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On an Elaboration of M. Kac's Theorem Concerning Eigenvalues of the Laplacian in a Region with Randomly Distributed Small Obstacles

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Abstract. We remove m-balls of centers $w_1,\dots,w_m$ with the same radius α/m from a bounded domain Ω in $\mathbf{R}^3$ with smooth boundary γ. Let $\mu_k(\alpha/m;w(m))$ denote the k-th eigenvalue of the Laplacian in $\Omega\backslash m$-balls under the Dirichlet condition. We consider $\mu_k(\alpha/m;w(m))$ as a random variable on a probability space $(w_1,\dots,w_m)\in\Omega\times\cdots\times\Omega$ and we examine a precise behaviour of $\mu_k(\alpha/m;w(m))$ as $m\to\infty$. We give an elaboration of. M. Kac's theorem.

1. Introduction

We consider a bounded domain Ω in $\mathbf{R}^3$ with smooth boundary γ. Let $B(\varepsilon;w)$ be the ball defined by $B(\varepsilon;w)=\{x\in\mathbf{R}^3;|x-w|<\varepsilon\}$. Let $0<\mu_1(\varepsilon;w(m))\leqq\mu_2(\varepsilon;w(m))\leqq\mu_3(\varepsilon;w(m))<\cdots$ be the eigenvalues of $-\Delta(=-\mathrm{div\ grad})$ in $\Omega_{\varepsilon,w(m)}=\Omega\backslash\bigcup_{i=1}^{m}B(\varepsilon;w_i^{(m)})$ under the Dirichlet condition on its boundary. Here $w(m)$ denotes the set of m-points $\{w_1^{(m)}\}_{i=1}^{m}$. We arrange $\mu_k(\varepsilon;w(m))$ repeatedly according to their multiplicities.

Let $V(x)\geqq 0$ be a C^1 function on $\bar{\Omega}$ satisfying

$$\int_\Omega V(x)\,dx=1.$$

Then, we consider Ω as the probability space with the probability $V(x)\,dx$. Let $\Omega^m=\prod_{i=1}^{m}\Omega$ be the probability space with the product measure.

The aim of this note is to prove the following:

Theorem 1. *Fix $\alpha>0$ and k. Then,*

$$\lim_{m\to\infty}\mathbb{P}(w(m)\in\Omega^m;m^{\tilde{\delta}}|\mu_k(\alpha/m;w(m))-\mu_k^V|<\varepsilon)=1 \tag{1.1}$$

for any $\varepsilon>0$ and $\tilde{\delta}\in[0,\frac{1}{4})$. Here *$\mu_k^V$ denotes the k^{th} eigenvalue of $-\Delta+4\pi\alpha V(x)$ in Ω under the Dirichlet condition on γ.*

Theorem 1 is an elaboration of the result of Kac [4] and Rauch–Taylor [13]. Kac [4] proved (1.1) when $\tilde{\delta}=0$, $V(x)=(\text{volume of } \Omega)^{-1}$ and Rauch and Taylor [13] proved (1.1) for general $V(x)$ when $\tilde{\delta}=0$. Kac used the theory of Wiener sausage to get his result. Rauch and Taylor gave their result by combining functional analysis of operators and the Feynmann–Kac formula. See also the very interesting papers of Papanicolaou–Varadhan [12] and Simon [14]. Our proof of Theorem 1 is different from [4, 13] in the point that we employ perturbational calculus using Green's function of $\Delta-\lambda$. For other related topics, see Bensoussan–Lions–Papanicolaou [1], Huruslov–Marchenko [3] and Lions [6].

Theorem 1 was announced in Ozawa [9]. See also Ozawa [10, 11].

Now we give a rough sketch of the proof of Thoerem 1. Let $G_m^{(\lambda)}(x,y;w(m))$ be the Green's function of $\Delta-\lambda$ in $\Omega_{\alpha/m,w(m)}$ under the Dirichlet condition on its boundary satisfying

$$(\Delta_x-\lambda)G_m^{(\lambda)}(x,y;w(m))=-\delta(x-y),\quad x,y\in\Omega_{\alpha/m,w(m)},$$
$$G_m^{(\lambda)}(x,y;w(m))=0,\quad x\in\Omega_{\alpha/m,w(m)}.$$

Let $G^{(\lambda)}(x,y)$ be the Green's function of $\Delta-\lambda$ defined by

$$(\Delta_x-\lambda)G^{(\lambda)}(x,y)=-\delta(x-y),\quad x,y\in\Omega,$$
$$G^{(\lambda)}(x,y)=0,\qquad x\in\gamma.$$

Hereafter, we abbreviate $G^{(\lambda)}(x,y)$ as $G(x,y)$, if there is no fear of confusion. Let $h_m^{(\lambda)}(x,y;w(m))$ be as follows:

$$\begin{aligned}h_m^{(\lambda)}(x,y;w(m))&=G(x,y)-(4\pi\alpha/m)e^{\lambda^{1/2}(\alpha/m)}\sum_{i=1}^{m}G(x,w_i)G(w_i,y)\\&\quad+\sum_{s=2}^{m}(-4\pi\alpha/m)^s e^{\lambda^{1/2}(\alpha/m)s}\sum_{(s)}G(x,w_{i_1})G(w_{i_1},w_{i_2})\\&\quad\ldots G(w_{i_{s-1}},w_{i_s})G(w_{i_s},y).\end{aligned}\tag{1.2}$$

Here the indices $(i_1,\ldots,i_s)$ in $\sum_{(s)}$ run over all $1\leqq i_1,\ldots,i_s\leqq m$ satisfying $i_1\neq i_2$, $i_2\neq i_3,\ldots,i_{s-1}\neq i_s$. A key to Theorem 1 is the fact that $h_m^{(\lambda)}$ is a nice approximation of $G_m^{(\lambda)}$ in a rough sense. This is discussed in Sect. 2.

Recall now that

$$\frac{1}{m}\sum_{i=1}^{m}G(x,w_i)G(w_i,y)\quad\text{tends to}\quad\int_\Omega G(x,z)V(z)G(z,y)\,dz$$

with probability one by the strong law of large numbers. See Kingman–Taylor [5], Hall–Heyde [2], etc. We take a sufficiently large λ and we fix it. Then, we know from probabilistic argument as above that $h_m^{(\lambda)}$ converges in a rough sense to the integral kernal function of the integral operator $(-\Delta+\lambda+4\pi\alpha V)^{-1}$. Of course, we need rigorous steps. Along this line we get Theorem 1.

From now on we show some technical points in our proof. The following conditions on $w(m)$, $m=1,2,\ldots$ are important in our study.

(C-1)$_\nu$ There exists a constant C_0 independent of m such that

$$w_1^{(m)} \in \Omega$$

$$\min_{i \neq j} |w_i^{(m)} - w_j^{(m)}| \geqq C_0 m^{-1+\nu}$$

hold. Here $\nu \in (0, \frac{1}{3})$ is a fixed constant.

(C-2) There exists a constant C_ξ^* independent of m (possibly depending on ξ) such that

$$\max_m m^{-2} \sum_{\substack{i,j=1 \\ i \neq j}}^m |w_i^{(m)} - w_j^{(m)}|^{-3+\xi} \leqq C_\xi^* < +\infty \tag{1.3}$$

holds for any $\xi > 0$.

(C-3) Let f_h, $h = 1, 2, 3, \dots$ be an arbitrary family of continuous functions on $\bar{\Omega}$ satisfying

$$\max_{x \in \bar{\Omega}} |f_h(x)| \leqq C_*^{-h} \cdot D^*$$

for some constant $C_* > 1$ and $D^* < \infty$. Then,

$$\lim_{m \to \infty} m^\beta \left(\sup_h C_*^{h/2} \left(\frac{1}{m} \sum_{i=1}^m f_h(w_i^{(m)}) - \int_\Omega f_h(x) V(x)\, dx \right) \right) = 0 \tag{1.4}$$

and

$$\lim_{m \to \infty} m^\beta \left(\sup_h C_*^{h/2} \left(\frac{1}{m} \sum_{i=1}^m \left\{ \frac{1}{m} \sum_{\substack{j=1 \\ j \neq i}}^m G^{(\lambda)}(w_i^{(m)}, w_j^{(m)}) f_h(w_j^{(m)}) \right.\right.\right.$$

$$\left.\left.\left. - (\mathbb{G}^{(\lambda)} V f_h)(w_i^{(m)}) \right\}^2 \right) \right) = 0 \tag{1.5}$$

hold for any fixed $\beta \in [0, \frac{1}{2})$ and $\lambda \geqq 0$. Here $\mathbb{G}^{(\lambda)}$ denotes the integral operator defined by

$$(\mathbb{G}^{(\lambda)} f)(x) = \int_\Omega G^{(\lambda)}(x, y) f(y)\, dy.$$

We can prove the following Proposition which is crucial to our step to prove Theorem 1. Let $\mathbb{H}_\infty^{(\lambda)}$ denote the operator given by

$$\mathbb{G}^{(\lambda)} + \sum_{s=1}^\infty (-4\pi\alpha)^s \mathbb{G}^{(\lambda)} (V \mathbb{G}^{(\lambda)})^s.$$

Let $\mathbb{G}_m^{(\lambda)}$ denote the operator given by

$$(\mathbb{G}_m^{(\lambda)} f)(x) = \int^{\Omega_{\alpha/m,\, w(m)}} G_m^{(\lambda)}(x, y; w(m)) f(y)\, dy, \quad x \in \Omega_{\alpha/m, w(m)}.$$

Proposition 1. *Fix* $\alpha > 0$ *and* k. *Let* φ_k^V *denote the* k^{th} *eigenfunction of* $-\Delta + 4\pi\alpha V(x)$ *in* Ω *under the Dirichlet condition on* γ *satisfying*

$$\int_\Omega \varphi_k^V(x)^2\, dx = 1.$$

Assume that $\{w(m)\}_{m=1}^{\infty}$ *satisfies* (C-1)$_v$, (C-2) and (C-3) *for any fixed* $\lambda > 0$. *Then*

$$\lim_{m\to\infty} \|m^{\delta}(\mathbb{G}_m^{(\lambda)} - \mathbb{H}_{\infty}^{(\lambda)})\varphi_k^V\|_{L^2(\Omega_{\alpha/m, w(m)})} = 0 \tag{1.6}$$

holds for any fixed $\tilde{\delta}\in[0, \frac{1}{4})$ *and for any sufficiently large* $\lambda > 0$.

In Sect. 4, we make a probabilistic consideration on (C-1)$_v$, (C-2), (C-3) and we finish our proof of Theorem 1 based on Proposition 1.

2. Construction of an Approximate Green's Function

We give preliminary Lemmas. Let b_{ij}, a_{jk}, a'_{kl} denote positive numbers. Then we have the following:

Lemma 1. *The inequality*

$$\sum_{\substack{i, j_1, \ldots, j_s = 1 \\ i \neq j_1, \ldots, j_{s-1} \neq j_s}}^{m} b_{ij_1} a_{j_1 j_2} \ldots a_{j_{s-2} j_{s-1}} a'_{j_{s-1} j_s} \leqq m^{1/2} \omega^{s-2} \left(\sum_{i=1}^{m} \left(\sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}^2 \right)^{1/2} \right) \cdot \left(\sum_{\substack{j,k=1 \\ j \neq k}}^{m} a'^2_{jk} \right)^{1/2} \tag{2.1}$$

holds for $s \geqq 2$, *where*

$$\omega = \left(\sum_{\substack{j,k=1 \\ j \neq k}}^{m} a_{jk}^2 \right)^{1/2}.$$

Therefore (2.1) does not exceed

$$m\omega^{s-2} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{m} b_{ij}^2 \right)^{1/2} \left(\sum_{\substack{j,k=1 \\ j \neq k}}^{m} a'^2_{jk} \right)^{1/2}. \tag{2.2}$$

Proof. By the iterative use of the Schwarz inequality we get (2.1) and (2.2). q.e.d.

From now on we abbreviate $\Omega_{\alpha/m, w(m)}$ as Ω_w. Also $B(\alpha/m; w_r)$ is written as B_r, if there is no fear of confusion. We have the following

Lemma 2. *Suppose that* $u \in C^{\infty}(\bar{\Omega}_w)$ *satisfies*

$$\begin{aligned} (-\Delta + \lambda)u(x) &= 0 \quad x \in \Omega_w, \\ u(x) &= 0, \quad x \in \gamma, \\ \max\{|u(x)|; x \in \partial B_r\} &= M_r(m), \quad r = 1, \ldots, m. \end{aligned} \tag{2.3}$$

Then, there exists a constant C_p *independent of* m *such that*

$$\|u\|_{L^p(\Omega_w)} \leqq C_p m^{-(3/p)} \sum_{r=1}^{m} M_r(m) \tag{2.4}$$

holds for any fixed $p > 3$.

Proof. By using the Hopf maximum principle we easily see that

$$|u(x)| \leqq C(\alpha/m) \sum_{r=1}^{m} |x - w_r|^{-1} M_r(m) \tag{2.5}$$

holds for a constant C independent of m. See [8]. Thus (2.4) follows. q.e.d.

For the sake of simplicity we abbreviate $G^{(\lambda)}(x, y)$ as $G(x, y)$. We put

$$S(x, y) = G(x, y) - G_*(x, y),$$

where

$$G_*(x, y) = (4\pi|x - y|)^{-1} e^{-\lambda^{1/2}|x - y|}.$$

Then $S(x, y) \in C^\infty(\Omega \times \Omega)$.

We have the following.

Lemma 3. *Assume that* $\{w(m)\}_{m=1}^{\infty}$ *satisfies* $(\text{C-1})_v$. *Then*

$$\max_{x \in \partial B_r} |G(x, w_i) - G(w_r, w_i)| \leqq C(\alpha/m)|w_i - w_r|^{-2}, \tag{2.6}$$

$$\max_{x \in \partial B_r} |S(x, w_r) G(w_r, w_i)| \leqq C|w_i - w_r|^{-2} \tag{2.7}$$

hold for a constant C independent of sufficiently large m.

Remark. C can be taken as independent of λ.

Proof. We know from $(\text{C-1})_v$ that $|w_r - w_i| \geqq 4(\alpha/m)$ holds for sufficiently large m. By the intermediate value theorem we get

$$\max_{x \in \partial B_r} |G(x, w_i) - G(w_r, w_i)| \leqq C(\alpha/m) \max_{y \in \bar{B}_r} |(\nabla_y G)(y, w_i)|.$$

Now we have (2.6) by Thoerem 8.6 in [7].

We want to prove (2.7). Let w^* be a point on γ such that $\operatorname{dist}(w_r, \gamma) = \operatorname{dist}(w_r, w^*)$. Then

$$\operatorname{dist}(w_r, \gamma)^{-1} G(w_r, w_i) = |w_r - w^*|^{-1} |G(w_r, w_i) - G(w^*, w_i)|. \tag{2.8}$$

By a simple consideration we see that (2.8) does not exceed $C_0|w_i - w_r|^{-2}$ for a constant C_0 independent of m. Here we also use Theorem 8.6 in [7]. Now we will show

$$\max_{x \in \partial B_r} |S(x, w_r)| \operatorname{dist}(w_r, \gamma) \leqq C_1. \tag{2.9}$$

Consider the case $\Omega = \mathbf{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbf{R}^3;\ x_1 > 0\}$. In this case $S(x, w_r) = -(4\pi|\tilde{x} - w_r|)^{-1} \exp(-\lambda^{1/2}|x - w_r|)$, where $\tilde{x} = (-x_1, x_2, x_3)$. Thus

$$|S(x, w_r)| \leqq C_2 \operatorname{dist}(w_r, \gamma)^{-1}.$$

We can apply the usual techniques in analyzing boundary value problems, for example, local parametrix ... etc., to study $S(x, w_r)$ and we get (2.9). In summing up these facts we get (2.7). q.e.d.

Now we come back to study $\mathbb{G}_m^{(\lambda)}$. Let $\mathbb{H}_m^{(\lambda)}$ be the integral operator given by

$$(\mathbb{H}_m^{(\lambda)} f)(x) = \int_{\Omega_w} h_m^{(\lambda)}(x, y; w(m)) f(y)\, dy, \quad x \in \Omega_w.$$

We here introduce the following decomposition (2.11) of $\mathbb{H}_m^{(\lambda)}f$. Fix r. We put

$$(I_r^s(\lambda)f)(x) = \sum_{(s)}{}' G(x, w_{i_1})G(w_{i_1}, w_{i_2})\dots G(w_{i_{s-1}}, w_{i_s})(\mathbb{G}^{(\lambda)}f)(w_{i_s}) - (4\pi\alpha/m)e^{\lambda^{1/2}(\alpha/m)} \cdot \sum_{(s)}{}' G(x, w_r)G(w_r, w_{i_1})\dots G(w_{i_{s-1}}, w_{i_s})(\mathbb{G}^{(\lambda)}f)(w_{i_s}) \tag{2.10}$$

for $s \geqq 1$. Here the indices in $\sum_{(s)}{}'$ run over all $1 \leqq i_1, \dots, i_s \leqq m$ such that $i_1 \neq r$, $i_2 \neq i_1, \dots, i_s \neq i_{s-1}$. Then it is easy to see that

$$(\mathbb{H}_m^{(\lambda)}f)(x) = (\mathbb{G}^{(\lambda)}f)(x) - (4\pi\alpha/m)e^{\lambda^{1/2}(\alpha/m)}G(x, w_r)(\mathbb{G}^{(\lambda)}f)(w_r) + \sum_{s=1}^{m} (-4\pi\alpha/m)^s e^{\lambda^{1/2}(\alpha/m)s}(I_r^s(\lambda)f)(x) + (-4\pi\alpha/m)^m e^{\lambda^{1/2}\alpha} \cdot \sum_{(m)}{}' G(x, w_{i_1})G(w_{i_1}, w_{i_2})\dots G(w_{i_{m-1}}, w_{i_m})(\mathbb{G}^{(\lambda)}f)(w_{i_m}). \tag{2.11}$$

Recall the definition of $S(x, y)$ and $G_*(x, y)$. It is easy to see that

$$(I_r^s(\lambda)f)(x)|_{x\in\partial B_r} = (L_r^s(\lambda)f)(x)|_{x\in\partial B_r} + (N_r^s(\lambda)f)(x)|_{x\in\partial B_r}, \tag{2.12}$$

where

$$(L_r^s(\lambda)f)(x)|_{x\in\partial B_r} = \sum_{(s)}{}' ((G(x, w_{i_1}) - G(w_r, w_{i_1}))G(w_{i_1}, w_{i_2})\dots G(w_{i_{s-1}}, w_{i_s})(\mathbb{G}^{(\lambda)}f)(w_{i_s}), \tag{2.13}$$

and

$$(N_r^s(\lambda)f)(x)|_{x\in\partial B_r} = (-4\pi\alpha/m)e^{\lambda^{1/2}(\alpha/m)} \cdot \sum_{(s)}{}' S(x, w_r)G(w_r, w_{i_1})\dots G(w_{i_{s-1}}, w_{i_s})(\mathbb{G}^{(\lambda)}f)(w_{i_s}). \tag{2.14}$$

Here we use the fact that $G_*(x, y) = (4\pi\alpha/m)^{-1}e^{-\lambda^{1/2}(\alpha/m)}$ when $|x - y| = \alpha/m$.

We have the following:

Lemma 4. *Assume that* $\{w(m)\}_{m=1}^{\infty}$ *satisfies* (C-1)$_v$. *Then*

$$\sum_{r=1}^{m} \max\{|I_r^s(\lambda)f(x)|; x\in\partial B_r\} \leqq C_p\alpha(1 + e^{2\lambda^{1/2}(\alpha/m)})\kappa(w(m); \lambda)^{s-1} \cdot\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m} |w_i - w_j|^{-4}\right)^{1/2} \|f\|_{L^p(\Omega_w)} \tag{2.15}$$

holds or a constant C_p *independent of* m, λ. *Here*

$$\kappa(w(m); \lambda) = \left(\sum_{\substack{i,j=1\\ i\neq j}}^{m} G(w_i, w_j)^2\right)^{1/2}, \tag{2.16}$$

and p *is a fixed constant satisfying* $p > 3$.

Proof. We apply Lemma 1 to (2.13), (2.14). We use the estimate

$$\max_{x\in\bar{\Omega}} (|\mathbb{G}^{(\lambda)}f(x)| + |\nabla_x\mathbb{G}^{(\lambda)}f(x)|) \leqq \tilde{C}_p\|f\|_{L^p(\Omega_w)}, \quad (p > 3) \tag{2.17}$$

to get the desired result. Here $\tilde{C}_p$ is independent of λ. q.e.d.

We put $\mathbb{Q}_m^{(\lambda)} = \mathbb{G}_m^{(\lambda)} - \mathbb{H}_m^{(\lambda)}$. Then it is easy to see that

$$(-\Delta_x + \lambda)\mathbb{Q}_m^{(\lambda)} f(x) = 0, \quad x \in \Omega_w,$$
$$\mathbb{Q}_m^{(\lambda)} f(x) = 0, \quad x \in \gamma,$$

for any $f \in C_0^\infty(\Omega_w)$. We have the following:

Lemma 5. *Assume that* $\{w(m)\}_{m=1}^\infty$ *satisfies* (C-1)$_\nu$. *Then there exists a constant* C_p *such that*

$$\|\mathbb{Q}_m^{(\lambda)}\|_{L^p(\Omega_w)} \leqq C_p \tau_p(w(m), \alpha, \lambda) \tag{2.18}$$

holds for any fixed $p > 3$. *Here*

$$\tau_p(w(m), \alpha, \lambda) = m^{-(3/p)}\{\alpha(1 + \exp(2\lambda^{1/2}(\alpha/m)))(1 + J_1) + J_2\}, \tag{2.19}$$

where

$$J_1 = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^m |w_i - w_j|^{-4}\right)^{1/2} \left\{\sum_{s=1}^m (4\pi\alpha/m)^s \exp(\lambda^{1/2}(\alpha s/m))\kappa(w(m); \lambda)^{s-1}\right\},$$

$$J_2 = (4\pi\alpha/m)^m \exp(\lambda^{1/2}\alpha)\kappa(w(m); \lambda)^{m-1} m\left(\sum_{\substack{i,j=1 \\ i \neq j}}^m |w_i - w_j|^{-2}\right)^{1/2}$$

Proof. Since we have Lemma 2 and (2.17), we must only examine

$$\sum_{r=1}^m \max\{|\mathbb{H}_m^{(\lambda)} f(x)|; x \in \partial B_r\}$$

to get a bound for $\|\mathbb{Q}_m^{(\lambda)} f\|_{L^p(\Omega_w)}$. Observing Lemmas 1, 4 and

$$|G(w_i, w_j)| \leqq C \exp(-\lambda^{1/2}|w_i - w_j|)|w_i - w_j|^{-1}, \tag{2.20}$$

we get (2.18).

We have the following:

Proposition 2. *Assume that* $\{w(m)\}_{m=1}^\infty$ *satisfies* (C-1)$_\nu$, (C-2). *Take an arbitrary fixed* $p \in (3, \infty)$ *and* $\rho > 0$. *Then there exists* $\lambda_0 > 0$ *and a constant* C_p *which is independent of* m, λ *such that*

$$\|\mathbb{Q}_m^{(\lambda)}\|_{L^p(\Omega_w)} \leqq C_p m^{-(3/p) + ((1-\nu)/2) + \rho} \tag{2.21}$$

holds for any $\lambda \in [\lambda_0, \infty)$.

Proof. We examine J_1. We have

$$|w_i - w_j|^{-4} = |w_i - w_j|^{-3+\xi}|w_i - w_j|^{-1-\xi}$$
$$\leqq \hat{C} m^{(1-\nu)(1+\xi)}|w_i - w_j|^{-3+\xi},$$

we get

$$\left(\sum_{\substack{i,j=1 \\ i \neq j}}^m |w_i - w_j|^{-4}\right)^{1/2} \leqq \hat{C} m^{(1-\nu)(1+\xi)/2} m C_\xi^*. \tag{2.22}$$

Recall (2.20). It is easy to see that

$$|G(w_i, w_j)| \leqq C\lambda^{-1/6}|w_i - w_j|^{-(4/3)}.$$

Thus

$$\kappa(w(m);\lambda) \leqq C''\lambda^{-1/6}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m} |w_i - w_j|^{-8/3}\right)^{1/2}$$

$$\leqq C''\lambda^{-1/6} m C_{1/3}^*. \tag{2.23}$$

By (2.22), (2.23) we have

$$J_1 \leqq \hat{C} m^{(1-\nu)(1+\xi)/2} C_\xi^* \left\{\sum_{s=1}^{m} (4\pi\alpha C''\lambda^{-1/6}C_{1/3}^*)^s \exp(\lambda^{1/2}(\alpha s/m))\right\}. \tag{2.24}$$

We also have the estimate for J_2. Since C'', $C_{1/3}^*$ are independent of λ, we get the desired result by taking $\xi > 0$ small enough. q.e.d.

Corollary 1. *Assume that* $\{w(m)\}_{m=1}^{\infty}$ *satisfies* (C-1)$_\nu$, (C-2). *Then there exists* $\lambda_0 > 0$ *and a constant* C *independent of* m *such that*

$$\|\mathbb{Q}_m^{(\lambda)}\|_{L^2(\Omega_w)} \leqq C m^{-1/2}$$

holds for any $\lambda \in [\lambda_0, \infty)$.

Proof. It is easy to see that

$$\int_{\Omega_w} \mathbb{Q}_m^{(\lambda)} u(x) v(x)\, dx = \int_{\Omega_w} u(x) \overline{\mathbb{Q}_m^{(\lambda)} v(x)}\, dx$$

for $u, v \in C_0^\infty(\Omega_w)$. Therefore

$$\|\mathbb{Q}_m^{(\lambda)}\|_{L^{p'}(\Omega_w)} = \|\mathbb{Q}_m^{(\lambda)}\|_{L^p(\Omega_w)}.$$

Here p' is defined by $p'^{-1} + p^{-1} = 1$. Since $p > 3$, $p' < \frac{3}{2}$. By the Riesz–Thorin interpolation theorem we get

$$\|\mathbb{Q}_m^{(\lambda)}\|_{L^2(\Omega_w)} \leqq \|\mathbb{Q}_m^{(\lambda)}\|_{L^p(\Omega)}.$$

Now we take $p \in (3, \infty)$ as close enough to 3. We get the desired result, since $\nu \in (0, \frac{1}{3})$ is fixed. q.e.d.

Let $\tilde{\mathbb{H}}_m^{(\lambda)}$ be the integral operator defined by

$$(\tilde{\mathbb{H}}_m^{(\lambda)} f)(x) = \int_\Omega h_m^{(\lambda)}(x, y; w(m))\, dy, \quad x \in \Omega.$$

Let χ_{Ω_w} (respectively $\tilde{\chi}_{\Omega_w}$) be the characteristic function of Ω_w (respectively $\Omega \backslash \bar{\Omega}_w$). Put $g_m(x) = \chi_{\Omega_w}((\tilde{\mathbb{H}}_m \varphi_k^V)(x) - \mathbb{H}_m(\chi_{\Omega_w}\varphi_k^V)(x))$. Then

$$g_m(x) = \chi_{\Omega_w}(\tilde{\mathbb{H}}_m^{(\lambda)}(\tilde{\chi}_{\Omega_w}(\varphi_k^V))(x).$$

We see that $\Delta g_m(x) = 0$ for $x \in \Omega_w$ and $g_m(x) = 0$ for $x \in \gamma$. To estimate g_m we need a bound for

$$\sum_{r=1}^{m} \max\{|g_m(x)|; x \in \partial B_r\}. \tag{2.25}$$

By a simple consideration we see that (2.25) does not exceed the term which is given as replacing f in the right hand side of (2.21) by $\tilde{\chi}_{\Omega_w}\varphi_k^V$. We know that

$$\|\mathbb{G}^{(\lambda)}(\tilde{\chi}_{\Omega_w}\varphi_k^V)\|_{C^1(\Omega)} \leqq C'\|\tilde{\chi}_{\Omega_w}\varphi_k^V\|_{L^4(\Omega)} \leqq \hat{C}m^{-1/2}. \tag{2.26}$$

Therefore, as in the proof of Lemma 5, we have the following:

Lemma 6. *Assume that* $\{w(m)\}_{m=1}^{\infty}$ *satisfies* (C-1)$_{\nu}$. *Then there exists* λ_0 *and a constant* C_λ *such that*

$$\|g_m\|_{L^2(\Omega_w)} \leqq C_\lambda m^{-1/2} \tag{2.27}$$

holds for $\lambda \in (\lambda_0, \infty)$.

Proof. We know that

$$\begin{aligned}\|g_m\|_{L^2(\Omega_w)} &\leqq \tilde{C}\|g_m\|_{L^4(\Omega_w)} \\ &\leqq \tilde{C}C_4\tau_4(w(m),\alpha,\lambda)\max_{\Omega}|\mathbb{G}^{(\lambda)}(\tilde{\chi}_{\Omega_w}\varphi_k^V)| \\ &\leqq CC_4 m^{-(3/4)+(1-\nu)/2)+\rho}\hat{C}m^{-1/2}\end{aligned}$$

by (2.17), (2.18), (2.21), (2.26). Since $\rho>0$ is arbitrary and $\nu\in(0,\frac{1}{3})$, we have the desired result.

3. Convergence of $\tilde{\mathbb{H}}_m^{(\lambda)}$ to $\mathbb{H}_\infty^{(\lambda)}$

In this section we will prove the following:

Proposition 4. *Fix* $\alpha>0$. *Assume that* $\{w(m)\}_{m=1}^{\infty}$ *satisfies* (C-1)$_\nu$, (C-2), (C-3). *Then there exists* λ *such that*

$$\lim_{m\to\infty} \|m^{\beta}(\tilde{\mathbb{H}}_m^{(\lambda)} - \mathbb{H}_\infty^{(\lambda)})\varphi_k^V\|_{L^2(\Omega)} = 0 \tag{3.1}$$

holds for any fixed $\beta\in[0,\frac{1}{4})$.

Proof. We examine the following term for $s\geqq 1$:

$$\begin{aligned}\not{h}_s(\lambda,u,v;w(m)) &= m^{-s}\sum_{(s)}(\mathbb{G}^{(\lambda)}v)(w_{i_1})G(w_{i_1},w_{i_2})\dots G(w_{i_{s-1}},w_{i_s})(\mathbb{G}^{(\lambda)}u)(w_{i_s}) \\ &\quad -\int_{\Omega}(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^s u)(x)v(x)\,dx \\ &= \sum_{h=1}^{s} J_{s,h}(\lambda,u,v;w(m)),\end{aligned} \tag{3.2}$$

where $\sum_{(1)}\dots$ means

$$\sum_{i=1}^{m}(\mathbb{G}^{(\lambda)}v)(w_i)(\mathbb{G}^{(\lambda)}u)(w_i),$$

and where

$$\begin{aligned}J_{s,s}(\lambda,u,v;w(m)) &= m^{-1}\sum_{i=1}^{m}(\mathbb{G}^{(\lambda)}v)(w_i)(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-1}u)(w_i) \\ &\quad -\int_{\Omega}(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^s u)(x)v(x)\,dx,\end{aligned} \tag{3.3}$$

$$J_{s,s-1}(\lambda,u,v;w(m)) = m^{-1}\sum_{i=1}^{m}(\mathbb{G}^{(\lambda)}v)(w_i)\left\{m^{-1}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m} G(w_{i_1},w_{i_2})\right.$$
$$\left.\cdot(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-2}u)(w_{i_2}) - (\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-1}u)(w_{i_1})\right\}, \tag{3.4}$$

$$J_{s,s-q}(\lambda,u,v;w(m)) = m^{-1}\sum_{i_1=1}^{m}(\mathbb{G}^{(\lambda)}v)(w_{i_1})\cdot\left(m^{-1}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m} G(w_{i_1},w_{i_2})\right.$$
$$\ldots\left(m^{-1}\sum_{\substack{i_q=1\\ i_q\neq i_{q-1}}}^{m} G(w_{i_{q-1}},w_{i_q})\left\{m^{-1}\sum_{\substack{i_{q+1}=1\\ i_{q+1}\neq i_q}}^{m} G(w_{i_q},w_{i_{q+1}})\right.\right.$$
$$\left.\cdot(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-q-1}u)(w_{i_{q+1}}) - (\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-q}u)(w_{i_q})\right\},$$
$$(1\leqq s-q, 2\leqq q). \tag{3.5}$$

We now put

$$\pi_{s-q}(\lambda,v;w(m)) = \left[m^{-1}\sum_{i=1}^{m}\left\{m^{-1}\sum_{\substack{j=1\\ j\neq i}}^{m} G(w_i,w_j)(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-q-1}v)(w_j)\right.\right.$$
$$\left.\left.-(\mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-q})(w_i)\right\}^2\right]^{1/2}. \tag{3.6}$$

By using Lemma 1 we get

$$|J_{s,s-q}(\lambda,u,v;w(m))| \leqq \left\{m^{-1}\sum_{i=1}^{m}(\mathbb{G}^{(\lambda)}u)(w_i)^2\right\}^{1/2}(m^{-1}\kappa(w(m);\lambda))^{q-1}$$
$$\cdot\pi_{s-q}(\lambda,v;w(m)) \tag{3.7}$$

for $q\geqq 1$ and

$$|J_{s,s}(\lambda,u,v;w(m))| \leqq \pi_s(\lambda,v;w(m)). \tag{3.8}$$

Therefore

$$\operatorname*{supremum}_{\|u\|_{L^2(\Omega)}\leqq 1}|\not{p}_s(\lambda,u,\varphi_k^V;w(m))| \leqq C\sum_{q=1}^{m}(m^{-1}\kappa(w(m);\lambda))^{q-1}\pi_{s-q}(\lambda,\varphi_k^V;w(m))$$
$$+\pi_s(\lambda,\varphi_k^V;w(m)) \tag{3.9}$$

holds for a constant C independent of λ. From now on we study π_s by using (C-3). We put

$$f''_{s-q,} = \mathbb{G}^{(\lambda)}(V\mathbb{G}^{(\lambda)})^{s-q-1}\varphi_k^V.$$

We see that there exists $\lambda_1\geqq 0$ such that

$$\sup_{x\in\Omega}|f''_{h,\lambda}(x)| \leqq C_4\|V\mathbb{G}^{(\lambda)}\|_{L^2(\Omega)}^{h-1}$$
$$\leqq C_5\lambda^{-h+1}$$
$$\leqq C_5(10^8\alpha^4)^{-h+1} \tag{3.10}$$

hold for any $\lambda \in [\lambda_1, \infty)$. We here fix $\lambda \geqq \lambda_1$ and take f_n in (C-3) as $f''_{h,\lambda}$. Taking into account of the assumption (C-3) we see that

$$\lim_{m\to\infty} m^{\beta/2} \sup_h 10^{2h}\alpha^h \pi_h(\lambda, \varphi_k^V; w(m)) = 0 \tag{3.11}$$

holds for any $\beta \in [0, \frac{1}{2})$, $\lambda \geqq \lambda_1$.

In summing up (2.23), (3.11) we get the following: Take an arbitrary $\varepsilon_0 > 0$ and fix it. Then there exist λ_0, m_0 and a constant C independent of m, s such that

$$\text{The term (3.9)} \leqq \varepsilon_0 \left\{ C \sum_{q=1}^{m} (10^{-3}\alpha^{-1})^{q-1} m^{-\beta/2} (10^{-2}\alpha^{-1})^{s-q} + (10^{-2}\alpha^{-1})^s m^{-\beta/2} \right\}$$

$$\leqq C\varepsilon_0 m^{-\beta/2} \alpha^{-s} 10^{-2s} 10^3 (\alpha s + 1) \tag{3.12}$$

holds for any $\lambda \geqq \lambda_0$, $m \geqq m_0$. Here β is a fixed constant in $(0, \frac{1}{2})$.

From now on we want to estimate

$$\langle (\tilde{\mathbb{H}}_m^{(\lambda)} - \mathbb{H}_\infty^{(\lambda)}) \varphi_k^V, u \rangle_{L^2} \equiv \langle \varphi_k^V, (\tilde{\mathbb{H}}_m^{(\lambda)} - \mathbb{H}_\infty^{(\lambda)}) u \rangle_{L^2},$$

where $\langle , \rangle_{L^2}$ denote the usual $L^2(\Omega)$ inner product. Recall the definition of $\tilde{\mathbb{H}}_m^{(\lambda)}$. Then we see that (3.13) is equal to $M_1 + M_2 + M_3$, where

$$M_1 = \sum_{s=1}^{m} \not{h}_s(\lambda, u, \varphi_k^V; w(m)) (-4\pi\alpha)^s \exp(\lambda^{1/2}(\alpha s/m)),$$

$$M_2 = \sum_{s=1}^{m} (\exp(\lambda^{1/2}(\alpha s/m)) - 1)(-4\pi\alpha)^s \langle \mathbb{G}^{(\lambda)} (V \mathbb{G}^{(\lambda)})^s u, \varphi_k^V \rangle_{L^2},$$

$$M_3 = -\sum_{s=m+1}^{\infty} (-4\pi\alpha)^s \langle \mathbb{G}^{(\lambda)} (V \mathbb{G}^{(\lambda)})^s u, \varphi_k^V \rangle_{L^2}. \tag{3.14}$$

By (3.12) we see that

$$m^{\beta/2} \operatorname*{supremum}_{\|u\|_{L^2(\Omega)} \leqq 1} |M_1| \leqq C\varepsilon_0 \left(\sum_{s=1}^{m} 10^{-2s} (4\pi)^s 10^3 (\alpha s + 1) \exp(\lambda^{1/2}\alpha) \right).$$

We divide M_2 into two parts.

$$M_2 = M_{21} + M_{22},$$

$$M_{21} = \sum_{s=1}^{[m^{1/2}]}, \quad M_{22} = \sum_{s=[m^{1/2}]+1}^{m},$$

where [ ] denotes the Gauss symbol. Take an arbitrary $\varepsilon_1 > 0$. Then we can take λ_2 sufficiently large so that

$$\left| \sum_{s=1}^{m} (4\pi\alpha)^s \langle \mathbb{G}^{(\lambda)} (V \mathbb{G}^{(\lambda)})^s u, \varphi_k^V \rangle_{L^2} \right| \leqq \varepsilon_1$$

holds for any $\lambda > \lambda_2$. Fix $\lambda^* \in [\lambda_2, \infty)$. Since

$$|\exp(\lambda^{1/2}(\alpha/m)[m^{1/2}]) - 1| \leqq C_5 m^{-1/2} \alpha \lambda^{1/2},$$

we know that there exists m_0 such that

$$m^{1/3} |M_{21}| \leqq \varepsilon_1$$

holds for any $m \geqq m_0$. Also we can see that there exists λ and m_0 such that $m(|M_{22}|+|M_3|) \leqq \varepsilon_1$ holds for any $m \geqq m_0$.

In summing up these facts we get Proposition 4. q.e.d.

Proposition 1 is an easy consequence of Proposition 4, Lemma 6, Corollary 1.

4. Probabilistic consideration on (C-1) ~ (C-3)

We recall a basic argument concerning the law of large numbers. Let $E(\cdot)$ denote the expectation. Let $g(x)$ be a square integrable function satisfying

$$E(g(\cdot))=0,$$

that is

$$\int_{\Omega} g(x)V(x)\,dx=0.$$

We consider

$$S_m(g;\cdot)=m^{-1}\sum_{i=1}^{m} g(w_i)$$

as the sum of independent random variables. We know

$$\mathbb{P}(w(m)\in\Omega^m; S_m(g;\cdot)^2 \geqq \varepsilon) \leqq \varepsilon^{-1}m^{-1}\|g\|^2_{L^2(\Omega)} \tag{4.1}$$

from

$$E(S_m(g;\cdot)^2)=m^{-1}\|g\|^2_{L^2(\Omega)}.$$

We now put

$$q_h(w(m))=m^{\beta}C_*^{h/2}\left(m^{-1}\sum_{i=1}^{m} f_h(w_i)-\int_{\Omega} f_h(x)V(x)\,dx\right), \tag{4.2}$$

and

$$\tilde{q}_h(w(m))=m^{\beta}C_*^{h/2}m^{-1}\sum_{i=1}^{m}\left\{m^{-1}\sum_{\substack{j=1\\ j\neq 1}}^{m} G^{(\lambda)}(w_i,w_j)f_h(w_j)\right.$$
$$\left.-(\mathbb{G}^{(\lambda)}Vf_h)(w_i)\right\}^2. \tag{4.3}$$

By (4.1) we have

$$\mathbb{P}(w(m)\in\Omega^m; |q_h(w(m))| \geqq \varepsilon) \leqq 4\varepsilon^{-2}C_*^{-h}m^{2\beta-1}|\Omega|D^{*^2}. \tag{4.4}$$

Here $|\Omega|$ = volume of Ω. Thus,

$$\mathbb{P}(w(m); \sup_h |q_h(w(m))| \leqq \varepsilon) \geqq 1-4\varepsilon^{-2}m^{2\beta-1}|\Omega|D^{*^2}\sum_{h=1}^{m} C_*^{-h}. \tag{4.5}$$

From now on we examine $\tilde{q}_h$. It is divided into three parts $m^{\beta}C_*^{h/2}(L_{1,h}+L_{2,h}+L_{3,h})$, where

$$L_{1,h}=m^{-1}\sum_{i=1}^{m}\left\{m^{-1}\sum_{\substack{j=1\\ j\neq i}}^{m} G^{(\lambda)}(w_i,w_j)f_h(w_j)\right\}^2, \tag{4.6}$$

$$L_{2,h} = -2m^{-2} \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} G^{(\lambda)}(w_i, w_j)(\mathbb{G}^{(\lambda)} V f_h)(w_i) f_h(w_j), \tag{4.7}$$

$$L_{3,h} = m^{-1} \sum_{i=1}^{m} (\mathbb{G}^{(\lambda)} V f_h)(w_i)^2. \tag{4.8}$$

We put

$$\langle L \rangle_h = \int_{\Omega} (\mathbb{G}^{(\lambda)} V f_h)(x)^2 \, dx.$$

It is easy to see that

$$\begin{aligned} \mathbb{P}(w(m) \in \Omega^m; |L_{3,h} - \langle L \rangle_h| \geqq \varepsilon) &\leqq 4\varepsilon^{-2} m^{-1} |\Omega| \max_{\Omega} |\mathbb{G}^{(\lambda)} V f_h|^2 \\ &\leqq 4\varepsilon^{-2} C_0 m^{-1} |\Omega| C_*^{-2h}. \end{aligned}$$

Therefore

$$\begin{aligned} P_{1,m} &\equiv P\left(w(m) \in \Omega^m; m^\beta \sup_h C_*^{h/2} |L_{3,h} - \langle L \rangle_h| \leqq \varepsilon \right) \\ &\geqq 1 - 4\varepsilon^{-2} C_0 |\Omega| m^{2\beta - 1} \sum_{h=1}^{\infty} C_*^{-h}. \end{aligned} \tag{4.9}$$

We see that $E(L_{2,h}) = -2\langle L \rangle_h$. By a similar argument to that above we get

$$\begin{aligned} \mathbb{P}(w(m) \in \Omega^m; m^\beta |L_{2,h} + 2\langle L \rangle_h| \geqq \varepsilon) &\leqq 16 C_0 \varepsilon^{-2} m^{2\beta - 2} \max_{\Omega} |\mathbb{G}^{(\lambda)} V f_h|^2 \max_{\Omega} |f_h|^2 \\ &\leqq 16 C_1 \varepsilon^{-2} m^{2\beta - 2} C_*^{-4h}. \end{aligned} \tag{4.10}$$

Thus

$$\begin{aligned} P_{2,m} &\equiv \mathbb{P}\left(w(m) \in \Omega^m; \mathrm{m}^\beta \sup_h C_*^{h/2} |L_{2,h} + 2\langle L \rangle_h| \leqq \varepsilon \right) \\ &\leqq 1 - 16 C_1 \varepsilon^{-2} m^{2\beta - 2} \sum_{h=1}^{m} C_*^{-3h}. \end{aligned} \tag{4.11}$$

Notice that $L_{1,h} = \dot{L}_{1,h} + \ddot{L}_{1,h}$, where

$$\dot{L}_{1,h} = m^{-3} \sum_{\substack{i,j,k=1 \\ i \neq j, j \neq k, k \neq i}} G^{(\lambda)}(w_i, w_j) G^{(\lambda)}(w_i, w_k) f_h(w_j) f_h(w_k),$$

$$\ddot{L}_{1,h} = m^{-3} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} G^{(\lambda)}(w_i, w_j)^2 f_h(w_j)^2.$$

Then we also see that

$$\begin{aligned} \lim_{m \to \infty} P_{3,m} &\equiv \lim_{m \to \infty} \mathbb{P}\left(w(m) \in \Omega^m; m^\beta \sup_h C_*^{h/2} |L_{1,h} - \langle L \rangle_h| \geqq \varepsilon \right) \\ &= 0 \end{aligned} \tag{4.12}$$

when $\beta\in[0,\frac{1}{2})$. Since

$$\sup_h |\tilde{q}_h| \leqq m^{\beta}\left\{\sup_h C_*^{h/2}|L_{1,h}-\langle L\rangle_h| + \sup_h C_*^{h/2}|L_{2,h}+2\langle L\rangle_h| + \sup_h C_*^{h/2}|L_{3,h}-\langle L\rangle_h|\right\},$$

we have

$$\mathbb{P}\left(w(m)\in\Omega^m; \sup_h |\tilde{q}_h(w(m))| \geqq 3\varepsilon \right) \leqq \max\{(1-P_{1,m}),(1-P_{2,m}),(1-P_{3,m})\}.$$

Hence we get

$$\lim_{m\to\infty} \mathbb{P}\left(w(m)\in\Omega^m; \sup_h |\tilde{q}_h(w(m))| \geqq 3\varepsilon \right) = 0 \tag{4.13}$$

if $\beta\in[0,\frac{1}{2})$ for any $\varepsilon>0$.

We easily see that

$$\lim_{m\to\infty} \mathbb{P}(w(m)\in\Omega^m;\ \text{(C-2) does not hold}) = 0, \tag{4.14}$$

since the probability of

$$\left| m^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{m} |w_i-w_j|^{-3+\xi} - \int_\Omega\int_\Omega |x-y|^{-3+\xi}\,dx\,dy \right| \geqq \varepsilon,$$

tends to zero as $m\to\infty$.

Finally we examine $(\text{C-1})_\nu$. By a simple combinatorial argument we have

$$\begin{aligned}
&\mathbb{P}\left(w(m)\in\Omega^m; \min_{i\neq j}|w_i-w_j| < C_0 m^{-1+\nu} \right)\\
&\leqq \binom{2}{m}\mathbb{P}((w_1,w_2)\in\Omega^2; |w_1-w_2| < C_0 m^{-1+\nu})\\
&\leqq \tilde{C}m^{3\nu-1}.
\end{aligned} \tag{4.15}$$

Thus (4.14) tends to zero as $m\to\infty$, if $\nu\in(0,\frac{1}{3})$.

We are now in a position to prove Theorem 1. In summing up these facts and Proposition 1, we have the following:

$$\lim_{m\to\infty} \mathbb{P}(w(m)\in\Omega^m; m^{\beta}\|(\mathbb{G}_m^{(\lambda)}-\mu_k^V)\varphi_k^V\|_{L^2(\Omega_w)} < \varepsilon) = 1 \tag{4.16}$$

for any fixed $\varepsilon>0$, $\beta\in[0,\frac{1}{4})$. We know from the spectral theory of self-adjoint compact operators that

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{P}(w(m))\in\Omega^m;\ &\text{there exists at least } \mathfrak{M}_k\text{-eigenvalues}\\
&\mu_{k_j}(w(m)), j=1,\dots,\mathfrak{M}_k \text{ of } \mathbb{G}_m^{(\lambda)}\\
&\text{satisfying } |\mu_{k_j}(w(m))-\mu_k^V| < 2\varepsilon m^{-\beta})\\
=1.&
\end{aligned} \tag{4.17}$$

Here $\mathfrak{M}_k$ denotes the multiplicity of φ_k^V. On the other hand, we know that Theorem 1 with $\tilde{\delta}=0$ holds. See Kac [4], Rauch–Taylor [13] and p. 235 of Simon [14]. By combining Theorem 1 with $\tilde{\delta}=0$ and (4.17), we get the theorem for general $\tilde{\delta}\in[0,\frac{1}{4})$.

The author hopes that Theorem 1 with $\tilde{\delta}=0$ can also be proved by using our perturbational calculus.

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