to bring together facts about harmonic analysis on a compact Lie group G which will be needed in Sects. 4 and 5. First, we recall some basic facts about the representation theory of G; our references for this material are the books of Wallach [28] and Zelobenko [31].

The irreducible unitary representations π_λ of G are naturally indexed by $\lambda \in \mathscr{L} \cap C$, where C is a convex cone in $\mathbb{R}^k$ called a Weyl chamber. Here k is the dimension of a maximal torus $\mathbb{T}^k$ of G, and $\mathbb{R}^k$ is identified with $T_e^*\mathbb{T}^k$. $\mathscr{L}$ is a lattice in $\mathbb{R}^k$. The entries $\pi_\lambda^{ij}(x)$ of the matrix π_λ are functions on G. The Peter-Weyl theorem implies

$$\{\sqrt{d_\lambda}\,\pi_\lambda^{ij}: \lambda \in \mathscr{L} \cap C,\ 1 \leqq i,j \leqq d_\lambda\} \text{ is an orthonormal basis of } L^2(G). \tag{3.1}$$

Here d_λ is the dimension of the representation space of π_λ. It is given by Weyl's formula

$$d_\lambda = \prod_{\alpha \in P} \langle \lambda + \delta, \alpha \rangle / \langle \delta, \alpha \rangle . \tag{3.2}$$

Here $\delta \in \mathbb{R}^k$ is half the sum of the positive roots, P is the set of positive roots, and the inner product is induced by the Killing form.

If P is any bi-invariant differential operator on G, then $\{\pi_\lambda^{ij}\}$ belong to an eigenspace of P, for any fixed λ. An example of this is $P = \Delta$, the Laplacian on G, endowed with a bi-invariant Riemannian metric (which induces a metric on $\mathbb{R}^k \approx T_e^*\mathbb{T}^k$). In this case we have

$$-\Delta \pi_\lambda^{ij} = (\|\lambda + \delta\|^2 - \|\delta\|^2)\pi_\lambda^{ij} . \tag{3.3}$$

As before, $\delta \in \mathbb{R}^k$ is half the sum of the positive roots. This result has the following important generalization, proved in Zelobenko [31, p. 369]:

Theorem 3.1. *Let $q_m(\lambda)$ be any homogeneous polynomial on $\mathbb{R}^k$, of degree m, which is invariant under the Weyl group. There exists a bi-invariant differential operator Q, of order m, such that*

$$Q\pi_\lambda^{ij} = q_m(\lambda + \delta)\pi_\lambda^{ij} . \tag{3.4}$$

Conversely, if Q is a bi-invariant differential operator of order m, then (3.4) *holds, for some $q_m(\lambda)$, polynomial of order m, invariant under the Weyl group (perhaps not homogeneous).*

The Weyl group is the group of linear transformations on $\mathbb{R}^k = T_e^*\mathbb{T}^k$ induced by inner automorphisms of G which leave $\mathbb{T}^k$ invariant. It is a finite group, generated by reflections across the walls of the Weyl chamber. For more details, see [28], or [31].

It follows from the proof of Theorem 3.1 that the relation between $q_m(\lambda)$ and $q_m(e, \xi)$, the principal symbol of Q, is the following. We think of $\lambda \in T_e^*\mathbb{T}^k$ included in $T_e^*(G)$, and then $q_m(\lambda) = q_m(e, \lambda)$. Since $q_m(e, \xi)$ is invariant under the coadjoint action of G on T_e^*G, this uniquely specifies $q_m(e, \xi)$. We note the assertion that restriction to $\mathbb{R}^k$, giving $q_m(e, \xi) \mapsto q_m(\lambda)$, is an isomorphism from the space of polynomials (homogeneous of degree m) on T_e^*G invariant under the coadjoint action of G onto the space of polynomials (homogeneous of degree m) on $\mathbb{R}^k$ invariant under the Weyl group, a theorem of Chevalley.

In view of Chevalley's theorem, results of Schwartz [17] or Mather [15] imply the following. The restriction map gives an isomorphism from the space of functions in $C^\infty(T_e^*G\backslash 0)$, homogeneous of degree m, invariant under the coadjoint group, onto the space of functions in $C^\infty(\mathbb{R}^k\backslash 0)$, homogeneous of degree m, invariant under the Weyl group. One then obtains the following pseudodifferential operator analogue of Theorem 3.1; for a proof, see Chap. XII, Sect. 6, of [21].

Theorem 3.2. *Let $q(\lambda)\in S^m(\mathbb{R}^k)$ be invariant under the Weyl group. Then there exists $Q\in OPS^m(G)$, bi-invariant, such that*

$$Q\pi_\lambda^{ij}=q(\lambda+\delta)\pi_\lambda^{ij}. \tag{3.5}$$

Conversely, for any bi-invariant $Q\in OPS^m(G)$, there is a $q(\lambda)\in S^m(\mathbb{R}^k)$, invariant under the Weyl group, such that (3.5) *holds. The principal symbol $q_m(x,\xi)$ of Q and the principal term $q_m(\lambda)$ in the expansion of q are related by the identity*

$$q_m(e,\lambda)=q_m(\lambda), \qquad \lambda\in\mathbb{R}^k\subset T_e^*G, \tag{3.6}$$

which uniquely determines the correspondence between $q_m(x,\xi)$ and $q_m(\lambda)$.

As an application of Theorem 3.2, we will analyze the behavior as $n\to\infty$ of the quantities

$$d_n^{-1}\,\mathrm{trace}\, e^{\pi_n(X)/n}, \qquad X\in\mathfrak{g}. \tag{3.7}$$

Here π_n is the irreducible representation of G corresponding to the point $n\lambda_1$ in $\mathscr{L}\cap C$, where $\lambda_1\in\mathscr{L}\cap C$ corresponds to an irreducible representation π_1 of G; $d_n=d_{\pi_n}$ is the dimension of the representation space of π_n, given by (3.2). The limit of (3.7) was obtained in [18] and was used in [11] as a preliminary step toward analyzing the limiting behavior of quantum partition functions. The result we obtain here is more precise since it is a complete asymptotic expansion.

It is convenient to alter the exponent in (3.7) slightly, replacing $1/n$ by $1/\|n\lambda_1+\delta\|$, so we will study the limiting behavior of

$$\varphi(n)=d_n^{-1}\,\mathrm{trace}\, e^{\pi_n(X)/\|n\lambda_1+\delta\|}, \qquad X\in\mathfrak{g}. \tag{3.8}$$

Now, consider the pseudodifferential operator

$$A=(-\Delta+\|\delta\|^2)^{1/2}\in OPS^1. \tag{3.9}$$

It follows from (3.3) that

$$A\pi_\lambda^{ij}=\|\lambda+\delta\|\pi_\lambda^{ij}, \tag{3.10}$$

so

$$e^{\pi_n(X)/\|n\lambda_1+\delta\|}=e^{A^{-1}X} \tag{3.11}$$

on the linear span of $\{\pi_{n\lambda_1}^{ij}: 1\leqq i,j\leqq d_n\}$, which is a direct sum of d_n copies of π_n. Note that $A^{-1}X\in OPS^0$, and hence, $e^{A^{-1}X}\in OPS^0(G)$.

Now one way to describe the quantity $\varphi(n)$ in (3.8) is as follows. Conjugate $e^{\pi_n(X)/\|n\lambda_1+\delta\|}$ by the action of $\pi_n(g)$, and average over $g\in G$. The resulting operator is a scalar, namely $\varphi(n)I$. Note that, with R_g denoting the right regular representation,

$$\int_G R_g^{-1} e^{A^{-1}X} R_g dg = T \in OPS^0(G) \tag{3.12}$$

is a bi-invariant operator. By Theorem 3.2 we have

$$T\pi_\lambda^{ij} = \tau(\lambda+\delta)\pi_\lambda^{ij} \tag{3.13}$$

for some $\tau \in S^0(\mathbb{R}^k)$, i.e.,

$$\tau(\lambda) \sim \tau_0(\lambda) + \tau_1(\lambda) + \dots, \tag{3.14}$$

with $\tau_j(\lambda)$ homogeneous of degree $-j$ in λ. Note, however, that

$$\varphi(n) = \tau(n\lambda_1 + \delta). \tag{3.15}$$

We have the following result.

Proposition 3.3. *For any given $X \in \mathfrak{g}$, as $n \to \infty$, there is an asymptotic expansion*

$$d_n^{-1} \operatorname{trace} e^{\pi_n(X)/\|n\lambda_1+\delta\|} \sim \tau_0(n\lambda_1+\delta) + \tau_1(n\lambda_1+\delta) + \tau_2(n\lambda_1+\delta) + \dots, \tag{3.16}$$

with $\tau_j(\lambda)$ homogeneous of degree $-j$ in λ. In particular, the leading term is

$$\tau_0(\lambda) = \int_G e^{i\langle \operatorname{Ad} gX, \lambda\rangle/\|\lambda\|} dg. \tag{3.17}$$

Here we regard $\lambda \in \mathbb{R}^k \subset \mathfrak{g}^*$.

Note that (3.17) arises from (3.12), in view of the formula for the principal symbol of a pseudodifferential operator conjugated by the action of a diffeomorphism. We can rewrite the inner product in the exponent of (3.17) as $\langle X, \operatorname{Ad}^* g\lambda\rangle$. As g ranges over G, $\operatorname{Ad}^* g\lambda$ ranges over a coadjoint orbit Γ_λ of G in $\mathfrak{g}^*$. Γ_λ is a dilate of a coadjoint orbit $\Gamma_{\lambda/\|\lambda\|} = \Gamma_\lambda^\#$, and we see that (3.17) is equivalent to

$$\tau_0(\lambda) = \int_{\Gamma_\lambda^\#} e^{i\langle X, \ell\rangle} d\mu_\lambda(\ell), \tag{3.18}$$

where $d\mu_\lambda(\ell)$ is the natural homogeneous probability measure on the orbit $\Gamma_\lambda^\#$. This is the form in which the limit was written in [18, 11]. In the latter paper it was shown that $\varphi(n) - \tau_0(n\lambda_1+\delta)$ is bounded by a constant times $n^{-1/2}$; the arguments of [11] can be improved to obtain the sharp bound implied by (3.16), namely a constant times n^{-1}.

It is clear that we could replace the vector field X by any left invariant pseudodifferential operator $X = \varkappa(x, D) \in OPS^1(G)$, and then the leading term in (3.16) is

$$\tau_0(\lambda) = \int_{\Gamma_\lambda^\#} e^{\varkappa(e,\ell)} d\mu_\lambda(\ell). \tag{3.19}$$

We will now re-derive the complete asymptotic expansion for $d_n^{-1} \operatorname{trace} e^{\pi_n(X)/n}$, equivalent to (3.16), using the study of maximal weight vectors. This tool figures into the method of coherent projections, which we will use in Sect. 5.

If a choice is made of the set of positive roots Δ^+ of $\mathfrak{g}$, as those which are positive on the Weyl chamber C, then the irreducible representation π_λ, $\lambda \in \mathscr{L} \cap C$, has λ as its highest weight, with respect to the induced ordering on $\mathscr{L}$, and there is a unique λ-weight vector ψ_λ up to a complex scalar. Choose ψ_λ of norm 1; it is called the highest weight vector. It is annihilated by all "raising operators", i.e., by all root vectors in $\mathfrak{g}_\mu$ for $\mu\in\Delta^+$. The following "generating function" plays an important role in the representation theory of G, including the Borel-Weil theorem (see [28, 31]):

$$\Psi_\lambda(g)=\langle\pi_\lambda(g)\psi_\lambda,\psi_\lambda\rangle\,,\qquad \lambda\in\mathscr{L}\cap C\,,\qquad g\in G\,. \tag{3.20}$$

Note that, although ψ_λ is defined only up to a phase $e^{i\theta}$, $\Psi_\lambda(g)$ is independent of this phase factor. Now given $\lambda,\mu\in\mathscr{L}\cap C$, $\lambda+\mu$ is the highest weight for the representation $\pi_\lambda\otimes\pi_\mu$, and the unique highest weight vector is $\psi_\lambda\otimes\psi_\mu$; this vector is hence contained in a copy of $\pi_{\lambda+\mu}$. From this one has the following simple but remarkable identity:

$$\Psi_\lambda(g)\Psi_\mu(g)=\Psi_{\lambda+\mu}(g)\,,\qquad \lambda,\mu\in\mathscr{L}\cap C\,. \tag{3.21}$$

This identity was proved by Zelobenko (see [31], Sect. 109). It was re-discovered and applied to the study of quantum partition functions by Gilmore [6], and also discussed by Simon [18]. It can be rephrased as follows. Let $e_1,\dots,e_k$ be the fundamental weights in $\mathscr{L}\cap C$, so any $\lambda\in\mathscr{L}\cap C$ can be written uniquely in the form

$$\lambda=\sum_{j=1}^{k}\ell_j e_j\,,\qquad \ell_j\geqq 0\,,\quad \text{integer}\,. \tag{3.22}$$

Then

$$\Psi_\lambda(g)=\prod_{j=1}^{k}\Psi_j(g)^{\ell_j}\,, \tag{3.23}$$

where

$$\Psi_j(g)=\Psi_{e_j}(g)\,. \tag{3.24}$$

This reduces the problem of determining $\Psi_\lambda(g)$ for all weights λ to the finite problem of determining the cases (3.24).

The relevance of $\Psi_\lambda(g)$ to the study of $\operatorname{trace} e^{\pi_n(X)/n}$ is provided by the identity

$$\begin{aligned}\operatorname{trace} e^{\pi_\lambda(Y)}&=\operatorname{trace}\int_G e^{\pi_\lambda(\operatorname{Ad}gY)}dg\\ &=d_\lambda\int_G\langle e^{\pi_\lambda(\operatorname{Ad}gY)}\psi_\lambda,\psi_\lambda\rangle dg\,,\end{aligned} \tag{3.25}$$

which follows from the fact that the first integral is scalar, by Schur's lemma. Now pick $\lambda_1\in\mathscr{L}\cap C$ and set $\lambda_n=n\lambda_1$, $d_n=d_{\lambda_n}$, $\pi_n=\pi_{\lambda_n}$, $\psi_n=\psi_{\lambda_n}$. It follows from (3.21) that

$$d_n^{-1}\operatorname{trace} e^{\pi_n(X)/n}=\int_G\langle e^{\pi_1(\operatorname{Ad}gX)/n}\psi_1,\psi_1\rangle^n dg\,. \tag{3.26}$$

This sort of identity was also exploited in [18] and [11]. Here we show how it yields a complete asymptotic expansion as $n\to\infty$. For the moment, set

$$B=\pi_1(\operatorname{Ad}gX)\,. \tag{3.27}$$

We are looking at

$$\begin{aligned}\tau_n(B) &= \langle e^{B/n}\psi_1, \psi_1\rangle^n \\ &= [1+n^{-1}\langle(B+n^{-1}B^2/2!+n^{-2}B^3/3!+\dots)\psi_1,\psi_1\rangle]^n \\ &= \exp n\log(1+n^{-1}K(1/n)),\end{aligned} \tag{3.28}$$

where

$$K(\varepsilon)=\langle(B+\varepsilon B^2/2!+\varepsilon^2B^3/3!+\dots)\psi_1,\psi_1\rangle. \tag{3.29}$$

It is convenient to write

$$K(\varepsilon)=b_1+\varepsilon L(\varepsilon), \tag{3.30}$$

with

$$b_1=\langle B\psi_1,\psi_1\rangle, \tag{3.31}$$

and

$$L(\varepsilon)=\langle(B^2/2!+\varepsilon B^3/3!+\varepsilon^2B^4/4!+\dots)\psi_1,\psi_1\rangle. \tag{3.32}$$

Thus,

$$\tau_n(B)=\exp[(b_1+\varepsilon L(\varepsilon))(1-\varepsilon K(\varepsilon)/2+\varepsilon^2K(\varepsilon)^2/3+\dots)], \tag{3.33}$$

with

$$\varepsilon=1/n. \tag{3.34}$$

in view of the expansion

$$\log(1+x)=x(1-x/2+x^2/3-\dots). \tag{3.35}$$

It is clear that the right side of (3.33) is analytic in ε near $\varepsilon=0$; it is routine to rearrange (3.33) in powers of ε. We work out the following couple of terms explicitly, as follows:

$$\begin{aligned}\tau_n(B) &= \exp[b_1+\varepsilon(L(\varepsilon)-b_1K(\varepsilon)/2)+\dots] \\ &= e^{b_1}\exp[n^{-1}(L(0)-b_1K(0)/2)]+O(n^{-2}) \\ &= e^{b_1}[1+(2n)^{-1}(\langle B^2\psi_1,\psi_1\rangle-\langle B\psi_1,\psi_1\rangle^2)+O(n^{-2})],\end{aligned} \tag{3.36}$$

since $K(0)=b_1=\langle B\psi_1,\psi_1\rangle$ and $L(0)=\frac{1}{2}\langle B^2\psi_1,\psi_1\rangle$.

Information on $b_1=\langle\pi_1(Y)\psi_1,\psi_1\rangle$, $Y=\mathrm{Ad}\,gX$, is provided by the identity

$$\langle\pi_\lambda(Y)\psi_\lambda,\psi_\lambda\rangle=i\langle Y,\lambda\rangle,\qquad Y\in\mathfrak{g},\qquad \lambda\in\mathscr{L}\cap C. \tag{3.37}$$

In fact, $\pi_\lambda(Y)\psi_\lambda=i\langle Y,\lambda\rangle\psi_\lambda$ for $Y\in T_eT^k$, since ψ_λ is a weight vector. Since it is the highest weight vector, $\pi_\lambda(Z)\psi_\lambda=0$ when $Z\in\mathbb{C}\mathfrak{g}$ is a root vector in $\mathfrak{g}_\mu$, for any positive root μ. By duality, if Z is a root vector for a negative root, $\pi_\lambda(Z)\psi_\lambda$ is orthogonal to ψ_λ. This proves (3.37). Thus, the principal term in (3.36) agrees with the principal term in (3.16):

$$\lim_{n\to\infty}\tau_n(B)=e^{b_1}=\exp i\langle\mathrm{Ad}\,gX,\lambda_1\rangle,$$

so

$$\lim_{n\to\infty} d_n^{-1}\,\mathrm{trace}\, e^{\pi_n(X)/n} = \int_G e^{i\langle \mathrm{Ad}\, gX, \lambda_1\rangle} dg\,.$$

In fact, (3.36) gives

$$\begin{aligned} &d_n^{-1}\,\mathrm{trace}\, e^{\pi_n(X)/n} \\ &\quad = \int_G e^{i\langle \mathrm{Ad}\, gX, \lambda_1\rangle}[1 + (\langle \pi_1(\mathrm{Ad}\, gX)^2 \psi_1, \psi_1\rangle + \langle \mathrm{Ad}\, gX, \lambda_1\rangle^2)/2n + O(n^{-2})]dg\,. \end{aligned} \tag{3.38}$$

We are motivated to generalize (3.37) to the study of

$$\langle \pi_\lambda(P)\psi_\lambda, \psi_\lambda\rangle\,, \qquad P \in \mathfrak{N}(\mathfrak{g})\,, \tag{3.39}$$

where $\mathfrak{N}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. In fact, if we identify P with a left-invariant differential operator on G, we have

$$\langle \pi_\lambda(P)\psi_\lambda, \psi_\lambda\rangle = P\Psi_\lambda(g)|_{g=e}\,. \tag{3.40}$$

In light of (3.23), we see that (3.40) is a polynomial in $(\ell_1, \ldots, \ell_k)$ of order $\deg P$; in other words, (3.40) is a polynomial in λ. In the special case arising in (3.38), we have, for $Y \in \mathfrak{g}$, λ of the form (3.22),

$$\begin{aligned} \langle \pi_\lambda(Y^2)\psi_\lambda, \psi_\lambda\rangle &= YY\left(\prod_{j=1}^{k} \Psi_j(g)^{\ell_j}\right)\Bigg|_{g=e} \\ &= \sum_j \ell_j(\ell_j - 1)(Y\Psi_j(e))^2 + \sum_{j\neq k} \ell_j \ell_k Y\Psi_j(e) Y\Psi_k(e) + \sum_j \ell_j Y^2 \Psi_j(e)\,. \end{aligned} \tag{3.41}$$

Note that $Y\Psi_j(e) = i\langle Y, e_j\rangle$, in view of (3.37), so the principal part of (3.41) is $-\langle Y, \lambda\rangle^2$.

One advantage of the latter analysis of $d_n^{-1}\,\mathrm{trace}\, e^{\pi_n(X)/n}$ is that it is more straightforward to be explicit about further terms in the asymptotic expansion, compared with (3.16), since it is not so easy to work with an explicit complete symbol calculus for pseudodifferential operators on G. An advantage of the first method is the uniformity of the expansion one obtains as $\lambda \to \infty$ in a Weyl chamber, not merely along a ray.

In fact, the two methods are not totally unrelated. In particular, the fact that, for any bi-invariant differential operator Q one has (3.4) with a polynomial $q_m(\lambda)$ follows from (3.40) and the subsequent observation, since in the bi-invariant case we have

$$q_m(\lambda + \delta) = Q\Psi_\lambda(e)\,. \tag{3.42}$$

One can obtain the leading term in $q_m(\lambda)$ by an argument parallel to the examination of (3.41), using (3.37). An additional argument would be required to check the Weyl group invariance of $q_m(\lambda)$.

We end this section with a parenthetical remark that one can also use the generating functions $\Psi_\lambda(g)$ to produce formulas for the character $\chi_\lambda(g)$ of an irreducible representation π_λ, and the dimension d_λ of its representation space. In fact, averaging conjugates of $\Psi_\lambda(g)$ clearly produces a scalar multiple of $\chi_\lambda(g)$; since

$\Psi_\lambda(e)=1$, we have

$$\begin{aligned}\chi_\lambda(x)&=d_\lambda\int_G \Psi_\lambda(g^{-1}xg)dg\\&=d_\lambda\int_G \Psi_1(g^{-1}xg)^{\ell_1}\dots\Psi_k(g^{-1}xg)^{\ell_k}dg\end{aligned}\tag{3.43}$$

for $x,g\in G$. Furthermore, since $\chi_\lambda(g)$ has L^2 norm 1 on G, we have

$$d_\lambda^{-2}=\int_G\left|\int_G \Psi_\lambda(g^{-1}xg)dg\right|^2 dx.\tag{3.44}$$

4. Asymptotics for Quantum Partition Functions via Microlocal Analysis

In this section only, we assume for simplicity that $P\to M$ is a product bundle. We aim to study

$$\lim_{n\to\infty}\hbar^\nu d_n^{-1}\,\mathrm{trace}\,e^{-tH_n}=\lim_{n\to\infty}\hbar^\nu d_n^{-1}Z_n(t),\tag{4.1}$$

where H_n is given by (1.4)–(1.5), i.e.,

$$H_n=-\hbar^2g^{-1/2}(\partial_j+\pi_n(A_j(x)))g^{jk}g^{1/2}(\partial_k+\pi_n(A_k(x)))-i\hbar\pi_n(A_0(x))+V(x).\tag{4.2}$$

Note that e^{-tH_n} operates on functions and distributions on M with values in the d_n-dimensional representation space of π_n. In the introduction we took n and $\hbar$ to be related by

$$\hbar=1/n.\tag{4.3}$$

As is Sect. 3, we will find it convenient to modify this slightly, setting instead

$$\hbar=\|n\lambda_1+\delta\|^{-1}.\tag{4.4}$$

Consider the following operator on distributions and functions on $G\times M$:

$$\Omega=-A^{-2}g^{-1/2}(\partial_j+A_j(x))g^{jk}g^{1/2}(\partial_k+A_k(x))-iA^{-1}A_0(x)+V(x).\tag{4.5}$$

Here $A=(-\Delta_G+\|\delta\|^2)^{1/2}$ is as in (3.9), an operator on $\mathscr{D}'(G)$, and, for $0\leqq j\leqq \nu$, the functions $A_j(x)$ on M with values in $\mathfrak{g}$, are considered as vector fields on G; thus, $\partial_j+A_j(x)$ is the horizontal lift, to a vector field on $G\times M$, of the vector field ∂_j on M, determined by the connection at hand. Thus, in (4.5), A^{-2} is composed with a second order differential operator and A^{-1} is composed with a first order differential operator. However, Ω is not a pseudodifferential operator on $G\times M$, as its symbol is singular at points in the cotangent bundle annihilating vectors tangent to the fibers of $G\times M\to M$. Nevertheless, we will be able to analyze $e^{-t\Omega}$ as a pseudodifferential operator on $G\times M$, for any $t>0$. Before we get into this, let us relate $\hbar^\nu d_n^{-1}Z_n(t)$ to $e^{-t\Omega}$.

The operator $e^{-t\Omega}$ commutes with the left action of G on $C^\infty(G\times M)$. If R_g denotes the right action, set

$$\Xi(t)=\int_G R_g^{-1}e^{-t\Omega}R_g dg,\tag{4.6}$$

so $\Xi(t)$ is an operator on $C^\infty(G\times M)$ commuting with the left and the right actions of G. Now let $\xi(t)=\mathrm{trace}_M\,\Xi(t)$ denote the trace relative to M of $\Xi(t)$; if the

Schwartz kernel function of $\Xi(t)$ is $\Xi(t; x, g, x', g')$, then, provided this relative trace exists (and we will see that it does in our case), the Schwartz kernel function of $\xi(t)$ is

$$\xi(t; g, g') = \int_M \Xi(t; x, g, x, g')\, d\,\mathrm{vol}(x). \tag{4.7}$$

Then $\xi(t)$ is a bi-invariant operator on $C^\infty(G)$, so there are uniquely specified scalars $\omega_t(\lambda)$ such that, for $\lambda \in \mathscr{L} \cap C$,

$$\xi(t)\pi_\lambda^{ij} = \omega_t(\lambda + \delta)\pi_\lambda^{ij}. \tag{4.8}$$

Now it is clear that

$$d_n^{-1} Z_n(t) = \omega_t(n\lambda_1 + \delta). \tag{4.9}$$

Consequently, the asymptotic analysis of (4.1) will follow from the asymptotic analysis of $\omega_t(\lambda)$ in (4.8), which in turn will follow from the analysis of $e^{-t\Omega}$ as a pseudodifferential operator on $G \times M$. We turn to this analysis.

The operator we want to exponentiate is

$$\Omega = A^{-2} L_0 + iA^{-1} A_0 + V = A^{-2} L + iA^{-1} A_0 + V_1, \tag{4.10}$$

where $L = L_0 - \Delta_G + \|\delta\|^2$ is an elliptic second order differential operator on $G \times M$ and $V_1 = V - 1$. Note that $A = (-\Delta_G + \|\delta\|^2)^{1/2}$ commutes with L and also with the first order differential operator A_0. If local product coordinates are chosen on $G \times M$, we get coordinates on $T^*(G \times M)$:

$$(x', x'', \xi', \xi''), \quad x' \in G, \quad x'' \in M, \quad \xi' \in T^*_{x'} G, \quad \xi'' \in T^*_{x''} M. \tag{4.11}$$

The symbol of $A^{-2} L = LA^{-2}$ has the form

$$\sigma_{LA^{-2}}(x, \xi) = \sum_j \ell_j(x, \xi) \beta_j(x', \xi') \tag{4.12}$$

with $\ell_j(x, \xi) \in S^2(M \times G)$, $\beta_j(x', \xi') \in S^{-2}(G)$. Outside any conic neighborhood of $\{\xi' = 0\}$ it belongs to $S^0(G \times M)$. $\{\xi' = 0\}$ describes the normal bundle to the fibers of $G \times M \to M$; we denote it by $\mathfrak{N}$. The fact that Ω has a singular symbol at $\mathfrak{N}$ will complicate our symbolic construction of $e^{-t\Omega}$. Before we get into this, we will begin with some more basic information on the nature of the operators $e^{-t\Omega}$.

If we let E_λ be the linear span of π_λ^{ij}, for $1 \leqq i, j \leqq d_\lambda$, then

$$L^2(G \times M) = \bigoplus_\lambda E_\lambda \otimes L^2(M), \tag{4.13}$$

with each summand invariant under $e^{-\sigma L}$, $\sigma \geqq 0$, and hence,

$$e^{-tA^{-2}L} = e^{-t\varepsilon^2 L} \quad \text{on} \quad E_\lambda \otimes L^2(M), \quad \varepsilon = \|\lambda + \delta\|^{-1}. \tag{4.14}$$

Note that $0 < \varepsilon \leqq \|\delta\|^{-1}$, so as long as $0 \leqq t \leqq T_0$, the right side of (4.14) is uniformly bounded. Thus, $e^{-tA^{-2}L}$ is bounded on $L^2(G \times M)$. More generally,

$$\|(I - \Delta_M - \Delta_G)^k e^{-t\varepsilon^2 L} (I - \Delta_M - \Delta_G)^{-k}\|_{\mathscr{L}(L^2)} \leqq C_k$$

for $0 \leqq t \leqq T_0$, so $e^{-tA^{-2}L}$ is bounded on the Sobolev space $H^{2k}(G \times M)$ for each k, and hence, on each Sobolev space $H^s(G \times M)$, $s \geqq 0$. It is easy to see that the perturbations $iA^{-1}A_0$ and V_1 are bounded on each $H^k(G\times M)$, so, for $t\geqq 0$,

$$e^{-t\Omega} \text{ is bounded on each Sobolev space } H^s(G\times M), \tag{4.15}$$

for $s\geqq 0$. By duality we get this result also for $s\leqq 0$. Note that

$$e^{-t\Omega}=e^{-t\varepsilon^2L-t\varepsilon A_0-tV_1} \quad \text{on} \quad E_\lambda\otimes L^2(M), \qquad \varepsilon=\|\lambda+\delta\|^{-1}. \tag{4.16}$$

Now we can obtain the following regularity theorem, which eventually will justify some formal symbolic constructions.

Lemma 4.1. *Suppose we have*

$$(\partial/\partial t+\Omega)u=h\in C^\infty([0,T_0]\times G\times M), \qquad u|_{t=0}=f\in C^\infty(G\times M).$$

Then $u\in C^\infty([0,T_0]\times G\times M)$.

Proof. In light of (4.15), this is an immediate consequence of Duhamel's principle, which gives

$$u(t)=e^{-t\Omega}f+\int_0^t e^{-(t-s)\Omega}h(s)ds.$$

Next we look at the action of $e^{-t\Omega}$ near the singular set $\xi'=0$. This is easier to understand for $e^{-t\Omega_0}$, where $\Omega_0=A^{-2}L$, so we first examine this. Note that $e^{-t\Omega_0}=e^{-t\varepsilon^2L}$ on $E_\lambda\otimes L^2(M)$. But clearly,

$$\|\Delta^\ell_{G\times M}e^{-t\varepsilon^2L}\Delta^k_{G\times M}\|_{\mathscr{L}(L^2)}\leqq C_{k\ell}(t\varepsilon^2)^{-k-\ell}. \tag{4.17}$$

Hence,

$$\|\Delta^\ell_{G\times M}e^{-t\Omega_0}\Delta^k_{G\times M}u\|_{L^2}\leqq C'_{k\ell}t^{-k-\ell}\|\Delta^{k+\ell}_G u\|_{L^2}. \tag{4.18}$$

We claim you get the same sort of estimate for $e^{-t\Omega}$:

$$\|\Delta^\ell_{G\times M}e^{-t\Omega}\Delta^k_{G\times M}u\|_{L^2}\leqq C'_{k\ell}t^{-k-\ell}\|\Delta^{k+\ell}_G u\|_{L^2}. \tag{4.19}$$

This would follow from the estimate

$$\|\Delta^\ell_{G\times M}e^{-t\varepsilon^2L-t\varepsilon A_0-tV_1}\Delta^k_{G\times M}\|_{\mathscr{L}(L^2)}\leqq C_{k\ell}t^{-k-\ell}\varepsilon^{-2k-2\ell}, \tag{4.20}$$

for $0<t\leqq T_0$, $0<\varepsilon\leqq E_0$. Now (4.20) is not an immediate consequence of (4.17), but the singular perturbation analysis of Sect. 2 enables us to prove (4.20). Recall the parametrix for $U(t,\varepsilon)=e^{-t\varepsilon^2L-t\varepsilon A_0-tV_1}$:

$$U(t,\varepsilon)f\sim\sum_{j\geqq 0}\int t^j\varepsilon^j\tilde{a}_j(t,x,\varepsilon\xi)e^{-t\tilde{\Gamma}(x,\varepsilon\xi)}\hat{f}(\xi)e^{ix\cdot\xi}d\xi. \tag{4.21}$$

Now applying the left side of (4.20) effectively throws in a factor of $|\xi|^{2k+2\ell}$ into (4.21), together with some lower order effects. Since $|\xi|^{2k+2\ell}=t^{k+\ell}|\varepsilon\xi|^{2k+2\ell}t^{-k-\ell}\varepsilon^{-2k-2\ell}$, the estimate (4.20) is an easy consequence of the analysis of (4.21) given in Lemmas 2.1 and 2.2. Thus, we have the estimate (4.19). From (4.19) we can also deduce the estimate

$$\|\Delta^\ell_{G\times M}e^{-t\Omega}u\|_{L^2}\leqq C_{k\ell}t^{-k-\ell}\|\Delta^{k+\ell}_G\Delta^{-k}_{G\times M}u\|_{L^2}. \tag{4.22}$$

From here we can deduce smoothness of $e^{-t\Omega}u$ whenever u has "wave front set" in a "subconic" neighborhood of $\{\xi'=0\}=\mathfrak{N}$, the bundle of normals to the fibers of $G\times M\to M$. Pick $a\in(0,\frac{1}{2})$, small, pick $\varphi\in C_0^\infty(\mathbb{R}^\mu)$ ($\mu=\dim G$), $\varphi=1$ in a neighborhood of the origin, and set

$$\psi(\xi)=\varphi(|\xi|^{-(1-a)}\xi')\in S^0_{1-a,0}\,. \tag{4.23}$$

Then ψ is the symbol of a pseudodifferential operator $\Psi\in OPS^0_{1-a,a}$. If there exists $a\in(0,\frac{1}{2})$ such that

$$\Psi u=u \bmod C^\infty(G\times M), \tag{4.24}$$

we say u has wave front set in a subconic neighborhood of $\mathfrak{N}$. This notion depends only on $\mathfrak{N}$, not on the choice of coordinate system. This is a special case of results in Appendix B of [24], on a refined wave front set $\widetilde{WF}$. A general $u\in\mathscr{D}'(G\times M)$ can be decomposed as $u=(I-\Psi)u+\Psi u=u_1+u_2$, where u_2 has wave front set in a subconic neighborhood of $\mathfrak{N}$, and, in an analogous sense, the wave front set of u_1 misses a subconic neighborhood of $\mathfrak{N}$.

Lemma 4.2. *Suppose $u\in\mathscr{D}'(G\times M)$ has wave front set in a subconic neighborhood of $\mathfrak{N}$. Then, for any $t>0$, $e^{-t\Omega}u\in C^\infty(G\times M)$.*

Proof. Let $k\to\infty$ and $\ell=[\alpha k]$, where $\alpha\in(0,1)$ is fixed, sufficiently small, compared with some given $a\in(0,\frac{1}{2})$. Then, with Ψ defined by (4.23), we have

$$\Phi_k=\Delta_G^{k+\ell}\Delta_{G\times M}^{-k}\Psi\,, \tag{4.25}$$

a sequence of pseudodifferential operators of type $(1-a,a)$, with orders going to $-\infty$ as $k\to\infty$. Hence, if $\Psi u=u \bmod C^\infty$, (4.22) gives estimates of $\|\Delta^\ell_{G\times M}e^{-t\Omega}u\|_{L^2}$ for arbitrarily large ℓ. This proves the lemma.

This lemma is equivalent to

$$e^{-t\Omega}\Psi:\mathscr{D}'(G\times M)\to C^\infty(G\times M) \tag{4.26}$$

for any $t>0$. By duality, one deduces

$$\Psi e^{-t\Omega}:\mathscr{D}'(G\times M)\to C^\infty(G\times M) \tag{4.27}$$

for any $t>0$. Thus, for any $t>0$, any $u\in\mathscr{D}'(G\times M)$, $e^{-t\Omega}u$ has wave front set which misses a subconic neighborhood of $\mathfrak{N}$, in the sense indicated above.

Our strategy for analyzing $e^{-t\Omega}$ is the following. Outside some subconic neighborhood of $\mathfrak{N}$, Ω agrees with the action of a pseudodifferential operator B in $OPS^a_{1-a,a}$, where $a>0$ can be taken arbitrarily small. We expect pseudodifferential operator techniques to analyze e^{-tB}, and then we hope that Lemma 4.2 will lead to $e^{-t\Omega}-e^{-tB}\in OPS^{-\infty}$, for any $t>0$.

We next turn to the question of constructing u such that

$$u(0)=f,\qquad \partial u/\partial t=-Bu \bmod C^\infty([0,T_0)\times X)\,, \tag{4.28}$$

where $X=G\times M$ and B is a pseudodifferential operator on X with a non-classical symbol. Whether such u agrees $\bmod C^\infty$ with the exact solution to $\partial u/\partial t=-Bu$ is a separate issue, which in our case of interest will be taken care of by Lemma 4.1. We attempt to construct u in the form

$$u=\int a(t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi\,, \tag{4.29}$$

in a local coordinate system. We will attempt to construct $a(t,x,\xi)$ as an asymptotic sum

$$a(t,x,\xi)\sim\sum_{j\geqq 0}a_j(t,x,\xi)\,, \tag{4.30}$$

where, hopefully, a_j are (non-classical) symbols with orders tending to $-\infty$ as $j\to\infty$. If u is given by (4.29), then

$$Bu=\int k(t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi\,, \tag{4.31}$$

where

$$k(t,x,\xi)=e^{-ix\cdot\xi}B(ae^{ix\cdot\xi})\,. \tag{4.32}$$

If $B(x,\xi)$ and $a(t,x,\xi)$ satisfy appropriate symbol estimates, one can write the asymptotic expansion

$$k(t,x,\xi)\sim\sum_{\alpha\geqq 0}(1/\alpha!)B^{(\alpha)}(x,\xi)a_{(\alpha)}(t,x,\xi)\,, \tag{4.33}$$

where $B^{(\alpha)}=D_\xi^\alpha B$, $a_{(\alpha)}=D_x^\alpha a$. Since we want $\partial a/\partial t+k\sim 0$, or

$$\partial a/\partial t+B(x,\xi)a(t,x,\xi)+\sum_{\alpha\geqq 1}(1/\alpha!)B^{(\alpha)}(x,\xi)a_{(\alpha)}(t,x,\xi)\sim 0\,, \tag{4.34}$$

it is natural to specify the terms a_j in the expansion (4.30) by

$$\partial a_0/\partial t=-B(x,\xi)a_0\,, \tag{4.35}$$

and, for $j\geqq 1$,

$$\partial a_j/\partial t+B(x,\xi)a_j=R_j(t,x,\xi)\,, \tag{4.36}$$

where

$$R_j(t,x,\xi)\sim-\sum_{\alpha\geqq 1}(1/\alpha!)B^{(\alpha)}(x,\xi)D_x^\alpha a_{j-1}(t,x,\xi)\,. \tag{4.37}$$

We have initial conditions

$$a_0(0,x,\xi)=1\,,\qquad a_j(0,x,\xi)=0\quad\text{for}\quad j\geqq 1\,, \tag{4.38}$$

so

$$a_0(t,x,\xi)=e^{-tB(x,\xi)} \tag{4.39}$$

and, for $j\geqq 1$,

$$a_j(t,x,\xi)=\int_0^t e^{-(t-s)B(x,\xi)}R_j(s,x,\xi)ds\,. \tag{4.40}$$

Suppose now that, for some $a\in[0,\tfrac{1}{2})$, we have

$$B(x,\xi)\in S^a_{1-a,a}\,.$$

We will assume $B(x, \xi)$ is bounded below, $\operatorname{Re} B(x, \xi) \geqq -C$. In fact, suppose

$$e^{-tB(x,\xi)} \in S^0_{1-a,a}, \quad \text{bounded, for} \quad 0 \leqq t \leqq T_0 .$$

Then we clearly have $a_0(t, x, \xi) \in S^0_{1-a,a}$. More generally, if $a_j(t, x, \xi) \in S^{\mu_j}_{1-a,a}$, the recursion (4.37)–(4.38) gives

$$B^{(\alpha)}(x, \xi) D^\alpha_x a_j(t, x, \xi) \in S^{\mu_j-(1-2a)|\alpha|+a}_{1-a,a},$$

and hence,

$$R_j \in S^{\mu_j-(1-3a)}_{1-a,a}.$$

Thus, (4.30) is asymptotic, provided $a < 1/3$. We have proved the following.

Proposition 4.3. *Suppose we have*

$$B(x, \xi) \in S^a_{1-a,a}, \tag{4.41}$$

and

$$e^{-tB(x,\xi)} \in S^0_{1-a,a}, \quad \textit{bounded, for} \quad 0 \leqq t \leqq T_0 . \tag{4.42}$$

Then (4.29)–(4.40) *give a construction of u satisfying* (4.28), *provided*

$$0 \leqq a < 1/3 . \tag{4.43}$$

The hypothesis (4.42) can be replaced by something more explicit, in light of the following.

Proposition 4.4. *Suppose* $B(x, \xi) \in S^\alpha_{1-\alpha,\alpha}$ *and* $\operatorname{Re} B(x, \xi) \geqq -C$. *Then* (4.42) *holds, with* $a = 2\alpha$.

Proof. We have $D_x e^{-tB} = -t(D_x B) e^{-tB}$, with $e^{-tB(x,\xi)}$ bounded and $D_x B \in S^{2\alpha}_{1-\alpha,\alpha}$. The rest of the estimates follow similarly.

Corollary 4.5. *Suppose we have*

$$B(x, \xi) \in S^a_{1-a,a} \tag{4.44}$$

and

$$\operatorname{Re} B(x, \xi) \geqq -C . \tag{4.45}$$

Then (4.29)–(4.40) *give a construction of u satisfying* (4.28), *provided*

$$0 \leqq a < 1/6 . \tag{4.46}$$

These propositions are not directly applicable to $e^{-t\Omega}$, since the operator Ω is too singular. Nevertheless, the basic construction (4.29)–(4.40) can still be made to work, with some modifications. It is more convenient to take a method which works in the classical case $B(x, \xi) \in S^0$, i.e.,

$$B(x, \xi) \sim B_0(x, \xi) + B_1(x, \xi) + \dots , \tag{4.47}$$

where each $B_j(x, \xi)$ is C^∞ on $\xi \neq 0$ and homogeneous of degree $-j$ in ξ. Of course, the construction (4.29)–(4.40) works in this case, but we will modify it slightly, so each $a_j(t, x, \xi)$ will actually be homogeneous, rather than in the symbol class S^{-j}.

To do this, replace (4.32)–(4.33) by

$$k(t,x,\xi)=e^{-ix\cdot\xi}B(ae^{ix\cdot\xi}), \tag{4.48}$$

with

$$k(t,x,\xi)\sim\sum_{j\geqq 0}\sum_{\alpha\geqq 0}(1/\alpha!)B_j^{(\alpha)}(x,\xi)a_{(\alpha)}(t,x,\xi). \tag{4.49}$$

Thus, our transport equations become

$$\partial a_0/\partial t=-B_0(x,\xi)a_0, \tag{4.50}$$

and, for $j\geqq 1$,

$$\partial a_j/\partial t+B_0(x,\xi)a_j=R_j^{\#}(t,x,\xi), \tag{4.51}$$

where

$$R_j^{\#}(t,x,\xi)=-\sum_{\substack{\ell+k+|\alpha|=j\\ 0\leqq\ell\leqq j-1}}(1/\alpha!)B_k^{(\alpha)}(x,\xi)D_x^\alpha a_\ell(t,x,\xi). \tag{4.52}$$

With the initial condition (4.38), we have

$$a_0(t,x,\xi)=e^{-tB_0(x,\xi)}, \tag{4.53}$$

and, for $j\geqq 1$,

$$a_j(t,x,\xi)=\int_0^t e^{-(t-s)B_0(x,\xi)}R_j^{\#}(s,x,\xi)ds. \tag{4.54}$$

It is easy to see that a_j is homogeneous of degree $-j$ in ξ. Inductively, one sees that

$$a_j(t,x,\xi)=\tilde{a}_j(t,x,\xi)e^{-tB_0(x,\xi)}, \tag{4.55}$$

where $\tilde{a}_j(t,x,\xi)$ is obtained from $B_\ell(x,\xi)$, $0\leqq\ell\leqq j$, through the process of taking x and ξ derivatives, and forming sums and products, and $\tilde{a}_j(t,x,\xi)$ is homogeneous in ξ of degree $-j$.

What happens if we apply the construction (4.52)–(4.55) with B replaced by Ω, given by (4.5)? We can write the "symbol" of Ω as a formal sum

$$\Omega(x,\xi)\sim\Omega_0(x,\xi)+\Omega_1(x,\xi)+\ldots \tag{4.56}$$

of the following nature. Each $\Omega_j(x,\xi)$ is C^∞ in x and ξ except at $\xi'=0$, and homogeneous of degree $-j$ in ξ. Near $\xi'=0$, $\Omega_j(x,\xi)$ has an algebraically bounded singularity, i.e., for any given j and k, if $|\alpha'|$ is taken sufficiently large, $(\xi')^{\alpha'}\Omega_j(x,\xi)$ is smooth of class C^k near $\xi'=0$. Let us denote by $\mathcal{S}^{-j}$ the set of such functions. From (4.55) we have

$$a_j(t,x,\xi)=\tilde{a}_j(t,x,\xi)e^{-t\Omega_0(x,\xi)}, \tag{4.57}$$

with

$$\tilde{a}_j(t,x,\xi)\in\mathcal{S}^{-j}. \tag{4.58}$$

Now $\Omega_0(x,\xi)$ tends to $+\infty$ at the singular set $\xi'=0$, and it is clear that $e^{-t\Omega_0(x,\xi)}=a_0(t,x,\xi)$ is C^∞ and vanishes to infinite order at $\xi'=0$, for any $t>0$. This infinite order vanishing cancels out the algebraically bounded singularity in $\tilde{a}_j$, and we deduce that the construction (4.52)–(4.55) applied to Ω in place of B yields

$$a_j(t,x,\xi)\in S^{-j}, \text{ vanishing to infinite order at } \xi'=0, \text{ for any } t>0. \tag{4.60}$$

It is not convenient to demonstrate the validity of the resulting approximating operator for $e^{-t\Omega}$ directly from the construction (4.48)–(4.55), as the product nature of Ω makes it hard to obtain (4.49), and the smoothness of (4.57) does not hold uniformly as $t\to 0$. Therefore, what it is convenient to do is apply a cut-off $I-\Psi$ to the initial data, where Ψ is as in (4.23)–(4.24). Pick $a<1/6$, sufficiently small. The formal construction (4.48)–(4.55), modified so $a_0(0,x,\xi)=1-\psi(\xi)$, gives (4.57)–(4.59), with $\tilde{a}_j(t,x,\xi)$ multiplied by $1-\psi(\xi)$. If we replace Ω by $B=\Omega(I-\Psi_1)$, where Ψ_1 is similarly defined, with a enlarged slightly, we see that u is constructed so that

$$(\partial/\partial t+B)u\in C^\infty, \qquad u|_{t=0}=(I-\Psi)f \bmod C^\infty. \tag{4.61}$$

On the other hand, since $(1-\psi(\xi))$ is a factor in each term in the parametrix for e^{-tB}, we see that

$$\Omega u-Bu\in C^\infty, \tag{4.62}$$

so

$$(\partial/\partial t+\Omega)u\in C^\infty, \qquad u|_{t=0}=(I-\Psi)f, \bmod C^\infty. \tag{4.63}$$

Consequently, we see that, $\bmod C^\infty$,

$$e^{-t\Omega}(I-\Psi)f=\int(1-\psi(\xi))a(t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi, \tag{4.64}$$

where $a(t,x,\xi)\in S^0$ vanishes to infinite order at $\xi'=0$, for $t>0$. Note that $\psi(\xi)a(t,x,\xi)$ has order $-\infty$ under such circumstances, so, for any fixed $t\geqq 0$,

$$e^{-t\Omega}(I-\Psi)f=\int a(t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)d\xi \bmod C^\infty, \tag{4.65}$$

with $a(t,x,\xi)$ as above. On the other hand, for any $t>0$ fixed, Lemma 4.2 implies $e^{-t\Omega}\Psi$ is a smoothing operator. This proves the following main result of this section.

Theorem 4.6. *For any fixed $t>0$, we have the classical pseudodifferential operator*

$$e^{-t\Omega}\in OPS^0(G\times M). \tag{4.66}$$

Its complete symbol, in a product coordinate system, is given by the same rule as for e^{-tB} with classical $B\in OPS^0$. Furthermore, its complete symbol vanishes to infinite order on the bundle $\mathfrak{N}$ of cotangent vectors normal to the fibers in $G\times M\to M$.

Recall we are interested in the relative trace of $e^{-t\Omega}$, as an operator on $\mathscr{D}'(G)$. We have the following general result.

Proposition 4.7. *Suppose P is a classical pseudodifferential operator*

$$P\in OPS^m(G\times M) \tag{4.67}$$

whose complete symbol vanishes to infinite order on $\mathfrak{N}$. Then the trace of P relative to M is well defined as a pseudodifferential operator on G:

$$\operatorname{trace}_M P=Q\in OPS^{m+\nu}(G), \qquad \nu=\dim M. \tag{4.68}$$

Proof. Since a smoothing operator on $\mathscr{D}'(G\times M)$ clearly has relative trace which is a smoothing operator on $\mathscr{D}'(G)$, we can suppose the Schwartz kernel function of P is supported near the diagonal and work in local coordinates. Say

$$Pu=\int p(x',x'',\xi',\xi'')\hat{u}(\xi)e^{ix\cdot\xi}d\xi\,. \tag{4.69}$$

Then

$$Qf(x')=\int q(x',\xi')\hat{f}(\xi')e^{ix'\cdot\xi'}d\xi'\,, \tag{4.70}$$

where

$$q(x',\xi')=\iint\limits_{T^*M} p(x',x'',\xi',\xi'')dx''d\xi''\,. \tag{4.71}$$

If $p(x',x'',\xi',\xi'')$ is homogeneous of degree $m-j$ in $\xi=(\xi',\xi'')$ and vanishes to infinite order at $\xi'=0$, it is easy to see that

$$q_j(x',\xi')=\iint p_j(x',x'',\xi',\xi'')dx''d\xi'' \tag{4.72}$$

is smooth and homogeneous of degree $m-j+v$ in ξ'. This establishes (4.68).

In the case $P=e^{-t\Omega}$, with $t>0$, we have

$$\mathrm{trace}_M e^{-t\Omega}=Q(t)\in OPS^v(G)\,, \tag{4.73}$$

with symbol

$$q(t,x',\xi')\sim q_0+q_1+\ldots\,, \tag{4.74}$$

and

$$q_0(t,x',\xi')=\int e^{-t\Omega_0(x,\xi)}dx''d\xi''\,. \tag{4.75}$$

Thus, with

$$\ell(x,\xi)=(\xi_j''+\sigma_j(x,\xi'))g^{jk}(x'')(\xi_k''+\sigma_k(x,\xi'))\,, \tag{4.76}$$

where $\sigma_j(x,\xi')$ is the symbol of $A_j(x)/i$, real valued and linear in ξ', we have

$$q_0(t,x',\xi')=\int\limits_{T^*M} e^{-t[\ell(x,\xi)\|\xi'\|^{-2}+\|\xi'\|^{-1}\sigma_0(x,\xi')+V(x'')]}dx''d\xi''\,. \tag{4.77}$$

If we make a change of variable, we can replace $\ell(x,\xi)$ by

$$\|\xi''\|^2=\xi_j''g^{jk}(x'')\xi_k''\,, \tag{4.78}$$

and write

$$\begin{aligned}q_0(t,x',\xi')&=\int\limits_{T^*M} e^{-t[\|\xi''\|^2\|\xi'\|^{-2}+\|\xi'\|^{-1}\sigma_0(x,\xi')+V(x'')]}dx''d\xi''\\&=\|\xi'\|^v\int\limits_{T^*M} e^{-t[\|\xi''\|^2+\sigma_0(x,\xi'/\|\xi'\|)+V(x'')]}dx''d\xi''\,,\end{aligned} \tag{4.79}$$

which gives

$$q_0(t,x',\xi')=\|\xi'\|^v(\pi/t)^{\frac{1}{2}v}\int\limits_M e^{-t[\sigma_0(x,\xi'/\|\xi'\|)+V(x'')]}d\,\mathrm{vol}_M(x'')\,. \tag{4.80}$$

Since σ_0 is the symbol of the operator $A_0(x'')$, we can write

$$q_0(t,x',\xi')=(\pi/t)^{\frac{1}{2}\nu}\|\xi'\|^\nu \int_M e^{-t[\langle A_0(x''),\xi'/\|\xi'\|\rangle+V(x'')]}d\,\mathrm{vol}_M(x'')\,. \tag{4.81}$$

If we average over G with respect to the action R_g as in (4.6), we get for the operator $\xi(t)$ of (4.7)–(4.9) that

$$\xi(t)\in OPS^\nu(G) \quad \text{for} \quad t>0\,, \tag{4.82}$$

and its principal symbol is

$$\xi_0(t,x',\xi')=(\pi/t)^{\nu/2}\|\xi'\|^\nu \int_G\int_M e^{-t\|\xi'\|^{-1}\langle A_0(x''),\mathrm{Ad}^*g\xi'\rangle-tV(x'')}d\,\mathrm{vol}(x'')dg\,. \tag{4.83}$$

In light of (4.8)–(4.9), we can read off the asymptotic behavior of the quantum partition function $Z_n(t)=d_n\omega_t(n\lambda_1+\delta)$. In fact,

$$\omega_t(\lambda)=\xi(t,e,\lambda)\sim\xi_0(t,e,\lambda)+\xi_1(t,e,\lambda)+\,\dots\,, \tag{4.84}$$

where

$$\xi_0(t,e,\lambda)=(\pi/t)^{\frac{1}{2}\nu}\|\lambda\|^\nu \int_G\int_M e^{-t\|\lambda\|^{-1}\langle A_0(x''),\mathrm{Ad}^*g\lambda\rangle-tV(x'')}d\,\mathrm{vol}(x'')dg\,, \tag{4.85}$$

and, generally, $\xi_j(t,e,\lambda)$ is smooth and homogeneous of degree $\nu-j$ in λ. We thus have Theorem A in this context. Let us formally state the result.

Theorem 4.8. *Let $Z_n(t)$ be the quantum partition function, given by* (4.1), (4.2), *with*

$$\hbar=\|n\lambda_1+\delta\|^{-1}\,. \tag{4.86}$$

Fix $t>0$. Then, as $\hbar\to 0$, there is an asymptotic expansion

$$d_n^{-1}Z_n(t)\sim\hbar^{-\nu}[a_0(t)+a_1(t)\hbar+a_2(t)\hbar^2+\dots]\,, \tag{4.87}$$

where $\nu=\dim M$. The coefficient $a_0(t)$ is given by the integral formula

$$\begin{aligned} a_0(t)&=(\pi/t)^{\frac{1}{2}\nu}\int_G\int_M e^{-t\|\lambda_1\|^{-1}\langle A_0(x''),\mathrm{Ad}^*g\lambda_1\rangle-tV(x'')}d\,\mathrm{vol}(x'')dg\\ &=(\pi/t)^{\frac{1}{2}\nu}\int_{\Gamma_{\lambda_1}}\int_M e^{-t\|\ell\|^{-1}\langle A_0(x''),\ell\rangle-tV(x'')}d\,\mathrm{vol}(x'')d\mu_{\Gamma_{\lambda_1}}(\ell)\,, \end{aligned} \tag{4.88}$$

where Γ_{λ_1} is the coadjoint orbit in $\mathfrak{g}^*$ containing $\lambda_1\in\mathfrak{g}^*$ and $d\mu_{\Gamma_{\lambda_1}}$ is the natural homogeneous probability measure on this orbit.

5. Coherent States and Uniform Parametrices

In this section we will give another derivation of the asymptotic expansion of

$$\hbar^\nu d_n^{-1}\,\mathrm{trace}\,e^{-tH_\hbar}=\hbar^\nu d_n^{-1}Z_n(t)\,. \tag{5.1}$$

This treatment will be based on a modification of the singular perturbation method of Sect. 2, and will incorporate the method of coherent projections. Here, $H_\hbar$ is given as in (1.4) and (1.5), by

$$H_\hbar=-\hbar^2 g^{-1/2}(\partial_j+\pi_n(A_j))g^{jk}g^{1/2}(\partial_k+\pi_n(A_k))-i\hbar\pi_n(A_0)+V\,. \tag{5.2}$$

As in the introduction, $\hbar$ and n are related by

$$\hbar = 1/n. \tag{5.3}$$

We choose $\lambda_1 \in \mathscr{L} \cap C$, and set $\lambda_n = n\lambda_1$, with associated representation $\pi_n = \pi_{\lambda_n}$, on a space $\mathfrak{h}_n$ of dimension $d_n = d_{\lambda_n}$.

We introduce the notion of coherent projections here; for more details see [18, 11]. Let Γ_{λ_1} denote the co-adjoint orbit in $\mathfrak{g}'$ containing λ_1. If $\lambda = \mathrm{Ad}^* g\lambda_1$, then the orthogonal projection onto the linear span of

$$\pi_n(g)^{-1}\psi_{n\lambda_1}, \tag{5.4}$$

a one dimensional subspace of $\mathfrak{h}_n$, is a projection which depends only on λ, and n; denote it $P_n(\lambda)$. Also if $d\mu_\Gamma$ is the unique homogeneous measure on Γ_{λ_1}, of total mass 1, then, as proved in [18], as an easy consequence of Schur's lemma,

$$d_n \int_{\Gamma_{\lambda_1}} P_n(\lambda) d\mu_\Gamma(\lambda) = I_{\mathfrak{h}_n}. \tag{5.5}$$

If f is a function of $x \in R^\nu$ with values in $\mathfrak{h}_n$, let

$$\begin{aligned} \hat{f}(\xi, \lambda) &= (2\pi)^{-\nu} \int P_n(\lambda) f(x) e^{-ix\cdot\xi} d\xi \\ &= P_n(\lambda)\hat{f}(\xi). \end{aligned} \tag{5.6}$$

We will construct a uniform parametrix for the initial value problem

$$\partial U_n/\partial t = -H_\hbar U_n, \qquad U_n|_{t=0} = I_{\hbar_n}, \tag{5.7}$$

such that, in local coordinates,

$$U_n(t)f(x) = d_n \iint a(t, \hbar, x, \xi, \lambda) e^{ix\cdot\xi} \hat{f}(\xi, \lambda) d\mu_\Gamma(\lambda) d\xi. \tag{5.8}$$

Expression in local coordinates will be patched together via a partition of unity, as in Sect. 2; we will not dwell further on this in this section. We will specify the amplitude $a(t, \hbar, x, \xi, \lambda)$ through a series of transport equations.

In parallel with (2.2), we can write

$$-H_\hbar = \hbar^2 L + \hbar X_n + V_n, \tag{5.9}$$

where

$$\begin{aligned} &L = \Delta, \quad X_n = 2\pi_n(A_j/n) g^{jk} \partial_k + g^{-1/2}(\partial_j g^{1/2} g^{jk} \pi_n(A_k/n)), \\ &V_n = -V + i\pi_n(A_0/n) + g^{jk}\pi_n(A_j/n)\pi_n(A_k/n). \end{aligned} \tag{5.10}$$

We may as well consider a slightly more general situation, where, in local coordinates,

$$L = g^{jk}(x)\partial_j \partial_k + b^j(x)\partial_j + c(x) \tag{5.11}$$

has scalar coefficients (g_{jk} still denoting the metric tensor),

$$X_n = \pi_n(B^j(x)/n)\partial_j + i\pi_n(B^0(x)/n) = X_n^{\#} + i\pi_n(B^0(x)/n), \tag{5.12}$$

and

$$V_n = -V + i\pi_n(A_0(x)/n) + g^{jk}(x)\pi_n(A_j(x)/n)\pi_n(A_k(x)/n). \tag{5.13}$$

Here $B^j(x)$ and $A_j(x)$ are given C^∞ functions of x with values in $\mathfrak{g}$.

We introduce the following quantities that will play a role in deriving the transport equations. Let

$$\tilde{X} = i\langle B^j(x), \lambda\rangle\partial_j - \langle B^0(x), \lambda\rangle = \tilde{X}^\# - \langle B^0(x), \lambda\rangle, \tag{5.14}$$

$$\tilde{W}_1 = -\langle A_0(x), \lambda\rangle, \tag{5.15}$$

$$\tilde{W}_2 = -g^{jk}(x)\langle A_j(x), \lambda\rangle\langle A_k(x), \lambda\rangle, \tag{5.16}$$

so $\tilde{X} = \tilde{X}(x, \lambda)$, etc. Following [11] we introduce the quantities

$$\Phi_{n,\lambda}(A) = -n^{1/2}(i\pi_n(A/n) + \langle A, \lambda\rangle), \qquad A \in \mathfrak{g}. \tag{5.17}$$

In [11] these functions were shown to describe the quantum fluctuations of (for example) the isospin in the SU(2) case around its classical limit. We will also need the following quantities:

$$Y = i\Phi_{n,\lambda}(B^j(x))\partial_j - \Phi_{n,\lambda}(B^0(x)) = Y^\# - \Phi_{n,\lambda}(B^0(x)), \tag{5.18}$$

$$Z_1 = -\Phi_{n,\lambda}(A_0(x)), \tag{5.19}$$

$$Z_2 = -g^{jk}(x)\Phi_{n,\lambda}(A_j(x))\Phi_{n,\lambda}(A_k(x)), \tag{5.20}$$

$$Z_3 = -2g^{jk}(x)\Phi_{n,\lambda}(A_j(x))\langle A_k(x), \lambda\rangle, \tag{5.21}$$

so $Y = Y_n(x, \lambda)$, etc. Then, for each $\lambda \in \Gamma_{\lambda_1}$, we have the identity

$$-H_\hbar = \hbar^2 L + \hbar(\tilde{X} + \hbar^{1/2} Y) - V + \tilde{W}_1 + \hbar^{1/2} Z_1 + \tilde{W}_2 + \hbar Z_2 + \hbar^{1/2} Z_3, \tag{5.22}$$

as long as (5.3) holds.

To derive the transport equations, we apply $\partial/\partial t + H_\hbar$ to $ae^{i\psi}$, where $\psi = x \cdot \xi$, as in Sect. 2. We use the fact that

$$L(ae^{i\psi})e^{-i\psi} = -\|\xi\|^2 a + 2i\langle\xi, \nabla a\rangle + i(M\psi)a + La, \tag{5.23}$$

where

$$\|\xi\|^2 = g^{jk}(x)\xi_j\xi_k, \qquad \langle\xi, v\rangle = g^{jk}(x)\xi_j v_k, \tag{5.24}$$

and

$$M\psi = b^j(x)\partial_j\psi = b^j(x)\xi_j. \tag{5.25}$$

Similarly we have

$$\begin{aligned} \tilde{X}(ae^{i\psi})e^{-i\psi} &= i(\tilde{X}^\#\psi)a + Xa, \\ Y(ae^{i\psi})e^{-i\psi} &= i(Y^\#\psi)a + Ya. \end{aligned} \tag{5.26}$$

Consequently, we require of the amplitude $a = a(t, \hbar, x, \xi, \lambda)$ that, in an appropriate sense,

$$\begin{aligned} &-\partial a/\partial t - \hbar^2\|\xi\|^2 a + 2i\hbar^2\langle\xi, \nabla a\rangle + i\hbar^2(M\psi)a + \hbar^2 La \\ &+ i\hbar^{3/2}(Y^\#\psi)a + \hbar^{3/2} Ya + i\hbar(\tilde{X}^\#\psi)a + \hbar\tilde{X}a + \hbar Z_2 a \\ &+ \hbar^{1/2} Z_1 a + \hbar^{1/2} Z_3 a - Va + \tilde{W}_1 a + \tilde{W}_2 a \sim 0. \end{aligned} \tag{5.27}$$

For awhile, we shall proceed from (5.27) treating $\hbar$ and n as independent variables. We will resume the identification (5.3) later.

We will produce the amplitude $a^n(t, \hbar, x, \xi, \lambda)$ in the form

$$a^n(t, h, x, \xi, \lambda) \sim \sum_{j \geqq 0} a_j^n(t, \hbar, x, \xi, \lambda), \tag{5.28}$$

where each a_j^n is defined by an inductive process. As in Sect. 2, we want to assign weights to the terms in the associated formal expansion of (5.27). We assign weights as follows:

$$-\partial a_j/\partial t - \hbar^2 \|\xi\|^2 a_j + i\hbar(\tilde{X}^\# \psi) a_j + (\tilde{W}_1 + \tilde{W}_2 - V) a_j \text{ has weight } -j, \tag{5.29}$$

and

$$\begin{aligned} &2i\hbar^2 \langle \xi, \nabla a_j \rangle + \hbar^2 L a_j + i\hbar^2 (M\psi) a_j - i\hbar^{3/2} (Y^\# \psi) a_j - \hbar^{3/2} Y a_j \\ &\quad - \hbar \tilde{X} a_j - \hbar Z_2 a_j - \hbar^{1/2} Z_1 a_j - \hbar^{1/2} Z_3 a_j \text{ has weight } -j-1. \end{aligned} \tag{5.30}$$

Our iterative procedure will consist of requiring the sums of all terms of weight $0, -1, -2$, etc., to vanish. Requiring the terms of weight 0 to sum to zero leads to the "first transport equation"

$$\partial a_0/\partial t = (-\hbar^2 \|\xi\|^2 + i\hbar(\tilde{X}^\# \psi) + \tilde{W}_1 + \tilde{W}_2 - V) a_0; \tag{5.31}$$

in light of (5.5) it is desirable to take the initial condition

$$a_0|_{t=0} = P_n(\lambda). \tag{5.32}$$

Thus

$$a_0^n(t, \hbar, x, \xi, \lambda) = e^{-t\tilde{\Gamma}(x, \hbar\xi, \lambda)} P_n(\lambda), \tag{5.33}$$

where

$$\tilde{\Gamma}(x, \zeta, \lambda) = \|\zeta\|^2 - i(\tilde{X}^\# \psi_1) - \tilde{W}_1 - \tilde{W}_2 + V, \qquad \psi_1 = x \cdot \zeta. \tag{5.34}$$

Note that this exponent is *scalar*, and independent of n. Part of the reason for introducing (5.14)–(5.17) was to arrange this. Note that, in case $H_\hbar$ has the form (5.2), we have

$$\tilde{\Gamma}(x, \zeta, \lambda) = \|\zeta + \langle A, \lambda \rangle\|^2 + \langle A_0, \lambda \rangle + V, \tag{5.35}$$

where

$$\langle A, \lambda \rangle = (\langle A_1, \lambda \rangle, \ldots, \langle A_v, \lambda \rangle). \tag{5.36}$$

This is the classical Hamiltonian with the momentum ζ and classical isospin λ in the SU(2) case; see e.g., [11]. For $j \geqq 1$, the transport equation becomes

$$\partial a_j/\partial t = -\tilde{\Gamma} a_j + \Omega_j, \tag{5.37}$$

where now

$$\begin{aligned} \Omega_j = &2i\hbar^2 \langle \xi, \nabla a_{j-1} \rangle + \hbar^2 L a_{j-1} + i\hbar^2 (M\psi) a_{j-1} + i\hbar^{3/2} (Y^\# \psi) a_{j-1} + \hbar^{3/2} Y a_{j-1} \\ &+ \hbar \tilde{X} a_{j-1} + \hbar Z_2 a_{j-1} + \hbar^{1/2} Z_1 a_{j-1} + \hbar^{1/2} Z_3 a_{j-1}. \end{aligned} \tag{5.38}$$

Again, the amplitude a_j^n is obtained as

$$a_j^n(t,\hbar,x,\zeta,\lambda)=\int_0^t e^{-(t-s)\tilde{\Gamma}}\Omega_j(s)ds\,. \tag{5.39}$$

We have the following analysis of the amplitude a_j^n, parallel to but necessarily more complicated than that of Lemma 2.1.

Lemma 5.1. *We can write*

$$a_j^n(t,\hbar,x,\xi,\lambda)=b_j^n(t,\hbar,x,\hbar\xi,\lambda)e^{-t\tilde{\Gamma}(x,\hbar\xi,\lambda)}P_n(\lambda)\,, \tag{5.40}$$

and

$$b_j^n(t,\hbar,x,\hbar\xi,\lambda)=\tilde{b}_j^n(t,\hbar,x,\omega,\sigma,\lambda)\,, \tag{5.41}$$

where $\tilde{b}_j^n$ is smooth in x,λ and a polynomial in ω,σ, and the following arguments:

$$\hbar t,\hbar\sigma,\hbar^{1/2}\sigma\Phi_{n,\lambda}(B_{jn\nu}(x)),\hbar^{1/2}t\Phi_{n,\lambda}(B_{jn\nu}(x)),\hbar t\Phi_{n,\lambda}(B_{jn\nu}(x))\Phi_{n,\lambda}(B'_{jn\nu}(x))\,, \tag{5.42}$$

for some smooth $B_{jn\nu}(x)$, $B'_{jn\nu}(x)$ with values in $\mathfrak{g}$. Every monomial in $\tilde{b}_j^n$ contains at least j factors of the form (5.42). *Here,*

$$\omega=t^{1/2}\hbar\xi\,,\qquad \sigma=t\hbar\xi\,, \tag{5.43}$$

and only even powers of ω appear.

Proof. As in Lemma 2.1, the proof proceeds by induction on j, the case $j=0$ following from (5.33). In view of (5.34), we can write

$$\begin{aligned} t\tilde{\Gamma}(x,\hbar\xi,\lambda)&=\|\omega\|^2-i(\tilde{X}^{\#}\psi_2)-t(\tilde{W}_1+\tilde{W}_2-V)\quad(\psi_2=x\cdot\sigma)\\ &=\|\omega\|^2+\varkappa(x,\sigma,\lambda,t)\\ &=\Gamma^{\#}(t,x,\omega,\sigma,\lambda)\,. \end{aligned} \tag{5.44}$$

Now, assume (5.40)–(5.42) true for a_j^n. The transformation from a_j^n to a_{j+1}^n is determined by (5.38) and (5.39). Note that

$$b_{j+1}^n(t,\hbar,x,\hbar\xi,\lambda)=\int_0^t\tilde{\Omega}_j(s,\hbar,x,\hbar\xi,\lambda)ds\,,$$

where $\Omega_j=\tilde{\Omega}_je^{-t\tilde{\Gamma}}$. Thus, this transformation can be analyzed as a sum of nine contributions, from (5.38). It is routine to verify that, if b_j^n has the form (5.41) and (5.42), then each of these nine contributions respects this form, and throws in an extra factor from among the five types listed in (5.42). This proves the lemma. Note in particular that the dependence on n comes entirely from the factors listed in (5.42).

In order to analyze the trace of (5.8) we will need to understand $\operatorname{tr}B_nP_n(\lambda)$ when $B_n\in\operatorname{End}\mathfrak{h}_n$ is a product of terms listed in (5.42). Also, in order to get good symbol estimates, we will want to estimate $\|B_nP_n(\lambda)\|_{\mathrm{HS}}^2=\operatorname{tr}(P_n(\lambda)B_n^*B_nP_n(\lambda))=\operatorname{tr}(C_nP_n(\lambda))$, where $C_n=B_n^*B_n$ is also a product of terms listed in (5.42). We will use the notation

$$\langle\langle B_n\rangle\rangle_{n,\lambda}=\operatorname{trace}_{\mathfrak{h}_n}(B_nP_n(\lambda))\,. \tag{5.45}$$

The following lemma will provide the needed information. This lemma has som points in common with lemmas from Sect. 4 of [11], particularly Lemma 4.7 o [11].

Lemma 5.2. *For any* $B_1, \ldots, B_k \in \mathfrak{g}$,

$$\langle\langle \Phi_{n,\lambda}(B_1) \ldots \Phi_{n,\lambda}(B_k) \rangle\rangle_{n,\lambda} \tag{5.46}$$

is zero for $k=1$ *and a polynomial of degree* $\leqq k-2$ *in* $n^{-1/2}$ *for* $k \geqq 2$, *whos coefficients depend smoothly on* λ. *Furthermore, if* k *is even (respectively, odd) the only even (respectively, odd) powers of* $n^{-1/2}$ *appear.*

Proof. The case $k=1$ is equivalent to (3.37). The case $k \geqq 2$ will exploit the identit (3.21) for the generating function

$$\Psi_{n\lambda_1}(g) = (\pi_{n\lambda_1}(g)\psi_{n\lambda_1}, \psi_{n\lambda_1}) = \operatorname{trace}(\pi_n(g)P_n(\lambda_1)), \tag{5.47}$$

namely that $\Psi_{n\lambda_1}(g) = \Psi_{\lambda_1}(g)^n$. Note that, if we set, for $\lambda \in \Gamma_{\lambda_1}$, $\lambda = \mathrm{Ad}^* g\lambda_1$,

$$\Psi_{n\lambda}(g_1) = \operatorname{trace}(\pi_n(g_1)P_n(\lambda)) = \langle\langle \pi_n(g_1) \rangle\rangle_{n,\lambda}, \tag{5.48}$$

since $P_n(\lambda) = \pi_n(g)^{-1} P_n(\lambda_1)\pi_n(g)$, we have

$$\Psi_{n\lambda}(g_1) = \Psi_{n\lambda_1}(g g_1 g^{-1}), \tag{5.49}$$

and hence, for any $\lambda \in \Gamma_{\lambda_1}$, $g \in G$,

$$\Psi_{n\lambda}(g) = \Psi_\lambda(g)^n. \tag{5.50}$$

We now set things up to apply (5.50). We have (5.46) equal to

$$i^{-k}(\partial/\partial z_1) \ldots (\partial/\partial z_k) \langle\langle \exp(i\Phi_{n,\lambda}(z_1 B_1)) \ldots \exp(i\Phi_{n,\lambda}(z_k B_k)) \rangle\rangle_{n,\lambda}|_{z_1 = \ldots = z_k = 0}. \tag{5.51}$$

Now the quantity being differentiated in (5.51) is equal to

$$\exp(-in^{1/2}\langle z_1 B_1 + \ldots + z_k B_k, \lambda\rangle) \langle\langle \exp \pi_n(n^{-1/2} z_1 B_1) \ldots \exp \pi_n(n^{-1/2} z_k B_k) \rangle\rangle_{n,\lambda}. \tag{5.52}$$

It follows from the Campbell-Hausdorff formula that there is a function T_k analytic in a neighborhood of the origin in $\mathfrak{g}^k$, taking values in $\mathfrak{g}$, such that, fc $X_j \in \mathfrak{g}$, $\|X_j\| < \delta$, $\exp T_k(X_1, \ldots, X_k) = (\exp X_1) \ldots (\exp X_k)$. Then (5.52) is equal to

$$\begin{aligned} \mathfrak{A}_n(z) &= \exp(-in^{1/2}\langle z_1 B_1 + \ldots + z_k B_k, \lambda\rangle) \langle\langle \exp \pi_n(T_k(n^{-1/2} z_1 B_1, \ldots, n^{-1/2} z_k B_k)) \rangle\rangle_n \\ &= [\exp(-i\langle n^{-1/2}(z_1 B_1 + \ldots + z_k B_k), \lambda\rangle) \\ &\quad \cdot \langle\langle \exp \pi_1(T_k(n^{-1/2} z_1 B_1, \ldots, n^{-1/2} z_k B_k)) \rangle\rangle_{1,\lambda}]^n . \end{aligned} \tag{5.53}$$

by virtue of (5.50). In order to compute (5.46), in light of (5.51) we want to pull th coefficient of $z_1, \ldots, z_k$ out of (5.53). Expansion of the right side of (5.53) gives

$$\begin{aligned} \mathfrak{A}_n(z) &= (1 + \mathfrak{B}(n^{-1/2} z))^n \quad [\mathfrak{B}(0) = 0] \\ &= \exp(n \log(1 + \mathfrak{B}(n^{-1/2} z))) \\ &= \exp(n[\mathfrak{B}(n^{-1/2} z) - \tfrac{1}{2}\mathfrak{B}(n^{-1/2} z)^2 + \mathfrak{B}(n^{-1/2} z)^3/3 - \ldots]). \end{aligned} \tag{5.54}$$

Note that

$$1+\mathfrak{B}(z)=\exp(-i\langle z_1B_1+\dots z_kB_k,\lambda\rangle)\langle\langle\exp\pi_1(T_k(z_1B_1+\dots z_kB_k))\rangle\rangle_{1,\lambda}.$$

Since $T_k(X_1,\dots,X_k)=X_1+\dots+X_k+O(\sum\|X_j\|^2)$, we have, with $z\cdot B=z_1B_1+\dots+z_kB_k$,

$$\begin{aligned}1+\mathfrak{B}(z)&=\exp(-i\langle z\cdot B,\lambda\rangle)\langle\langle\exp\pi_1(z\cdot B)\rangle\rangle_{1,\lambda}+O(|z|^2)\\&=1-i\langle z\cdot B,\lambda\rangle+\langle\langle\pi_1(z\cdot B)\rangle\rangle_{1,\lambda}+O(|z|^2)\\&=1+O(|z|^2)\end{aligned}$$

in light of (3.37). In other words, $\mathfrak{B}(z)=O(|z|^2)$. Say

$$\mathfrak{B}(z)=z_jz_\ell E^{j\ell}(z)\quad\text{(summation convention)},$$

with $E^{j\ell}(z)$ analytic for z near 0 in $\mathbb{C}^k$. Then (5.54) becomes

$$\mathfrak{A}_n(z)=\exp(z_jz_\ell E^{j\ell}(n^{-1/2}z)+\dots)=1+\sum_{|\alpha|\leqq 2}\kappa_\alpha(n)z^\alpha,\tag{5.55}$$

where $\kappa_\alpha(n)$ is a polynomial in $n^{-1/2}$ of degree $\leqq|\alpha|-2$, with only even (respectively, odd) powers appearing when $|\alpha|$ is even (respectively, odd). Examining the coefficient of $z_1\dots z_k$ proves the lemma.

Our first application of Lemma 5.2 will be to get symbol estimates for the amplitudes a_j^n which were given a qualitative analysis in Lemma 5.1. On the space $\operatorname{End}\mathfrak{h}_n$, the Hilbert-Schmidt (HS) norm is defined by $\|T\|_{\mathrm{HS}}^2=\operatorname{trace}(T^*T)$.

Lemma 5.3. *Fix positive T and E. Then, for $0\leqq t\leqq T$ and $0\leqq\hbar\leqq E$ we have*

$$\|D_x^\beta D_\xi^\alpha a_j^n(t,\hbar,x,\xi,\lambda)\|_{\mathrm{HS}}\leqq C_{j\alpha\beta}(\hbar t)^{j/2}(1+|\xi|)^{-|\alpha|}e^{-t\|\hbar\xi\|^2/2},\tag{5.56}$$

and

$$\|D_x^\beta D_\xi^\alpha a_j^n(t,\hbar,x,\xi,\lambda)\|_{\mathrm{HS}}\leqq C'_{j\alpha\beta}(1+|\xi|)^{-j/2-|\alpha|},\tag{5.57}$$

with $C_{j\alpha\beta}$ and $C'_{j\alpha\beta}$ independent of λ and of n.

Proof. From Lemma 5.2 we conclude that all terms of the form $\Phi_{n,\lambda}(B_1)\dots\Phi_{n,\lambda}(B_kP_n(\mathrm{g})$ have uniformly bounded Hilbert-Schmidt norms as $n\to\infty$. Thus the estimate (5.56) follows from (5.40)–(5.42), together with the formula (5.34) for the exponent $\tilde{\Gamma}$. The estimate (5.57) follows from (5.56). Note that applying $D_t^\mu D_\hbar^\nu$ to $D_x^\beta D_\xi^\alpha a_j^n$ increases the order in ξ by at most $2(\mu+\nu)$ units.

Since we want to compute traces, we will also need to estimate the trace norms of various symbols. Sufficiently good estimates for our purposes will follow from the simple observation that

$$T\in\operatorname{End}\mathfrak{h}_n\ \Rightarrow\ \|T\|_{\mathrm{tr}}\leqq d_n^{1/2}\|T\|_{\mathrm{HS}},\tag{5.58}$$

where $d_n=\dim\mathfrak{h}_n$. Here the trace norm is defined by $\|T\|_{\mathrm{tr}}=\operatorname{trace}(T^*T)^{1/2}$. Recall that Weyl's formula for d_n is given by (3.2). We deduce that

$$d_n\leqq Cn^k,\tag{5.59}$$

where k is the dimension of the maximal torus in G. Thus, if (5.3) is satisfied, we have

$$\|D_x^\beta D_\xi^\alpha a_j^n(t,\hbar,x,\xi,\lambda)\|_{\mathrm{tr}} \leqq C_{j\alpha\beta}(\hbar t)^{j/2-k}(1+|\xi|)^{-|\alpha|}e^{-t\|\hbar\xi\|^2/2}$$
$$\leqq C'_{j\alpha\beta}(1+|\xi|)^{k-\frac{1}{2}j-|\alpha|}, \quad \text{if} \quad j\geqq 2k. \tag{5.60}$$

Let us consider a partial sum of the expansion (5.28):

$$A_n^\ell(t,\hbar,x,\xi,\lambda)=\sum_{j=0}^{\ell} a_j^n(t,\hbar,x,\xi,\lambda). \tag{5.61}$$

Form

$$W_n^\ell(t)f(x)=d_n\iint A_n^\ell(t,\hbar,x,\xi,\lambda)e^{ix\cdot\xi}\hat{f}(\xi,\lambda)d\mu_\Gamma(\lambda)d\xi. \tag{5.62}$$

We see that

$$W_n(0)=I, \tag{5.63}$$

by virtue of (5.5), and, with $\hbar=1/n$,

$$(\partial/\partial t-H_\hbar)W_n(t)f(x)=d_n\iint B_n^\ell(t,\hbar,x,\xi,\lambda)e^{ix\cdot\xi}P_n(\lambda)\hat{f}(\xi)d\mu_\Gamma(\lambda)d\xi, \tag{5.64}$$

where B_n is a sum of $\partial a_\ell^n/\partial t$ and expressions of the form (5.39) with a_{j-1} replaced by $a_{\ell-1}^n$ and a_ℓ^n. We see from Lemma 5.3 that

$$\|D_x^\beta D_\xi^\alpha B_n^\ell(t,\hbar,x,\xi,\lambda)\|_{\mathrm{HS}} \leqq C_{\ell\alpha\beta}(\hbar t)^{\frac{1}{2}\ell-1}(1+|\xi|)^{-|\alpha|}e^{-t\|\hbar\xi\|^2/2}$$
$$\leqq C'_{\ell\alpha\beta}(1+|\xi|)^{-\frac{1}{2}\ell+1-|\alpha|} \quad \text{if} \quad \ell\geqq 2. \tag{5.65}$$

We next need a replacement for Lemma 2.3. For an element $u\in C^\infty(M,\mathfrak{h}_n)$, let $\Lambda^s=(1-\Delta)^{s/2}$, a scalar elliptic operator of order s, and define

$$\|u\|_s^2=\int_M \|\Lambda^s u\|_{\mathfrak{h}_n}^2 dx, \tag{5.66}$$

and

$$(u,v)_s=\int_M (\Lambda^s u,\Lambda^s v)_{\mathfrak{h}_n}dx, \tag{5.67}$$

where $\| \ \|_{\mathfrak{h}_n}$ and $(\, ,)_{\mathfrak{h}_n}$ denote the norm and inner product on $\mathfrak{h}_n$. Similarly, if $V\in C^\infty(M,\mathrm{End}\,\mathfrak{h}_n)$, let

$$\|V\|_s^2=\int_M \|\Lambda^s V\|_{\mathrm{HS}}^2 dx. \tag{5.68}$$

For starters, we want to estimate the operator norm of

$$e^{-tH_\hbar}: H^s(M,\mathfrak{h}_n)\to H^s(M,\mathfrak{h}_n), \qquad \hbar=1/n, \tag{5.69}$$

obtaining an estimate *independent of n*. We will prove the following.

Lemma 5.4. *With $\hbar=1/n$, we have the operator norm estimate on* (5.69):

$$\|e^{-tH_\hbar}\|_{\mathscr{L}(H^s)}\leqq Ae^{Bt}, \tag{5.70}$$

with A and B independent of n (perhaps depending on s).

Proof. Recall $H_\hbar$ is given by

$$-H_\hbar = \hbar^2 L + \hbar X_n + V_n. \tag{5.71}$$

Since L is scalar, we can apply Gårding's inequality to get

$$\operatorname{Re}(-Lu, u)_s \geqq C_0 \|u\|_{s+1}^2 - C_1 \|u\|_s^2, \tag{5.72}$$

with C_0 and C_1 independent of n. In order to extend the reasoning used to derive (2.37), let us note that

$$X \in \mathfrak{g} \Rightarrow \|\pi_n(X/n)\| \leqq C\|X\|, \quad \text{independent of } n. \tag{5.73}$$

This follows from the fact that $X(-\Delta_G)^{-1/2} \in OPS^0(G)$, if we fit all π_n inside the regular representation of G and use (3.3). Another proof of (5.73) is given in [11]. Using this and the form (5.12) and (5.13) of X_n and V_n in (5.71), it is elementary to derive, at least for s an even integer, that

$$\begin{aligned}
\operatorname{Re}(H_\hbar u, u)_s &= \hbar^2 \operatorname{Re}(-Lu, u)_s + \hbar \operatorname{Re}(-X_n u, u)_s + (-V_n u, u)_s \\
&\geqq C_0 \hbar^2 \|u\|_{s+1}^2 - K\hbar \|u\|_{s+1} \|u\|_s - C_2 \|u\|_s^2 \\
&\geqq C_0 \hbar^2 \|u\|_{s+1}^2 - [\tfrac{1}{2} C_0 \hbar^2 \|u\|_{s+1}^2 + 2K^2 C_0^{-1} \|u\|_s^2] - C_2 \|u\|_s^2 \\
&\geqq \tfrac{1}{2} C_0 \hbar^2 \|u\|_{s+1}^2 - (C_2 + 2K^2 C_0^{-1}) \|u\|_s^2 \\
&\geqq -C\|u\|_s^2.
\end{aligned} \tag{5.74}$$

Here C_0, C_2, K, and C are all positive and independent of n (hence of $\hbar$). From this, (5.70) is an immediate consequence, at least if s is an even positive integer. The result follows for all positive real s, by interpolation, and then for all real s, by duality.

We now can obtain our energy estimates.

Lemma 5.5. *Let $v_n(t,x)$ take values in $\mathfrak{h}_n$ and satisfy (with $\hbar = 1/n$)*

$$(\partial/\partial t + H_\hbar) v_n = g_n(t,x), \quad v_n(0,x) = f_n(x). \tag{5.75}$$

Then, for $0 \leqq t \leqq T$, we have

$$\sup_t \|v_n(t,\cdot)\|_{H^s} \leqq C_1 \|f_n\|_{H^s} + C_2 \sup_t \|g_n(t,\cdot)\|_{H^s}, \tag{5.76}$$

where the H^s norm is defined by (5.66), *and C_1 and C_2 are independent of n. One has similar estimates on*

$$\sup_t \|D_t^\mu v_n(t,\cdot)\|_{H^{s-2\mu}}. \tag{5.77}$$

Proof. In view of Duhamel's principle,

$$v_n = e^{-tH_\hbar} f_n + \int_0^t e^{-(t-\tau)H_\hbar} g_n(\tau, x) d\tau,$$

the estimate (5.76) follows directly from Lemma 5.4. Analogous estimates for (5.77) follow easily. Let us remark we also have the same sort of estimates when g_n, f_n, and v_n take values in $\operatorname{End}\mathfrak{h}$, given the Hilbert-Schmidt norm.

We apply this to $U_n(t)-W_n^\ell(t)$. We have

$$U_n(0)-W_n^\ell(0)=0, \tag{5.78}$$

and

$$(\partial/\partial t+H_\hbar)(U_n(t)-W_n^\ell(t))=B_n^\ell(t), \tag{5.79}$$

defined by the right side of (5.64). Using (5.68), we see that, for

$$u_n\in H^{-\frac{1}{2}(v+1)}(M,\mathfrak{h}_n), \tag{5.80}$$

$$\|B_n^\ell(t)u_n\|_{H^{\frac{1}{2}(v+1)}}\leqq C_\ell(\hbar t)^{\frac{1}{2}\ell-v-2}\|u_n\|_{H^{-\frac{1}{2}(v+1)}}, \quad \text{for} \quad \ell>2v+4, \tag{5.81}$$

with a similar estimate for $u\in H^{-\frac{1}{2}(v+1)}(M,\operatorname{End}\mathfrak{h}_n)$; C_ℓ is independent of n. Recall $v=\dim M$. If we take $u_n=\delta_y\otimes I_{\mathfrak{h}_n}$, we have

$$\|u_n\|_{H^{-\frac{1}{2}(v+1)}}\leqq Cd_n^{1/2}, \tag{5.82}$$

so, for $0\leqq t\leqq T$,

$$\|B_n(t)\delta_y\|_{H^{\frac{1}{2}(v+1)}}\leqq C'_\ell d_n^{1/2}(\hbar t)^{\frac{1}{2}\ell-v-2},$$

and hence, by Lemma 5.5,

$$\|(U_n(t)-W_n^\ell(t))\delta_y\|_{H^{\frac{1}{2}(v+1)}}\leqq C'_\ell d_n^{1/2}(\hbar t)^{\frac{1}{2}\ell-v-2}. \tag{5.83}$$

By the Sobolev imbedding theorem, we have, for $\ell>2v+4$,

$$\sup_{y,x\in M}\|(U_n(t)-W_n^\ell(t))\delta_y(x)\|_{\mathrm{HS}}\leqq C_\ell d_n^{1/2}(\hbar t)^{\frac{1}{2}\ell-v-2}.$$

In light of (5.58) and (5.59), this gives, for $0\leqq t\leqq T$, $\hbar=1/n$,

$$\begin{aligned}\sup_{y,x\in M}\|(U_n(t)-W_n^\ell(t))\delta_y(x)\|_{\mathrm{tr}}&\leqq C_\ell d_n(\hbar t)^{\frac{1}{2}\ell-v-2}\\&\leqq C'_\ell\hbar^{\frac{1}{2}\ell-v-2-k},\end{aligned} \tag{5.84}$$

which in turn yields the important estimate

$$|\operatorname{trace}(U_n(t)-W_n(t))|\leqq C_\ell n^{-(\frac{1}{2}\ell-v-2-k)}. \tag{5.85}$$

Things wind down fairly quickly from here. We have only to analyze the traces of the terms $V_j^n(t)$ in $W_n^\ell(t)=V_0^n(t)+\ldots+V_\ell^n(t)$:

$$\begin{aligned}V_j^n(t)f(x)&=d_n\iint a_j^n(t,\hbar,x,\xi,\lambda)e^{ix\cdot\xi}P_n(\lambda)\hat{f}(\xi)d\mu_\Gamma(\lambda)d\xi\\&=d_n\iint b_j^n(t,\hbar,x,\hbar\xi,\lambda)e^{-t\tilde{\Gamma}(x,\hbar\xi,\lambda)}e^{ix\cdot\xi}P_n(\lambda)\hat{f}(\xi)d\mu_\Gamma(\lambda)d\xi.\end{aligned} \tag{5.86}$$

In analogy with (2.47), we have

$$\begin{aligned}\operatorname{trace}V_j^n(t)&=d_n\iiint \operatorname{tr}b_j^n(t,\hbar,x,\hbar\xi,\lambda)e^{-t\tilde{\Gamma}(x,\hbar\xi,\lambda)}P_n(\lambda)d\mu_\Gamma(\lambda)d\xi\,dx\\&=d_n\hbar^{-v}\iiint \operatorname{tr}b_j^n(t,\hbar,x,\zeta,\lambda)e^{-t\tilde{\Gamma}(x,\zeta,\lambda)}P_n(\lambda)d\mu_\Gamma(\lambda)d\zeta\,dx.\end{aligned} \tag{5.87}$$

Now, by (5.41)–(5.42), we see that $b_j(t,\hbar,x,\zeta,\lambda)$ is a polynomial in $\hbar^{1/2}=n^{-1/2}$ containing at least $\hbar^{j/2}$ in each term, and there are an odd number of factors $\Phi_{n,\lambda}(B_{jnv}(x))$ in a term if and only if the exponent of $\hbar$ is not an integer. In light of Lemma 5.2, any such term in $b_j^n(t,\hbar,x,\zeta,\lambda)$ contributes to (5.87) a multiple of $(d_n\hbar^{-v})\hbar^J$, where $J\geqq j/2$ is an *integer*. In other words,

$$\operatorname{trace} V_j^n(t) \sim d_n \hbar^{-\nu}(\alpha_{j0}(t)\hbar^{[\frac{1}{2}(j+1)]} + \alpha_{j1}(t)\hbar^{[\frac{1}{2}(j+1)]+1} + \ldots), \tag{5.88}$$

which, combined with (5.85), proves our main result:

Theorem 5.6. *If $H_\hbar$ is given by* (5.9), *with $\hbar = 1/n$, we have, as $n \to \infty$,*

$$\hbar^\nu d_n^{-1} \operatorname{trace} e^{-tH_\hbar} \sim a_0(t) + a_1(t)\hbar + a_2(t)\hbar^2 + \ldots, \tag{5.89}$$

and $a_0(t)$ is given by

$$a_0(t) = d_n^{-1} \operatorname{trace} V_0^n(t) = \iiint e^{-t\tilde{\Gamma}(x,\zeta,\lambda)} d\zeta\, d\mu_\Gamma(\lambda) d\operatorname{vol}(x). \tag{5.90}$$

Of course, this result coincides with Theorem 4.8. Note that, after the fact, we can sharpen up (5.85), replacing $n^{-(\frac{1}{2}\ell-\nu-2-k)}$ by $n^{-[\frac{1}{2}\ell]-1}$.

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