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Maxwell Equations in Conformal Invariant Electrodynamics

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Abstract. We consider a conformal invariant formulation of quantum electrodynamics. Conformal invariance is achieved with a specific mathematical construction based on the indecomposable representations of the conformal group associated with the electromagnetic potential and current. As a corollary of this construction modified expressions for the 3-point Green functions are obtained which both contain transverse parts. They make it possible to formulate a conformal invariant skeleton perturbation theory.

It is also shown that the Euclidean Maxwell equations in conformal electrodynamics are manifestations of its kinematical structure: in the case of the 3-point Green functions these equations follow (up to constants) from the conformal invariance while in the case of higher Green functions they are equivalent to the equality of the kernels of the partial wave expansions. This is the manifestation of the mathematical fact of a (partial) equivalence of the representations associated with the potential, current and the field tensor.

1. Introduction

The development of conformal invariant quantum field theory in the last decade (see, e.g. [1, 2] and references cited therein) has demonstrated that the traditional approach to the conformal invariant formulation of massless quantum electrodynamics leads to a purely longitudinal version of the theory. To show this, consider the transformation law for the electromagnetic potential A_μ under conformal transformations. We will use the Euclidean formulation [3] of quantum electrodynamics. Conformal transformations of Euclidean coordinates are obtained from the following transformations (and translations)

$$x_\mu \to \chi x_\mu, \qquad x_\mu \to R x_\mu = -x_\mu/x^2, \tag{1.1}$$

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where λ is any positive number. For the potential A_μ we have the transformation law

$$\lambda : A_\mu(x) \to V_\lambda A_\mu(x) = \lambda A_\mu(\lambda x), \qquad R : A_\mu(x) \to V_R A_\mu(x) = \frac{1}{x^2} g_{\mu\nu}(x) A_\nu(Rx), \quad (1.2)$$

where $g_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2$. From this it follows that the conformal invariant propagator

$$D_{\mu\nu}(x_{12}) = \langle 0 | T A_\mu(x_1) A_\nu(x_2) | 0 \rangle$$

must satisfy the functional equation

$$D_{\mu\nu}(x_1 - x_2) = \frac{1}{x_1^2 x_2^2} g_{\mu\mu'}(x_1) g_{\nu\nu'}(x_2) D_{\mu'\nu'}(Rx_1 - Rx_2). \quad (1.3)$$

The only solution to this functional equation is the longitudinal function

$$D_{\mu\nu}^{\text{long}}(x_{12}) \sim \partial_\mu \partial_\nu \log x_{12}^2 . \quad (1.4)$$

We obtain, therefore, a trivial variant of electrodynamics containing purely longitudinal photons. Such a theory has been considered in [4] (see also [5]) for purposes of method. Various attempts have been tried to overcome this difficulty, (see [2] and references cited therein). However, the final solution of the problem had not been obtained until now.

The reason for the difficulty mentioned above lies, as it will be shown, in the unusual structure of the conformal group representations corresponding to the field A_μ. The fact is that in the usual approach an irreducible unitary representation of the conformal group is attributed to each field and the propagator of such a field is identified as the kernel of an invariant scalar product in the space of the representation. It turns out to be impossible, however, to act in such a way if one adopts the transformation law (1.2) for A_μ. In this case we obtain an indecomposable representation and the corresponding invariant kernel turns out to be degenerate [8, 9]. Indeed, in the space of functions with the transformation law (1.2) there is an invariant subspace consisting of longitudinal functions; if $A_\mu(x) = \partial_\mu \varphi(x)$, then $A'_\mu(x) = \frac{1}{x^2} g_{\mu\nu}(x) A_\nu(Rx) = \partial_\mu \varphi(Rx)$. However, the complement to this subspace is not invariant, and this fact indicates the indecomposability of the representation. To work with such representations demands application of special mathematical apparatus which is reviewed in Sect. II (see also [7, Chap. III] and [10]).

In the present paper we formulate a new approach to the solution of the problem mentioned above in which fully reducible representations obtained from the indecomposable ones (1.2) are used instead of the latter. As it is shown in Sect. II this results in a new transformation law for the potential and the current. In particular, we obtain

$$A_\mu(x) \to V'_R A_\mu(x) = (1 - P^\ell) V_R (1 - P^\ell) A_\mu(x) + V_R P^\ell A_\mu(x), \quad (1.5)$$

where $P^{\ell}=\partial_\mu\partial_\nu/\Box$, instead of (1.2), see also [6, 7]. Analogously the operators V'_g in the new representation are defined for each conformal transformation g:

$$V'_g=(1-P^{\ell})V_g(1-P^{\ell})+V_gP^{\ell}, \tag{1.6}$$

where V_g is the corresponding operator of the indecomposable representation. [It is easy to verify that the operators V'_g give rise to a representation: $V'_{g_1}V'_{g_2}=V'_{g_1g_2}$. To do this it is sufficient to use the relation $(1-P^{\ell})V_gP^{\ell}A_\mu=0$, which follows from the mentioned invariance of the subspace of longitudinal functions under the transformation V_g.] The new representation V'_g is reducible and an irreducible representation acts in each of the invariant subspaces $P^{\ell}A_\mu(x)$ and $(1-P^{\ell})A_\mu(x)$, see Sect. II for more details.

The modified conformal invariant photon propagator corresponding to the new representation (1.5), (1.6) looks, as it can be easy verified, as follows (see also [6, 7])

$$D_{\mu\nu}(x_{12})=\frac{1}{4\pi^2}\left[\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)\frac{1}{x^2}-\eta\frac{\partial_\mu\partial_\nu}{\Box}\frac{1}{x^2}\right], \tag{1.7}$$

where η is an arbitrary constant and the normalization factor $1/4\pi^2$ is chosen for the sake of convenience.[1] Thus, use of the representation (1.6) solves the problem of conformal invariant electrodynamics. The new transformation law (1.5) evidently results in modified expressions for conformal invariant 3-point Green functions and vertices. These expressions can be found in Sect. III.

The main result of Sect. III is the analysis of the Maxwell equations for the Euclidean Green functions. We will show that the Maxwell equations for the 3-point Green functions are consequences (as well as the equations for Euclidean classical fields, see [7]) of the conformal invariance. Then it will be shown that in the case of higher Green functions the Maxwell equations are equivalent to the condition of the equality of the kernels in the partial wave expansions. It means that the fields A_μ, j_μ, $F_{\mu\nu}$ appear in the conformal invariant theory as an unified object and the Maxwell equations serve as a manifestation of equivalence of the corresponding representations of the Euclidean conformal group.[2]

In Sect. IV skeleton perturbation theory will be formulated and a proof will be given of its conformal invariance (see also [6, 7]).

II. The Structure of Conformal Group Representations

First we introduce a space H_A of test functions for the vector potential. Let H_A be the space of the representation T_A of the conformal group which is created from a vacuum by a quantum field A_μ. It consists of the vector functions with the transformation law

$$R: f_\mu(x)\to\frac{1}{(x^2)^3}g_{\mu\nu}(x)f_\nu(Rx), \tag{2.1}$$

1 A nontrivial propagator with somewhat different longitudinal part was exhibited in [11]

2 An interpretation of the Maxwell equations in Minkowski space as an intertwining relation was proposed by Jakobsen and Vergne [12]. See also Chap. VII of [2]

and the function $D^{\text{long}}_{\mu\nu}$ (1.4) defines an invariant hermitian form in this space. In the space H_A (do not confuse with the space of classical fields of which it was said in the introduction) there exists an invariant subspace consisting of the transverse functions $f^{\text{tr}}_\mu(x)$, $\partial_\mu f^{\text{tr}}_\mu(x)=0$. An invariant hermitian form having as a kernel the conformal invariant propagator (1.4) vanishes on this subspace. According to the known mathematical theorems, in this situation one cannot introduce an invariant scalar product and consequently, the Green functions $D^{\text{long}}_{\mu\nu}$, having the property of being non-degenerate on the full space H_A. This is the cause of the difficulty mentioned above.

A way out is as follows. Consider the quotient space

$$H^{\ell}_A = H_A/H^{\text{tr}}_A. \tag{2.2}$$

The equivalence classes consisting of all functions in the space H_A differing by a transverse function are the elements of this space. The subspace H^{ℓ}_A is isomorphic to the space of longitudinal functions. To the space H_A we associate the direct sum of spaces

$$H_A \to H = H^{\text{tr}}_A \oplus H^{\ell}_A. \tag{2.3}$$

It is known that an irreducible unitary representation of the conformal group acts in each of the spaces H^{tr}_A and H^{ℓ}_A. Denote these representations as T^{tr}_A and T^{ℓ}_A. Evidently, the unitary representation $T^{\text{tr}}_A \oplus T^{\ell}_A$ acts in the space H; thus there exists an invariant scalar product in this space. The kernel of this scalar product is an invariant function to be identified with the photon propagator (1.7).

To elaborate on this program it is suitable to choose in each equivalence class from the space H^{ℓ}_A the longitudinal representative. Then the space H can be realized as the space of vector functions

$$f_\mu(x) = f^{\text{tr}}_\mu(x) + f^{\ell}_\mu(x), \tag{2.4}$$

where $f^{\text{tr}}_\mu(x) \in H^{\text{tr}}_A$, f^{ℓ}_μ is the longitudinal representative of an equivalence class from H^{ℓ}_A. With such a realization each function $f_\mu \in H$ has an unusual transformation law. The transverse part f^{tr}_μ transforms according to (2.1) while the longitudinal one, being the representative of an equivalence class, has the following transformation law accomplished in two steps

$$f^{\ell}_\mu(x) \to f'^{\ell}_\mu(x) = g_{\mu\nu}(x)\frac{1}{(x^2)^3} f^{\ell}_\nu(Rx), \tag{2.5}$$

and then

$$f'^{\ell}_\mu(x) \to (\partial_\mu\partial_\nu/\Box) f'^{\ell}_\nu. \tag{2.6}$$

The last step is connected with the fact that the property of longitudinally has been lost after the conformal transformation, see e.g. (2.5), therefore it is necessary to go again to the longitudinal representative of a given equivalence class.

Consider an invariant scalar product on H. The transverse function $D^{\text{tr}}_{\mu\nu}(x)$ $\sim (\delta_{\mu\nu} - \partial_\mu\partial_\nu/\Box)\frac{1}{x^2}$ can be chosen as the kernel of an invariant scalar product on

$H_A^{\rm tr}$. One can directly verify that the scalar product

$$(f,\varphi)=\int dx_1 dx_2 f_\mu^{\rm tr}(x_1)D_{\mu\nu}^{\rm tr}(x_{12})\varphi_\nu^{\rm tr}(x_2)$$

(where $f_\mu^{\rm tr}$ and $\varphi_\mu^{\rm tr}$ are transverse functions from $H_A^{\rm tr}$) is conformal invariant, see [8] and Sect. IV for more details. The kernel $D_{\mu\nu}^{\rm long}$ of the scalar product on H_A^ℓ coinciding with (1.4) is the longitudinal function. A superposition $D_{\mu\nu}^{\rm tr}+D_{\mu\nu}^{\rm long}$ can serve as the kernel of an invariant scalar product on the whole space H if one realizes the latter as it has been described above. As it has been pointed out, this kernel can be identified with the photon Green function. Thus, in the described realization of the space H we obtain a conformal invariant formulation of quantum electrodynamics in which the photon propagator is as in (1.7).

Let us now consider the properties of the representation corresponding to the current j_μ. This representation is indecomposable. Denote as H_j the space of the representation T_j. It consists of the vector functions φ_μ with the following transformation law

$$\lambda:\varphi_\mu(x)\to\lambda\varphi_\mu(\lambda x),\quad R:\varphi_\mu(x)\to\frac{1}{x^2}g_{\mu\nu}(x)\varphi_\nu(Rx). \tag{2.7}$$

It can be shown that H_j has an invariant subspace H_j^ℓ consisting of the longitudinal functions. Let us introduce the quotient space $H_j^{\rm tr}=H_j/H_j^\ell$ consisting of the equivalence classes. This space is isomorphic to the space of the transverse functions. The irreducible unitary representations $T_j^{\rm tr}$ and T_j^ℓ act [8, 9] in $H_j^{\rm tr}$ and H_j^ℓ. Define analogously to (2.3) the space

$$\tilde{H}=H_j^{\rm tr}\oplus H_j^\ell. \tag{2.8}$$

Choosing in each equivalence class from $H_j^{\rm tr}$ the transverse representative we obtain a suitable realization of the space $H_j^{\rm tr}$ to be used later. With such a realization the space of the vector functions

$$\varphi_\mu(x)=\varphi_\mu^{\rm tr}(x)+\varphi_\mu^\ell(x), \tag{2.9}$$

where $\varphi_\mu^\ell\in H_j^\ell$, $\varphi_\mu^{\rm tr}$ are the transverse representatives of the equivalence classes from $H_j^{\rm tr}$, can be chosen as the space $\tilde{H}$. Like the functions f_μ^ℓ [see (2.5) and (2.6)] they are transformed in two steps

$$R:\varphi_\mu^{\rm tr}(x)\to\varphi_\mu'^{\rm tr}(x)=g_{\mu\nu}(x)\frac{1}{x^2}\varphi_\nu^{\rm tr}(Rx), \tag{2.10}$$

$$\varphi_\mu'^{\rm tr}(x)\to(\delta_{\mu\nu}-\partial_\mu\partial_\nu/\Box)\varphi_\nu'^{\rm tr}(x). \tag{2.11}$$

The transformation (2.11) is due to the fact that the transversality is violated after the R-transformation (2.10), and it is necessary to go back to the transverse representative of a given class again. As a result, we obtain for the vector functions (2.9) the transformation law (1.5).

It is essential that the relations of equivalence [8]

$$T_j^{\rm tr}\sim T_A^{\rm tr},\qquad T_j^\ell\sim T_A^\ell \tag{2.12}$$

take place between the irreducible unitary representations $T_j^{\rm tr}$ and $T_A^{\rm tr}$, T_j^ℓ and T_A^ℓ introduced above. The corresponding spaces are isomorphic. This isomorphism can be written down with the aid of the kernels $D^{\mathrm{tr}}_{\mu\nu}$ and $D^{\mathrm{long}}_{\mu\nu}$. We have

$$f^{\mathrm{tr}}_{\mu}(x_1)=\int dx_2 (D^{-1}_{\mu\nu})^{\mathrm{tr}} F_\nu(x_2)=(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)F_\nu(x_2), \tag{2.13}$$

where $f^{\mathrm{tr}}_{\mu}\in H^{\mathrm{tr}}_{A}$, $F_\nu(x)$ is any representative of the equivalence class from H^{tr}_{j};

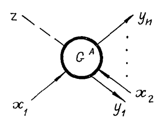
$$\varphi^{\ell}_{\mu}(x_1)=\int dx_2 D^{\mathrm{long}}_{\mu\nu}(x_{12})\phi_\nu(x_2)\sim\int dx_2(\partial^{x_1}_{\mu}\partial^{x_1}_{\nu}\ln x^2_{12})\phi_\nu(x_2), \tag{2.14}$$

where $\varphi^{\ell}_{\nu}\in H^{\ell}_{j}$, $\phi_\nu(x)$ is any representative of the equivalence class from H^{ℓ}_{A}. Invariant kernels[3] $(D^{-1}_{\mu\nu})^{\mathrm{tr}}$ and $D^{\mathrm{long}}_{\mu\nu}$ appear as intertwining operators for equivalent representations (2.12). The relations (2.13) and (2.14) expressing the isomorphism of the spaces of these representations are conformal invariant. These relations play an important role in the analysis of the Maxwell equations.

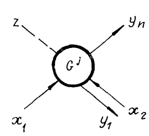
III. Maxwell Equations

In this section it will be shown that the Maxwell equations for each pair of Euclidean Green functions

$$G^{A}_{\mu}(z|x_1\ldots x_n, y_1\ldots y_n)=\langle 0|TA_\mu(z)\psi(x_1)\ldots\bar\psi(y_n)|0\rangle \tag{3.1}$$



$$G^{j}_{\mu}(z|x_1\ldots x_n, y_1\ldots y_n)=\langle 0|Tj_\mu(z)\psi(x_1)\ldots\bar\psi(y_n)|0\rangle \tag{3.2}$$



constitute one of the ways to express the relations of partial equivalence of representations, see (2.12) and (2.13), (2.14). We start with 3-point Green functions. Consider the Green function $B^{d,\ell}_{\mu}(z|x_1x_2)$, including a potential A_μ with scale dimension $\Delta_A=1$ and two different spinor fields with dimensions d and ℓ. As known, there exist two invariant structures with such quantum numbers

$$C^{d,\ell}_{1,\mu}(x_3|x_1x_2)\sim\frac{\hat{x}_{13}}{(x^2_{13})^{\frac{d-\ell+2}{2}}}\gamma_\mu\frac{\hat{x}_{32}}{(x^2_{23})^{\frac{\ell-d+2}{2}}}\frac{1}{(x^2_{12})^{\frac{\ell+d-1}{2}}}, \tag{3.3}$$

$$C^{d,\ell}_{2,\mu}(x_3|x_1x_2)\sim\frac{\hat{x}_{12}}{(x^2_{12})^{\frac{\ell+d+1}{2}}}\frac{1}{(x^2_{13})^{\frac{d-\ell}{2}}(x^2_{23})^{\frac{\ell-d}{2}}}\left(\frac{(x_{13})_\mu}{x^2_{13}}-\frac{(x_{23})_\mu}{x^2_{23}}\right). \tag{3.4}$$

3 Contrary to $D^{\mathrm{tr}}_{\mu\nu}$ the transverse kernel $(D^{-1}_{\mu\nu})^{\mathrm{tr}}(x_{12})=(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\delta(x_{12})$ is conformal invariant:

$$(D^{-1}_{\mu\nu})^{\mathrm{tr}}(x_{12})=(x^2_1x^2_2)^{-3}g_{\mu\mu'}(x_1)g_{\nu\nu'}(x_2)(D^{-1}_{\mu'\nu'}(Rx_1-Rx_2))^{\mathrm{tr}}.$$

It defines an invariant scalar product on the space H^{tr}_{j}

The function $C^{d,\ell}_{2,\mu}$ is longitudinal. It[4] can be considered as a function belonging to H^{ℓ}_j. It should be noted that one cannot construct a transverse superposition from the functions $C^{d,\ell}_{1,\mu}$ and $C^{d,\ell}_{2,\mu}$. This is due to the structure of the space H^{tr}_j which consists of the equivalence classes. The function $C^{d,\ell}_{1,\mu}$ can be considered as a representative of such a class. According to Sect. II it is necessary to select the transverse representative in each class; only in this case the choice (1.7) of propagator can be justified (see Sect. IV for more details). It is necessary, therefore, to make the substitution

$$C^{d,\ell}_{1,\mu}(z|x_1x_2)\to(\delta_{\mu\nu}-\partial^z_\mu\partial^z_\nu/\Box^z)C^{d,\ell}_{1,\nu}(z|x_1x_2). \tag{3.5}$$

Thus, the most general expression for the Green function is [6, 7]

$$B^{d,\ell}_\mu=(\delta_{\mu\nu}-\partial_\mu\partial_\nu/\Box)C^{d,\ell}_{1,\nu}+C^{d,\ell}_{2,\mu}. \tag{3.6}$$

Consider the Green function $\tilde{B}^{d,\ell}_\mu(z|x_1x_2)$, including the current j_μ (the current dimension is $\Delta_j=3$) and two spinors. In this case there also exist two invariant structures one of which is transverse

$$\begin{aligned}\tilde{C}^{d,\ell}_{1,\mu}(x_3|x_1x_2)\sim\Bigg\{&\frac{\hat{x}_{13}}{(x^2_{13})^{\frac{d-\ell+4}{2}}}\gamma_\mu\frac{x_{32}}{(x^2_{23})^{\frac{\ell-d+4}{2}}}\frac{1}{(x^2_{12})^{\frac{\ell+d-3}{2}}}\\&-\frac{\hat{x}_{12}}{(x^2_{12})^{\frac{\ell+d-1}{2}}}\frac{1}{(x^2_{13})^{\frac{d-\ell+2}{2}}}\frac{1}{(x^2_{23})^{\frac{\ell-d+2}{2}}}\\&\left[\frac{(x_{13})_\mu}{x^2_{13}}-\frac{(x_{23})_\mu}{x^2_{23}}\right]\Bigg\},\\\tilde{C}^{d,\ell}_{2,\mu}(x_3|x_1x_2)\sim&\frac{\hat{x}_{12}}{(x^2_{12})^{\frac{\ell+d-1}{2}}}\frac{1}{(x^2_{13})^{\frac{d-\ell+2}{2}}}\frac{1}{(x^2_{23})^{\frac{\ell-d+2}{2}}}\left[\frac{(x_{13})_\mu}{x^2_{13}}-\frac{(x_{23})_\mu}{x^2_{23}}\right].\end{aligned} \tag{3.7}$$

The transverse function $\tilde{C}^{d,\ell}_{1,\mu}$ belongs to the space H^{tr}_A, while $\tilde{C}^{d,\ell}_{2,\mu}$ is a representative of equivalence class from H^{ℓ}_A. As in Sect. II let us choose the longitudinal representative in each class

$$\tilde{C}^{d,\ell}_{2,\mu}(z|x_1x_2)\to\frac{\partial^z_\mu\partial^z_\nu}{\Box^z}\tilde{C}^{d,\ell}_{2,\nu}(z|x_1x_2). \tag{3.8}$$

As a result we find for $\tilde{B}^{d,\ell}_\mu$,

$$\tilde{B}^{d,\ell}_\mu=\tilde{C}^{d,\ell}_{1,\mu}+\frac{\partial_\mu\partial_\nu}{\Box}\tilde{C}^{d,\ell}_{2,\nu}. \tag{3.9}$$

The functions B_μ and $\tilde{B}_\mu$ [more precisely, their convolutions, see the footnote after (3.4)], are elements of the spaces introduced in the preceding section: $B_\mu\in\tilde{H}$, $\tilde{B}_\mu\in H$. Since the representations acting in H and $\tilde{H}$ are equivalent, these functions can be connected via relations of the type (2.13) and (2.14). To obtain these

4 More generally, it is its convolution $\int dx_1dx_2\bar{\chi}(x_1)C^{d,\ell}_{2,\mu}\chi(x_2)$, where χ and $\bar{\chi}$ are spinors, that can be considered as a function belonging to H^{ℓ}_j

relations it should be noted that by a suitable choice of normalizations the functions $C_{1,2}$ and $\tilde{C}_{1,2}$ satisfy the conformal invariant amputation relations

$$\begin{aligned}\tilde{C}^{d,\ell}_{1,\mu}(z_1|x_1x_2)&=\int(D^{-1}_{\mu\nu}(z_{12}))^{\mathrm{tr}}C^{d,\ell}_{1,\nu}(z_2|x_1x_2)dz_2\\&=-(\delta_{\mu\nu}\Box^{z_1}-\partial^{z_1}_\mu\partial^{z_1}_\nu)C^{d,\ell}_{1,\nu}(z_1|x_1x_2),\end{aligned}\tag{3.10}$$

$$\begin{aligned}C^{d,\ell}_{2,\mu}(z_1|x_1x_2)&=\int D^{\mathrm{long}}_{\mu\nu}(z_{12})\tilde{C}^{d,\ell}_{2,\nu}(z_2|x_1x_2)\\&\sim\int dz_2\partial^{z_1}_\mu\partial^{z_1}_\nu\ln z^2_{12}\cdot\tilde{C}^{d,\ell}_{2,\nu}(z_2|x_1x_2).\end{aligned}\tag{3.11}$$

Here (3.10) is analogous to the relation (2.13) expressing the isomorphism of the spaces H^{tr}_A and H^{tr}_j while (3.11) is analogous to the relation (2.14). The function $\tilde{C}_{2,\mu}$ enters the right-hand side of (3.11). In view of the longitudinality of the kernel $D^{\mathrm{long}}_{\mu\nu}$ one can keep only the longitudinal part of this function, i.e. it is possible to pass to a longitudinal representative (3.8) and then to take the inverse of the obtained relation

$$\begin{aligned}\frac{\partial^{z_1}_\mu\partial^{z_1}_\nu}{\Box^{z_1}}\tilde{C}^{d,\ell}_{2,\nu}(z_1|x_1x_2)&=\int dz_2(D^{-1}_{\mu\nu}(z_{12}))^{\mathrm{long}}C^{d,\ell}_{2,\nu}(z_2|x_1x_2)\\&\sim\partial^{z_1}_\mu\partial^{z_1}_\nu C^{d,\ell}_{2,\nu}(z_1|x_1x_2).\end{aligned}\tag{3.12}$$

Evidently, one can pass to full Green functions (3.5) in the right-hand side of (3.10) and (3.12)

$$\tilde{C}_{1,\mu}(z|x_1x_2)\sim(\delta_{\mu\nu}\Box^z-\partial^z_\mu\partial^z_\nu)B_\nu(z|x_1x_2),$$
$$(\partial^z_\mu\partial^z_\nu/\Box^z)\tilde{C}_{2,\nu}(z|x_1x_2)\sim\partial^z_\mu\partial^z_\nu B_\nu(z|x_1x_2).$$

Substituting these relations into (3.9) we obtain

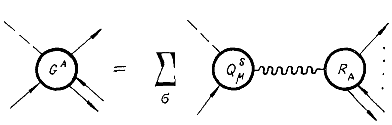
$$\left\{-(\delta_{\mu\nu}\Box^z-\partial^z_\mu\partial^z_\nu)+\frac{1}{\eta}\partial^z_\mu\partial^z_\nu\right\}B^{d,\ell}_\nu(z|x_1x_2)=\tilde{B}^{d,\ell}_\mu(z|x_1x_2).\tag{3.13}$$

Two arbitrary parameters – an overall normalization (set equal to 1 below) and the constant $1/\eta$ enter this relation. This arbitrariness is due to the possibility of an arbitrary choice of the normalizations of kernels $D^{\mathrm{tr}}_{\mu\nu}$ and $D^{\mathrm{long}}_{\mu\nu}$. Putting $\ell=d$ in (3.13) we obtain the Maxwell equations for 3-point Green functions in the generalized Feynman gauge.

$$\left\{-(\delta_{\mu\nu}\Box^z-\partial^z_\mu\partial^z_\nu)+\frac{1}{\eta}\partial^z_\mu\partial^z_\nu\right\}G^A_\nu(z|x_1x_2)=G^j_\mu(z|x_1x_2).\tag{3.14}$$

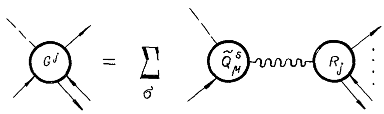
Thus, we have demonstrated that these equations follow from conformal invariance.

Consider higher Green functions (2.1) and (2.2). One can represent them as partial wave expansions (see [1, 2] and references therein). For G^A_μ we have



(3.15)

Here the quantum numbers $\sigma=(\ell,s)$, where ℓ is dimension s is spin, are attributed to the internal line, $\sum\limits_\sigma=\dfrac{1}{2\pi i}\displaystyle\int_{2-i\infty}^{2+i\infty}d\ell\sum_s$, Q^s_μ is an invariant 3-point function. A dotted leg of these vertices corresponds to the vector potential A_μ while the solid line corresponds to the spinor field ψ. For $s=1/2$ they coincide with functions (3.5): $Q_\mu^{1/2}(x_3|x_1x_2)=B_\mu(x_3(x_3|x_1x_2)$. In close analogy with (3.15) we have for the Green function



(3.16)

The current j_μ is associated with the dotted leg of the invariant functions $\tilde{Q}_\mu^s$. For $s=1/2$ they coincide with the invariant functions (3.9):

$$\tilde{Q}_\mu^{1/2}(x_3|x_1x_2)=\tilde{B}_\mu(x_3|x_1x_2).$$

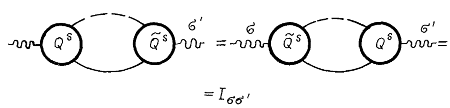
It is essential for future use that the normalization of the functions Q_μ^s and $\tilde{Q}_\mu^s$ can be chosen in such a way that they can be connected by an amputation relation (as in the case of B_μ and $\tilde{B}_\mu$ this follows from a partial equivalence of representations):

$$Q_\mu^s(x_3|x_1x_2)=\int dx_4 D_{\mu\nu}(x_{34})\tilde{Q}_\nu^s(x_4|x_1x_2). \tag{3.17}$$

The inverse relation is

$$\left\{-(\delta_{\mu\nu}\Box^{x_3}-\partial_\mu^{x_3}\partial_\nu^{x_3})+\frac{1}{\eta}\partial_\mu^{x_3}\partial_\nu^{x_3}\right\}Q_\nu^s(x_3|x_1x_2)=\tilde{Q}_\mu^s(x_3|x_1x_2). \tag{3.18}$$

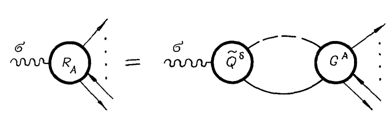
For $s=1/2$ it reduces to (3.13). The normalization is further fixed by the orthogonality relation



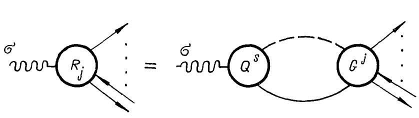
(3.19)

where a dotted internal line corresponds to a δ-function and $\sum_{\sigma'} I_{\sigma\sigma'}f(\sigma')=f(\sigma)$ for any f.

Now we are ready to find the quantities R and $\tilde{R}$ entering (3.15) and (3.16). Making use of (3.19) we have



(3.20)



(3.21)

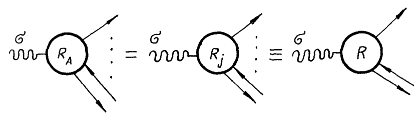
where the dotted internal line corresponds to a δ-function. Suppose that the Green functions G_μ^A and G_μ^j satisfy the Maxwell equations (in the generalized Feynman gauge):

$$\left\{-(\delta_{\mu\nu}\Box^{z}-\partial_\mu^{z}\partial_\nu^{z})+\frac{1}{\eta}\partial_\mu^{z}\partial_\nu^{z}\right\}G_\nu^A(z|x_1\dots y_n)=G_\mu^j(z|x_1\dots y_n). \tag{3.22}$$

Let us represent the δ-function corresponding to a dotted internal line on the right-hand side of (3.20) as

$$\delta_{\mu\nu}\delta(x_1-x_2)=\int dx_3 D_{\mu\varrho}(x_{13})D^{-1}_{\varrho\nu}(x_{32}).$$

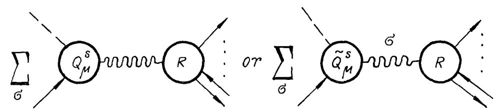
Making use of the Maxwell equation (3.22) and relation (3.17), one can transform the right-hand side of (3.20) to the right-hand side of (3.21). It means that the equality



(3.23)

follows from the Maxwell equations. An opposite proposition is also true: if one postulates (3.23) then the Maxwell equations follow from conformal invariance.

It is essential that all the dynamical information is contained in the quantity R. Thus the Green functions G^A_μ and G^j_μ constitute a unified object and can be written both in the following equivalent forms:



(3.24)

The Maxwell equations connect these forms and express the equivalence of the corresponding conformal group representations.

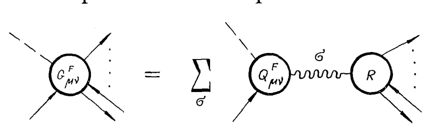
The Green functions containing the electromagnetic tensor $F_{\mu\nu}$ can be treated in an analogous way. Let T_F be the irreducible representation corresponding to this field and H_F be the space of this representation. It is known [8,9] that the representation T_F is equivalent to the representations T^{tr}_A and T^{tr}_j introduced above:

$$T_F \sim T^{\mathrm{tr}}_A, \qquad T_F \sim T^{\mathrm{tr}}_j. \tag{3.25}$$

In particular, the kernel of the intertwining operator connecting T_F with T^{tr}_j is

$$\langle 0|TF_{\mu\nu}(x_1)j_\tau(x_2)|0\rangle \sim (\partial_\mu\delta_{\nu\tau}-\partial_\nu\delta_{\mu\tau})\delta(x_{12}).$$

The Green function $G^F_\mu(z|x_1\ldots y_n)=\langle 0|TF_{\mu\nu}(z)\psi(x_1)\ldots\bar\psi(y_n)|0\rangle$ can be represented as above in the form of the partial wave expansion:



(3.26)

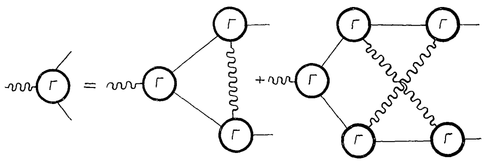
where a dotted line corresponds to the field $F_{\mu\nu}$. Demanding that the quantity R coincides with the analogous one in (3.24) one can show that the Green function $G^F_{\mu\nu}$ satisfies the Maxwell equations

$$G^F_{\mu\nu}=\partial^z_\mu G^A_\nu-\partial^z_\nu G^A_\mu, \qquad -\partial^z_\nu G^F_{\mu\nu}=G^j_\mu+\partial^z_\mu\partial^z_\nu G^A_\nu. \tag{3.27}$$

These equations are a consequence of the equivalence relations (3.25), and the Green function $G^F_{\mu\nu}$ appears to be one more form of the transverse part of (3.24).

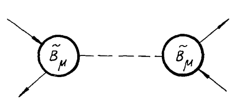
IV. Invariance of Skeleton Theory

In order to clarify the question of the existence of conformal behaviour in electrodynamics it is necessary also to verify the selfconsistency of the equations for the vertex and the propagator. As known, these equations play the role of conditions of existence of conformal solutions in QFT. From these equations it is possible to calculate the values of the scale dimension of the spinor field and a coupling constant. In particular, the skeleton equation for the vertex has the form



where conformal invariant vertices and propagators enter the right-hand side. First of all, it is necessary to prove conformal invariance (in the sense of Sects. II and III) of the graphs on the right-hand side. Then, as known [1, 2], this equation becomes an algebraic relation determining the parameters of the theory (in the present case, the parameters d and g). Another condition appears from the equation for the propagator. More detailed analysis of bootstrap program in electrodynamics is given in [6]. Here we confine ourselves to the proof of the conformal invariance of the skeleton theory.

Consider any skeleton graph. As is known, the integrals over internal fermionic lines are conformal invariant provided these lines join conformal invariant subgraphs. Therefore, it is sufficient to consider the integral over an internal photonic line



(4.1)

$$=\int dx_5 dx_6 \tilde{B}_\mu^{\ell_1, d_1}(x_1 x_2|x_5) D_{\mu\nu}(x_{56}) \tilde{B}_\nu^{\ell_2, d_2}(x_6|x_3 x_4),$$

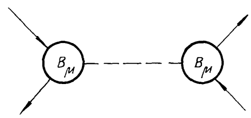
where $\tilde{B}_\mu^{\ell, d}$ is the function (3.5). This expression represents the invariant scalar product on H (because $\tilde{B}_\mu \in H$, see Sect. III) and, consequently, is conformal invariant. Indeed, substituting into (4.1) the explicit expressions (1.7) and (3.9) we obtain

$$\begin{aligned}(4.1)=&\int dx_5 dx_6 \tilde{C}_{1,\mu}(x_1 x_2|x_5) D_{\mu\nu}^{\mathrm{tr}}(x_{56}) \tilde{C}_{1,\nu}(x_6|x_3 x_4)\\ &+\int dx_5 dx_6 \tilde{C}_{2,\mu}(x_1 x_2|x_5) D_{\mu\nu}^{\mathrm{long}}(x_{56}) \tilde{C}_{2,\nu}(x_6|x_3 x_4).\end{aligned} \tag{4.2}$$

The first term is an invariant scalar product in the space H_A^{tr}. Its invariance is provided by the transversality of the invariant functions $\tilde{C}_{1,\mu}$, since the kernel $D_{\mu\nu}^{\mathrm{tr}}$ by itself is not invariant under the R-transformation. Terms in $D_{\mu\nu}^{\mathrm{tr}}$ appearing after a R-transformation and breaking the invariance are longitudinal and disappear when substituted into (4.2). The invariance of the second term, which is a scalar product on H_A^ℓ, is manifest since only conformal invariant functions enter the integrand.

The special role of the projection operator $\partial_\mu \partial_\nu / \Box$ in (3.9) should be noted. Due to this operator the cross terms breaking the invariance do not appear when passing from (4.1) to (4.2). But in the final expression, in the second term of (4.2), this operator can be omitted due to the longitudinality of the kernel $D_{\mu\nu}^{\mathrm{long}}$.

Integral (4.1) can also be expressed in terms of the functions B_μ and inverse propagator $D_{\mu\nu}^{-1}$. In this case we have

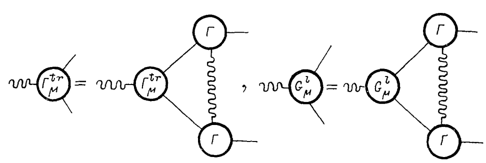


(4.3)

$$
\begin{aligned}
&= \int dx_5 dx_6 B_\mu^{d_1\ell_1}(x_1x_2|x_5) D_{\mu\nu}^{-1}(x_{56}) B_\nu^{d_2\ell_2}(x_6|x_3x_4) \\
&= \int dx_5 dx_6 C_{1,\mu}(x_1x_2|x_5)\,(D_{\mu\nu}^{-1}(x_{56}))^{\mathrm{tr}} C_{1,\nu}(x_6|x_3x_4) \\
&\quad + \int dx_5 dx_6 C_{2,\mu}(x_1x_2|x_5)\,(D_{\mu\nu}^{\mathrm{long}}(x_{56}))^{-1} C_{2,\nu}(x_6|x_3x_4)\,. \qquad (4.4)
\end{aligned}
$$

Integral (4.3) is the invariant scalar product in the space $\tilde{H}$. The presence in (3.5) of the projection operator $(\delta_{\mu\nu} - \partial_\mu\partial_\nu/\Box)$ is essential when passing to (4.4). In the final result it is omitted due to the transversality of the invariant function $(D_{\mu\nu}^{-1})^{\mathrm{tr}}$. The invariance of the second term [with non-invariant kernel $(D_{\mu\nu}^{-1})^{\mathrm{long}}$] is provided by the longitudinality of the function $C_{2,\mu}$. Thus the integral (4.3) is conformal invariant.

In conclusion we note that since the vertex Γ_μ (or the Green function G_μ) contains two independent structures, the skeleton equation shown in the beginning of this section is in fact equivalent to a couple of equations which may be suitably presented as equations for conformal invariant functions Γ_μ^{tr} and G_μ^{ℓ} [6]. In the three-vertex approximation we have



It can be shown that the second of these equations is equivalent to the fermion self-energy equation.

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