Commun. Math. Phys. 92, 179 – 194 (1983)

Communications in Mathematical Physics

Leading Large Order Asymptotics for $(\phi^4)_2$ Perturbation Theory*

Stephen Breen**
Mathematics Department, University of British Columbia Vancouver, B.C., Canada V6T 1Y4

Abstract. We develop a rigorous semiclassical expansion to compute the radius of convergence of the Borel transform for the pressure in $(\phi^4)_2$ field theory. This result gives a partial justification for the Lipatov method of finding large order perturbation theory asymptotics in quantum field theory.

1. Introduction

The Lipatov method is an interesting formal technique for finding the large order behavior of the perturbation coefficients in certain divergent perturbation series. The real power of this method is that it is applicable to the perturbation series which occur in quantum field theory. The basic idea of the approach, initiated by Lipatov [1] and extensively developed by Brezin et al. [2–4], is to use a path integral representation for the k^{th} perturbation coefficient in the perturbation series and then to do a formal semiclassical expansion of the path integral as $k \to \infty$. Knowledge of the large order behavior of perturbation theory may be combined with summability methods, such as Padé or Borel summation, to do numerical calculations (see [5] and other articles in the same volume for more details). Of particular interest for field theory are the calculations of critical exponents done by Le Guillou and Zinn-Justin [6–8] based on the perturbation theory asymptotics of [2]. Our result described in the abstract gives a partial justification, for $(\phi^4)_2$ field theory, of these large order asymptotics and of the Lipatov method of deriving them.

In order to state our result and describe its connection with the Lipatov method calculations, consider a $(\phi^4)_2$ field theory with partition function

$$Z_X(\lambda) = \int e^{-\lambda V(\phi)} d\mu_\Lambda^X$$

in which $V(\phi) = \int_\Lambda :\phi^4(x): d^2x, \Lambda = [-T/2, T/2]^2$, and $X = p$ (periodic), D

---

* Research supported by the Natural Sciences and Engineering Research Council of Canada

** Present address: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA

(Dirichlet). The mean-zero Gaussian measure $d\mu_\Lambda^X$ has covariance

$$\int \phi(f)\phi(g)\, d\mu_\Lambda^X = \langle f, (-\Delta_X + 1)^{-1} g \rangle,$$

where $\langle , \rangle$ is the $L^2(\Lambda)$ inner product, Δ_X is the Laplacian with $X = p, D$ boundary conditions on Λ, and the Wick ordering in $V(\phi)$ is with respect to $d\mu_\Lambda^X$. The $(\phi^4)_2$ pressure is then defined as

$$p(\lambda) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \ln Z_X(\lambda)$$

($p(\lambda)$ is independent of the choice of boundary conditions for $Z_X(\lambda)$ [9]). The pressure has a divergent perturbation series [10]

$$p(\lambda) \sim \sum_{k=0}^{\infty} a_k \lambda^k, \tag{1.2}$$

which is known to be Borel summable [11]. In particular, the Borel transform $B(t)$ has the representation

$$B(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \tag{1.3}$$

for t in a disk of non-zero radius R.

Next, we will need the functional $S(\phi)$ defined by

$$S(\phi) = \tfrac{1}{2} \int_{\mathbb{R}^2} [(\nabla\phi)^2(x) + \phi^2(x)]\, d^2x - \ln \int_{\mathbb{R}^2} \phi^4(x)\, d^2x \tag{1.4}$$

for ϕ in the Sobolev space $W^{1,2}(\mathbb{R}^2)$, which is the completion of $C_0^\infty(\mathbb{R}^2)$ in the norm

$$\|\phi\|_{1,2}^2 = \int_{\mathbb{R}^2} [(\nabla\phi)^2(x) + \phi^2(x)] d^2$$

We can now state our main result:

Theorem 1.1. *Let R be the radius of convergence of the Borel transform defined in (1.3). Then*

$$R^{-1} = \lim_{k\to\infty} \left|\frac{a_k}{k!}\right|^{1/k} = \exp[-\inf S(\phi) + 2]$$

in which the infimum is over all $\phi \in W^{1,2}(\mathbb{R}^2)$.

Remark. It will be shown in Lemma 2.1 that $S(\phi)$ is bounded below and attains its infimum.

In order to see how this result follows from a semiclassical expansion, we note that $Z_X(\lambda)$ also has a divergent perturbation series

$$Z_X(\lambda) \sim \sum_{k=0}^{\infty} b_k^X(\Lambda)\lambda^k$$

in which

$$b_k^X(\Lambda) = \frac{(-1)^k}{k!} \int V^k(\phi) d\mu_\Lambda^X = \frac{(-1)^k k^{2k}}{k!} \int V^k(\phi)/\sqrt{k} d\mu_\Lambda^X, \tag{1.5}$$

where the Wick ordering in $V(\phi/\sqrt{k})$ has been scaled as in (1.14). The connection between the two series will be shown by using

$$a_k^X(\Lambda) = \frac{1}{|\Lambda| k!} \frac{d^k}{d\lambda^k}\bigg|_{\lambda=0} \ln Z_X(\lambda) \tag{1.6}$$

for $X = p$, D, which are the finite volume analogs of the perturbation coefficients a_k in (1.2).

Before stating the next results, we define the functionals

$$S_X(\phi) = \tfrac{1}{2} \int_\Lambda [(\nabla_X \phi)^2(x) + \phi^2(x)]\, d^2x - \ln \int_\Lambda \phi^4(x)\, d^2x, \tag{1.7}$$

where ∇_X is the gradient obeying $X = p$, D boundary conditions.

Theorem 1.2. *If $X = p$, D, then*

$$\lim_{k\to\infty} \left(\int V^k(\phi/\sqrt{k})\, d\mu_\Lambda^X\right)^{1/k} = e^{-\inf S_X(\phi)}. \tag{1.8}$$

Remarks. 1. The infimum in (1.8) is taken over $\phi \in W_0^{1,2}(\Lambda)$ for $X = D$, and over $\phi \in W^{1,2}(\Lambda)$ for $X = p$. If we consider the norm

$$\int_\Lambda [(\nabla_X \phi)^2(x) + \phi^2(x)]\, d^2x,$$

then $W_0^{1,2}(\Lambda)$ is defined to be the completion of $C_0^\infty(\Lambda)$ in the above norm for $X = D$, and $W^{1,2}(\Lambda)$ is the completion of the periodic C^1 functions in this norm for $X = p$.

2. Our proof of (1.8) could easily accommodate other choices of boundary conditions. However, $X = p$, D are the most useful choices for proving Theorem 1.1.

Corollary 1.3. *Let $X = p$, D. Then*

$$\lim_{k\to\infty} \left|\frac{a_k^X(\Lambda)}{k!}\right|^{1/k} = \exp[-\inf S_X(\phi) + 2]. \tag{1.9}$$

Remark. (1.9) will follow from (1.8) by showing that

$$\left|\frac{a_k^X(\Lambda)}{k!}\right|^{1/k} \sim \left|\frac{b_k^X(\Lambda)}{k!}\right|^{1/k}$$

as $k \to \infty$. The e^2 factor in (1.9) comes from Sterling's formula applied to the $[k^{2k}/(k!)^2]^{1/k}$ term in $|b_k^X(\Lambda)/k!|^{1/k}$.

Theorem 1.1 is an immediate consequence of Corollary 1.3 and the following bracketing inequality.

Lemma 1.4. [12, Lemma 2] *For all k and T,*

$$(-1)^k a_k^D(\Lambda) \leqq (-1)^k a_k \leqq (-1)^k a_k^p(\Lambda). \tag{1.10}$$

Proof of Theorem 1.1. Combining Corollary 1.3 and Lemma 1.4 yields

$$\exp[-\inf S_D(\phi) + 2] \leqq \lim_{k\to\infty} \left|\frac{a_k}{k!}\right|^{1/k} \leqq \exp[-\inf S_p(\phi) + 2]. \tag{1.11}$$

In Sect. 3, we will show that $\lim_{|\Lambda|\to\infty}(\inf S_X(\phi)) = \inf S(\phi)$, and this will complete the proof of Theorem 1.1. □

While the exact knowledge of the radius of convergence R for $B(t)$ is of use in the numerical calculations of [6–8], we feel that the real merit of Theorem 1.1 is as a partial justification for the formal but quite believable full asymptotic formula

$$a_k \overset{k+\infty}{\sim} ab^k k^c \frac{k^{2k}}{k!}, \tag{1.12}$$

which follows from the Lipatov method calculations of [2]. If (1.12) is true, then we have computed $b = R^{-1} \cdot e^{-2}$. We hope that the methods of this paper may serve as a starting point for a proof of (1.12).

Our proof of Theorem 1.2 will be by a Laplace-type asymptotic expansion for the functional integral in (1.8). The asymptotic expansion may be seen on an intuitive level by dropping the Wick counterterms in $V(\phi)$, so that *formally* we have

$$\int V^k(\phi/\sqrt{k})\, d\mu_\Lambda^X = \int e^{-kS_X(\phi/\sqrt{k})}\, d\phi,$$

where $d\phi$ is a non-existent measure. Therefore, (1.8) is equivalent to the formal computation

$$\lim_{k\to\infty} \left(\int e^{-kS_X(\phi/\sqrt{k})}\, d\phi\right)^{1/k} = e^{-\inf S_X(\phi)}, \tag{1.13}$$

which is clearly a Laplace expansion.

In a field theory which requires renormalization, the treatment of counterterms in a Laplace expansion is an interesting question. For a general polynomial interaction $V(\phi, \lambda)$ which includes counterterms, the calculations of [2] proceed on the principle: amongst terms in $V(\phi, \lambda)$ which are of the same order in λ, those which are lower order in ϕ will not contribute to the leading order asymptotics (see the remark below). In particular, the calculation (1.13) will not be affected by counterterms. While this is plausible for super renormalizable $(\phi^4)_d$, theories (our result proves it for $d=2$), the fact that one must be careful is shown by two interesting examples of Herbst and Simon [13]. They construct two anharmonic oscillator-type hamiltonians of the form $p^2 + x^2 - 1 + V(x, \lambda)$, where V is a polynomial and $p = -id/dx$, which have ground state energies whose perturbation coefficients $a_k = 0$, for all k. However, if one follows the above principle and drops lower order terms in x to a given order in λ from $V(x, \lambda)$, the resulting hamiltonians have groundstate energies with perturbation coefficients which grow like $k!$ (see [13] for details).

Remark. While counterterms are not supposed to affect (1.13), the calculations of [2, 7] show that there is a renormalization effect on the constant a in (1.12). That is, the constant a is usually given by a determinant and counterterms will modify this into a renormalized determinant (the $\det_n$ of [19, p. 107]).

Our method of dealing with the Wick counterterms in $V(\phi)$ will be to use the lattice approximation to $(\phi^4)_2$ [14, 15], but with the lattice spacing δ depending on k, $\delta = O(k^{-\varepsilon})$ for small ε. The interaction in $V(\phi/\sqrt{k})$ will then be of the form

$$:\left(\frac{\phi_\delta}{\sqrt{k}}\right)^4: = \left(\frac{\phi_\delta}{\sqrt{k}}\right)^4 - 6\left(\frac{\phi_\delta}{\sqrt{k}}\right)^2 \frac{c_\delta}{k} + 3\left(\frac{c_\delta}{k}\right)^2, \tag{1.14}$$

where ϕ_δ is a lattice field and c_δ is the Wick constant. As is well known, $c_\delta = O(\ln \delta^{-1})$, so the new Wick constant c_δ/k will obey

$$\frac{c_\delta}{k} = O\left(\frac{\ln k}{k}\right).$$

We will be able to use this to show that only the $(\phi_\delta/\sqrt{k})^4$ term in (1.14) contributes to the Laplace expansion in (1.8).

The outline of the paper is as follows. In Sect. 2 we will derive upper and lower bounds which will prove Theorem 1.2. Corollary 1.3 will then be proven in Sect. 3, where we will also show that $\inf S_X(\phi) \to \inf S(\phi)$ as $|\Lambda| \to \infty$, which is needed in the proof of Theorem 1.1.

2. Leading Order Asymptotics for $b_k^X(\Lambda)$

We must first establish a number of definitions for the lattice approximation, mostly following [9]. Let $\Lambda_\delta = \Lambda \cap L_\delta$, where $L_\delta = \{\delta n | n = (n_1, n_2) \in \mathbb{Z}^2\}$ and δ is the lattice spacing. The lattice measure $d\mu_{\Lambda,\delta}^X$, $X = p, D$, is defined as the mean-zero Gaussian measure with covariance

$$G_X^\delta \equiv (A_X^\delta)^{-1} \equiv (-\Delta_X^\delta + 1)^{-1},$$

where Δ_X^δ is the finite difference Laplacian with $X = p, D$ boundary conditions. Explicitly,

$$d\mu_{\Lambda,\delta}^X = e^{-(1/2)\langle q, A_X^\delta q\rangle} dq N^X,$$

where N^X is the appropriate normalization and $\langle q, q\rangle = \sum_{n \in \Lambda_\delta} \delta^2 q_n^2$. It is convenient to be able to define the lattice theory in terms of the continuum theory by the identification $q_n = \phi(f_{n,\delta}^X)$, so that

$$\int \phi(f_{n,\delta}^X)\phi(f_{m,\delta}^X)\, d\mu_\Lambda^X = (G_X^\delta)_{nm} = \int q_n q_m \, d\mu_{\Lambda,\delta}^X$$

(see [9, Sect. IX.1] for $f_{n,\delta}^X$). Next, let

$$V_\delta(\phi) = \delta^2 \sum_{n \in \Lambda_\delta} :\phi_\delta^4(n): = \delta^2 \sum_{n \in \Lambda_\delta} :q_n^4:$$

in which $\phi_\delta(n) = \phi(f_{n,\delta}^X)$. The above Wick order will always correspond to the measure $d\mu_\Lambda^X$ with which we are integrating $V_\delta(\phi)$ (i.e., the Wick lattice constant $c_\delta(n) = (G_X^\delta)_{nn}$). We will also need

$$V_{0,\delta}(q) = \delta^2 \sum_{n \in \Lambda_\delta} q_n^4$$

and its continuum counterpart

$$V_0(\phi) = \int_\Lambda \phi^4(x)\, d^2x$$

($V_0(\phi)$ will only be used when ϕ is a function). Our last definition is

$$S_X^\delta(q) = \tfrac{1}{2}\langle q, A_X^\delta q\rangle - \ln V_{0,\delta}(q), \quad X = p, D.$$

We begin with an elementary lemma.

Lemma 2.1. *The functionals $S_X(\phi)$, $S_X^\delta(q)$, for $X = p$, D, and $S(\phi)$ all attain their infimums.*

Remark. It will be useful to have functions ϕ^c, q^c, and η^c for which

$$S_X(\phi^c) = \inf S_X(\phi),$$
$$S_X^\delta(q^c) = \inf S_X^\delta(q).$$
$$S(\eta^c) = \inf S(\phi).$$

We are suppressing the dependence of ϕ^c and q^c on the different boundary conditions, as this will not be important for our purposes. The above functions are not necessarily unique, as $S_p(\phi)$ and $S(\phi)$ are invariant under translations, and $S_p^\delta(q)$ is invariant under translations mod δ.

Proof. The proof will be given for $S_D(\phi)$ as the other cases are virtually identical. By the Sobolev inequality [17]

$$\|\phi\|_4 \leqq \text{const}\left(\int_\Lambda [(\nabla_D\phi)^2(x) + \phi^2(x)]d^2x\right)^{1/2}$$

($\|\cdot\|_4$ is the $L^4(\Lambda)$ norm), it follows that $S_D(\phi)$ is bounded below. The Rellich–Kondrachov Theorem [17] then shows that a weakly convergent sequence in $W_0^{1,2}(\Lambda)$ has a strongly convergent subsequence in $L^4(\Lambda)$, so if $\{\phi_n\}$ is a minimizing sequence for $S_D(\phi)$ we have

$$S_D(\phi^c) \leqq \lim_{n\to\infty} \inf S_D(\phi_n) = \inf S_D(\phi),$$

where ϕ^c is the function to which $\{\phi_n\}$ converges weakly and we have used the weak lower semicontinuity of the $W_0^{1,2}(\Lambda)$ norm. This proves the lemma for $S_D(\phi)$.

The only case which needs further comment is that of $S(\phi)$, since $W^{1,2}(\mathbb{R}^2)$ is relatively compact in $L^4(\Lambda)$ only for Λ bounded. However, as in [16, Prop. 3.4] we may use symmetric rearrangement to obtain a minimizing sequence for $S(\phi)$ of symmetric monotone decreasing functions. By combining Lemma 1 of [19] and Theorem 2.22 of [17], this sequence will be relatively compact in $L^4(\mathbb{R}^2)$ and the proof proceeds as before. □

As mentioned in the Introduction, we will prove Theorem 1.2 by using the lattice approximation, but with the lattice spacing δ depending on k. The explicit dependence we choose is

$$\delta = T/(2[k^\varepsilon] + 1),$$

where $[\cdot]$ is the greatest integer function and $\varepsilon < \frac{1}{2}$. The above choice of δ is consistent with the number of lattice sites in Λ being an integer and with the usual convention that the sides of Λ lie midway between lattice sites. Note that $\delta = O(k^{-\varepsilon})$ as $k \to \infty$.

Proof of Theorem 1.2. Lower bound. We consider separately the cases of k odd and even. For k even, the proof is quite simple and does not even require the lattice approximation. Therefore, let $k = 2j$, with j a positive integer. We will obtain the lower bound by translating to the minimum of $S_X(\phi)$ and then using Jensen's inequality. If we translate $\phi \to \phi + \sqrt{k}\phi^c$, then

$$\begin{aligned}\int V^k(\phi/\sqrt{k})\,d\mu_A^X &= e^{-(k/2)\langle\phi^c, A_X\phi^c\rangle}\int V^k(\phi/\sqrt{k}+\phi^c)e^{-\sqrt{k}\langle\phi, A_X\phi^c\rangle}\,d\mu_A^X \\ &= e^{-(k/2)\langle\phi^c, A_X\phi^c\rangle}\int [V^2(\phi/\sqrt{k}+\phi^c)e^{-j_X^{-1}\cdot\sqrt{k}\langle\phi, A_X\phi^c\rangle}]^j\,d\mu_A^{X^{-1}},\end{aligned} \tag{2.1}$$

where $A_X = -\Delta_X + 1$. Since the term inside the square brackets in (2.1) is non-negative, we may apply Jensen's inequality to obtain

$$\int [V^2(\phi/\sqrt{k}+\phi^c)e^{-j^{-1}\cdot\sqrt{k}\langle\phi, A_X\phi^c\rangle}]^j\,d\mu_A^X \geqq [\int V^2(\phi/\sqrt{k}+\phi^c)e^{-(2/\sqrt{k})\langle\phi, A_X\phi^c\rangle}\,d\mu_A^X]^j. \tag{2.2}$$

Combining (2.1) and (2.2) yields

$$\begin{aligned}(\int V^k(\phi\cdot\sqrt{k})\,d\mu_A^X)^{1/k} &\geqq e^{-(1/2)\langle\phi^c, A_X\phi^c\rangle}[\int V^2(\phi/\sqrt{k}+\phi^c)e^{-(2/\sqrt{k})\langle\phi, A_X\phi^c\rangle}\,d\mu_A^X]^{1/2} \\ &= e^{-(1/2)\langle\phi^c, A_X\phi^c\rangle}V_0(\phi^c)[1+O(1/k)]^{1/2} \\ &= e^{-S_X(\phi^c)}[1+O(1/k)],\end{aligned} \tag{2.3}$$

and this is the lower bound for $k = 2j$.

In order to obtain the lower bound for k odd, it will be necessary to use the lattice approximation with our k dependent choice of δ. We will start with a lemma for which we define the set

$$A = \{\phi \mid |V(\phi/\sqrt{k}) - V_\delta(\phi/\sqrt{k})| < k^{-\beta}\}.$$

In the following, C will be used to denote various k independent constants.

Lemma 2.2. *Let* $\delta = T/(2[k^\varepsilon]+1)$, $0 < \varepsilon < \frac{1}{2}$, *and choose* $0 < \beta < \varepsilon\theta$. *Then*

(i) $\mu_A^X\{\phi \mid \phi \in A^\sim\} \leqq e^{-Ck^{1+\eta}}$,

(ii) $\int \chi_{A^\sim}(\phi)V^k(\phi/\sqrt{k})\,d\mu_A^X \leqq e^{-Ck^{1+\eta}}$

where $\eta = (\varepsilon\theta - \beta)/2$ *and* θ *is defined in (2.5).*

Proof of Lemma 2.2. The proof of part (i) is based on the Nelson semiboundedness argument [14, sect. V.2]. From the definition of the set A, we see that

$$\begin{aligned}\mu_A^X\{\phi \mid \phi \in A^\sim\} &= \mu_A^X\{\phi \mid |V(\phi) - V_\delta(\phi)| \geqq k^{2-\beta}\} \\ &\leqq k^{-(2-\beta)p}\int |V(\phi) - V_\delta(\phi)|^p\,d\mu_A^X \\ &\leqq k^{-(2-\beta)p}(p-1)^{2p}\,\|V(\phi) - V_\delta(\phi)\|_2^p \\ &\leqq C_1^p k^{-(2-\beta)p}p^{2p}k^{-\varepsilon\theta p},\end{aligned} \tag{2.4}$$

where we have used hypercontractivity [14, Theorem I.22] in the third line and the standard estimate [15, sect. 9.5]

$$\|V(\phi) - V_\delta(\phi)\|_2 = O(\delta^\theta), \tag{2.5}$$

with $\theta < 1$ in the last line ($\|\ \|_p$ is the $L^p(d\mu_A^X)$ norm). The choice

$$p = C_1^{-1/2}\cdot e^{-1}\cdot k^{(2-\beta+\varepsilon\theta)/2} = C_1^{-1/2}\cdot e^{-1}\cdot k^{1+\eta}$$

turns the final inequality of (2.4) into the claimed estimate of part (i).

For part (ii), Schwartz inequality and hypercontractivity yield

$$\begin{aligned}\int \chi_{A^\sim}(\phi)V^k(\phi/\sqrt{k})\,d\mu_A^X &\leqq \mu_A^X\{\phi\,|\,\phi\in A^\sim\}^{1/2}\left(\int V^{2k}(\phi/\sqrt{k})\,d\mu_A^X\right)^{1/2}\\ &\leqq \mu_A^X\{\phi\,|\,\phi\in A^\sim\}^{1/2}(2k-1)^{2k}\,\|V(\phi/\sqrt{k})\|_2^k\\ &\leqq \mu_A^X\{\phi\,|\,\phi\in A^\sim\}^{1/2}2^{2k}\,\|V(\phi)\|_2^k,\end{aligned}$$

and the estimate of part (i) completes the proof. □

Part (ii) of the lemma gives us

$$\begin{aligned}\int V^k(\phi/\sqrt{k})\,d\mu_A^X &= \int \chi_A(\phi)V^k(\phi/\sqrt{k})\,d\mu_A^X + 0(e^{-Ck^{1+\eta}})\\ &= e^{-(k/2)\langle\phi^c,A_X\phi^c\rangle}\int \chi_{A^t}(\phi)V^k(\phi/\sqrt{k}+\phi^c)e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X\\ &\quad + 0(e^{-Ck^{1+\eta}})\end{aligned}\tag{2.6}$$

where we have translated, $\phi\to\phi+\sqrt{k}\phi^c$, in the second line of (2.6) and

$$\chi_{A^t}(\phi)=\chi_A(\phi+\sqrt{k}\phi^c).$$

If $\phi\in A^t$, then

$$V(\phi/\sqrt{k}+\phi^c)\geqq V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta},$$

and so

$$V^k(\phi/\sqrt{k}+\phi^c)\geqq [V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta}]^k,$$

since k is odd. Combining this with (2.6) yields

$$\begin{aligned}\int V^k(\phi/\sqrt{k})\,d\mu_A^X \geqq e^{-(k/2)\langle\phi^c,A_X\phi^c\rangle}&\int \chi_{A^t}(\phi)[V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta}]^k\\ &\cdot e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X + O(e^{-Ck^{1+\eta}}).\end{aligned}\tag{2.7}$$

Remark. Since $\phi_\delta(n)=\phi(f_{n,\delta}^X)$, the summand in $V_\delta(\phi/\sqrt{k}+\phi^c)$ is $\phi(f_{n,\delta}^X)/\sqrt{k}+\langle\phi^c,f_{n,\delta}^X\rangle$.

Next, we define the set

$$B=\{\phi\,|\,V_\delta(\phi/\sqrt{k}+\phi^c)\geqq k^{-\beta}\},$$

so that (2.7) may be written as

$$\begin{aligned}\int V^k(\phi/\sqrt{k})\,d\mu_A^X \geqq e^{-(k/2)\langle\phi^c,A_X\phi^c\rangle}\Big\{&\int \chi_{A^t\cap B}(\phi)[V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta}]^k\\ &\cdot e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X - \int\chi_{A^t\cap B^\sim}(\phi)[k^{-\beta}-V_\delta(\phi/\sqrt{k}+\phi^c)]^k\\ &\cdot e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X\Big\},\end{aligned}\tag{2.8}$$

and we are omitting the $O(e^{-Ck^{1+\eta}})$ term from (2.7) for the remainder of the proof. We may now use our previous Jensen's inequality argument to obtain

$$\begin{aligned}&\int \chi_{A^t\cap B}(\phi)[V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta}]^k e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X\\ &\geqq \left[\int \chi_{A^t\cap B}(\phi)(V_\delta(\phi/\sqrt{k}+\phi^c)-k^{-\beta})e^{-k^{-1/2}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X\right]^k\mu_A^X\{\phi\,|\,\phi\in A^t\cap B\}^{1-k},\end{aligned}\tag{2.9}$$

and we will show at the end of the proof of the lower bound that

$$\mu_A^X\{\phi\,|\,\phi\in A^t\cap B\}=1-O(e^{-Ck^{1/4}}).\tag{2.10}$$

By straightforward estimates such as (2.10) and

$$\begin{aligned}\int \chi_{A^t\cap B}(\phi)e^{-k^{-1/2}\langle\phi,A_X\phi^c\rangle}d\mu_A^X &= \int e^{-k^{-1/2}\langle\phi,A_X\phi^c\rangle}d\mu_A^X \\ &\quad - \int \chi_{(A^t\cap B)^\sim}(\phi)e^{-k^{-1/2}\langle\phi,A_X\phi^c\rangle}d\mu_A^X \\ &= e^{-(1/2)k^{-1}\langle\phi,A_X\phi^c\rangle} - O(e^{-Ck^{1/4}}) \\ &= 1 - O(1/k),\end{aligned}$$

we may simplify the last line of (2.9) so that

$$\begin{aligned}&\int \chi_{A^t\cap B}(\phi)[V_\delta(\phi/\sqrt{k}+\phi^c) - k^{-\beta}]^k e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X \\ &\quad \geqq V_{0,\delta}^k(\phi^c)[1-O(k^{-\beta})]^k[1-O(e^{-Ck^{1/4}})]^{1-k}.\end{aligned} \tag{2.11}$$

The second term in (2.8) may be estimated by using

$$V_\delta(\phi/\sqrt{k}+\phi^c) \geqq -O\left(\left(\frac{\ln k}{k}\right)^2\right)$$

to obtain

$$0 < k^{-\beta} - V_\delta(\phi/\sqrt{k}+\phi^c) \leqq k^{-\beta} + O\left(\left(\frac{\ln k}{k}\right)^2\right)$$

for $\phi \in A^t \cap B^\sim$. This yields

$$\begin{aligned}&\int \chi_{A^t\cap B^\sim}(\phi)[k^{-\beta} - V_\delta(\phi\sqrt{k}+\phi^c)]^k e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X \\ &\leqq \left[k^{-\beta} + O\left(\left(\frac{\ln k}{k}\right)^2\right)\right]^k \int \chi_{A^t\cap B^\sim}(\phi)e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X \\ &= O(k^{-\beta k}),\end{aligned} \tag{2.12}$$

since

$$\int \chi_{A^t\cap B^\sim}(\phi)e^{-\sqrt{k}\langle\phi,A_X\phi^c\rangle}\,d\mu_A^X \leqq e^{(k/2)\langle\phi^c,A_X\phi^c\rangle}.$$

Combining (2.8), (2.11), and (2.12) gives us

$$\int V^k(\phi/\sqrt{k})\,d\mu_A^X \geqq e^{-k/2\langle\phi^c,A_X\phi^c\rangle}V_{0,\delta}^k(\phi^c)[1-O(k^{-\beta})]^k\{1-O(e^{-Ck^{1/4}})-O(k^{-\beta k})\},$$

and so

$$\left(\int V^k(\phi/\sqrt{k})\,d\mu_A^X\right)^{1/k} \geqq e^{-1/2\langle\phi^c,A_X\phi^c\rangle}V_{0,\delta}(\phi^c)[1-0(k^{-\beta})]. \tag{2.13}$$

By the remark following (2.7),

$$V_{0,\delta}(\phi^c) = \delta^2 \sum_{n\in\Lambda_\delta} (\langle\phi^c, f_{n,\delta}^X\rangle)^4,$$

and so (2.13) will yield the correct lower bound as it easily follows from [9, sect. IX.1] that $V_{0,\delta}(\phi^c) \to V_0(\phi^c)$ as $k\to\infty$ (or $\delta\to 0$).

It remains to prove (2.10). This will follow from Lemma 2.2(i) and from

$$\mu_A^X\{\phi\,|\,\phi\in B^\sim\} = O(e^{-Ck^{1/4}}). \tag{2.14}$$

Since $V_{0,\delta}(\phi^c)\to V_0(\phi^c)$ as $k\to\infty$, assume that k is large enough so that $k^{-\beta} < V_{0,\delta}(\phi^c)$.

This yields

$$\begin{aligned}\mu_\Lambda^X\{\phi\,|\,\phi\in B^{\sim}\} &= \mu_\Lambda^X\{\phi\,|\,V_\delta(\phi/\sqrt{k}+\phi^c)-V_{0,\delta}(\phi^c)<k^{-\beta}-V_{0,\delta}(\phi^c)\}\\ &= \mu_\Lambda^X\{\phi\,|\,|V_{0,\delta}(\phi^c)-V_\delta(\phi/\sqrt{k}+\phi^c)|>V_{0,\delta}(\phi^c)-k^{-\beta}\}\\ &\leqq [V_{0,\delta}(\phi^c)-k^{-\beta}]^{-p}\int|V_{0,\delta}(\phi^c)-V_\delta(\phi/\sqrt{k}+\phi^c)|^p\,d\mu_\Lambda^X\\ &\leqq e^{-Ck^{1/4}}\end{aligned}$$

by the same argument as in the proof of Lemma 2.2(i), but now using

$$\int|V_{0,\delta}(\phi^c)-V_\delta(\phi/\sqrt{k}+\phi^c)|^2\,d\mu_\Lambda^X\leqq Ck^{-1}$$

instead of (2.5).

Upper Bound. We will again use our k dependent choice of δ for the upper bound. First, we combine

$$\int V^k(\phi/\sqrt{k})\,d\mu_\Lambda^X\leqq\int|V(\phi/\sqrt{k})|^k\,d\mu_\Lambda^X=\|V(\phi/\sqrt{k})\|_k^k$$

with the estimate

$$\begin{aligned}|\,\|V(\phi/\sqrt{k})\|_k-\|V_\delta(\phi/\sqrt{k})\|_k| &\leqq \|V(\phi/\sqrt{k})-V_\delta(\phi/\sqrt{k})\|_k\\ &\leqq (k-1)^2\|V(\phi/\sqrt{k})-V_\delta(\phi/\sqrt{k})\|_2\\ &\leqq \|V(\phi)-V_\delta(\phi)\|_2\\ &= O(\delta^\theta)\\ &= O(k^{-\varepsilon\theta})\end{aligned}\tag{2.15}$$

to obtain

$$\left(\int V^k(\phi/\sqrt{k})\,d\mu_\Lambda^X\right)^{1/k}\leqq\left(\int|V_\delta(\phi/\sqrt{k})|^k\,d\mu_\Lambda^X\right)^{1/k}+O(k^{-\varepsilon\theta}).\tag{2.16}$$

We have used hypercontractivity in the second line of (2.15) and (2.5) in the fourth line. The same argument may again be applied to find that

$$\begin{aligned}|\,\|V_\delta(\phi/\sqrt{k})\|_k-\|V_{0,\delta}(\phi/\sqrt{k})\|_k| &\leqq \|V_\delta(\phi/\sqrt{k})-V_{0,\delta}(\phi/\sqrt{k})\|_k\\ &\leqq (k-1)\|V_\delta(\phi/\sqrt{k})-V_{0,\delta}(\phi/\sqrt{k})\|_2\\ &= \frac{k-1}{k^2}\|V_\delta(\phi)-V_{0,\delta}(\phi)\|_2\\ &= O\left(\frac{(\ln k)^2}{k}\right),\end{aligned}\tag{2.17}$$

where the fact that $V_\delta(\phi)-V_{0,\delta}(\phi)$ is only quadratic in ϕ has been used in the hypercontractive estimate. From (2.17), we obtain

$$\left(\int|V_\delta(\phi/\sqrt{k})|^k\,d\mu_\Lambda^X\right)^{1/k}\leqq\left(\int V_{0,\delta}^k(\phi/\sqrt{k})\,d\mu_\Lambda^X\right)^{1/k}+O\left(\frac{(\ln k)^2}{k}\right).\tag{2.18}$$

Now by using $q_n=\phi(f_{n,\delta}^X)$ and the definition of $d\mu_{\Lambda,\delta}^X$, we may write

$$\begin{aligned}\int V_{0,\delta}^k(\phi/\sqrt{k})\,d\mu_\Lambda^X &= \int V_{0,\delta}^k(q/\sqrt{k})\,d\mu_{\Lambda,\delta}^X\\ &= \int e^{k\ln V_{0,\delta}(q/\sqrt{k})}\,d\mu_{\Lambda,\delta}^X\\ &= \int e^{-kS_X^\delta(q/\sqrt{k})}\,dqN^X,\end{aligned}\tag{2.19}$$

and by scaling $q \to \sqrt{k}q$, we obtain

$$\int e^{-kS_X^\delta(q/\sqrt{k})} dqN^X = k^{n/2} \int e^{-kS_X^\delta(q)} dqN^X, \tag{2.20}$$

where $n = (2[k^\varepsilon] + 1)^2 = O(k^{2\varepsilon})$. If we assume $k \geqq 1$, then

$$k[S_X^\delta(q) - S_X^\delta(q^c)] \geqq S_X^\delta(q) - S_X^\delta(q^c),$$

and this yields

$$\begin{aligned} \int e^{-kS_X^\delta(q)} dqN^X &= e^{-kS_X^\delta(q^c)} \int e^{-k[S_X^\delta(q) - S_X^\delta(q^c)]} dqN^X \\ &\leqq e^{-kS_X^\delta(q^c)} e^{S_X^\delta(q^c)} \int e^{-S_X^\delta(q)} dqN^X. \end{aligned} \tag{2.21}$$

The integral in the last line of (2.21) may be estimated as

$$\begin{aligned} \int e^{-S_X^\delta(q)} dq\, N^X &= \int V_{0,\delta}(q)\, d\mu_{\Lambda,\delta}^X \\ &= O(\ln^2(k)), \end{aligned}$$

where the last line follows from $(G_X^\delta)_{nn} = O(\ln k)$ by our choice of δ.

Combining together (2.19)–(2.21) gives us the upper bound

$$\int V_{0,\delta}^k(\phi/\sqrt{k})\, d\mu_\Lambda^X \leqq Ce^{-kS_X^\delta(q^c)[1 - O(k^{2\varepsilon - 1}\ln k)]}, \tag{2.22}$$

in which we have used $k^{n/2} = \exp[O(k^{2\varepsilon} \cdot \ln k)]$ and $\varepsilon < \frac{1}{2}$. Therefore we obtain from (2.16), (2.18), and (2.22)

$$\lim_{k\to\infty} \left(\int V^k(\phi/\sqrt{k})\, d\mu_\Lambda^X\right)^{1/k} \leqq \exp\left[-\lim_{k\to\infty} S_X^\delta(q^c)\right]. \tag{2.23}$$

We will show at the end of this section that $S_X^\delta(q^c) \to S_X(\phi^c)$ as $k \to \infty$ (or $\delta \to 0$). This will yield the desired upper bound

$$\lim_{k\to\infty} \left(\int V^k(\phi/\sqrt{k})\, d\mu_\Lambda^X\right)^{1/k} \leqq e^{-S_X(\phi^c)}, \tag{2.24}$$

which finishes the proof of Theorem 1.2. □

The next lemma supplies the step from (2.23) to the upper bound (2.24).

Lemma 2.3. *As $\delta \to 0$, $S_X^\delta(q^c) \to S_X(\phi^c)$ for $X = p, D$.*

Proof. If we let $\phi_\delta^c(n) \equiv \phi^c(n\delta)$ for $n \in \Lambda_\delta$, then $S_X^\delta(\phi_\delta^c) \geqq S_X^\delta(q^c)$. The minimizing function ϕ^c is smooth [18–20], so $S_X^\delta(\phi_\delta^c) \to S_X(\phi^c)$ as $\delta \to 0$, since this is just saying that the Reimann sums converge. Therefore,

$$\lim_{\delta\to 0} S_X^\delta(q^c) \leqq S_X(\phi^c). \tag{2.25}$$

Now q^c is a critical point of $S_X^\delta(q)$, so it will obey the equation

$$(-\Delta_X^\delta + 1)q^c = 4(q^c)^3 / \sum_{n\in\Lambda\delta} \delta^2 (q_n^c)^4. \tag{2.26}$$

Taking the inner product of (2.26) with q^c yields

$$\langle q^c, (-\Delta_X^\delta + 1)q^c \rangle = 4. \tag{2.27}$$

It is useful to think of q^c as a piecewise constant continuum function $q^c(x)$ defined by $q^c(x) = q_n^c$ when x is in the square of side δ centered at the lattice point

δn. It follows from this definition that $\delta^2 \sum_n |q_n^c|^p = \int_\Lambda d^2x |q^c(x)|^p$, and so the continuum identification along with (2.27) shows that there exists $\phi, \Psi \in L^2(\Lambda)$ for which a subsequence $q^c \to \phi$ and $(-\Delta_X^\delta + 1)^{1/2} q^c \to \Psi$ both weakly in $L^2(\Lambda)$ as $\delta \to 0$. Next, we consider

$$\begin{aligned}
\|q^c - G_X^{1/2}\Psi\|_2 &= \|(G_X^\delta)^{1/2}(-\Delta_X^\delta + 1)^{1/2} q^c - G_X^{1/2}\Psi\|_2 \\
&\leqq \|(G_X^\delta)^{1/2}((-\Delta_X^\delta + 1)^{1/2} q^c - \Psi)\|_2 + \|((G_X^\delta)^{1/2} - G_X^{1/2})\Psi\|_2 \\
&\leqq \|(G_X^\delta)^{1/2} - G_X^{1/2}\| \, \|(-\Delta_X^\delta + 1)^{1/2} q^c - \Psi\|_2 \\
&\quad + \|G_X^{1/2}((-\Delta_X^\delta + 1)^{1/2} q^c - \Psi)\|_2 + \|((G_X^\delta)^{1/2} - G_X^{1/2})\Psi\|_2 .
\end{aligned}$$

The norm convergence of $(G_X^\delta)^{1/2}$ (see for example [22, Lemma 3.9 and Appendix B]) and the compactness of $G_X^{1/2}$ imply that for a subsequence $q^c \to G_X^{1/2}\Psi$ strongly in $L^2(\Lambda)$. This shows that $\Psi = (-\Delta_X + 1)^{1/2}\phi$ and so $q^c \to \phi$ in L^2 with $\|(-\Delta_X + 1)^{1/2}\phi\|_2$ finite (i.e. ϕ belongs to the appropriate Sobolev space, as defined in Remark 1 following Theorem 1.2). We also have L^4 convergence of $q^c \to \phi$ since we may show that $\|q^c\|_\infty$ is uniformly bounded as $\delta \to 0$. This is a consequence of (2.26), as

$$\begin{aligned}
\|q^c\|_\infty &= 4 \sup_{x \in \Lambda} \left| \int_\Lambda G_X^\delta(x,y)(q^c)^3(y)\, d^2y \right| \Big/ \|q^c\|_4^4 \\
&\leqq 4 \sup_{x \in \Lambda} \left| \int_\Lambda |G_X^\delta(x,y)|^4 \, d^2y \right|^{1/4} \Big/ \|q^c\|_4 ,
\end{aligned} \tag{2.28}$$

where we have used $\|G_X^\delta(x,\cdot)(q^c)^3\|_1 \leqq \|G_X^\delta(x,\cdot)\|_4 \|(q^c)^3\|_{4/3}$. The G_X^δ term in the last line of (2.28) is bounded [15, Proposition 9.5.7], and $\|q^c\|_4 \nrightarrow 0$, since $S_X^\delta(q^c)$ is bounded. The same argument also shows that $\|\phi\|_\infty$ is finite. Therefore, the L^4 convergence and (2.25) easily prove that for a subsequence

$$\lim_{\delta \to 0} S_X^\delta(q^c) = S_X(\phi) = S_X(\phi^c). \tag{2.29}$$

However, given an arbitrary subsequence of $\{S_X^\delta(q^c)\}$, the above argument and (2.25) will allow us to extract a sub-subsequence for which (2.29) holds and so the original sequence is Cauchy. □

3. Leading Order Asymptotics for $a_k^X(\Lambda)$

The purpose of this section is to prove Corollary 1.3 by showing that the large order behavior of $a_k^X(\Lambda)$ follows from that of $b_k^X(\Lambda)$.

Proof of Corollary 1.3. The proof is very similar to that of [16, Lemma 2.2], to which we will refer for details. We will drop the X and Λ notation from $a_k^X(\Lambda)$ and $b_k^X(\Lambda)$, since this dependence will play no part in the proof. To begin with, we may use (1.6) and the Taylor series for $\ln(1+x)$ to obtain

$$a_k = \frac{1}{|\Lambda|} \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m} B(k,m), \tag{3.1}$$

in which

$$B(k,m)=\sum_{\substack{k_1+\ldots+k_m=k\\ k_i\geq 1}}\prod_{i=1}^{m} b_{k_i}.$$

Our goal is to show that $|a_k|^{1/k}\sim|b_k|^{1/k}$ as $k\to\infty$. If we left

$$b_k^*=\frac{(-1)^k}{k!}\int|V(\phi)|^k d\mu,$$

then a_k may be written as

$$a_k=\frac{1}{|\Lambda|}b_k\left[1+(b_k^*/b_k)\sum_{m=2}^{k}\frac{(-1)^{m-1}}{m}(b_k^*)^{-1}B(k,m)\right]. \tag{3.2}$$

The proof of [16, Lemma 2.2] then yields

$$\sum_{m=2}^{k}\frac{(-1)^{m-1}}{m}(b_k^*)^{-1}B(k,m)=-0(1/k). \tag{3.3}$$

We have multiplied and divided by b_k^* in (3.2) as it is important for the proof of (3.3) that the functional integral in the denominator of $B(k,m)/b_k^*$ is an $L^k(d\mu)$ norm (see (2.8)–(2.10) of [16]). We also note that as in [16, Lemma 2.2] we may show that $|b_k^*|$ is log convex in k, and since $|b_k|\leqq|b_k^*|$ this will suffice for adopting the proof of [16, Lemma 2.2] to showing (3.3). Briefly, the log convexity of $|b_k^*|$ shows that

$$|b_j^*b_{k-j}^*|\leqq|b_2^*b_{k-2}^*|$$

for $2\leqq j\leqq k-2$, and so

$$|B(k,2)|\leqq\sum_{j=1}^{k-1}|b_j^*b_{k-j}^*|\leqq 2|b_1^*b_{k-1}^*|+(k-3)|b_2^*b_{k-2}^*|\leqq C|b_{k-1}^*|.$$

This bound and

$$B(k,m)=\sum_{n=1}^{k-m+1}b_nB(k-n,m-1)$$

then yields an inductive proof that

$$|B(k,m)|\leqq C^{m-1}|b_{k-m+1}^*|. \tag{3.4}$$

The remainder of the proof involves showing that

$$\frac{|b_{k-m+1}^*|}{|b_k^*|}\leqq(C/k)^{m-1},$$

which together with (3.4) yields (3.3).

In order to complete the proof of Corollary 1.3, we may use (3.2) and (3.3) to obtain

$$|a_k|^{1/k}=\left|\frac{b_k}{|\Lambda|}\right|^{1/k}[1-(b_k^*/b_k)O(1/k)]^{1/k}. \tag{3.5}$$

Since b_k^*/b_k is positive, dropping the $O(1/k)$ term gives us

$$|a_k|^{1/k} \leqq \left|\frac{b_k}{|\Lambda|}\right|^{1/k} \tag{3.6}$$

Applying the inequality $(1-x)^{1/k} \geqq (1-x^{1/k})^{1/k}$ for $0 \leqq x \leqq 1$ (to (3.5) yields the lower bound

$$|a_k|^{1/k} \geqq \left|\frac{b_k}{|\Lambda|}\right|^{1/k} [1-(b_k^*/b_k)^{1/k}[0(1/k)]^{1/k}]^{1/k}, \tag{3.7}$$

where we note that (3.2) and (3.3) show that $0 \leqq (b_k^*/b_k)O(1/k) \leqq 1$. The upper and lower bounds of Sect. 2 then yield

$$\lim_{k\to\infty} (b_k^*/b_k)^{1/k} = 1,$$

and so

$$\lim_{k\to\infty} [1-(b_k^*/b_k)^{1/k}[O(1/k)]^{1/k}]^{1/k} = 1. \tag{3.8}$$

While it should certainly be true that $(b_k^*/b_k) \sim 1$ as $k \to \infty$, we find it difficult to show this with the bounds of Sect. 2 for k odd. This is why we have used (3.7).

It is worth nothing that the stronger upper bound

$$(-1)^k a_k \leqq \frac{(-1)^k}{|\Lambda|} b_k$$

follows from the Feynman graph representation for a_k and b_k (see for example [15, sect. 8.1–8.3]), in which $(-1)^k b_k/|\Lambda|$ is the sum of all graphs with k vertices, 4 lines attached to each vertex, and no line beginning and ending at the same vertex. The graphical representation for $(-1)^k a_k$ is identical, but with the additional restriction that all graphs are connected and this yields the inequality.

We may now combine (3.6), (3.7), and (3.8) to obtain $|a_k|^{1/k} \sim |b_k|^{1/k}$ as $k \to \infty$. This may be used with Theorem 1.2 and the remark following the statement of Corollary 1.3 to prove (1.9). □

The final step in the proof of Theorem 1.1 is to apply the following lemma to (1.11).

Lemma 3.1. *Let $X = p, D$ Then* $\lim_{|\Lambda|\to\infty} (\inf S_X(\phi)) = \inf S(\phi)$.

Proof. We know from Lemma 2.1 that there are functions ϕ^c and η^c for which $S_X(\phi^c) = \inf S_X(\phi)$ and $S(\eta^c) = \inf S(\phi)$ (we are suppressing the dependence of ϕ^c on the boundary conditions). The proof will be given for $X = D$, since the periodic case is similar. If we define

$$\bar{\phi}^c(x) = \begin{cases} \phi^c(x), & x \in \Lambda \\ 0, & x \notin \Lambda, \end{cases}$$

then the smoothness of ϕ^c [18–20] implies that $\nabla_D \phi^c(x) = \nabla \phi^c(x)$, $x \in \Lambda$, and so $\bar{\phi}^c \in W^{1,2}(\mathbb{R}^2)$ with $S(\bar{\phi}^c) = S_D(\phi^c)$. Since $S(\bar{\phi}^c) \geqq S(\eta^c)$, this yields

$$\lim_{|\Lambda|\to\infty} S_D(\phi^c) \geqq S(\eta^c). \tag{3.9}$$

Now it follows from [19] that we may take η^c to be a positive, spherically symmetric, monotone decreasing function which decays exponentially at infinity. Next, define $\bar{\eta}^c = \rho_\Lambda \eta^c$, where ρ_Λ is a C_0^∞ function with support contained in Λ which is identically one in the smaller box $[-T/2+1, T/2-1]^2$, and $\rho_\Lambda(x) \leqq 1$ for all x. We have that $\bar{\eta}^c \in W_0^{1,2}(\Lambda)$ and that $S(\bar{\eta}^c) = S_D(\bar{\eta}^c) \geqq S_D(\phi^c)$. This implies that

$$\lim_{|\Lambda| \to \infty} S_D(\phi^c) \leqq S(\eta^c), \tag{3.10}$$

since $S(\bar{\eta}^c) \to S(\eta^c)$ as $|\Lambda| \to \infty$ follows easily from the uniform fall off of η^c (see Lemma 1 of [19]). This proves the lemma for $X = D$ and it is not hard to show that $\lim_{|\Lambda| \to \infty} (\inf S_p(\phi) - \inf S_D(\phi)) = 0$. □

Acknowledgement. I wish to thank Tom Spencer for a number of helpful discussions.

References

1. Lipatov, L. N.: Calculation of the Gell–Mann–Low function in scalar theory with strong non-linearity. Sov. Phys. JETP **44**, 1055–1062 (1976); Divergence of the perturbation-theory series and pseudoparticles. JETP Lett. **25**, 104–107 (1977); Divergence of the perturbation-theory series and the quasi-classical theory. Sov. Phys. JETP **45**, 216–223 (1977)
2. Brezin, E., LeGuillou, J. C., Zinn-Justin, J.: Perturbation theory at large order. I. The ϕ^{2N} interaction. Phys. Rev. **D15**, 1544–1557 (1977)
3. Brézin, E., LeGuillou, J. C., Zinn-Justin, J.: Perturbation theory at large order. II. Role of the vacuum instability. Phys. Rev. **D15**, 1558–1564 (1977)
4. Brézin, E., Parisi, G., Zinn-Justin, J.: Perturbation theory at large orders for a potential with degenerate minima. Phys. Rev. **D16**, 408–412 (1977)
5. Simon, B.: Large orders and summability of eigenvalue perturbation theory: a mathematical overview. Int. J. Q. Chem. **21**, 3–25 (1982)
6. LeGuillou, J. C., Zinn-Justin, J.: Critical exponents for the n-vector model in three dimensions from field theory. Phys. Rev. Lett. **39**, 95–98 (1977)
7. Brézin, E., Parisi, G.: Critical exponents and large-order behaviour of perturbation theory. J. Stat. Phys. **19**, 269–292 (1978)
8. LeGuillou, J. C., Zinn-Justin, J.: Critical exponents from field theory. Phys. Rev. **B21**, 3976–3998 (1980)
9. Guerra, F., Rosen, L., Simon, B.: Boundary conditions in the $P(\phi)_2$ Euclidean field theory. Ann. Inst. H. Poincaré Sect. **A25**, 231–334 (1976)
10. Jaffe, A.: Divergence of perturbation theory for Boson. Commun. Math. Phys. **1**, 127–149 (1965)
11. Eckmann, J.-P., Magnen, J., Sénéor, R.: Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories. Commun. Math. Phys. **39**, 251–271 (1975)
12. Spencer, T.: The Lipatov argument. Commun. Math. Phys. **74**, 273–280 (1980)
13. Herbst, I. W., Simon, B.: Some remarkable examples in eigenvalue perturbation theory. Phys. Lett. **B78**, 304–306 (1978)
14. Simon, B.: The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton: Princeton University Press 1974
15. Glimm, J., Jaffe, A.: Quantum physics. Berlin, Heidelberg, New York: Springer 1981
16. Breen, S.: Large order perturbation theory for the anharmonic oscillator. Thesis (Rutgers University, 1982), and preprint
17. Adams, R.: Sobolev spaces. London, New York: Academic Press 1975
18. Ambrosetti, A., Rabinowitz, P. H.: Dual Variational Methods in Critical point Theory and Applications. J. Funct. Anal. **14**, 349–381 (1973)

19. Strauss, W. A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)
20. Morrey, C. Jr.: Multiple integrals in the calculus of variations. New York: Springer 1966
21. Simon, B.: Trace ideals and their applications. London, New York: Cambridge University Press 1979
22. Brydges, D., Fröhlich, J., Seiler, E.: Construction of quantized gauge fields. II. Convergence of the lattice approximation. Commun. Math. Phys. **71**, 159–205 (1980)

Communicated by B. Simon
Received May 4, 1982, in revised form June 7, 1983