Commun. Math. Phys. 92, 507–524 (1984)

Communications in
Mathematical
Physics

Classical and Quantum Algebras of Non-Local Charges in σ Models

H. J. de Vega, H. Eichenherr, and J. M. Maillet

Laboratoire de Physique Théorique et Hautes Energies*, Université Pierre et Marie Curie, Tour 16 – 1er étage, 4, Place Jussieu, F-75230 Paris Cedex 05, France

Abstract. We investigate the algebras of the non-local charges and their generating functionals (the monodromy matrices) in classical and quantum non-linear σ models. In the case of the classical chiral σ models it turns out that there exists no definition of the Poisson bracket of two monodromy matrices satisfying antisymmetry and the Jacobi identity. Thus, the classical non-local charges do not generate a Lie algebra. In the case of the quantum $O(N)$ non-linear σ model, we explicitly determine the conserved quantum monodromy operator from a factorization principle together with $\mathscr{P}$, $\mathscr{T}$, and $O(N)$ invariance. We give closed expressions for its matrix elements between asymptotic states in terms of the known two-particle S-matrix. The quantum R-matrix of the model is found. The quantum non-local charges obey a quadratic Lie algebra governed by a Yang-Baxter equation.

I. Introduction

The notion of complete integrability in field theory involves the existence of an infinite number of commuting conserved quantities. In addition to these usually local quantities, some models possess an infinite number of non-local conserved charges which do not commute among themselves. This raises the important question whether the integrability of such field theories can be related to the existence of an infinite dimensional non-abelian dynamical symmetry algebra. For finitely many degrees of freedom dynamical symmetries are well known (e.g. the Coulomb problem and the harmonic oscillator). In field theory, the non-linear σ models are good candidates to possess this kind of structure.

To construct such a dynamical algebra one must find the Poisson brackets of non-local charges [1–4] in the classical field theory and the corresponding commutators in the quantum field theory. The monodromy matrix of the associated linear system [3–5] (Lax pair) serves as the generating functional of the non-local charges. This is a system of linear differential equations having the field

* Laboratoire associé au CNRS No. LA 280

equations as a compatibility condition. The monodromy matrix connects the solutions of the linear system at plus and minus (spatial) infinity. For a large class of integrable models the Poisson bracket of monodromy matrices can be expressed in an elegant way using the so-called r-matrix [6]. This r-matrix must solve the classical Yang-Baxter equations in order that the Jacobi identity holds for Poisson brackets. In all these cases, the monodromy matrix directly provides action and angle variable for the classical theory. In contrast to this, angle variables for the σ models have been unknown up to now: because of conformal invariance these models are lacking a frequency scale. So, the associated linear problem does not have Jost solutions which oscillate at infinity. The whole monodromy matrix $T(\lambda)$ is time-independent and all its matrix elements are conserved charges.

To obtain the canonical algebra of these non-local charges in a closed form we have investigated the Poisson bracket $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}$ of their generating functionals. We have done this for the chiral σ models where the canonical formalism is particularly simple. To start with we compute the canonical transformation generated by $T(\lambda)$ on the field g and momentum π. From this we find that $T(\lambda)$ commutes with the densities of energy, momentum and conformal charges. Poisson brackets between monodromy matrices can be efficiently obtained from the Poisson brackets of the currents entering the linear system [6, 7]. In the case of the non-linear σ models this current algebra is non-ultralocal since it contains derivatives of the delta function.

A careful analysis of $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}$ leads to the conclusion that this object is not uniquely defined. Moreover, there is no definition consistent with the basic properties of Poisson brackets, namely, antisymmetry and the Jacobi identity. This disease is related to the short distance singularities of the current algebra (non-ultralocality) and the absence of a mass scale. On the level of the algebra of canonical transformations induced by $T(\lambda)$, a related problem shows up: The commutator of two such transformations is not generated by any function on phase space, in particular not by a function of the monodromy matrices.

A natural way to regularize short-distance singularities is to introduce a spatial lattice such that integrability is preserved. An integrable formulation of the chiral σ model on a space-time lattice is known [8, 9]. Unfortunately, no consistent integrable space discretization with continuous time is presently available.

It is known that there exists an infinite-dimensional Lie algebra of symmetry transformations acting on the space of solutions of the chiral σ model [10]. This algebra has turned out to be a loop algebra. The non-locality of these symmetries raised the question whether they are related to the non-local charges and in particular, whether they are canonically generated by them. Since these transformations do not preserve the basic Poisson brackets [11], the latter cannot be true. In conclusion, this loop algebra of symmetry transformations is restricted to solution space and cannot be extended to phase space. Moreover, the classical non-local charges do not form a loop algebra since they do not even form a Lie algebra.

It is known that the first non-local charge exists as a renormalized, conserved operator in the quantum O(N) non-linear σ model [12]. Since by dimensional transmutation a mass scale is generated in the quantum σ models, one can hope that the problems found in classical theory are absent in the quantum case. In fact, we have obtained an explicit formula for the quantum monodromy operator $T(\lambda)$ which generates the quantum non-local charges. Moreover, we derive an expression for the commutator $[T(\lambda) \underset{,}{\otimes} T(\mu)]$ consistent with all the properties of a Lie algebra.

To determine the quantum monodromy operator we have used $\mathscr{P}$, $\mathscr{T}$, and $O(N)$ invariance, time independence of $T(\lambda)$ and a factorization principle for the action of $T(\lambda)$ on multiparticle states. In the classical theory $T(\lambda)$ for a configuration consisting of several separated lumps there is an appropriately ordered product of the $T(\lambda)$ for each lump. In the quantum theory we assume analogously that $T(\lambda)$ for an asymptotic k-particle state is an appropriately ordered product of the $T(\lambda)$ for each one-particle state [13]. From this factorization principle and the conservation of $T(\lambda)$ in two-particle scattering one gets a set of homogeneous functional equations for the one-particle matrix elements of $T(\lambda)$. These matrix elements fulfill additional bilinear equations due to $\mathscr{P}$ and $\mathscr{T}$ invariance. The whole set of functional equations has the same form as the equations determining the two-body S-matrix (factorization, unitarity and crossing). This observation leads to the result that the quantum monodromy operators can be expressed in closed form in terms of the two-body S-matrix. Explicitly we find for the one-particle matrix elements of $T_{ab}(\lambda)$ between states with isospin labels c and d and rapidities θ and θ'

$$\langle \theta, d | T_{ab}(\lambda) | \theta', c \rangle = \delta(\theta - \theta') S_{ad,bc}(\theta + \gamma(\lambda)).$$

Here $\gamma(\lambda)$ is a new spectral parameter arising in a natural way from the factorization equations. Due to the factorization principle, the matrix elements of $T_{ab}(\lambda)$ between k-particle states are appropriate products of k S-matrices, in complete formal analogy with the monodromy matrix in the statistical mechanics of an inhomogenous vertex model on a one-dimensional lattice of k sites, the $S_{ad,bc}(\theta + \gamma(\lambda))$ being identified with the statistical weights. In this way we have obtained an explicit solution for the monodromy operator in the quantum $O(N)$ non-linear σ model including all the higher non-local charges. This solution matches with the known first two charges [12] obtained from the classical ones by renormalization. This comparison provides an expression for $\gamma(\lambda)$ at small λ. The commutator algebra of the non-local charges follows immediately from the above mentioned analogy with vertex models: Since the statistical weights, i.e. the S-matrix, satisfy a Yang-Baxter equation, the algebra is governed by the same Yang-Baxter equation. Hence it is a quadratic algebra in the sense that the commutator of two charges contains terms quadratic in the charges:

$$\begin{aligned}[T_{ac}(\lambda), T_{bd}(\lambda')] = & \frac{2\pi i}{N-2} \frac{1}{\gamma(\lambda) - \gamma(\lambda')} (T_{bc}(\lambda) T_{ad}(\lambda') - T_{bc}(\lambda') T_{ad}(\lambda)) \\ & + \frac{2\pi i}{N-2} \frac{1}{\gamma(\lambda) - \gamma(\lambda') - i\pi} (\delta_{cd} T_{be}(\lambda') T_{ae}(\lambda) - \delta_{ab} T_{ec}(\lambda) T_{ed}(\lambda')).\end{aligned}$$

A complete set of infinitesimal generators of such an algebra therefore includes products of any number of charges.

Our results amount to the determination of the quantum R-matrix for the $O(N)$ σ model. For theories like the non-linear Schrödinger or sine-Gordon this is the key to the exact solution of the quantum model. In the present case the situation is different: Since $T(\lambda)$ as in the classical case is conserved, it does not provide creation and annihilation operators.

II. On the Canonical Algebra of Non-Local Charges for Classical Chiral Fields

II.1. Chiral Fields and Loop Algebras

The chiral field $g(t,x)$ takes values in a compact Lie group G with Lie algebra $\mathfrak{g}$. The Lagrangian reads

$$\mathscr{L} = -\tfrac{1}{2}\operatorname{tr} A_\mu A^\mu\,, \tag{1}$$

where $A_\mu = g^{-1}\partial_\mu g$ obeys the equations of motion

$$\partial_\mu A^\mu = 0\,, \tag{2}$$

and the zero curvature condition

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0\,. \tag{3}$$

We assume the boundary conditions

$$\lim_{x\to\pm\infty} A_\mu(t,x) = 0\,, \qquad \lim_{x\to\pm\infty} g(t,x) = g_\pm\,. \tag{4}$$

The hamiltonian formulation is given in terms of the canonical variables $g(t,x)$ and $\pi(t,x) = \partial_0(g^{-1})^T$ by the Hamiltonian

$$H = \tfrac{1}{2}\int_{-\infty}^{+\infty} dx\, \operatorname{tr}(-\pi^T g \pi^T g + \partial_x g\, \partial_x(g^{-1}))\,, \tag{5}$$

and the Poisson brackets

$$\begin{gathered} \{g(t,x) \overset{\otimes}{,} g(t,y)\} = \{\pi(t,x) \overset{\otimes}{,} \pi(t,y)\} = 0\,, \\ \{g(t,x) \overset{\otimes}{,} \pi^T(t,y)\} = \delta(x-y)P\,, \end{gathered} \tag{6}$$

where we have used the tensor product notation

$$(A\otimes B)_{ac,bd} = A_{ab}B_{cd}\,, \qquad \{A \overset{\otimes}{,} B\}_{ac,bd} = \{A_{ab}, B_{cd}\}\,, \tag{7}$$

and the permutation operator $P_{ab,cd} = \delta_{ad}\delta_{cb}$, with the property $P(A\otimes B)P = B\otimes A$. The algebra of the currents A_μ follows from Eqs. (6):

$$\begin{gathered} \{A_0(x) \overset{\otimes}{,} A_0(y)\} = \delta(x-y)\,[\mathbb{1}\otimes A_0(x), P]\,, \\ \{A_1(x) \overset{\otimes}{,} A_0(y)\} = \delta(x-y)\,(\mathbb{1}\otimes A_1(x))P - \delta'(x-y)\,(g^{-1}(x)\,g(y)\otimes\mathbb{1})P\,, \\ \{A_1(x) \overset{\otimes}{,} A_1(y)\} = 0\,. \end{gathered} \tag{8}$$

The main object of our interest is the monodromy matrix $T(X, Y|\lambda)$ which fulfills the equations [3–5]

$$\frac{\partial}{\partial X^\mu} T(X, Y|\lambda) = -L_\mu(X,\lambda)\, T(X, Y|\lambda)\,, \tag{9}$$

$$\frac{\partial}{\partial Y^{\mu}} T(X, Y|\lambda) = T(X, Y|\lambda) L_{\mu}(Y, \lambda), \tag{10}$$

$$T(X, X|\lambda) = \mathbb{1}, \tag{11}$$

where

$$L_{\mu}(X, \lambda) = \frac{\lambda}{\lambda^2 - 1}(\lambda A_{\mu}(X) - \varepsilon_{\mu\nu} A^{\nu}(X)), \tag{12}$$

$$X = (t, x),\ Y = (t', y),\ g_{00} = -g_{11} = 1,\ \varepsilon_{01} = 1.$$

Equations (2), (3) are the compatibility conditions of Eqs. (9), or equivalently, of Eqs. (10). We also introduce

$$\begin{aligned} T^{+}(X, \lambda) &= \lim_{y \to +\infty} T(X, Y|\lambda), \\ T^{-}(Y, \lambda) &= \lim_{x \to -\infty} T(X, Y|\lambda), \\ T(\lambda) &= \lim_{\substack{x \to -\infty \\ y \to +\infty}} T(X, Y|\lambda). \end{aligned} \tag{13}$$

Since $L_{\mu}(X, \lambda)$ vanishes for $|x| \to \infty$, the monodromy matrix $T(X, Y|\lambda)$ becomes independent of $t(t')$ when $x(y)$ goes to infinity. For later use, we note the transformation behaviour of these quantities under $\mathscr{P}$ (parity) and $\mathscr{T}$ (time reversal):

$$\mathscr{P}: \begin{cases} A_0(X) \to A_0(\tilde{X}) \\ A_1(X) \to -A_1(\tilde{X}) \\ L_0(X, \lambda) \to L_0(\tilde{X}, -\lambda) \\ L_1(X, \lambda) \to -L_1(\tilde{X}, -\lambda), \end{cases} \qquad \mathscr{P}\mathscr{T}: \begin{cases} A_0(X) \to -A_0(-X) \\ A_1(X) \to -A_1(-X) \\ L_0(X, \lambda) \to -L_0(-X, \lambda) \\ L_1(X, \lambda) \to -L_1(-X, \lambda), \end{cases}$$

with $\tilde{X} = (t, -X)$, and consequently, from Eqs. (9), (10), and (11)

$$\mathscr{P}: T(X, Y|\lambda) \to T(\tilde{X}, \tilde{Y}|-\lambda) \quad \text{and} \quad T(\lambda) \to T^{-1}(-\lambda), \tag{14}$$

$$\mathscr{P}\mathscr{T}: T(X, Y|\lambda) \to T(-X, -Y|\lambda) \quad \text{and} \quad T(\lambda) \to T^{-1}(\lambda). \tag{15}$$

The classical equations of motion of the chiral σ models are known to possess an infinite-dimensional Lie algebra of symmetry transformations which has the structure of the loop algebra $\mathfrak{g} \otimes \mathbb{R}[\lambda]$ [10]. To define these transformations, we use the functions [14]

$$\overset{(-)}{V}_a(X, \lambda) = [T^{-}(X, \lambda)]^{-1} t_a T^{-}(X, \lambda),$$

$$\overset{(+)}{V}_a(X, \lambda) = T^{+}(X, \lambda)\, t_a [T^{+}(X, \lambda)]^{-1},$$

where the t_a are a basis of $\mathfrak{g}$ with

$$[t_a, t_b] = f_{abc} t_c \quad \text{and} \quad \operatorname{tr}(t_a t_b) = \delta_{ab}.$$

Now the transformation

$$g \to g - \delta \overset{(\pm)}{M}{}^{\lambda}_{a} g, \tag{16}$$

where $(\overset{(\pm)}{M}{}^{\lambda}_{a}g)(X)=g(X)\overset{(\pm)}{V}_{a}(X,\lambda)$ is an infinitesimal symmetry of the equations of motion. This means that $g-\delta\overset{(\pm)}{M}{}^{\lambda}_{a}g$ is a solution of (2) up to terms of order δ^2 if g satisfies (2). For the generators

$$\overset{(\pm)}{M}{}^{n}_{a}=\frac{1}{n!}\frac{d^n}{d\lambda^n}\overset{(\pm)}{M}{}^{\lambda}_{a}|_{\lambda=0},$$

one finds the loop algebra commutation relations

$$[\overset{(\pm)}{M}{}^{m}_{a},\overset{(\pm)}{M}{}^{n}_{b}]=f_{abc}\overset{(\pm)}{M}{}^{m+n}_{c}. \tag{17}$$

The transformation $\overset{(\pm)}{M}{}^{1}_{a}$ reads explicitly

$$(\overset{(\pm)}{M}{}^{1}_{a}g)(X)=\mp g(X)\int\limits_{-\infty}^{+\infty}dz\theta(\pm(z-x))[A_0(t,z),t_a]. \tag{18}$$

On the other hand, the chiral models possess infinitely many conserved non-local charges $Q^{(n)}$ [1–4] the generating functional of which is the time independent monodromy matrix $T(\lambda)$

$$T(\lambda)=\exp Q(\lambda),\qquad Q(\lambda)=\sum_{1}^{\infty}\lambda^n Q^{(n)}. \tag{19}$$

Defining $Q^{(n)}_{a}=\operatorname{tr}t_aQ^{(n)}$, we find that the transformation generated by $Q^{(2)}_{a}$ is

$$\{Q^{(2)}_{a}\underset{,}{\otimes}g(X)\}=\tfrac{1}{2}g(X)\int\limits_{-\infty}^{+\infty}dz\varepsilon(z-x)[A_0(t,z),t_a]. \tag{20}$$

It has been noticed [11] that the transformations (18) and (20) do not coincide. In fact, the transformation (20), being generated by a charge, is canonical, whereas (18) is not because it does not preserve, e.g., the Poisson bracket

$$\{g(t,x)\underset{,}{\otimes}g(t,y)\}=0.$$

Both are related by

$$-2\{Q^{(2)}_{a}\underset{,}{\otimes}g(X)\}=(\overset{(+)}{M}{}^{1}_{a}g+\overset{(-)}{M}{}^{1}_{a})(X).$$

The $\overset{(+)}{M}{}^{1}_{a}$ and $\overset{(-)}{M}{}^{1}_{a}$, taken separately, generate two isomorphic loop algebras which, however, do not commute:

$$[\overset{(+)}{M}{}^{1}_{a},\overset{(-)}{M}{}^{1}_{b}]\neq 0.$$

Thus $Q^{(2)}_{a}$ does not necessarily generate a loop algebra. To find the general algebraic structure, we have investigated the Poisson bracket of two generating functionals $\{T(\lambda)\underset{,}{\otimes}T(\mu)\}$.

II.2. Poisson Brackets and Monodromy Matrices

First of all, let us look at the infinitesimal canonical transformations which are induced on g and π by the monodromy matrix $T(\lambda)$:

$$\{T_{ab}(\lambda),g_{cd}(x)\}=-\frac{\delta T_{ab}(\lambda)}{\delta\pi_{cd}(x)}=-\int\limits_{-\infty}^{+\infty}dz\sum_{i,j}\frac{\delta T_{ab}(\lambda)}{\delta L_{ij}(z,\lambda)}\frac{\delta L_{ij}(z,\lambda)}{\delta\pi_{cd}(x)}. \tag{21}$$

Using the formula [7]

$$\frac{\delta T_{ab}(\lambda)}{\delta L_{ij}(z,\lambda)} = T^{-}_{ai}(z,\lambda)\, T^{+}_{jb}(z,\lambda)\,, \tag{22}$$

and

$$\frac{\delta L_{ij}(z,\lambda)}{\delta \pi_{cd}(x)} = -\frac{\lambda}{\lambda^2-1} g_{cj}(z)\,\delta_{id}\,\delta(x-z)\,, \tag{23}$$

we find

$$\{T(\lambda)\underset{,}{\otimes} g(x)\} = \frac{\lambda}{\lambda^2-1}[T^{-}(x,\lambda)\otimes g(x)\, T^{+}(x,\lambda)]\,P\,. \tag{24}$$

Similarly, the action of the canonical transformation on the momentum π is given by

$$\begin{aligned}\{T(\lambda)\underset{,}{\otimes}\pi^T(x)\} = -\frac{\lambda}{\lambda^2-1}[&T^{-}(x,\lambda)\,\pi^T(x)\otimes T^{+}(x,\lambda) + \lambda T^{-}(x,\lambda)\, g^{-1}(x) \\ &\otimes A_1(x)\, T^{+}(x,\lambda) + \lambda\,\partial_x(T^{-}(x,\lambda)\, g^{-1}(x)\otimes T^{+}(x,\lambda))]\,P\,.\end{aligned} \tag{25}$$

The classical chiral field theory is invariant under the conformal group which is generated by the charges

$$Q_{\pm}(F) = -\frac{1}{2}\int_{-\infty}^{+\infty} dx F\left(\frac{t\pm x}{2}\right)\mathrm{tr}(A_0\pm A_1)^2(t,x), \qquad \{H, Q_{\pm}(F)\}=0\,. \tag{26}$$

From Eqs. (8) we derive the conformal algebra

$$\begin{aligned}\{Q_{\pm}(F), Q_{\pm}(G)\} &= Q_{\pm}(W_{\pm}[F,G])\,,\\ \{Q_{\pm}(F), Q_{\mp}(G)\} &= 0\,,\end{aligned} \tag{27}$$

where

$$W_{\pm}(F,G) = F\,\partial_{\pm}G - G\,\partial_{\pm}F, \qquad \partial_{\pm} = \partial_t \pm \partial_x\,.$$

From Eqs. (24), (25) we get

$$\{T(\lambda)\underset{,}{\otimes}\mathrm{tr}(A_0\pm A_1)(x)\} = 0\,. \tag{28}$$

So the canonical transformations generated by $T(\lambda)$ commute with the conformal transformations, and, in particular, with energy-momentum.

As basis for the discussion of $\{T(\lambda)\underset{,}{\otimes} T(\mu)\}$ we find it useful first to compute the Poisson bracket of two monodromy matrices on finite, overlapping intervals $[a,b]$ and $[c,d]$ with $a<b$, $c<d$, $K=\max(a,c)<L=\min(b,d)$:

$$\{T_{ij}(a,b|\lambda),\, T_{k\ell}(c,d|\mu)\} = \int_a^b dx\int_c^d dy\,\frac{\delta T_{ij}(a,b|\lambda)}{\delta L_{1mn}(x,\lambda)}\{L_{1mn}(x,\lambda),\, L_{1rs}(y,\mu)\}\frac{\delta T_{k\ell}(c,d|\mu)}{\delta L_{1rs}(y,\mu)}\,.$$

For reasons which become clear later, we require $a\neq c$, $b\neq d$. Employing the method which has been developed in [7], we start from the equation

$$\begin{aligned}\{T(a,b|\lambda)\underset{,}{\otimes} T(c,d|\mu)\} = \int_a^b dx\int_c^d dy[&T(a,x|\lambda)\otimes T(c,y|\mu)]\,\{L_1(x,\lambda)\underset{,}{\otimes} L_1(y,\mu)\}\\ &\cdot[T(x,b|\lambda)\otimes T(y,d|\mu)]\,,\end{aligned} \tag{29}$$

where all the time arguments are equal and have been suppressed. The non-ultralocality of the Lax pair (9), (10), i.e. the presence of spatial derivatives of the field $g(x)$, is reflected by the δ' functions in the expression

$$\{L_1(x,\lambda) \underset{,}{\otimes} L_1(y,\mu)\} = [\alpha(\lambda)\alpha(\mu)]^{-1}[\delta(x-y)(\alpha(\lambda)(\mathbb{1}\otimes L_1(x,\lambda)) - \alpha(\mu)(L_1(x,\mu)\otimes\mathbb{1})) \\ -\lambda\partial_x\delta(x-y)(g^{-1}(x)g(y)\otimes\mathbb{1}) + \mu\partial_y\delta(y-x)(\mathbb{1}\otimes g^{-1}(y)g(x))]P, \tag{30}$$

where $\alpha(\lambda) = (\lambda^2-1)/\lambda$. Inserting (30) into (29) and evaluating the δ functions leads to

$$\begin{aligned}
&[\alpha(\lambda)\alpha(\mu)]\{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \\
&\quad = \int_K^L dx[(T(a,x|\lambda)\otimes T(c,x|\mu))(\alpha(\lambda)(\mathbb{1}\otimes L_1(x,\lambda)) \\
&\qquad - \alpha(\mu)(L_1(x,\mu)\otimes\mathbb{1}))(T(x,d|\mu)\otimes T(x,b|\lambda)) \\
&\qquad + \lambda(\partial_x T(a,x|\lambda)\otimes T(c,x|\mu))(T(x,d|\mu)\otimes T(x,b|\lambda)) \\
&\qquad + \lambda(T(a,x|\lambda)\otimes T(c,x|\mu))(T(x,d|\mu)\otimes\partial_x T(x,b|\lambda)) \\
&\qquad - \mu(T(a,x|\lambda)\otimes\partial_x T(c,x|\mu))(T(x,d|\mu)\otimes T(x,b|\lambda)) \\
&\qquad - \mu(T(a,x|\lambda)\otimes T(c,x|\mu))(\partial_x T(x,d|\mu)\otimes T(x,b|\lambda)) \\
&\qquad - \lambda(T(a,x|\lambda)\otimes T(c,x|\mu))(A_1(x)\otimes\mathbb{1})(T(x,d|\mu)\otimes T(x,b|\lambda)) \\
&\qquad + \mu(T(a,x|\lambda)\otimes T(c,x|\mu))(\mathbb{1}\otimes A_1(x))(T(x,d|\mu)\otimes T(x,b|\lambda))]P \\
&\qquad - \lambda\int_c^d dy(T(a,x|\lambda)\otimes T(c,y|\mu))(T(y,d|\mu)\otimes T(x,b|\lambda))\delta(x-y)P|_{x=a}^{x=b} \\
&\qquad + \mu\int_a^b dx(T(a,x|\lambda)\otimes T(c,y|\mu))(T(y,d|\mu)\otimes T(x,b|\lambda))\delta(x-y)P|_{y=c}^{y=d}.
\end{aligned} \tag{31}$$

With the help of Eqs. (9), (10) and the relation

$$A_1(x) = (\lambda-\mu)^{-1}[\alpha(\lambda)L_1(x,\lambda) - \alpha(\mu)L_1(x,\mu)],$$

the integrand I of the first integral in (31) turns out to be a complete derivative:

$$I = h(\lambda,\mu)\partial_x[(T(a,x|\lambda)\otimes T(c,x|\mu))(T(x,d|\mu)\otimes T(x,b|\lambda))],$$
$$h(\lambda,\mu) = \lambda(1-\alpha(\lambda)/(\lambda-\mu)) = -\mu(1-\alpha(\mu)/(\mu-\lambda)).$$

Taking into account the ordering of a, b, c, and d in the boundary terms, we arrive at

$$\begin{aligned}
&\{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \\
&\quad = [\alpha(\lambda)\alpha(\mu)]^{-1}\{h(\lambda,\mu)(T(a,L|\lambda)\otimes T(c,L|\mu))(T(L,d|\mu)\otimes T(L,b|\lambda)) \\
&\qquad - h(\lambda,\mu)(T(a,K|\lambda)\otimes T(c,K|\mu))(T(K,d|\mu)\otimes T(K,b|\lambda)) \\
&\qquad - \lambda\theta(d-b)\theta(b-c)(T(a,b|\lambda)\otimes T(c,b|\mu))(T(b,d|\mu)\otimes\mathbb{1}) \\
&\qquad + \lambda\theta(d-a)\theta(a-c)(\mathbb{1}\otimes T(c,a|\mu))(T(a,d|\mu)\otimes T(a,b|\lambda)) \\
&\qquad + \mu\theta(b-d)\theta(d-a)(T(a,d|\lambda)\otimes T(c,d|\mu))(\mathbb{1}\otimes T(d,b|\lambda)) \\
&\qquad - \mu\theta(b-c)\theta(c-a)(T(a,c|\lambda)\otimes\mathbb{1})(T(c,d|\mu)\otimes T(c,b|\lambda))\}P.
\end{aligned} \tag{32}$$

To obtain an expression for $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}$ we have to look at the limit where a, c tend to $-\infty$ and b, d tend to $+\infty$. However, we see that already for finite a, b, c, d the Poisson bracket (32) is discontinuous at the points $a=c$ and $b=d$. Thus the limit depends on the order in which a, c and b, d go to infinity:

$$\lim_{\substack{b \to +\infty \\ c \to -\infty}} \left(\lim_{\substack{d \to +\infty \\ a \to -\infty}} \{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \right)$$

$$= -\frac{\lambda}{(\lambda-\mu)\,\alpha(\mu)} [T(\lambda) \otimes T(\mu), P], \tag{33.1}$$

$$\lim_{\substack{d \to +\infty \\ a \to -\infty}} \left(\lim_{\substack{b \to +\infty \\ c \to -\infty}} \{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \right)$$

$$= \frac{\mu}{(\mu-\lambda)\,\alpha(\lambda)} [T(\lambda) \otimes T(\mu), P], \tag{33.2}$$

$$\lim_{\substack{d \to +\infty \\ c \to -\infty}} \left(\lim_{\substack{b \to +\infty \\ a \to -\infty}} \{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \right)$$

$$= \left(\frac{\mu}{(\mu-\lambda)\,\alpha(\lambda)} T(\lambda) \otimes T(\mu) + \frac{\lambda}{(\lambda-\mu)\,\alpha(\mu)} T(\mu) \otimes T(\lambda) \right) P, \tag{33.3}$$

$$\lim_{\substack{b \to +\infty \\ a \to -\infty}} \left(\lim_{\substack{d \to +\infty \\ c \to -\infty}} \{T(a,b|\lambda) \underset{,}{\otimes} T(c,d|\mu)\} \right)$$

$$= \left(-\frac{\lambda}{(\lambda-\mu)\,\alpha(\mu)} T(\lambda) \otimes T(\mu) - \frac{\mu}{(\mu-\lambda)\,\alpha(\lambda)} T(\mu) \otimes T(\lambda) \right) P. \tag{33.4}$$

The origin of these discontinuities is the short distance singularity of the classical current algebra displayed by the δ' functions in Eq. (30). These δ' distributions act in Eq. (29) on monodromy matrices which do not vanish at the end of the interval. They give rise to the boundary terms in Eq. (31) and to the step functions in Eq. (32). The conclusion is that the Poisson bracket $\{T(a,b|\lambda) \underset{,}{\otimes} T(a,b|\mu)\}$ is not a uniquely defined object, be the interval $[a,b]$ finite or not.

Poisson brackets must fulfill the antisymmetry property

$$P\{A \underset{,}{\otimes} B\} P = -\{B \underset{,}{\otimes} A\} \tag{34}$$

and the Jacobi identity. None of the four expressions (33.1)–(33.4) satisfies separately Eq. (34). If instead we take the infinite volume limit letting $a=c$, $b=d$, $a \to -\infty$, $b \to +\infty$ and defining $\theta(0)$ to be some number $0 \leqq \theta(0) \leqq 1$, we obtain a one-parameter family of solutions[1]:

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \{T(a,b|\lambda) \underset{,}{\otimes} T(a,b|\mu)\} = f(\lambda,\mu)\,[T(\lambda) \otimes T(\mu), P], \tag{35}$$

where

$$f(\lambda,\mu) = \frac{\lambda\mu[1-\lambda\mu-\theta(0)(\lambda-\mu)^2]}{(\lambda-\mu)(\lambda^2-1)(\mu^2-1)}.$$

1 This result has been obtained independently by L. D. Faddeev and V. E. Korepin (private communication)

This result is antisymmetric for any choice of $\theta(0)$. The only antisymmetric expression one can get taking linear combinations of Eqs. (33.1)–(33.4) coincides with Eq. (35) for $\theta(0)=1/2$. In order to fulfill the Jacobi identity, $f(\lambda,\mu)$ must be of the form

$$f(\lambda,\mu)=1/(\beta(\lambda)-\beta(\mu)), \tag{36}$$

where $\beta(\lambda)$ is an arbitrary function of λ (this is the general solution with the form $f(\lambda,\mu)P$ of the classical Yang-Baxter equations [6]). However, for no value of $\theta(0)$ one can recast $f(\lambda,\mu)$ from Eq. (35) in the form of Eq. (36). In conclusion, there is no definition of the Poisson bracket of two monodromy matrices $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}$ consistent with the basic properties of Lie algebras.

Usually, to a well behaved Poisson bracket algebra of charges there corresponds an isomorphic commutator algebra of infinitesimal canonical transformations generated by these charges. Thus, one could try to define the Poisson bracket $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}$ by its connection with the commutator $[N(\lambda) \underset{,}{\otimes} N(\mu)]$, where $N(\lambda)$ denotes the canonical transformation induced by $T(\lambda)$. For the chiral field g we have [see Eq. (24)]

$$N(\lambda)\otimes g(x)=\{T(\lambda) \underset{,}{\otimes} g(x)\}=\frac{\lambda}{\lambda^2-1}(T^-(x,\lambda)\otimes g(x)\,T^+(x,\lambda))P, \tag{37}$$

and the action of $N(\lambda)$ on π has been given in Eq. (25). Observe that these transformations are uniquely defined and can be applied repeatedly on $g(x)$ and $\pi(x)$. To obtain the commutator $[N(\lambda) \underset{,}{\otimes} N(\mu)]$, we first compute

$$N(\lambda)\otimes T^-(x,\mu)=\frac{\mu}{\alpha(\lambda)(\mu-\lambda)}[T^-(x,\lambda)\otimes T^-(x,\mu)\,T^+(x,\lambda)-T^-(x,\mu)\otimes T(\lambda)]P, \tag{38}$$

$$N(\lambda)\otimes T^+(x,\mu)=-\frac{\mu}{\alpha(\lambda)(\mu-\lambda)}[T^-(x,\lambda)\,T^+(x,\mu)\otimes T^+(x,\lambda)-T(\lambda)\otimes T^+(x,\mu)]P \tag{39}$$

These expressions fulfill the correct differential equations and boundary conditions, e.g. for Eq. (38):

$$\begin{aligned}\partial_x(N(\lambda)\otimes T^-(x,\mu))&=(N(\lambda)\otimes T^-(x,\mu))(\mathbb{1}\otimes L_1(x,\mu))\\&\quad+(\mathbb{1}\otimes T^-(x,\mu))(N(\lambda)\otimes L_1(x,\mu)),\\ \lim_{x\to-\infty}&(N(\lambda)\otimes T^-(x,\mu))=0.\end{aligned}$$

From Eqs. (37), (38), and (39) we then get the result

$$\begin{aligned}[N_{ab}(\lambda),N_{cd}(\mu)]\,g_{ij}(x)&=\frac{\mu}{\alpha(\lambda)(\mu-\lambda)}[T_{cb}(\lambda)(N_{ad}(\mu)\,g_{ij}(x))-(a\leftrightarrow c,b\leftrightarrow d)]\\&\quad+(\lambda\leftrightarrow\mu).\end{aligned} \tag{40}$$

An analogous formula holds for the application on $\pi_{k\ell}$. To define e.g. $\{\{T_{ab}(\lambda),T_{cd}(\mu)\},g_{ij}(x)\}$ through Eq. (40), there must exist a function $\mathscr{F}_{abcd}(\lambda,\mu|g,\pi)$ such that

$$[N_{ab}(\lambda),N_{cd}(\mu)]\,g_{ij}(x)=\{\mathscr{F}_{abcd}(\lambda,\mu|g,\pi),g_{ij}(x)\}. \tag{41}$$

In turn, this would require the integrability condition

$$\frac{\delta}{\delta\pi_{k\ell}(y)}\frac{\delta}{\delta g_{ij}(x)}\mathscr{F}=\frac{\delta}{\delta g_{ij}(x)}\frac{\delta}{\delta\pi_{k\ell}(y)}\mathscr{F},$$

or equivalently,

$$\{\{\mathscr{F},\pi_{k\ell}(y)\},g_{ij}(x)\}=\{\{\mathscr{F},g_{ij}(x)\},\pi_{k\ell}(y)\} \tag{42}$$

to hold. However, (42) is true only for $\lambda+\mu=0$. Thus, in general no such function $\mathscr{F}$ exists. In other words, $[N(\lambda)\underset{,}{\otimes}N(\mu)]$ is a canonical transformation which is not generated by any function on phase space, in particular, not by a non-local charge.

One of our starting points was the question whether the classical algebra of the non-local charges is a loop algebra or some other type of infinite-dimensional Lie algebra. Our result shows that this question is not posed correctly: It is not even a Lie algebra.

II.3. Integrable Chiral Fields on the Lattice

Since the short distance singularities of the current algebra make it impossible to properly define $\{T(\lambda)\underset{,}{\otimes}T(\mu)\}$, it is natural to introduce a space discretization in order to regularize the short distance behaviour. It is crucial that such a lattice version be integrable in the sense that it admits a Lax pair. A formulation with both space and (euclidean) time discretized is known [8]:

$$\mathscr{L}=\sum_{\mu=1,2}\operatorname{tr}(g_{\mathbf{n}+\mu}\,g_{\mathbf{n}}^{-1}). \tag{43}$$

Here $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2$, $\mathbf{1}=(1,0)$, $\mathbf{2}=(0,1)$, and $g_{\mathbf{n}}\in\mathrm{GL}(N,\mathbb{R})$. One can write the system of difference-difference equations [9]

$$X_{\mathbf{n}+\mu}(\lambda)-(1+\lambda\varepsilon_{\mu\nu})X_{\mathbf{n}}(\lambda)+\lambda\varepsilon_{\mu\nu}(\mathbb{1}+J_{\mathbf{n}-\nu,\nu})X_{\mathbf{n}-\nu}(\lambda)=0,$$

where

$$J_{\mathbf{n},\mu}=g_{\mathbf{n}+\mu}\,g_{\mathbf{n}}^{-1}-\mathbb{1}.$$

The corresponding compatibility condition precisely gives the equations of motion associated to the Lagrangian (43): $\Delta_\mu J_{\mathbf{n},\mu}=0$. Here $\Delta_\mu\Psi_{\mathbf{n}}=\Psi_{\mathbf{n}}-\Psi_{\mathbf{n}-\mu}$.

An infinite number of conserved currents follows by expanding

$$K_{\mathbf{n},\mu}=\varepsilon_{\mu\nu}\Delta_\nu X_{\mathbf{n}+\mu}(\lambda)$$

into a power series in λ. In the limit where both space and time become continuous one recovers the euclidean version of the chiral model [Eq. (1)]. However, keeping space discrete and letting the time spacing Δ tend to zero, one gets

$$\mathscr{L}_n(t)=\operatorname{tr}[g_{n+1}(t)\,g_n^{-1}(t)]+\frac{\Delta^2}{2}\operatorname{tr}A_0^2(t,n)+O(\Delta^3). \tag{44}$$

Here $g_n(t)=g_{\mathbf{n}}$ with $t=n_2\Delta$ and $A_0(t,n)=\dfrac{dg_n(t)}{dt}g_n^{-1}(t)$. It must be noted that the $O(\Delta^3)$ term in Eq. (44) contains up to three time derivatives. If one drops in Eq. (44)

the $O(\Delta^3)$ terms and then sets $t=\Delta t'$, one gets the system investigated in [15]:

$$\mathcal{L}_n(t')=\mathrm{tr}(g_{n+1}(t')\,g_n^{-1}(t'))+\tfrac{1}{2}\,\mathrm{tr}\,A_0'^2(t',n). \tag{45}$$

Moreover, if one now lets the space be continuous ($x=n\delta$, $\delta\to 0$) in Eq. (45), Eq. (1) is recovered provided a *singular* scale transformation is carried out, namely $x'=x/\delta$. In conclusion, a consistent integrable space discretization of the chiral model with continuous time is not presently available.

III. Non-Local Charges and their Algebra in the Quantum O(N) Non-Linear σ Model

The quantum O(N) non-linear σ model has been studied in several approaches ($1/N$ expansion, perturbative expansion, factorizable S-matrix method). From these investigations, the following picture emerges: the quantum spectrum contains a N-plet of massive scalars transforming under the fundamental representation of O(N) [16]. The mass is dynamically generated through dimensional transmutation. The scattering of particles is governed by a factorizable S-matrix; the two-body S-matrix reads [17]

$$\begin{aligned}{}_{\mathrm{out}}\langle\theta_1c_1,\theta_2c_2|\theta_1'c_1',\theta_2'c_2'\rangle_{\mathrm{in}}&=\delta(\theta_1-\theta_1')\delta(\theta_2-\theta_2')S_{c_1c_2,c_1'c_2'}(\theta_2-\theta_1)\\&\quad+\begin{pmatrix}\theta_1\leftrightarrow\theta_2\\ c_1\leftrightarrow c_2\end{pmatrix},\\ S_{c_1c_2,c_1'c_2'}(\theta)&=\delta_{c_1c_2}\delta_{c_1'c_2'}\sigma_1(\theta)+\delta_{c_1c_1'}\delta_{c_2c_2'}\sigma_2(\theta)+\delta_{c_1c_2'}\delta_{c_2c_1'}\sigma_3(\theta),\end{aligned} \tag{46}$$

where e.g. $|\theta_1'c_1',\theta_2'c_2'\rangle_{\mathrm{in}}$ is an incoming two-particle state with isospin indices c_1', c_2' and rapidities θ_1',θ_2' such that $\theta_2'>\theta_1'$ (similarly $\theta_2>\theta_1$ in the out-state). The states are normalized according to

$$\langle\theta,c|\theta',d\rangle=\delta(\theta-\theta')\delta_{cd}.$$

From factorizability, unitarity, analyticity, crossing and absence of bound states one gets

$$\begin{gathered}\sigma_1(\theta)=-\frac{2\pi i\Delta}{i\pi-\theta}\sigma_2(\theta),\qquad \sigma_2(\theta)\,\sigma_2(-\theta)=\frac{\theta^2}{\theta^2+\left(\dfrac{2\pi}{N-2}\right)^2},\\ \sigma_3(\theta)=-\frac{2\pi i\Delta}{\theta}\sigma_2(\theta),\\ \sigma_2(\theta)=\frac{\Gamma\left(\Delta+\dfrac{\theta}{2\pi i}\right)\Gamma\left(\dfrac{1}{2}+\dfrac{\theta}{2\pi i}\right)\Gamma\left(\Delta+\dfrac{1}{2}-\dfrac{\theta}{2\pi i}\right)\Gamma\left(1-\dfrac{\theta}{2\pi i}\right)}{\Gamma\left(\dfrac{\theta}{2\pi i}\right)\Gamma\left(\Delta+\dfrac{1}{2}+\dfrac{\theta}{2\pi i}\right)\Gamma\left(\dfrac{1}{2}-\dfrac{\theta}{2\pi i}\right)\Gamma\left(\Delta+1-\dfrac{\theta}{2\pi i}\right)},\qquad \Delta=(N-2)^{-1}.\end{gathered} \tag{47}$$

It is known that the first non-local charge exists in the quantum σ model as a renormalized conserved operator which implies factorization and absence of particle production [12]. Moreover we will assume that the whole classical monodromy matrix has a quantum counterpart

$$T_{ab}(\lambda)=\delta_{ab}\mathbb{1}+\lambda T^{(1)}_{ab}+\lambda^2 T^{(2)}_{ab}+\dots, \tag{48}$$

where the $T^{(n)}_{ab}$ are quantum non-local conserved charges which are obtained from the classical ones by renormalization. In particular:

$$\begin{aligned} T^{(1)}&=-\int_{-\infty}^{+\infty} dx\, A_0(x),\\ T^{(2)}&=-\tfrac{1}{2}\int_{-\infty}^{+\infty} dx\int_{-\infty}^{+\infty} dy\,\varepsilon(x-y)\,A_0(x)\,A_0(y)-Z\int_{-\infty}^{+\infty} dx\,A_1(x)+\tfrac{1}{2}\left[\int_{-\infty}^{+\infty} dx\,A_0(x)\right]^2. \end{aligned} \tag{49}$$

Here $(A_\mu)_{ab}=2n_a\overleftrightarrow{\partial}_\mu n_b$, $1\leqq a,\ b\leqq N$, and n_a is the usual sigma field, Z is a renormalization factor. The action of $T^{(1)}$ and $T^{(2)}$ on an asymptotic k-particle state $|\theta_1c_1\dots\theta_kc_k\rangle_{\substack{\text{in}\\ \text{out}}}$ reads [12]

$$T^{(1)}_{ab}|\theta_1c_1\dots\theta_kc_k\rangle_{\substack{\text{in}\\ \text{out}}}=\sum_{\{d\}}|\theta_1d_1\dots\theta_kd_k\rangle_{\substack{\text{in}\\ \text{out}}}\sum_{j=1}^{k}2i(I^{(j)}_{ab})_{\{d\},\{c\}}, \tag{50}$$

$$\begin{aligned} &T^{(2)}_{ab}|\theta_1c_1\dots\theta_kc_k\rangle_{\substack{\text{in}\\ \text{out}}}\\ &=\sum_{\{d\}}|\theta_1d_1\dots\theta_kd_k\rangle_{\substack{\text{in}\\ \text{out}}}\Bigg\{-4\sum_{\substack{i<j\text{ for in}\\ i>j\text{ for out}}}(I^{(j)}_{ae}I^{(i)}_{eb})_{\{d\},\{c\}}\\ &\quad+\sum_{i=1}^{k}\left[2(N-2)\left(\frac{\theta_i}{i\pi}(I^{(i)}_{ab})_{\{d\},\{c\}}-\delta_{ad_i}\delta_{bc_i}\prod_{j\neq i}\delta_{d_jc_j}\right)-2\delta_{ab}\mathbb{1}_{\{d\},\{c\}}\right]\Bigg\}, \end{aligned} \tag{51}$$

where the matrix $I^{(j)}_{ab}$ acts on the isospin indices of the j^{th} particle only:

$$(I^{(j)}_{ab})_{\{d\},\{c\}}=(\delta_{ad_j}\delta_{bc_j}-\delta_{ac_j}\delta_{bd_j})\prod_{i\neq j}\delta_{d_ic_i}.$$

Here as everywhere in the sequel, $\theta_i<\theta_j$ for $i<j$.

In this chapter we shall show that the quantum $T(\lambda)$ can be determined from $\mathscr{P}$, $\mathscr{T}$ and $O(N)$ invariance, time independence and from the assumption that its action on k-particle states obeys a certain factorization law. Moreover, the commutator $[T(\lambda)\overset{\otimes}{,}T(\mu)]$ turns out to be uniquely determined by these principles.

The factorization law is based on the following property of the *classical* monodromy matrix [1]: let the field $n(x)$ form two separated waves, i.e.

$$n(x)=\begin{cases} n_1(x) & \text{for}\quad x\leqq A\\ n_2(x) & \text{for}\quad x\geqq B\\ \text{constant} & \text{for}\quad A<x<B.\end{cases}$$

Then

$$T_{ab}(\lambda;n)=T_{ae}(\lambda;n_1)\,T_{eb}(\lambda;n_2), \tag{52}$$

where $T(\lambda;n_i)$ denotes the classical monodromy matrix evaluated for $n_i(x)$ only. In [13] a quantum version of Eq. (52) has been introduced and studied. The action of the quantum $T(\lambda)$ in the in and out Fock spaces is assumed to be

$$T_{ab}(\lambda)|\theta_1 c_1 \dots \theta_k c_k\rangle_{\text{out}} = \sum_{a_1 \dots a_{k-1}=1}^{N} T_{aa_1}(\lambda)|\theta_1 c_1\rangle\, T_{a_1a_2}(\lambda)|\theta_2 c_2\rangle \dots T_{a_{k-1}b}(\lambda)|\theta_k c_k\rangle, \quad (53)$$

$$T_{ab}(\lambda)|\theta_1 c_1 \dots \theta_k c_k\rangle_{\text{in}} = \sum_{a_1 \dots a_{k-1}=1}^{N} T_{a_1b}(\lambda)|\theta_1 c_1\rangle\, T_{a_2a_1}(\lambda)|\theta_2 c_2\rangle \dots T_{aa_{k-1}}(\lambda)|\theta_k c_k\rangle. \quad (54)$$

Inserting Eq. (48) into Eqs. (53), (54) we obtain power series expansions of these equations, reproducing to order λ^1 the property that isospin is an additive quantum number. To second order in λ we get for, say, an outgoing 2-particle state

$$\begin{aligned} T^{(2)}_{ab}|\theta_1 c_1, \theta_2 c_2\rangle_{\text{out}} &= (T^{(2)}_{ab}|\theta_1 c_1\rangle)\,|\theta_2 c_2\rangle + |\theta_1 c_1\rangle\,(T^{(2)}_{ab}|\theta_2 c_2\rangle) \\ &\quad + \sum_{e=1}^{N} (T^{(1)}_{ae}|\theta_1 c_1\rangle)(T^{(1)}_{eb}|\theta_2 c_2\rangle). \end{aligned} \quad (55)$$

In fact, this formula is fulfilled by $T^{(1)}$ and $T^{(2)}$ as given in Eqs. (50), (51). Actually, a corresponding equation for in and out states with any particle number holds. We take this as additional justification to use the factorization law (53), (54) as the starting point of our considerations. Due to the isospin covariance of $T(\lambda)$ and its conservation, its one-particle matrix elements are of the form

$$\begin{aligned} \langle \theta d|\, T_{ab}(\lambda)\, |\theta' c\rangle &= \delta(\theta - \theta')\, T_{ad,bc}(\lambda, \theta), \\ T_{ad,bc}(\lambda, \theta) &= \delta_{ad}\delta_{bc} f_1(\lambda, \theta) + \delta_{ab}\delta_{cd} f_2(\lambda, \theta) + \delta_{ac}\delta_{bd} f_3(\lambda, \theta), \\ T_{ad,bc}(\lambda, \theta) &= T_{bc,ad}(\lambda, \theta) = T_{da,cb}(\lambda, \theta), \end{aligned} \quad (56)$$

where the functions f_1, f_2, f_3 are to be determined. It has been shown in [13] that the conservation of $T_{ab}(\lambda)$ in $2 \to 2$ particle scattering, i.e.

$$\begin{aligned} &\sum_{d_1 d_2} S_{c_1c_2,d_1d_2}(\theta_2 - \theta_1)_{\text{in}}\langle \theta_1 d_1, \theta_2 d_2|\, T_{ab}(\lambda)\, |\theta'_1 c'_1, \theta'_2 c'_2\rangle_{\text{in}} \\ &\quad = \sum_{d_1 d_2} {}_{\text{out}}\langle \theta_1 c_1, \theta_2 c_2|\, T_{ab}(\lambda)\, |\theta'_1 d_1, \theta'_2 d_2\rangle_{\text{out}}\, S_{d_1d_2,c'_1c'_2}(\theta'_2 - \theta'_1) \end{aligned}$$

leads to the following relations:

$$\begin{aligned} &[f_2(\theta_2) f_3(\theta_1) + f_1(\theta_2) f_2(\theta_1)]\, \sigma_1(\theta_2 - \theta_1) = f_1(\theta_2) f_3(\theta_1)\, \sigma_2(\theta_2 - \theta_1), \\ &[f_2(\theta_2) f_3(\theta_1) - f_3(\theta_2) f_2(\theta_1)]\, \sigma_3(\theta_2 - \theta_1) = f_3(\theta_2) f_3(\theta_1)\, \sigma_2(\theta_2 - \theta_1), \\ &[f_1(\theta_2) f_2(\theta_1) - f_2(\theta_2) f_1(\theta_1)]\, \sigma_3(\theta_2 - \theta_1) = f_1(\theta_2) f_1(\theta_1)\, \sigma_2(\theta_2 - \theta_1), \\ &[f_1(\theta_2) f_3(\theta_1) - f_3(\theta_2) f_1(\theta_1)]\, \sigma_3(\theta_2 - \theta_1) = [f_3(\theta_2) f_2(\theta_1) + f_2(\theta_2) f_1(\theta_1) \\ &\quad + f_3(\theta_2) f_3(\theta_1) + N f_3(\theta_2) f_1(\theta_1) + f_1(\theta_2) f_1(\theta_1)]\, \sigma_1(\theta_2 - \theta_1) \\ &\quad + f_3(\theta_2) f_1(\theta_1)\, \sigma_2(\theta_2 - \theta_1), \end{aligned} \quad (57)$$

where the argument λ in the f_i has been suppressed. Inserting Eqs. (47) into

Eqs. (57), one arrives at the following set of functional equations:

$$\begin{aligned} f_3(\lambda,\theta) &= -\frac{2\pi}{N-2}\frac{i}{\gamma(\lambda)+\theta} f_2(\lambda,\theta), \\ f_1(\lambda,\theta) &= \frac{2\pi}{N-2}\frac{i}{\gamma(\lambda)+\theta-i\pi} f_2(\lambda,\theta). \end{aligned} \tag{58}$$

Here γ is a function of λ only.

The f_i are further restricted by the invariance of the theory under $\mathscr{P}$ (parity) and $\mathscr{T}$ (time reversal), $\mathscr{C}$ (charge conjugation) acting trivially in this model. The quantum analogue of Eq. (15) is the existence of an antiunitary operator $\tau = \mathscr{P}\mathscr{T}$ such that

$$\tau T(\lambda)\tau^{-1} = T^{-1}(\lambda),$$

or

$$T_{ac}(\lambda)\tau\, T_{cb}(\lambda)\tau^{-1} = \delta_{ab}\mathbb{1}.$$

Projecting this equation on one-particle states one gets

$$\delta_{ab}\delta_{fd} = T_{af,ce}(\lambda,\theta)\, T^*_{ce,bd}(\lambda,\theta). \tag{59}$$

In a similar way parity invariance implies

$$\mathscr{P}T(\lambda)\mathscr{P}^{-1} = T^{-1}(-\lambda),$$

or

$$T_{ac}(-\lambda)\mathscr{P}T_{cb}(\lambda)\mathscr{P}^{-1} = \delta_{ab}\mathbb{1}.$$

This gives on one-particle states

$$\delta_{ab}\delta_{fd} = T_{af,ce}(-\lambda,-\theta)\, T_{ce,bd}(\lambda,\theta). \tag{60}$$

Inserting Eq. (56) into Eqs. (59) and (60), we find the following nontrivial relations:

$$0 = f_2(\lambda,\theta) f_3^*(\lambda,\theta) + f_3(\lambda,\theta) f_2^*(\lambda,\theta), \tag{61}$$

$$1 = |f_2(\lambda,\theta)|^2 + |f_3(\lambda,\theta)|^2, \tag{62}$$

$$0 = f_2(\lambda,\theta) f_3(-\lambda,-\theta) + f_3(\lambda,\theta) f_2(-\lambda,-\theta), \tag{63}$$

$$1 = f_2(\lambda,\theta) f_2(-\lambda,-\theta) + f_3(\lambda,\theta) f_3(-\lambda,-\theta). \tag{64}$$

Taking the factorization Eqs. (58) into account, Eqs. (61) and (63) imply $\gamma^*(\lambda) = \gamma(\lambda)$ and $\gamma(-\lambda) = -\gamma(\lambda)$, respectively.

We also get from Eqs. (62) and (64)

$$|f_2(\lambda,\theta)|^2 = \frac{(\gamma(\lambda)+\theta)^2}{(\gamma(\lambda)+\theta)^2 + \left(\dfrac{2\pi}{N-2}\right)^2} = f_2(\lambda,\theta) f_2(-\lambda,-\theta). \tag{65}$$

Comparing Eqs. (47) with Eqs. (58) and (65), we see that they just differ by the shift $\theta \to \theta + \gamma(\lambda)$. So $\sigma_2(\theta + \gamma(\lambda))$ is a particular solution of Eq. (65). The general solution reads

$$f_i(\lambda,\theta)=\sigma_i(\theta+\gamma(\lambda))e^{i\phi(\lambda,\theta)}, \qquad i=1,2,3, \tag{66}$$

where $\phi(\lambda,\theta)$ is a real function of λ and θ for real λ and θ with

$$\phi(-\lambda,-\theta)=-\phi(\lambda,\theta).$$

In this way, we have explicitly determined the quantum monodromy matrix up to a phase: In one-particle states,

$$T_{ad,bc}(\lambda,\theta)=S_{ad,bc}(\theta+\gamma(\lambda))e^{i\phi(\lambda,\theta)}, \tag{67}$$

and we recall that the action of $T_{ab}(\lambda)$ on arbitrary k-particle states is given by the factorization laws (53), (54). We notice that the crossing relations for the S-matrix together with Eq. (60) lead to the equation

$$T_{ac}(\lambda)\,T_{bc}(\hat{\lambda})=\delta_{ab}\mathbb{1} \quad \text{for} \quad \gamma(\hat{\lambda})=\gamma(\lambda)+i\pi,$$

which can be considered as the quantum analogue of the orthogonality of the classical monodromy matrix.

As a check, we compare now the action of $T_{ab}(\lambda)$ on one-particle states as given by Eqs. (50), (51) with our expression (67). One gets from Eqs. (50), (51):

$$\begin{aligned}T_{ad,bc}(\lambda,\theta)=\delta_{ab}\delta_{cd}(1-2\lambda^2)-\delta_{ac}\delta_{bd}2i\lambda\left[1-\frac{\lambda}{\pi}(N-2)\theta\right]\\+\delta_{ad}\delta_{bc}2i\lambda\left[1+\frac{\lambda}{\pi}(N-2)(i\pi-\theta)\right]+O(\lambda^3).\end{aligned} \tag{68}$$

On the other hand,

$$\lim_{\gamma\to\infty}S_{ad,bc}(\theta+\gamma)=\delta_{ab}\delta_{cd},$$

so we expand Eq. (67) into a power series in γ^{-1} around $\gamma^{-1}=0$. From Eq. (47) and Stirling's formula we have

$$\begin{aligned}&\sigma_1(\theta+\gamma)=2\pi i/(N-2)\,(\gamma^{-1}+(i\pi-\theta)\gamma^{-2}+O(\gamma^{-3})),\\&\sigma_2(\theta+\gamma)=1-2(\pi/(N-2))^2\gamma^{-2}+O(\gamma^{-3}),\\&\sigma_3(\theta+\gamma)=2\pi i/(N-2)\,(-\gamma^{-1}+\theta\gamma^{-2}+O(\gamma^{-3})).\end{aligned} \tag{69}$$

We recover Eq. (68) if we identify in Eq. (67)

$$\gamma(\lambda)=\frac{\pi}{N-2}\lambda^{-1}+O(\lambda^1) \quad \text{and} \quad \phi(\lambda,\theta)=O(\lambda^3). \tag{70}$$

This latter equation suggests that $\phi(\lambda,\theta)$ vanishes identically. As it is the case for the two-body S-matrix, we would then obtain the minimal solution for the monodromy matrix as well.

Our next task is to compute the algebra of the quantum monodromy operators $T_{ab}(\lambda)$. We observe that their matrix elements between k-particle states read

$$
\begin{aligned}
&{}_{\mathrm{out}}\langle \theta'_1 c'_1 \dots \theta'_k c'_k | T_{ab}(\lambda) | \theta_1 c_1 \dots \theta_k c_k \rangle_{\mathrm{out}} \\
&= \delta(\theta'_1 - \theta_1) \dots \delta(\theta'_k - \theta_k) \sum_{a_1 \dots a_{k-1}=1}^{N} S_{ac'_1, a_1 c_1}(\theta_1 + \gamma(\lambda)) \\
&\quad \cdot S_{a_1 c'_2, a_2 c_2}(\theta_2 + \gamma(\lambda)) \, S_{a_{k-1} c'_k, b c_k}(\theta_k + \gamma(\lambda)) \\
&\equiv T_{a\{c'\}, b\{c\}}(\{\theta\}, \gamma(\lambda)) \,.
\end{aligned}
\tag{71}
$$

This formula formally coincides with the monodromy matrix in statistical mechanics for a line of k sites [18]. The $S_{ab,cd}$ being identified with the statistical weights, the indices a, b, c, d label the link states. From the Yang-Baxter equations for the S-matrix

$$
S_{aa',cc'}(\gamma-\gamma')\, S_{ca'',bc''}(\gamma)\, S_{c'c'',b'b''}(\gamma') = S_{a'a'',c'c''}(\gamma')\, S_{ac'',cb''}(\gamma)\, S_{cc',bb'}(\gamma-\gamma') \,, \tag{72}
$$

one can derive for $T_{a\{c'\},b\{c\}}$ the commutation relation

$$
S_{ij,k\ell}(\gamma(\lambda)-\gamma(\lambda'))\, [T(\lambda) \otimes T(\lambda')]_{k\ell,mn} = [T(\lambda') \otimes T(\lambda)]_{ji,\ell k}\, S_{k\ell,mn}(\gamma(\lambda)-\gamma(\lambda')) \,. \tag{73}
$$

Here the tensor product is over the O(N) indices, and in Fock space the operator product is understood. In this derivation, the phase ϕ has been assumed to vanish. A non-zero $\phi(\lambda,\theta)$ produces on both sides of Eq. (73) the same factor $\exp\left(i \sum_{n=1}^{K} \phi(\lambda,\theta_n)\right)$, thus leaving the commutation relation (73) unchanged. Finally, insertion of Eqs. (46) and (47) into Eq. (73) leads to the result

$$
\begin{aligned}
[T_{ac}(\lambda), T_{bd}(\lambda')] &= \frac{2\pi i}{N-2} \frac{1}{\gamma(\lambda)-\gamma(\lambda')} (T_{bc}(\lambda)\, T_{ad}(\lambda') - T_{bc}(\lambda')\, T_{ad}(\lambda)) \\
&\quad + \frac{2\pi i}{N-2} \frac{1}{\gamma(\lambda)-\gamma(\lambda')-i\pi} (\delta_{cd} T_{be}(\lambda')\, T_{ae}(\lambda) - \delta_{ab} T_{ec}(\lambda)\, T_{ed}(\lambda')) \,.
\end{aligned}
\tag{74}
$$

The Yang-Baxter equation (72) guarantees that the algebra defined by Eq. (74) satisfies the Jacobi identity – in contrast to the classical case. It must be pointed out that the commutator of two monodromy operators $T_{a,b}(\lambda)$ is not a linear but a quadratic expression on the T's [Eq. (74)]. Thus the coefficients $T^{(n)}_{ab}$, $n = 1 \dots \infty$ [see Eq. (48)] do not form a basis of the Lie algebra. One must also include products of the $T^{(n)}_{ab}$ with any number of factors in order to close the algebra. One may ask whether a suitable function of the non-local charges satisfies some simpler algebra without quadratic terms on the right-hand side. A natural choice would be $Q(\lambda) \equiv \log T(\lambda)$ [Eq. (19)]. To the leading orders in λ we get

$$
\begin{aligned}
Q^{(1)} &= T^{(1)} \\
Q^{(2)} &= T^{(2)} - \tfrac{1}{2}(T^{(1)})^2 \,, \quad \text{etc.}.
\end{aligned}
$$

It follows from Eq. (74) for the first non-local charge

$$
\begin{aligned}
[Q^{(2)}_{ac}, Q^{(2)}_{bd}] &= \frac{2\pi i}{N-2} \{ -\tfrac{1}{4} Q^{(1)}_{ba} Q^{(1)^2}_{cd} + \tfrac{1}{4} Q^{(1)^2}_{ba} Q^{(1)}_{cd} + \tfrac{1}{4} Q^{(1)}_{bc} Q^{(1)^2}_{ad} - \tfrac{1}{4} Q^{(1)^2}_{bc} Q^{(1)}_{ad} \\
&\quad + \delta_{bc} [Q^{(3)}_{ad} - \tfrac{1}{12} Q^{(1)^3}_{ad}] - \delta_{ad} [Q^{(3)}_{bc} - \tfrac{1}{12} Q^{(1)^3}_{bc}] \\
&\quad + \delta_{cd} [Q^{(3)}_{ba} - \tfrac{1}{12} Q^{(1)^3}_{ba}] - \delta_{ba} [Q^{(3)}_{cd} - \tfrac{1}{12} Q^{(1)^3}_{cd}] \} \,.
\end{aligned}
$$

Hence, by this transformation one does not get rid of the non-linear terms on the right hand side.

An abelian subalgebra follows as usual by taking the trace in group space, $T_{aa}(\lambda)$: From Eq. (73)

$$[T_{aa}(\lambda), T_{bb}(\lambda')]=0.$$

Expanding Eq. (74) around $\gamma^{-1}=0$ to order γ^{-1} and using Eqs. (70), (48) we find

$$[T^{(1)}_{ac}, T_{bd}(\lambda)]=2i\{\delta_{bc}T_{ad}(\lambda)-\delta_{ad}T_{bc}(\lambda)+\delta_{cd}T_{ba}(\lambda)-\delta_{ab}T_{cd}(\lambda)\}.$$

Since $T^{(1)}_{ac}$ are the isospin operators, it follows that the antisymmetric part of $T_{bd}(\lambda)$ transforms under the adjoint representation of $O(N)$.

Acknowledgements. One of us (H. J. de Vega) would like to thank L. D. Faddeev, V. E. Korepin, and A. B. Zamolodchikov for useful discussions.

References

1. Lüscher, M., Pohlmeyer, K.: Scattering of massless lumps and non-local charges in the two-dimensional classical non-linear σ-model. Nucl. Phys. B **137**, 46 (1978)
2. Brézin, E., Itzykson, C., Zinn-Justin, J., Zuber, J.B.: Remarks about the existence of non-local charges in two-dimensional models. Phys. Lett. **82** B, 442 (1979)
3. de Vega, H.J.: Field theories with an infinite number of conservation laws and Bäcklund transformations in two dimensions. Phys. Lett. **87** B, 233 (1979)
4. Eichenherr, H., Forger, M.: On the dual symmetry of the non-linear sigma models. Nucl. Phys. B **155**, 381 (1979)
5. Zakharov, V.E., Mikhailov, A.V.: Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Sov. Phys. JETP **47**, 1017 (1978)
6. Faddeev, L.D.: Les Houches lectures 1982, Saclay preprint T/82/76
   Kulish, P.P., Skylanin, E.K.: Lectures Notes in Physics, Vol. 151. Berlin, Heidelberg, New York: Springer 1982
7. Izergin, A.G., Korepin, V.E.: The inverse scattering method approach to the quantum Shabat-Mikhailov model. Commun. Math. Phys. **79**, 303 (1981)
8. Polyakov, A.M.: String representations and hidden symmetries for gauge fields. Phys. Lett. **82** B, 247 (1979)
9. de Vega, H.J., Zuber, J.B.: Unpublished
10. Dolan, L.: Kac-Moody algebra is hidden symmetry of chiral models. Phys. Rev. Lett. **47**, 1371 (1981)
    Ueno, K.: Kyoto University preprint RIMS-374 (1981)
11. Davies, M.C., Houston, P.J., Leinaas, J.M., Macfarlane, A.J.: Hidden symmetries as canonical transformations for the chiral model. Phys. Lett. **119** B, 187 (1982)
12. Lüscher, M.: Quantum non-local charges and absence of particle production in the two-dimensional nonlinear σ-model. Nucl. Phys. B **135**, 1 (1978)
13. Zamolodchikov, Al.: Dubna preprint E2-11485 (1978)
14. Devchand, C., Fairlie, D.B.: A generating function for hidden symmetries of chiral models. Nucl. Phys. B **194**, 232 (1982)
15. Korepin, V.E.: Zapisky Nauchny Seminarov **101**, 90 (1980)
16. Brézin, E., Zinn-Justin, J., Le Guillou, J.C.: Renormalization of the nonlinear σ model in $2+\varepsilon$ dimensions. Phys. Rev. D **14**, 2615 (1976)
17. Zamolodchikov, A.B., Zamolodchikov, Al.B.: Factorized S-Matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. Ann. Phys. **120**, 253 (1979)
18. See, for example, Babelon, O., de Vega, H.J., Viallet, C.-M.: Exact solutions of the $Z_{n+1}\times Z_{n+1}$ symmetric generalization of the XXZ model. Nucl. Phys. B **200** FS4, 266 (1982)

Communicated by K. Osterwalder

Received May 20, 1983