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Convergence of Grand Canonical Gibbs Measures*

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Abstract. We prove existence of Gibbs states for a large class of continuum many-body potentials through a limit process of hard-core approximations to the potential. Dobrushin uniqueness techniques [1] for the decay of correlations are then extended to a very general class of continuum potentials.

Introduction

Ruelle [11] proved existence of grand canonical Gibbs states for a large class of pair potentials in continuum statistical mechanics. In this paper we prove, by a different method, the existence of Gibbs states for a large class of continuum many-body potentials. The approach we take is to add a "hard-core N-body component" φ_N to a given potential V. φ_N has the effect of restricting the number of particles that can accumulate in a spherical region of space of diameter r_0 to no more than N. Existence of a Gibbs state σ_N for the potential $V+\varphi_N$ is easily established for each positive integer N. Using some standard theorems on convergence of probability measures, we prove that the Gibbs states $\{\sigma_N\}$ converge to a Gibbs state σ for the potential V.

We then apply these methods to extend results on the decay of averaged two point correlation functions established by Gross [4], Künsch [6], Föllmer [3], and the author [5] via Dobrushin uniqueness techniques [1]. In addition we show that the grand canonical pressure P_N for $V+\varphi_N$ converges to the pressure P for V as N approaches infinity.

Section 1. Notation and Definitions

For a Borel measurable subset $\Lambda \subset \mathbb{R}^d$, let $X(\Lambda)$ denote the set of all locally finite subsets of Λ. $X(\Lambda)$ represents configurations of identical particles in Λ. We let $\varnothing$ denote the empty configuration. Let B_Λ be the σ-field on $X(\Lambda)$ generated by all sets of the form $\{s \in X(\Lambda) : |s \cap B| = m\}$, where B runs over all bounded Borel subsets of Λ, m runs over the set of nonnegative integers, and

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$|\cdot|$ denotes cardinality. For later convenience we let $(\Omega, S) = (X(\mathbb{R}^d), B_{\mathbb{R}^d})$. The measurable space (Ω, S) is the same one considered by Preston [9], Ruelle [11], and the author [5] in their studies of Gibbs measures. For any Borel subset Λ of $\mathbb{R}^d$, there is a natural isomorphism between (Ω, S) and $(X(\Lambda), B_\Lambda) \times (X(\Lambda^c), B_{\Lambda^c})$, where Λ^c is the complement of Λ (see [9, 11]). We will identify these spaces and write

$$(\Omega, S) = (X(\Lambda), B_\Lambda) \times (X(\Lambda^c), B_{\Lambda^c}). \tag{1.1}$$

Similarly if $\{\Lambda_i\}$ is a set of unit cells partitioning $\mathbb{R}^d$, we make the natural identification

$$(\Omega, S) = \prod_{i=1}^{\infty} (X(\Lambda_i), B_{\Lambda_i}). \tag{1.2}$$

Let $\tilde{B}_\Lambda$ denote the inverse projection of the σ-field B_Λ under the identification (1.1), so that $\tilde{B}_\Lambda$ is a σ-field on Ω.

A many-body potential V is an S-measurable map from the set of finite configurations Ω_F in Ω to $(-\infty, \infty]$ of the form

$$V(x) = \sum_{N=1}^{\infty} \sum_{\substack{y \subset x \\ |y| = N}} \phi_N(x), \tag{1.3}$$

where the function ϕ_N on configurations of cardinality N is called an N-body potential.

Definition 1.1. For a configuration $x = (x_1, x_2, \ldots, x_N)$ in Ω_F, let

$$\varphi_N(x) = \begin{cases} \infty \text{ if } \max\limits_{ij} \|x_i - x_j\| < r_0 \\ 0 \quad \text{otherwise,} \end{cases} \tag{1.4}$$

where $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^d$ and $r_0 > 0$ is fixed. A potential V which can be expressed in the form

$$V(x) = V'(x) + \varphi_N(x)$$

for some potential V', and all $x \in \Omega_F$ is called a potential with hard-core N-component.

We define the S-measurable set $R_\Lambda \subset \Omega$ associated with the potential V exactly as in [9, p. 97] (see also [5]), and we let $V_\Lambda(x|s)$, as in [5], be the energy of the configuration $x \in X(\Lambda)$ interacting with the configuration $s \in R_\Lambda \cap X(\Lambda^c)$ outside Λ.

The following definition of the finite volume Gibbs measure $\mu_\Lambda(dx|s)$ with external configuration $s \in \Omega$ and Λ a bounded Borel set is similar to that given in [5]. Let $X_N(\Lambda)$ denote the configurations of cardinality N in Λ, and let $T: \Lambda^N \to X_N(\Lambda)$ be the map which takes the ordered N-tuple $(x_1, \ldots, x_N)$ to the (unordered) set $\{x_1, \ldots, x_N\}$. In a natural way T defines an equivalence relation on Λ^N and $X_N(\Lambda)$ may be regarded as the set of equivalence classes induced by T. For $n = 1, 2, 3, \ldots$, let $d^n x$ be the projection of nd-dimensional Lebesque measure onto $X_N(\Lambda)$ under the projection $T: \Lambda^N \to X_N(\Lambda)$. The measure d^0x assigns mass 1 to $X_0(\Lambda)=\{\emptyset\}$. Define, as in [5].

$$\nu_\Lambda(dx)=\sum_{n=0}^{\infty}\frac{z^n}{n!}d^nx, \tag{1.5}$$

where z is chemical activity. Corresponding to the potential V and $s\in R_\Lambda\cap X(\Lambda^c)$, define

$$\mu_\Lambda(dx|s)=\frac{\exp(-\beta V_\Lambda(x|s))}{Z_\Lambda(s)}\nu_\Lambda(dx), \tag{1.6}$$

where β is inverse temperature and $Z_\Lambda(s)$ makes $\mu_\Lambda(dx|s)$ a probability measure on $X(\Lambda)$. Note that $1\leqq Z_\Lambda(S)<\infty$. If $s\cap\Lambda^c\notin R_\Lambda$, define $\mu_\Lambda(dx|s)$ to be the zero measure.

Definition 1.2. Let $\{\pi_\Lambda\}$ denote the specification associated with β, z, and the potential V (see [9, p. 16]) defined by

$$\pi_\Lambda(s,A)=\int_{A'}\mu_\Lambda(dx|s), \tag{1.7}$$

where $A\in S$ and $A'=\{x\in X(\Lambda):x\cup(s\cap\Lambda^c)\in A\}$.

Definition 1.3. A probability measure σ on (Ω, S) is a Gibbs state for the specification $\{\pi_\Lambda\}$ if

$$\sigma(\pi_\Lambda(s,A))=\sigma(A) \tag{1.8}$$

for every $A\in S$ and bounded Borel set $\Lambda\subset\mathbb{R}^d$.

Definition 1.4. A function $f:\Omega\to\mathbb{R}$ is a cylinder function if there exists a bounded set $B\subset\mathbb{R}^d$ such that $f(s)=f(s\cap B)$ for every $s\in\Omega$. A set $A\in S$ is a cylinder set if the indicator function for A is a cylinder function.

We note that any function on $X(\Lambda)$, where Λ is a bounded Borel set, has a natural extension to a cylinder function on Ω via (1.1).

Definition 1.5. Let $F(\Omega)$ be the class of real valued, bounded, S-measurable functions on Ω, which can be uniformly approximated by cylinder functions.
Following Preston [9], we make the following definitions.

Definition 1.6. Let C_n be the hypercube in $\mathbb{R}^d$ of length $2n$ centered at the origin, and let v_n denote the volume of $C_n\backslash C_{n-1}$. Let

$$U_{nm}=\{s\in\Omega:|s\cap C_n\backslash C_{n-1}|\leqq mv_n\}.$$

For a given potential V, we let

$$D=\{s\in\Omega:V(y)<\infty \quad \text{for all} \quad y\in\Omega_F \quad \text{with} \quad y\subset s\}.$$

Define

$$U_m=D\bigcap_{n\geqq 1}U_{nm},$$

and

$$U_\infty=\bigcup_{m\geqq 1}U_m.$$

Definition 1.7. A Gibbs state σ corresponding to a potential V, inverse temperature β, and chemical activity z is tempered if $\sigma(U_\infty)=1$.

Section 2. Gibbs States for Potentials with Hard-Core Components

In this section we prove the existence of a Gibbs state, for any β and z, associated with a potential with a hard-core N-component. We will refer to the following conditions on a potential V in what follows.

Condition 2.1

a) *V is stable*, i.e., $V(x) \geqq -D|x|$ *for some* $D>0$ *and all* $x \in \Omega_F$.

b) *For any bounded Borel set* Λ, *any* $x \in X(\Lambda) \cap U_\infty$, *and any* $m \geqq 1$, $|V_\Lambda(x|s) - V_\Lambda(x|s \cap C_k)| \leqq \varepsilon_s(k)|x|$, *where* $\varepsilon_s(k)$ *converges uniformly to zero for all* $s \in U_m \cap X(\Lambda^c)$ *as* $k \to \infty$.

c) $\displaystyle\sum_{N=2}^{\infty} \sum_{\substack{y \subset x \cup s \\ |y|=N \\ y \cap x \cap s \neq \phi}} \phi_N(y) \geqq -c|x||s|$ *for some* $c>0$ *and all* $x, s \in \Omega_F$.

Condition 2.2 *V has a hard-core N-component for some* $N \geqq 2$.

To prove existence of Gibbs states associated with potentials satisfying Condition 2.1 and 2.2 we verify hypotheses to theorems of Preston [9, chapter 3]. For convenience we express Preston's results in a slightly modified form using our notation.

Theorem 2.1 (*Preston*). *Suppose the following three conditions hold for the specification* $\{\pi_\Lambda\}$:

A) *Given* $\varepsilon>0$, *there exists an* $m \geqq 1$ *such that* $\pi_\Lambda(s, U_m) \geqq 1-\varepsilon$ *for all bounded Borel sets* $\Lambda \subset \mathbb{R}^d$ *and all* $s \in U_\infty$.

B) *For any bounded Borel set* $\Lambda \subset \mathbb{R}^d$ *and any* $\varepsilon>0$, *there exists a probability measure* ω *on* $(\Omega, \tilde{B}_\Lambda)$ *and a* $\delta>0$ *such that if* $A \in \tilde{B}_\Lambda$ *with* $\omega(A)<\delta$, *then* $\pi_C(s, A)<\varepsilon$ *for any bounded Borel set* $C \supset \Lambda$ *and all* $s \in U_\infty$.

C) *For any bounded Borel set* $\Lambda \subset \mathbb{R}^d$ *and any cylinder set* $A \in S$ *and any* $m \geqq 1$, $\pi_\Lambda(\cdot, A)$, *as a function on* U_m, *is the uniform limit of a sequence of bounded S-measurable cylinder functions.*

Then the set of tempered Gibbs states for $\{\pi_\Lambda\}$ *is non-empty. Furthermore, for each* $s \in U_\infty$ *and all cylinder sets* $A \in S$,

$$\lim_{k \to \infty} \pi_{C_k}(s, A) = \sigma(A), \tag{2.1}$$

where σ *is a tempered Gibbs state (possibly depending on s).*

We verify now that potentials satisfying Conditions 2.1 and 2.2 satisfy hypotheses A), B), and C) of Theorem 2.1 for any β and z.

Lemma 2.1. *Let* $\Lambda \subset \mathbb{R}^d$ *be a bounded Borel set. If Condition 2.1 holds, then for all* $m \geqq 1$ *there exists a constant* $D_m>0$, *depending only on* V, Λ, *and* m, *such that*

$$V_\Lambda(x|s) \geqq -D_m|x| \tag{2.2}$$

for all $x \in X(\Lambda) \cap U_\infty$ *and all* $s \in U_m \cap X(\Lambda^c)$. *If Condition 2.2 is also satisfied, then there exists a* $D > 0$ *depending only on* V *and* Λ *such that*

$$V_\Lambda(x|s) \geqq -D|x| \tag{2.3}$$

for all $x \in X(\Lambda) \cap U_\infty$ *and all* $s \in U_\infty \cap X(\Lambda^c)$.

Proof: Condition 2.1b implies that we can choose k large enough so that $C_k \supset \Lambda$ and

$$V_\Lambda(x|s) \geqq V_\Lambda(x|s \cap C_k) - |x|$$

for all $x \in X(\Lambda) \cap U_\infty$ and $s \in X(\Lambda^c) \cap U_m$. The proof of (2.2) now follows from Condition 2.1.c. Condition 2.2 implies that for m sufficiently large $U_m = U_\infty$ and hence (2.3) follows. This concludes the proof.

Remark 2.1. Lemma 2.1 improves Theorem 3.1 of [5].

Remark 2.2. When V satisfies Condition 2.1, Lemma 2.1 guarantees that $R_\Lambda \supset U_\infty$, and therefore $\pi_\Lambda(s, \cdot)$, is a probability measure for any $s \in U_\infty$ (see Sect. 1).

Corollary 2.1. *Conditions 2.1 and 2.2 imply that hypothesis A of Theorem 2.1 holds for any* β *and* z.

Proof. For $s \in U_\infty \cap X(\Lambda^c)$ and any z and β,

$$\pi_\Lambda(s, U_m) = \int_{U'_m} \mu_\Lambda(dx|s),$$

where $U'_m = \{x \in X(\Lambda): x \cup s \in U_m\}$. As in the proof of Lemma 2.1, $U_m = U_\infty$ for m sufficiently large. Hence $U'_m = U'_\infty$. But $U'_\infty = X(\Lambda)$. Thus

$$\pi_\Lambda(s, U_m) = 1$$

for m sufficiently large because $\mu_\Lambda(\cdot|s)$ is a probability measure on $(X(\Lambda), B_\Lambda)$. This completes the proof.

Corollary 2.2. *Conditions 2.1 and 2.2 imply that hypothesis* B *of Theorem 2.1 holds for any* β *and* z.

Proof. Recall that

$$\pi_\Lambda(t, A) = \int_{A'} \frac{\exp[-\beta V_\Lambda(x|t)]}{Z_\Lambda(t)} \nu_\Lambda(dx),$$

so that

$$\pi_\Lambda(t, dx) = \sum_{n=0}^{\infty} \frac{\exp[-\beta V_\Lambda(x|t)]}{Z_\Lambda(t)} \frac{z^n}{n!} d^n x, \tag{2.4}$$

when restricted to $\tilde{B}_\Lambda$. By Condition 2.2 $V_\Lambda(x|t) = +\infty$ for all $|x|$ sufficiently large.

Thus

$$\pi_A(t, dx) = \sum_{n=0}^{M} \frac{\exp[-\beta V_A(x|t)]}{Z_A(t)} \frac{z^n}{n!} d^n x$$

for some positive integer M.

Let $\omega(A) = v_A(A')$ for $A \in \tilde{B}_A$. By consistency of the specification $\{\pi_A\}$ (see [9, p. 90]), and Corollary 2.1,

$$\pi_C(s, A) = \int_{U_\infty} \pi_A(t, A) \pi_C(s, dt).$$

Thus

$$\pi_C(s, A) \leqq \sup_{t \in U_\infty} \pi_A(t, A).$$

By (2.4), Lemma 2.1, and since $Z_A(s) \geqq 1$,

$$\begin{aligned}\pi_C(s, A) &\leqq \sup_{t \in U_\infty} \int_{A'} \frac{\exp[-\beta V_A(x|t)]}{Z_A(t)} v_A(dx) \\ &\leqq e^{\beta DM} v_A(A') = e^{\beta DM} \omega(A),\end{aligned}$$

where the right side is independent of C. This concludes the proof.

Lemma 2.2. *Condition 2.1 implies hypothesis* C *of Theorem 2.1 for any* β *and* z.

Proof. Let $m \geqq 1$ and $s \in U_m \cap X(A^c)$. Define

$$f_k(s) = \int_{A'} \frac{\exp[-\beta V_A(x|s \cap C_k)]}{Z_A(s \cap C_k)} v_A(dx).$$

Then $f_k(s)$ is a bounded S-measurable cylinder function on Ω. Also

$$|f_k(s) - \pi_A(s, A)| \leqq \int_{X(A)} \left| \frac{\exp[-\beta V_A(x|s \cap C_k)]}{Z_A(s \cap C_k)} - \frac{\exp[-\beta V_A(x|s)]}{Z_A(s)} \right| v_A(dx). \tag{2.5}$$

From Lemma 2.1, Eq. (2.4), and since $Z_A(t) \geqq 1$ for all t, it follows that given $\varepsilon > 0$, we can choose $j \geqq m$ sufficiently large so that

$$\int_{X(A) \cap U_j^c} \left| \frac{\exp[-\beta V_A(x|s \cap C_k)]}{Z_A(s \cap C_k)} - \frac{\exp[-\beta V_A(x|s)]}{Z_A(s)} \right| v_A(dx) < \varepsilon. \tag{2.6}$$

Combining (2.5) with (2.4) gives

$$|f_k(s) - \pi_A(s, A)| \leqq \int_{X(A) \cap U_j} \left| \frac{\exp[-\beta V_A(x|s \cap C_k)]}{Z_A(s \cap C_k)} - \frac{\exp[-\beta V_A(x|s)]}{Z_A(s)} \right| v_A(dx) + \varepsilon. \tag{2.7}$$

Recall that $U_m \subset U_j$ when $m \leqq j$. It follows from Condition 2.1b that the integral on the right side of (2.7) converges uniformly to zero for all $s \in U_m \cap X(\Lambda^c)$ as $k \to \infty$. Thus by choosing k sufficiently large, $|f_k(s) - \pi_\Lambda(s, A)| \leqq 2\varepsilon$ uniformly in $s \in U_m \cap X(\Lambda^c)$. This completes the proof.

Combining the above results we have the following theorem.

Theorem 2.2. *Let V satisfy Conditions 2.1 and 2.2. Then the set of tempered Gibbs states for V at any inverse temperature β and activity z is nonempty. Furthermore, formula (2.1) holds.*

Proof. The proof follows directly from Theorem 2.1, Corollary 2.1, Corollary 2.2, and Lemma 2.2.

Remark 2.3. It can be shown that, for any β and z, the set of Gibbs states for a potential V satisfying Conditions 2.1 and 2.2 satisfies a sequential compactness property, and that if V is translation invariant, then the set of translation invariant Gibbs states for V is nonempty. This follows from Theorem 3.4 and Chapter 4 of Preston [9].

Remark 2.4. In [5] we proved the high temperature uniqueness of the Gibbs state for a class of many-body potentials satisfying the hypotheses of Theorem 2.2, via Dobrushin uniqueness techniques [1]. The proof in [5] for uniqueness (as well as the proofs for decay of correlations) depended on the existence of the Gibbs states considered there. Thus Theorem 2.2 validates the results of [5].

Section 3. Gibbs States for Potentials without Hard-Core Components

In this section we prove the existence, at any temperature and activity, of a tempered Gibbs state for any many-body potential V satisfying Conditions 2.1 and 3.1 (given below), but not necessarily satisfying Condition 2.2, i.e., we do not assume that V has a hard-core N-component. The method of proof is via the limiting process described in the introduction. We will refer to the following condition.

Condition 3.1. *The specification $\{\pi_\Lambda\}$ corresponding to V, β, z satisfies the following. Given $\varepsilon > 0$,*

$$\pi_{C_k}(\varnothing, U_m) \geqq 1 - \varepsilon \tag{3.1}$$

for all m sufficiently large and all $k = 1, 2, 3, \ldots$.

Remark 3.1. Condition 3.1 may be interpreted to mean that the probability, corresponding to free boundary conditions, of configurations of particles in C_k with high density is small. Condition 3.1 is fulfilled for all β and z by any super-stable lower regular many-body potential, as shown by Ruelle [11] and Preston [9, p. 108].

Let a potential V satisfying Conditions 2.1 and 3.1 be given and let

$$V^N = V + \varphi_N, \tag{3.2}$$

where φ_N is given by Definition 1.1. It is easy to check that V^N satisfies Conditions 2.1 and 2.2. For fixed β and z, let σ_N be the Gibbs state for V^N, shown to exist in Theorem 2.2, satisfying

$$\sigma_N(A) = \lim_{k\to\infty} \pi^N_{C_k}(\varnothing, A), \tag{3.3}$$

for every cylinder set $A \in S$, where $\{\pi^N_A\}$ is the specification corresponding to V^N, β, z. It follows from (3.3) that

$$\sigma_N(f) = \lim_{k\to\infty} \pi^N_{C_k}(\varnothing, f) \tag{3.4}$$

for every bounded measurable cylinder function f on Ω, since such functions can be expressed as uniform limits of simple functions.

For simplicity in notation in what follows, let

$$\pi^N_k(\cdot) = \pi^N_{C_k}(\varnothing, \cdot). \tag{3.5}$$

Lemma 3.1. *Let V satisfy Conditions 2.1 and 3.1 with β and z fixed. Let f be a bounded measurable cylinder function on Ω. Then given $\varepsilon > 0$,*

$$|\pi^N_k(f) - \pi_{C_k}(\varnothing, f)| < \varepsilon \tag{3.6}$$

for all N sufficiently large, uniformly in k.

Proof. From (1.6) and (1.7),

$$\pi^N_k(f) = \int_{X(C_k)} f(x) \frac{\exp[-\beta V^N(x)]}{Z^N_{C_k}(\varnothing)} \nu_{C_k}(dx),$$

where $Z^N_{C_k}(\varnothing)$ is the normalizing constant in (1.6) corresponding to the potential V^N. To simplify notation in what follows, let us denote $Z^N_k = Z^N_{C_k}(\varnothing)$, $Z_k = Z_{C_k}(\varnothing)$, and $\nu_k = \nu_{C_k}$. With this notation,

$$|\pi^N_k(f) - \pi_{C_k}(\varnothing, f)| \leqq \|f\|_\infty \int_{X(C_k)} \left| \frac{\exp[-\beta V^N(x)]}{Z^N_k} - \frac{\exp[-\beta V(x)]}{Z_k} \right| \nu_k(dx). \tag{3.7}$$

By the triangle inequality and since $Z_k \geqq Z^N_k$ and $\exp[-\beta V^N(x)] \leqq \exp[-\beta V(x)]$,

$$\int_{X(C_k)} \left| \frac{\exp[-\beta V^N(x)]}{Z^N_k} - \frac{\exp[-\beta V(x)]}{Z_k} \right| \nu_k(dx) \leqq \int_{X(C_k)} \left(\frac{\exp[-\beta V^N(x)]}{Z^N_k} - \frac{\exp[-\beta V^N(x)]}{Z_k} \right) \nu_k(dx) + \int_{X(C_k)} \left(\frac{\exp[-\beta V(x)]}{Z_k} - \frac{\exp[-\beta V^N(x)]}{Z_k} \right) \nu_k(dx). \tag{3.8}$$

Performing the integrations on the right side of (3.8) and using the definitions of Z^N_k

and Z_k, we have

$$\int_{X(C_k)} \left| \frac{\exp[-\beta V^N(x)]}{Z_k^N} - \frac{\exp[-\beta V(x)]}{Z_k} \right| \nu_k(dx) \leqq 2\left(1 - \frac{Z_k^N}{Z_k}\right). \tag{3.9}$$

It follows from the definitions of φ_N and U_m (which depends on V) that given any m we can choose N large enough so that

$$\frac{Z_k^N}{Z_k} \geqq \int_{U'_m} \frac{\exp[-\beta V(x)]}{Z_k} \nu_k(dx) = \pi_{C_k}(\emptyset, U_m). \tag{3.10}$$

The desired conclusion now follows from (3.7), (3.9) and Condition 3.1.

Lemma 3.2. *Let V satisfy Conditions 2.1 and 3.1 with β and z fixed. Then the sequence $\{\sigma_N(f)\}$ is a Cauchy sequence for every bounded measurable cylinder function f on Ω.*

Proof. For any M, N, j, and k

$$|\pi_k^N(f) - \pi_j^M(f)| \leqq |\pi_k^N(f) - \pi_j^N(f)| + |\pi_j^N(f) - \pi_j^M(f)|. \tag{3.11}$$

By Lemma 2.1 $\{\pi_k^N(f)\}$ is a Cauchy sequence in N (uniformly in k) for each bounded measurable cylinder function f. Thus

$$|\pi_j^N(f) - \pi_j^M(f)| \to 0 \quad \text{as} \quad M, N \to \infty,$$

and the rate of convergence is independent of j. By Theorem 2.2 $\{\pi_k^N(f)\}$ is a Cauchy sequence in k for each fixed N and hence with N fixed,

$$|\pi_k^N(f) - \pi_j^N(f) \to 0 \quad \text{as} \quad k, j \to \infty.$$

Thus by first choosing large values of M and N, and then sufficiently large values of j and k, the left side of (3.11) can be made arbitrarily small.

By the triangle inequality,

$$|\sigma_N(f) - \sigma_M(f)| \leqq |\sigma_N(f) - \pi_k^N(f)| + |\pi_k^N(f) - \pi_j^N(f)| + |\pi_j^M(f) - \sigma_M(f)|. \tag{3.12}$$

From Theorem 2.2 and the preceding arguments, it follows that by choosing M and N sufficiently large and then k and j depending on M and N, each term on the right side of (3.12) can be made arbitrarily small. It follows that $\{\sigma_N(f)\}$ is a Cauchy sequence. This completes the proof.

Theorem 3.1. *With the same hypotheses as in Lemma 2.2, there exists a unique probability measure σ on (Ω, S) such that*

$$\lim_{N\to\infty} \sigma_N(f) = \sigma(f)$$

for every function $f \in F(\Omega)$.

Proof. As in (1.2) we write

$$(\Omega, S) = \prod_{i=1}^{\infty} (X(\Lambda_i), B_{\Lambda_i}). \tag{3.13}$$

For any subset K of the positive integers we let $\tilde{B}^K = \tilde{B}_{\bigcup_{i\in K} A_i}$ so that $\tilde{B}^K$ is the σ-field on Ω generated by the factors

$$\prod_{i\in K}(X(A_i), B_{A_i})$$

in (3.13). Let us denote the restriction of σ_N to $\tilde{B}^K$ by σ_N^K. By Lemma 3.2 $\{\sigma_N^K(A)\}$ is a Cauchy sequence for each $A \in \tilde{B}^K$. Thus by the Vitali–Hahn–Saks Theorem [2, p. 160], there exists a unique countably additive probability measure σ^K on $(\Omega, \tilde{B}^K)$ such that

$$\lim_{N\to\infty} \sigma_N^K(A) = \sigma^K(A)$$

for every $A \in \tilde{B}^K$. Since linear combinations of indicator functions of sets in $\tilde{B}^K$ are sup-norm dense in the set of bounded $\tilde{B}^K$-measurable functions, we also have $\sigma_N^K(g) \to \sigma^K(g)$ for every $\tilde{B}^K$-measurable bounded function g. The family of measures $\{\sigma^K\}$, where K runs over finite subsets of the positive integers, is consistent. Furthermore, for each factor $(X(A_i), B_{A_i})$ in (3.13), B_{A_i} is generated by a locally compact topology on $X(A_i)$. Thus by Kolmogorov's consistency Theorem (see for example [7, p. 251]), there exists a unique probability measure σ on (Ω, S), whose marginal distributions consist of $\{\sigma^K\}$. Furthermore $\sigma_N(g) \to \sigma(g)$ for every bounded measurable cylinder function g. Since such functions are sup-norm dense in $F(\Omega)$, it follows that $\sigma_N(f) \to \sigma(f)$ for every $f \in F(\Omega)$. This completes the proof.

Corollary 3.1. *With the same assumptions as in Theorem 3.1,* $\sigma(U_\infty) = 1$, *and for any bounded measurable cylinder function* g,

$$\lim_{k\to\infty} \pi_{C_k}(\emptyset, g) = \sigma(g).$$

Proof. By the triangle inequality,

$$|\sigma(g) - \pi_{C_k}(\emptyset, g)| \leqq |\sigma(g) - \sigma_N(g)| + |\sigma_N(g) - \pi_k^N(g)| + |\pi_k^N(g) - \pi_{C_k}(\emptyset, g)|. \tag{3.14}$$

By first choosing N and then k sufficiently large, each term on the right side of (3.14) can be made arbitrarily small by Theorem 3.1, Theorem 2.2, and Lemma 3.1. Hence

$$\pi_{C_k}(\emptyset, g) \to \sigma(g) \tag{3.15}$$

for every measurable cylinder function g. From (3.15)

$$\pi_{C_k}(\emptyset, A) \to \sigma(A)$$

for any cylinder set A. Since U_m is a countable intersection of cylinder sets and since by Condition 3.1, given any $\varepsilon > 0$

$$\pi_{C_k}(\emptyset, U_m) > 1 - \varepsilon$$

provided m is sufficiently large, it follows that $\sigma(U_m) > 1 - \varepsilon$ for m sufficiently large. Thus since ε is arbitrary and $U_\infty = \bigcup_{m\geqq 1} U_m$, $\sigma(U_\infty) = 1$. This concludes the proof.

Corollary 3.2. *With the same assumptions as in Theorem 3.1, σ is a tempered Gibbs state for V, β, and z.*

Proof. Let Λ be a bounded Borel set in $\mathbb{R}^d$. By Lemma 2.2, for any cylinder set $A \in S$, $\pi_\Lambda(s, A)$ can be uniformly approximated on U_m, for each m, by a bounded measurable cylinder function. It follows from Condition 3.1, Corollary 3.1, and the triangle inequality that

$$\lim_{k \to \infty} \pi_{C_k}(\emptyset, \pi_\Lambda(s, A)) = \sigma(\pi_\Lambda(s, A)). \tag{3.16}$$

But by the consistency of the specification $\{\pi_\Lambda\}$ we can write, as in the proof of Corollary 2.2,

$$\pi_{C_k}(\emptyset, \pi_\Lambda(s, A)) = \pi_{C_k}(\emptyset, A) \tag{3.17}$$

for all k large enough so that $\Lambda \subset C_k$. Combining (3.16) and (3.17) and letting $k \to \infty$ gives $\sigma(A) = \sigma(\pi_\Lambda(s, A))$ for any bounded measurable cylinder set A. A standard argument in measure theory shows that $\sigma(A) = \sigma(\pi_\Lambda(s, A))$ for any $A \in S$. Temperedness follows from Corollary 3.1. This completes the proof.

Remark 3.2. If σ_N is translation invariant for each N, then the Gibbs state σ of Corollary 3.2 is also translation invariant.

Section 4. Applications

In this section we give two examples in which behaviour of a potential V^N with hard-core N-component implies similar behaviour for a related potential V without any hard-core component. The basic tool for this is the convergence result of Sect. 3, namely, if σ_N is the Gibbs state for $V^N = V + \varphi_N$ described in Sect. 3, then

$$\sigma_N(f) \to \sigma(f) \tag{4.1}$$

for $f \in F(\Omega)$, where σ is a tempered Gibbs state for V.

We begin with an extension of a theorem on the decay of correlations given in [5, p. 244].

Theorem 4.1. *Let $d(\cdot, \cdot)$ be a translation invariant semimetric on $\mathbb{Z}^d \subset \mathbb{R}^d$. Let V^N be a translation invariant potential satisfying Conditions 2.1 and 2.2 and Condition 3.1* a *and* b *of* [5]. *Then for sufficiently small values of β and z, there exists a unique Gibbs state σ_N for V^N satisfying*

$$|\sigma_N(fg) - \sigma_N(f)\sigma_N(g)| \leqq C_N e^{-d(a,b)} \| f \|_a \| g \|_b \tag{4.2}$$

for all $a, b \in \mathbb{Z}^d$, $f, g \in F(\Omega)$ such that $\| f \|_c$, $\| g \|_c < \infty$ for each $c \in \mathbb{Z}^d$, and some constant C_N depending only on z, β and V^N. The norms $\| \cdot \|_c$ for $c \in \mathbb{Z}^d$ are defined in [5].

Remark 4.1. This slightly improves the result in [5] by removing the restriction that the N-body components of V for $N \geqq 3$ be of finite range and by guaranteeing existence of the Gibbs state σ_N.

Proof of Theorem 4.1. Condition 3.2 of [5] holds by Lemma 2.1 of this paper and by Remark 3.1 of [5]. Also, by Condition 2.1 and Lemma 2.1 of this paper, the left side of inequality (3.11) of [5] converges to zero as $n \to \infty$, establishing the conclusion of Theorem 3.2 of [5] for potentials considered here. It follows from this, that the conditions given on the top of page 244 of [5] are satisfied and hence the conclusions of Theorem 4.1 of [5] hold for potentials considered here. Existence of σ_N follows from Theorem 2.2. This concludes the proof.

Remark 4.2. The above proof extends the stronger result given by (4.3) of [5] to potentials satisfying the hypotheses of Theorem 4.1, but we will have no need of this result in what follows.

A variant of inequality (4.2) can now be extended to a Gibbs state for a potential V without a hard-core component. In the theorem below we let $V^N = V + \varphi_N$; σ_N and σ are the same as in (4.1).

Theorem 4.2. *Let V satisfy Condition 3.1* a *and* b *of* [5] *and Conditions 2.1 and 3.1. Given $f, g \in F(\Omega)$ such that $\|f\|_c, \|g\|_c < \infty$ for all $c \in \mathbb{Z}^d$ and $\varepsilon > 0$, N can be chosen sufficiently large and β and z sufficiently small so that*

$$|\sigma(fg) - \sigma(f)\sigma(g)| \leqq C_N e^{-d(a,b)} \|f\|_a \|g\|_b + \varepsilon \tag{4.3}$$

for all $a, b \in \mathbb{Z}^d$. The quantities C_N, $\|\cdot\|_a$, $\|\cdot\|_b$, and $d(\cdot,\cdot)$ are the same as in Theorem 4.1.

Proof. The proof follows directly from (4.1) and Theorem 4.1.

Remark 4.3. Theorem 4.2 extends Dobrushin uniqueness techniques for the decay of correlations to potentials on the continuum without hard-core and without any positivity restrictions. We point out that smaller values for the constants C_N than those given in [5] could be obtained through the analyses given by Künsch [6] and Föllmer [3].

As a second application of the techniques of Sect. 3, we show that the grand canonical pressure P_N, corresponding to the potential $V^N = V + \varphi_N$, converges to the grand canonical pressure P corresponding to the potential V, as $N \to \infty$.

The pressure $P(z, \beta)$ for z, β and a potential V is given by

$$\beta P(z,\beta) = \lim_{k \to \infty} \frac{1}{|C_k|} \log Z_{C_k}, \tag{4.4}$$

where $Z_{C_k} = Z_{C_k}(\emptyset)$ is the partition function defined in (1.6), and the hypercube C_k with volume $|C_k|$ is given in Definition 1.6.

Theorem 4.3. *Let V satisfy Conditions 2.1 and 3.1. Assume that the grand canonical pressure $P_N(z, \beta)$ corresponding to $V^N = V + \varphi_N$ exists for every N. Then the grand canonical pressure $P(z, \beta)$ corresponding to V exists and*

$$\lim_{N \to \infty} P_N(z,\beta) = P(z,\beta). \tag{4.5}$$

Proof. As in the proof of Lemma 3.1, let $Z_k^N = Z_{C_k}^N(\emptyset)$ and $Z_k = Z_{C_k}(\emptyset)$. We first show that $\{P_N(z,\beta)\}$ is a Cauchy sequence. By the triangle inequality,

$$|\beta P_N(z,\beta)-\beta P_M(z,\beta)| \leqq \left|\beta P_N(z,\beta)-\frac{1}{|C_k|}\log Z_k^N\right| + \left|\frac{1}{|C_k|}\log Z_k^N - \frac{1}{|C_k|}\log Z_k^M\right| + \left|\frac{1}{|C_k|}\log Z_k^M - \beta P_M(z,\beta)\right|. \tag{4.6}$$

The second term on the right of (4.6) can be rewritten as

$$\left|\frac{1}{|C_k|}\log Z_k^N - \frac{1}{|C_k|}\log Z_k^M\right| = \frac{1}{|C_k|}\left|\log\frac{Z_k^N}{Z_k^M}\right|.$$

As in (3.10), for sufficiently large N and M, this is small uniformly in k. Now choose k large enough and depending on M and N so that the first and third terms on the right side of (4.6) are small. Let $P(z,\beta)=\lim_{N\to\infty} P_N(z,\beta)$. It remains to show that

$$P(z,\beta)=\lim_{k\to\infty}\frac{1}{|C_k|}\log Z_k. \tag{4.7}$$

This holds by the following application of the triangle inequality.

$$\left|\frac{1}{|C_k|}\log Z_k - \beta P(z,\beta)\right| \leqq \left|\frac{1}{|C_k|}\log Z_k - \frac{1}{|C_k|}\log Z_k^N\right| + \left|\frac{1}{|C_k|}\log Z_k^N - \beta P_N(z,\beta)\right| + |\beta P_N(z,\beta)-\beta P(z,\beta)|.$$

By choosing N first and then k we see that (4.7) holds. This concludes the proof.

Remark 4.4. The existence of the grand canonical pressure P is already known for superstable, lower regular potentials V (see Ruelle [11]). We include Theorem 4.3 to establish further the relationship between V and V^N.

Theorem 3.1, Corollary 3.2, Theorem 4.2 and Theorem 4.3 show that some statistical mechanical properties of continuum potentials may be studied by investigating related potentials with hard-core components. These hard-core approximations give rise to models which more closely resemble standard lattice gas models. It may be expected then that some techniques successful in the study of standard lattice models could be extended to yield results on continuum models via this process, as was demonstrated above in the case of Dobrushin uniqueness methods.

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