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Universal Properties of Maps of the Circle with ε-Singularities

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Abstract. Following the work of Collet, Eckmann, and Lanford on the Feigenbaum conjecture, we study the structure of the renormalization transformation introduced in [12] upon maps of the circle with critical points of the form $x|x|^{\varepsilon}$.

1. Introduction

There appears to be a remarkable relationship between the seemingly universal features of certain bifurcations which destroy invariant tori of dissipative systems with particular incommensurate frequencies and the scaling properties of certain families of analytic mappings of the circle $T^1 = \mathbb{R}/\mathbb{Z}$. An explanation of this in terms of a renormalization transformation $\mathcal{T}$ is proposed in [12] and [4]. The evidence for this is numerical. Following the work of Collet et al. on the Feigenbaum conjecture [3] we study the action of $\mathcal{T}$ on a space of analytic functions of $x|x|^{\varepsilon}$, where $\varepsilon \geqq 0$ is small. The physically interesting case is $\varepsilon = 2$.

1.1. Motivation

A *cubic critical map* is any analytic homeomorphism of the circle which has a single critical point which is cubic. We will be interested here in the behaviour of (analytic) diffeomorphisms and critical maps related to the rotation number

$$\sigma = (\sqrt{5} - 1)/2 = 1/(1 + 1/(1 + 1/1 + \dots .$$

Similar results hold for any rotation number with a periodic (or eventually periodic) continued fraction, but to ease the exposition we stuck to the simplest case here. The rational approximants to σ are the numbers q_n/q_{n+1}, where

$$q_0/q_1 = 1/1, \quad q_1/q_2 = 1/(1+1), \quad q_2/q_3 = 1/(1 + 1/(1+1)), \dots ,$$

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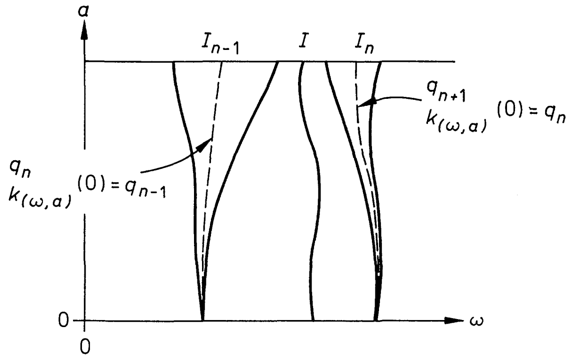

Fig. 1

i.e. $q_0=1$, $q_1=1$ and $q_{n+1}=q_n+q_{n-1}$. Two results about diffeomorphisms (which are elementary consequences of results of [5]) and cubic critical maps (from numerical experiments [12, 13]) are these:

1. Let k be the lift to $\mathbb{R}$ of an analytic circle diffeomorphism or cubic critical map whose rotation number $\varrho(k)$ is σ. Let $\alpha^{(n)}=-1/(k^{q_{n+1}}(0)-k^{q_n}(0)-q_{n-2})$, and let k_n be the function on $[0,1]$ given by

$$k_n(x)\equiv\alpha^{(n)}(k^{q_n}(x/\alpha^{(n)})-q_{n-1}).$$

Then $\alpha(k)=\lim\limits_{n\to\infty}\alpha^{(n+1)}/\alpha^{(n)}$, and $\zeta(k)=\lim\limits_{n\to\infty}k_n$ exist, and (i) $\alpha(k)=-\sigma^{-1}\simeq-1.6180\ldots$ and $\zeta(k)(x)\equiv x+\sigma$ if k is a diffeomorphism, while (ii) $\alpha(k)\simeq-1.2886\ldots$ and $\zeta(k)$ is a nontrivial function of x^3 if k is a cubic critical map, and ζ is independent of k.

2. Let k_μ be a 1-parameter family of (lifts of) diffeomorphisms or a 1-parameter family of cubic critical maps such that $\varrho(k_0)=\sigma$. Then if k_μ is transverse to the manifold $\varrho=\sigma$ and if numbers μ_n are chosen so that $k^{q_{n+1}}_{\mu_n}(0)=q_n$, there exists a $\delta<0$ such that $\lim\limits_{n\to\infty}\delta^n_{\mu_n}$ exists and is non-zero. For diffeomorphisms $\delta=-\sigma^{-2}=-2.6180\ldots$, but numerical experiments reveal that for cubic critical maps $\delta\simeq-2.8336\ldots$. Note that the objects δ, α, and ζ are essentially independent of the system being studied.

Consider for example the 2-parameter family

$$k_{(\omega,a)}(x)=x+\omega-(a/2\pi)\sin 2\pi x,$$

where $0\leqq\omega$, $a\leqq1$. The equation $\varrho(k_{(\omega,a)})=\sigma$ defines a smooth curve I which is approximated by the "tongues" I_n on which $\varrho(k_{(\omega,a)})=q_n/q_{n+1}$ [1, 5, 6]. For $0\leqq a<1$, the tongues accumulate on $\varrho=\sigma$ at a rate $(-\sigma^2)^n\simeq(-0.382)^n$. For $a=1$ this rate is approximately $(-0.353)^n$.

1.2. Renormalization Analysis

We now discuss briefly and heuristically how one can understand these results in terms of a renormalization transformation $\mathscr{T}$.

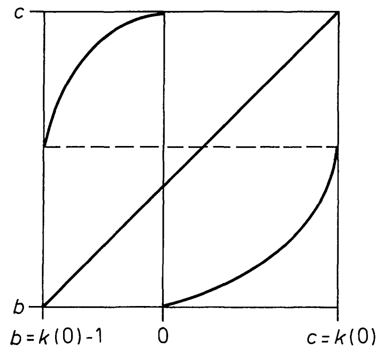

Fig. 2

Let ϕ be a cubic critical map or diffeomorphism of the circle whose rotation number is approximately σ, and represent ϕ as a map of the interval $[b, c] = [k(0)-1, k(0)]$ as in Fig. 2. Note that $\varrho(\phi) \simeq \sigma$ implies that $[b, 0]$ gets mapped into $[0, c]$. Now we construct a new map $\tilde{\phi}$ of the circle in the following way (see Fig. 3). Let $\hat{\phi} : [b, \phi(b)] \to [b, \phi(b)]$ be defined by $\hat{\phi} = \phi^2$ on $[b, 0]$ and $\hat{\phi} = \phi$ on $[0, \phi(b)]$. Now take $\tilde{\phi}(x) \equiv \alpha\hat{\phi}(x/\alpha)$, where $\alpha = -1/(\phi(b)-b)$. Note that α is chosen to be negative since this is necessary if we are to hope for such a process to coverge upon iteration.

It is easy to check that $\varrho(\tilde{\phi}) = \varrho(\phi)^{-1} - 1$; thus ϱ is invariant under this process if and only if $\varrho(\phi) = \sigma$ and a map with rotation number q_n/q_{n+1} is sent to one with q_{n-1}/q_n.

The problem is how to set this up as a renormalization scheme so that the scaling properties, etc. can be deduced from the structure of the dynamics near a fixed point. We note that even though ϕ was analytic, the map $\tilde{\phi}$ is not; it has discontinuities in its derivatives at 0 and at the end-points of the interval. A convenient way to deal with this problem is to enlarge the space and work with pairs (ξ, η) of analytic homeomorphisms of the line which when glued together define a map of the circle.

In fact, one considers the set[1] $\mathcal{N}$ of pairs (ξ, η) such that

(a) $0 < \xi(0) = \eta(0) + 1 < 1$;

(b) $\xi\eta(0) = \eta\xi(0)$;

(c)[2] $(\xi\eta - \eta\xi)'(0) = 0$; and

(d) if $\xi'(x) \leqq 0$ or $\eta'(x) \leqq 0$ for some $x \in (\eta(0), \xi(0))$, then $x = 0$ and $\xi'(0) = \eta'(0) = \xi''(0) = \eta''(0) = 0$ and $\xi'''(0)$ and $\eta'''(0)$ are non-zero.

We let $\mathcal{N}_{\text{crit}}$ denote the subset consisting of those (ξ, η) such that $\xi'(0) = \eta'(0) = 0$. Let $\mathcal{N}_0$ consist of those pairs (ξ, η) in $\mathcal{N}$ such that

(e) $\xi\eta(0) > 0$.

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1 For a precise definition suitable to our purposes see Sect. 2.1

2 This condition is introduced to ensure that the fixed points of $\mathcal{T}$ are hyperbolic. It is automatically satisfied by those (ξ, η) coming from analytic circle maps because then ξ and η commute. In fact, it would be quite natural to restrict to the class of commuting pairs, but so far as the results of this paper are concerned this is unnecessary

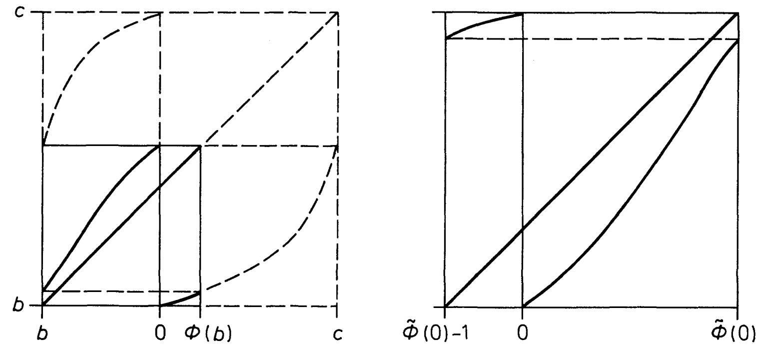

Fig. 3

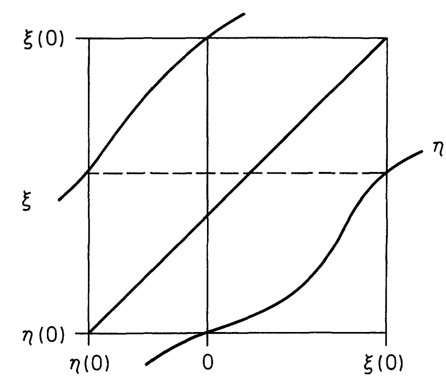

Fig. 4

To each $(\xi,\eta)\in\mathcal{N}$ we can associate a mapping $\phi=\phi(\xi,\eta)$ of the circle by defining ϕ to be ξ on $[\eta(0),0]$ and η on $[0,\xi(0)]$. In this way we can define the rotation number $\varrho(\xi,\eta)=\varrho(\phi(\xi,\eta))$ of (ξ,η).

We now consider the mapping $\mathcal{T}:\mathcal{N}_0\to\mathcal{N}$ corresponding to $\phi\to\tilde{\phi}$. It is given by $\mathcal{T}(\xi,\eta)=(\alpha\eta\alpha^{-1},\alpha\eta\xi\alpha^{-1})$, where $\alpha=-1/(\xi\eta(0)-\eta(0))$.

If $k:\mathbb{R}\to\mathbb{R}$ is the lift of ϕ chosen so that $0<k(0)<1$, then $(\xi,\eta)=(k,k-1)$, and when defined [e.g. when $\varrho(k)=\sigma$],

$$\mathcal{T}^n(\xi,\eta)=(\alpha^{(n)}k_n(\alpha^{(n)})^{-1},\alpha^{(n)}k_{n+1}(\alpha^{(n)})^{-1}),$$

where $k_n=k^{q_n}-q_{n-1}$. One fixed point of $\mathcal{T}$ is $\xi_0(x)=x+\sigma$, $\eta_0(x)=x-\sigma^2$.

The main claim made in [12], based upon numerical studies, is that there is only one other fixed point (ξ_1,η_1) of $\mathcal{T}$ in $\mathcal{N}$ and that this point has the following properties:

(a) ξ_1 and η_1 are analytic functions of x^3;

(b) $\lim_{n\to\infty}\alpha^{(n+1)}/\alpha^{(n)}=\alpha_1=\xi_1(0)/(\xi_1(0)-1)\simeq-1.2886\ldots$;

(c) (ξ_1,η_1) is hyperbolic with a 2-dimensional unstable manifold and a stable manifold consisting of those (ξ,η) in $\mathcal{N}_{\mathrm{crit}}$ with $\varrho(\xi,\eta)=\sigma$;

(d) one of the unstable eigendirections is tangent to $\mathcal{N}_{\mathrm{crit}}$ and has eigenvalue $\delta\simeq-2.8336\ldots$, the other has eigenvalue α_1^2 and describes the effect of adding a small linear term to the fixed point. That is, it describes the cross-over from behaviour associated with critical maps to that associated with diffeomorphisms. The scaling structure of a 2-parameter family of analytic cricle mappings which is transverse to the stable manifold of (ξ_1, η_1) can now be related to that of the universal family given by the 2-dimensional unstable manifold of (ξ_1, η_1).

Clearly one would like to give a proof of these facts. The first step would be to study the action of $\mathscr{T}$ on analytic functions of x^3, but this is a hard problem which is probably best addressed with the assistance of a computer as in Lanford's work [10]. In this paper we follow the approach of [3] and study $\mathscr{T}$ on analytic functions of $x|x|^\varepsilon$ for small ε. By means of perturbation methods we show that for sufficiently small ε, $\mathscr{T}$ has a hyperbolic fixed point on the space of analytic functions of $x|x|^\varepsilon$. Since we have a fixed point for $\varepsilon=0$, our proofs are much simpler than those in [3]. Of course the case $\varepsilon=2$ is not covered by our results. Nevertheless the results provide strong evidence that the conjecture and numerical results of [12] are correct.

2. Preliminaries and Statement of Results

We are going to consider functions $(\xi, \eta) = (\xi(f), \eta(g)) \in \mathscr{N}$ of the form

$$\xi(x) = f(x^{(\varepsilon)}), \qquad \eta(x) = g(x^{(\varepsilon)}),$$

where $x^{(\varepsilon)} = x|x|^\varepsilon$, and f and g are functions analytic in certain domains of the complex plane.

2.1. Definition and Properties of the Function Space

Choose numbers c_1, c_2, c_3, and c_4 with the following properties: $c_4 \in (\sigma, 1)$; $c_3 \in \left(1 - \frac{1}{\sigma} c_4, 0\right)$; $c_1 \in \left(-\frac{1}{\sigma} c_4, -\sigma c_4\right)$ and $c_2 \in \left(-\sigma c_3, -\frac{1}{\sigma} c_3\right)$. Then if $I_1 = (c_1, c_2)$ and $I_2 = (c_3, c_4)$, $-\sigma \bar{I}_1 \subset I_2$, $-\sigma \bar{I}_2 \subset I_1$ and $-\sigma \bar{I}_2 + \sigma \subset (0, c_4) \subset I_2$. Let Ω_1 and Ω_2 be, respectively, the open disks in $\mathbb{C}$ on the diameters I_1 and I_2. Since $x \to -\sigma x$ and $x \to -\sigma x + \sigma$ are similarities we have

$$-\sigma \bar{\Omega}_1 \subset \Omega_2 \,; \tag{1a}$$

$$-\sigma \bar{\Omega}_2 \subset \Omega_1 \,; \tag{1b}$$

and

$$-\sigma \bar{\Omega}_2 + \sigma \subset \Omega_2 \cap (\mathbb{C} \backslash \mathbb{R}_-) \,, \tag{1c}$$

where $\mathbb{R}_-$ is the negative real axis. For $i=1,2$ let $\mathscr{A}_i$ be the Banach space of functions analytic in Ω_i, continuous on $\bar{\Omega}_i$ and taking real values on I_i. Let $\mathscr{B} = \mathscr{A}_1 \times \mathscr{A}_2$. If for $f \in \mathscr{A}_1$ and $g \in \mathscr{A}_2$ we let $\|f\|$ and $\|g\|$ denote the sup norms of f and g on $\bar{\Omega}_1$ and $\bar{\Omega}_2$ respectively, then we take the norm $\|(f, g)\| = \|f\| + \|g\|$ on $\mathscr{B}$.

Let $f_0(z) = z + \sigma$ and $g_0(z) = z - \sigma^2$, and let $\mathscr{U}$ be the neighbourhood of (f_0, g_0) in $\mathscr{B}$ defined in Sect. 2.3.

We are especially interested in those pairs $(f,g)\in\mathscr{U}$ which for a given small ε satisfy the following conditions analogous to (a)–(e) in Sect. 1.2:

(a′) $0<f(0)=g(0)+1<1$;

(b′) $g(f(0)^{(\varepsilon)})=f(g(0)^{(\varepsilon)})$; and

(c′) $g'(f(0)^{(\varepsilon)})|f(0)|^{\varepsilon}f'(0)=f'(g(0)^{(\varepsilon)})|g(0)|^{\varepsilon}g'(0)$.

(d′) $f'>0$ on I_1 and $g'>0$ on I_2.

In particular, condition (d′) is the ε-analogue of (d). A straightforward calculation using the Implicit Function Theorem shows that the conditions define a submanifold $\mathscr{M}^{\varepsilon}$ of $\mathscr{B}$. These calculations will be outlined in Sect. 3.2.

In Sect. 3.2 we shall also define an open subset $\mathscr{M}_0^{\varepsilon}$ of $\mathscr{M}^{\varepsilon}$ containing (f_0,g_0) and satisfying the following condition arising from (e):

(e′) $f(g(0)^{(\varepsilon)})>0$.

For small $|\varepsilon|$, let $\mathscr{N}^{\varepsilon}$ (respectively, $\mathscr{N}_0^{\varepsilon}$) denote the set of (ξ,η) of the form $\xi(x)=f(x^{(\varepsilon)})$ and $\eta(x)=g(x^{(\varepsilon)})$, where $(f,g)\in\mathscr{M}^{\varepsilon}$ (respectively, $\mathscr{M}_0^{\varepsilon}$). By identifying $\mathscr{N}^{\varepsilon}$ with $\mathscr{M}^{\varepsilon}$ and $\mathscr{N}_0^{\varepsilon}$ with $\mathscr{M}_0^{\varepsilon}$ one may regard $\mathscr{N}^{\varepsilon}$ and $\mathscr{N}_0^{\varepsilon}$ as Banach manifolds. We define $\mathscr{N}=\mathscr{N}^0$.

2.2. *Definition of $\mathscr{T}_{\varepsilon}$ and Statement of Results*

The mapping $\mathscr{T}_{\varepsilon}:\mathscr{N}_0^{\varepsilon}\to\mathscr{N}^{\varepsilon}$ is defined by $\mathscr{T}_{\varepsilon}(\xi,\eta)=(\alpha\eta\alpha^{-1},\alpha\eta\xi\alpha^{-1})$, where $\alpha=-1/(\eta\xi(0)-\eta(0))$.

Theorem 1. *The mapping $\mathscr{T}_{\varepsilon}$ is C^{∞}.*

The rotation $(\xi_0,\eta_0)=(f_0,g_0)$, where $f_0(z)=z+\sigma$ and $g_0(z)=z-\sigma^2$, is obviously a fixed point for $\mathscr{T}_0$. Below we analyse its hyperbolic structure and its dependence upon ε. Then we can deduce this:

Theorem 2. *For $|\varepsilon|$ sufficiently small, $\mathscr{T}_{\varepsilon}$ has a fixed point $(\xi_{\varepsilon},\eta_{\varepsilon})$ in $\mathscr{N}_0^{\varepsilon}$, where $\xi_{\varepsilon}(z)=f_{\varepsilon}(z^{(\varepsilon)})$ and $\eta_{\varepsilon}(z)=g_{\varepsilon}(z^{(\varepsilon)})$ and $(f_{\varepsilon},g_{\varepsilon})\in\mathscr{M}_0^{\varepsilon}$. The function $f_{\varepsilon}(z)$ (respectively, $g_{\varepsilon}(z)$) is jointly C^{∞} in ε and z for $|\varepsilon|$ sufficiently small and $z\in\Omega_1$ (respectively, Ω_2). The linear operator $d\mathscr{T}_{\varepsilon}(\xi_{\varepsilon},\eta_{\varepsilon})$ is compact. Its spectrum consists of countably many distinct eigenvalues of finite multiplicity. One of these is $\delta_{\varepsilon}=(-1/\sigma^2)+O(\varepsilon)$; this eigenvalue is simple. The remainder of the spectrum is contained in a disk of the form $|z|\leqq\sigma_{\varepsilon}=\sigma+O(\varepsilon)$. In fact δ_{ε} and σ_{ε} are C^{∞} functions of ε.*

Numerical evidence indicates that $\delta_{\varepsilon}=(-1/\sigma^2)+O(\varepsilon^2)$. However we have not been able to prove this. An expression for the derivative $\frac{d}{d\varepsilon}(f_{\varepsilon},g_{\varepsilon})$ at $\varepsilon=0$ is easier to obtain and is given at the end of proof of Theorem 2. Kadanoff in [8] also addresses this question and presents a formal first order perturbation theory calculation concerning the existence of the ε-derivative of $(f_{\varepsilon},g_{\varepsilon})$.

To describe some of the relevant local and global dynamics of $\mathscr{T}=\mathscr{T}_{\varepsilon}$, we need to recall the following definitions which are complicated a little by the fact that $\mathscr{T}$ is not invertible (cf. [3]). For example, there do not necessarily exist global stable and unstable manifolds.

1. A *stable manifold* of $\zeta_{\varepsilon}=(\xi_{\varepsilon},\eta_{\varepsilon})$ is a smooth submanifold W^s of $\mathscr{N}_0^{\varepsilon}$ such that

(a) $\mathscr{T}W^s\subset W^s$,

(b) if $\zeta = (\xi, \eta) \in W^s$, then $\lim_{j \to \infty} \mathcal{T}^j \zeta = \zeta_\varepsilon$, and

(c) for any $\zeta \in W^s$, the range of $d\mathcal{T}(\zeta)$ is not contained in the tangent space to W^s at $\mathcal{T}(\zeta)$.

2. An *unstable manifold* of ζ_ε is a smooth submanifold W^u of $\mathcal{N}^\varepsilon$ such that

(a) $\mathcal{T}(W^u \cap \mathcal{N}_0^\varepsilon) \supset W^u$,

(b) if $\zeta \in W^u$ there is a sequence ζ_j converging to ζ_ε such that $\zeta = \mathcal{T}^j \zeta_j$, and

(c) if $\zeta \in W^u \cap \mathcal{N}_0^\varepsilon$, the tangential derivative of $\mathcal{T}$ along W^u at ζ is non-zero.

Theorem 3. *For $|\varepsilon|$ sufficiently small $(\xi_\varepsilon, \eta_\varepsilon)$ has a stable manifold W_ε^s of codimension one and a one-dimensional unstable manifold W_ε^u. If $(\xi, \eta) \in W_\varepsilon^s$ then $\varrho(\xi, \eta) = \sigma$. For some neighbourhood $\mathcal{V}_0$ of $(\xi_\varepsilon, \eta_\varepsilon)$*

$$W_\varepsilon^s \cap \mathcal{V}_0 = \{(\xi, \eta) \in \mathcal{V}_0 : \varrho(\xi, \eta) = \sigma\}.$$

For $|\varepsilon|$ sufficiently small, W_ε^u is C^1-close to the curve $\lambda \to (\xi_\varepsilon + \lambda, \eta_\varepsilon + \lambda)$.

Remark. Using Herman's theorem on the existence of an analytic conjugacy [5], one can assert the following stronger result for analytic diffeomorphisms (the case $\varepsilon = 0$): *If $\varepsilon = 0$ and $(\xi, \eta) = (f, g) = (f(\phi), g(\phi))$, where ϕ is an analytic diffeomorphism of the circle, then the following conditions are equivalent:*

(i) $\varrho(\xi, \eta) = \sigma$,

(ii) *$\mathcal{T}^n(\xi, \eta)$ is well-defined for all n, and for sufficiently large n, $\mathcal{T}^n(\xi, \eta) \in \mathcal{V}_0$ and $\mathcal{T}^n(\xi, \eta) \to (\xi_0, \eta_0)$ as $n \to \infty$.*

Let $\mathcal{P}(p/q, \varepsilon)$ denote the set of $(\xi, \eta) \in \mathcal{N}^\varepsilon$ such that $\phi = \phi(\xi, \eta)$ has 0 as a periodic point with rotation number $p/q \in \mathbb{Q}$.

Theorem 4. *For $|\varepsilon|$ sufficiently small and n sufficiently large, $\mathcal{P}(q_n/q_{n+1}, \varepsilon)$ meets W_ε^u in a single point and the intersection is transversal.*

The next theorem states that the expanding eigenvalue δ_ε of $d\mathcal{T}_\varepsilon(f_\varepsilon, g_\varepsilon)$ appears as a universal invariant in one-parameter families of circle mappings with ε-singularities.

Theorem 5. *For $|\varepsilon|$ sufficiently small there is a neighbourhood $\mathcal{V}$ of $(\xi_\varepsilon, \eta_\varepsilon)$ with the following properties:*

(a) *Suppose $\gamma(\mu) = (\xi(\mu), \eta(\mu)) \in \mathcal{V}$ is a continuously differentiable curve which crosses W_ε^s transversally at $\gamma(\mu_\infty)$. Then for sufficiently large n, $\gamma(\mu)$ meets $\mathcal{P}(q_n/q_{n+1}, \varepsilon)$ at a unique point $\gamma(\mu_n)$, and*

$$\lim_{n \to \infty} \delta_\varepsilon^n(\mu_\infty - \mu_n)$$

exists and is non-zero.

(b) *Let $\alpha_\varepsilon = \xi_\varepsilon(0)/(\xi_\varepsilon(0) - 1)$, and suppose that $\phi = \phi(\xi, \eta)$ for some $(\xi, \eta) \in \mathcal{V}$, and that $k : \mathbb{R} \to \mathbb{R}$ is the lift of ϕ with $k(0) \in (0, 1)$. Then $\varrho(\xi, \eta) = \sigma$ implies that*

$$\lim_{n \to \infty} \alpha_\varepsilon^n(k^{q_n}(0) - q_{n-1})$$

exists and is non-zero.

Theorem 6. *If ε is sufficiently small, $(\xi, \eta) \in W_\varepsilon^s$ and $\phi = \phi(\xi, \eta)$, then ϕ is C^0-conjugate to the rotation R_σ.*

Proof. Recall from the classical theory of homeomorphisms that ϕ is C^0-conjugate to R_σ if and only if there does not exist an interval $I \subset T^1$ such that $\phi^n I \cap I = \emptyset$ for all $n \geqq 0$. Call such an interval a Denjoy interval.

Suppose $(\xi, \eta) \in W^s_\varepsilon \cap \mathscr{V}$ and let $\phi = \phi(\xi, \eta)$. By Theorem 5 (b), $\phi^{q_n}(0)$ accumulates on 0 from both sides, and so 0 is not contained in a Denjoy interval. Consequently, $\phi(0)$ and $\phi^2(0)$ are not in Denjoy intervals. Let $l(\xi, \eta)$ denote the length of a largest Denjoy interval. By the above, any Denjoy interval I of this length must be contained in $(\eta(0), \eta\xi(0))$ or $(\eta\xi(0), \eta(0))$. By choosing ε and $\mathscr{V}$ sufficiently small we may assume that (ξ, η) is so near (ξ_0, η_0) and $\alpha = -1/(\eta\xi(0) - \xi(0))$ so near $-1/\sigma$ that on $[\eta\xi(0), \xi(0)]$, $|\phi'| > \delta/|\alpha|$ for some $\delta > 1$. It follows that if all Denjoy intervals I of length $l(\xi, \eta)$ are contained in $(\eta\xi(0), \xi(0))$, then $\phi(I)$ is a Denjoy interval of length $\geqq \delta/|\alpha|\, l(\xi, \eta)$ contained in $(\eta(0), \eta\xi(0))$. Then $\alpha\phi(I)$ is a Denjoy interval of length $\geqq \delta l(\xi, \eta)$ for $\mathscr{T}(\xi, \eta)$. Thus in any case

$$l(\mathscr{T}(\xi, \eta)) \geqq \max(\delta, |\alpha|)\, l(\xi, \eta) \geqq \delta_1 l(\xi, \eta),$$

where $\delta_1 > 1$ is independent of (ξ, η). Applying $\mathscr{T}$ a number of times leads to a contradiction.

The result now extends to all $(\xi, \eta) \in W^s_\varepsilon$, because if $\phi(\xi, \eta)$ has a Denjoy interval then so does $\phi(\mathscr{T}^n(\xi, \eta))$, and for sufficiently large $n, \mathscr{T}^n(\xi, \eta) \in W^s_\varepsilon \cap \mathscr{V}$. □

2.3. *Proof of Theorem 1*

If Ω is a relatively compact region in $\mathbb{C}$, then we denote by $\mathscr{A}(\Omega)$ the Banach space of functions analytic on Ω and continuous on $\bar{\Omega}$ with the sup norm. The disks Ω_i, $i = 1, 2$, and the spaces $\mathscr{A}_i$ were defined in Sect. 2.1. The following remark allows one to check quite easily that $\mathscr{A}(\Omega)$ is indeed a Banach space and also plays an important role in the following.

Remark. For any $r \in (0, 1)$ and for $i = 1, 2$, we let $\Omega_i(r)$ be the disk concentric with Ω_i and with diameter equal to $r \cdot \mathrm{diam}(\Omega_i)$. We let $\|\cdot\|_r$ denote the sup norm on the appropriate Banach space corresponding to $\Omega_1(r)$ and $\Omega_2(r)$. A simple application of the Cauchy integral formula gives the following bound on the derivatives of $(f, g) \in \mathscr{B}$:

$$\|f^{(n)}\|_r,\ \|g^{(n)}\|_r,\ \|(f^{(n)}, g^{(n)})\|_r \leqq K(n, r)\, \|(f, g)\|. \tag{2}$$

A similar inequality holds when g maps Ω_2 into the Banach space of bounded linear functionals $\mathrm{Lin}(E, \mathbb{C})$, where E is a Banach space and where $z \to g(z)(x)$ is analytic for each $x \in E$:

$$\|g^{(n)}\|_r \leqq K(n, r)\, \|g\|, \tag{3}$$

where $\|\cdot\|_r$ denotes the sup norm on the space of functions $\Omega_2(r) \to \mathrm{Lin}(E, \mathbb{C})$.

To prove Theorem 1 we use the following facts:

1. Assume that the neighbourhood $\mathscr{U}$ of (f_0, g_0) is chosen so that if $(f, g) \in \mathscr{U}$, then

$$f(0) \in (0, c_4) \tag{4}$$

and

$$\|g' - g_0'\|_{\mathscr{A}(\Omega_2(r))} < 1, \tag{5}$$

where r is chosen so that $\Omega_2(r)$ contains $[0, \sigma]$. Then, $a = g(f(0)^{(\varepsilon)}) - g(0) > 0$ if $(f, g) \in \mathscr{U}$.

2. $(\varepsilon, f, g) \to f(0)^{(\varepsilon)} = f(0)^{1+\varepsilon}$ is a C^∞ map of $\mathbb{R} \times \mathscr{U}$ into $\mathbb{R}_+$, since $f \to f(0)$ is a bounded linear map and $(\varepsilon, z) \to z^{1+\varepsilon}$ is C^∞ on $\mathbb{R} \times (\mathbb{C} \backslash \mathbb{R}_-)$.

3. If $0 < r < 1$, $j = 1$ or 2 and $n \in \mathbb{N}$, then the map $\mathscr{A}_j \to \mathscr{A}(\Omega_j(r))$ given by $h \to h^{(n)} | \bar{\Omega}_j(r)$ is C^∞. This is because it is linear and bounded by the estimate (3).

4. Let Ω_3, Ω_4, and Ω_5 be relatively compact open regions in $\mathbb{C}$ such that $\bar{\Omega}_3 \subset \Omega_4$. Suppose that

(a) h and k are C^∞ mappings of an open subset $\mathscr{V}$ of a Banach space $\mathscr{E}$ into $\mathscr{A}(\Omega_4)$ and $\mathscr{A}(\Omega_5)$ respectively, and

(b) $k(x)(\bar{\Omega}_5) \subset \Omega_3$ for every $x \in \mathscr{V}$. Then $x \to h(x) \circ k(x)$ is a C^∞ mapping of $\mathscr{V}$ into $\mathscr{A}(\Omega_5)$. This fact is a consequence of (2) and (3).

5. Combining 2 and 4 we see that $(\varepsilon, f, g) \to g(f(0)^{(\varepsilon)})$ is C^∞ and hence that $(\varepsilon, f, g) \to a$ is C^∞.

6. We can choose $\mathscr{U}$ so that there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that if $(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, then

$$a > \delta \tag{6}$$

and

$$-a^{1+\varepsilon} \bar{\Omega}_1 \subset \Omega_2(r) \tag{7}$$

for some $r < 1$. This is because $a = \sigma$ when $(\varepsilon, f, g) = (0, f_0, g_0)$, a is continuous, and $\sigma \bar{\Omega}_1 \subset \Omega_2$.

7. By 6, $F(z) = -\dfrac{1}{a} g(-a^{1+\varepsilon} z) \in \mathscr{A}_1$.

8. Since $a > \delta$, when $(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, $a^{1+\varepsilon}$ is C^∞ on this set. But then [11, Proposition 14, p. 9] $(\varepsilon, f, g) \to -a^{1+\varepsilon} z$ is a C^∞ mapping $(-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \to \mathscr{A}_1$. Therefore using (7) and 4 it follows that

$$\boxed{(\varepsilon, f, g) \to F \text{ is } C^\infty \text{ on } (-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \to \mathscr{A}_1}\,.$$

9. Since a is continuous we can choose ε_0 and $\mathscr{U}$ so that for some $r < 1$ we have both (7) and

$$-a^{1+\varepsilon} \bar{\Omega}_2 \subset \Omega_1(r), \tag{8}$$

whenever $(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$. Then by repeating the argument of 8 with g, (8) and $\mathscr{A}_2$ in place of f, (7) and $\mathscr{A}_1$ we prove that on $(-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, $(\varepsilon, f, g) \to f(-a^{1+\varepsilon} z)$ is C^∞.

10. In particular, from 9 we may assume that $\|f(-a^{1+\varepsilon} z) - f_0(-\sigma z)\| < \delta_1$, where $2\delta_1$ is the distance between $-\sigma \bar{\Omega}_2 + \sigma$ and the disk with diameter $[0, 1]$. Therefore, for all $(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, $f(-a^{1+\varepsilon} z)^{1+\varepsilon}$ is well defined, and by 4 the mapping $(\varepsilon, f, g) \to f(-a^{1+\varepsilon} z)^{1+\varepsilon}$ is C^∞ on $(-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \to \mathscr{A}_2$.

11. In particular, we may assume that for $(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, we have

$$\|f(-a^{1+\varepsilon} z)^{1+\varepsilon} - f_0(-\sigma z)\| < \delta_2, \tag{9}$$

where $2\delta_2$ is the distance between $-\sigma \bar{\Omega}_2 + \sigma$ and the boundary of Ω_2. That is, if we let $r = (w - \delta_2)/w$, where $w = \operatorname{diam}(\Omega_2)$, we have by (1c) of Sect. 2.1 that for

$(\varepsilon, f, g) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$, $(f(-a^{1+\varepsilon}\bar{\Omega}_2))^{1+\varepsilon} \subset \Omega_2(r)$. This allows us to use 4 once more to conclude that the mapping $(-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \to \mathscr{A}_2$ given by $(\varepsilon, f, g) \to g((f(-a^{1+\varepsilon}z))^{1+\varepsilon})$ is C^∞. Combining this with 5 and (10) we deduce that if $G(z) = -\frac{1}{a} g((f(-a^{1+\varepsilon}z))^{1+\varepsilon})$ then

$$\boxed{(\varepsilon, f, g) \to G \text{ is a } C^\infty \text{ mapping of } (-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \text{ into } \mathscr{A}_2}\,.$$

12. Putting these facts together one proves this:

Proposition 2.1. *Suppose that f_0, g_0, F, G, and $\mathscr{B}$ are as defined above. Then there is a neighbourhood $(-\varepsilon_0, \varepsilon_0) \times \mathscr{U}$ of $(0, f_0, g_0)$ in $\mathbb{R} \times \mathscr{B}$ defined by conditions* (4) *to* (9) *above on which the mapping $\mathscr{T} : (-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \to \mathbb{R} \times \mathscr{B}$ given by $\mathscr{T}(\varepsilon, f, g) = (\varepsilon, F, G)$ is well-defined and C^∞.*

In Sect. 3.2 we shall identify $\mathscr{M}^\varepsilon$, and thus also $\mathscr{N}^\varepsilon$, with the intersection of $\{\varepsilon\} \times \mathscr{U}$ with a C^∞ submanifold $\mathscr{M}$ invariant under $\mathscr{T}$. Theorem 1 follows when this observation is combined with Proposition 2.1.

3. Structure of the Fixed Point $(\xi_0, \eta_0) = (f_0, g_0)$ and the Proof of Theorem 2

Consider $\mathscr{T}_0$: $\mathscr{M}_0^0 \to \mathscr{M}^0$ and let $f_0(z) = z + \sigma$ and $g_0(z) = z - \sigma^2$ as above. Then $\mathscr{T}_0(f_0, g_0) = (f_0, g_0)$. Our main aim in this section is to describe the hyperbolic structure of (f_0, g_0) under $\mathscr{T}_0$.

The tangent space to $\mathscr{M}^0$ at (f_0, g_0) is the linear subspace $\mathscr{B}_T$ of $\mathscr{B}$ consisting of those $(X, Y) \in \mathscr{B}$ which satisfy

(a″) $X(0) = Y(0)$,

(b″) $X(-\sigma^2) = Y(\sigma)$,

(c″) $Y'(0) - X'(0) = Y'(\sigma) - X'(-\sigma^2)$.

These are the infinitesimal conditions corresponding to (a′), (b′), and (c′) of Sect. 2.1.

Let T be the derivative $d\mathscr{T}_0(f_0, g_0)$ of $\mathscr{T}_0$ at (f_0, g_0). A simple calculation proves that $T : \mathscr{B} \to \mathscr{B}$ is given by

$$T(X, Y) = \left(-X(0) + Y(0) - Y(\sigma) - \frac{1}{\sigma} Y(-\sigma z), \sigma X(0) + \sigma Y(\sigma) - \sigma Y(0) \right.$$
$$\left. - \frac{1}{\sigma} X(-\sigma z) - \frac{1}{\sigma} Y(-\sigma z + \sigma)\right).$$

Lemma 1. *$d\mathscr{T}_\varepsilon(f, g)$ is compact.*

Proof. We give the proof for $T = d\mathscr{T}_0(f_0, g_0)$. To give the general proof we would have to compute $d\mathscr{T}_\varepsilon$ first. The proof would then be similar to that given below for $\varepsilon = 0$. In any case, when we apply this lemma to prove the uniqueness of the expanding eigenvalue we only need to know that $d\mathscr{T}_0$ is compact.

Choose $r < 1$ so that $-\sigma\bar{\Omega}_1 \subset \Omega_2(r)$, $-\sigma\bar{\Omega}_2 \subset \Omega_1(r)$ and $-\sigma\bar{\Omega}_2 + \sigma \subset \Omega_2(r)$. Let $(X_i, Y_i) \in \mathscr{B}$ be a sequence such that $\|(X_i, Y_i)\| \leqq 1$. For $(X, Y) \in \mathscr{B}$, let $(\tilde{X}, \tilde{Y})$ denote $T(X, Y)$. Using (5) and (7) we have

$$\|\tilde{X}_i(z)-\tilde{X}_i(w)\| = \frac{1}{\sigma}\|Y_i(-\sigma z)-Y_i(-\sigma w)\| < (\|Y_i'\|_{\bar{\Omega}_2(r)})\cdot|z-w|$$
$$\leqq K(1,r)\cdot\|Y_i\|\cdot|z-w| \leqq K(1,r)\cdot|z-w|.$$

Here $\|\cdot\|_{\bar{\Omega}_2(r)}$ denotes the sup norm on $\bar{\Omega}_2(r)$. Also

$$\|\tilde{Y}_i(z)-\tilde{Y}_i(w)\| \leqq \frac{1}{\sigma}\|X_i(-\sigma z)-X_i(-\sigma w)\| + \frac{1}{\sigma}\|Y_i(-\sigma z+\sigma)-Y_i(-\sigma w+\sigma)\|$$
$$\leqq (\|X_i'\|_{\bar{\Omega}_1(r)})\cdot|z-w| + (\|Y_i'\|_{\bar{\Omega}_2(r)})\cdot|z-w|$$
$$\leqq 2K(1,r)\cdot|z-w|.$$

Therefore $(\tilde{X}_i, \tilde{Y}_i)$ is an equicontinuous family, and moreover since $\|(X_i, Y_i)\| \leqq 1$, $(\tilde{X}_i, \tilde{Y}_i)$ is a bounded sequence. Therefore, by Ascoli's theorem, the sequence $(\tilde{X}_i, \tilde{Y}_i)$ has a Cauchy subsequence. □

3.1. The Structure of the Fixed Point (ξ_0, η_0)

The spectrum of a compact linear map is a countable set of eigenvalues with no accumulation point different from zero (see Kato [9, Theorem III, 6.26]). We will compute the spectrum explicitly for the mapping T. In addition we need the following result about the effect of perturbations on the spectrum of T. Let T_ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be a continuous family of bounded linear maps of a Banach space $\mathscr{B}$ into itself, and suppose $T_0 = T$ is compact.

Proposition 3.1. *For every given eigenvalue $\lambda_1 \neq 0$ of T there are numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that if $|\varepsilon| < \varepsilon_1$ then the disk of radius δ_1 about λ_1 contains a finite number of eigenvalues of T_ε whose total multiplicity equals the multiplicity of λ_1 for T. Moreover, if D is an open disk centered at 0 and if $D_1, \ldots, D_m$ are disjoint open disks about the finite number of eigenvalues $\lambda_i \in D_i$ of T not contained in D, then ε_1 may be chosen so small that $|\varepsilon| < \varepsilon_1$ implies that every D_i, $i = 1, \ldots, m$, contains eigenvalues of T_ε to a total multiplicity of λ_i for T, while the rest of the spectrum of T_ε lies in D.*

Proof. See Kato [9, Sect. IV, Chap. 3.5].

To calculate the spectrum of T explicitly we let $\mathscr{P} \subset \mathscr{B}$ denote the space of pairs $(X, Y) \in \mathscr{B}$, where X and Y are polynomials. Let $\mathscr{P}_n \subset \mathscr{P}$ be the finite dimensional space of pairs (X, Y) with $\deg(X)$ and $\deg(Y) \leqq n$. Then $\mathscr{P} = \bigcup_{n \geqq 0} \mathscr{P}_n$ and $\mathscr{B} = \bar{\mathscr{P}}$. Clearly $T(\mathscr{P}_n) \subset \mathscr{P}_n$ for any n. It is also clear that the pairs $(1, -\sigma)$ and $(1, 1)$ together with the pairs $(\sigma z^n, z^n)$ and $(z^n, -\sigma z^n)$ for $n = 1, 2, 3, \ldots$ constitute a basis for $\mathscr{P}_n$. A simple calculation shows that

$$T(1, -\sigma) = (0, 0),$$

$$T(1, 1) = -\frac{1}{\sigma^2}(1, 1),$$

$$T(\sigma z^n, z^n) = -(-\alpha)^{n-2}(\sigma z^n, z^n) \pmod{\mathscr{P}_{n-1}},$$

$$T(z^n, -\sigma z^n) = (-\sigma)^n(z^n, -\sigma z^n) \pmod{\mathscr{P}_{n-1}}.$$

We will now deduce that the spectrum of T on $\mathscr{B}$ consists of the eigenvalues 0, $-\frac{1}{\sigma^2}, \frac{1}{\sigma}, -1, \pm\sigma, \pm\sigma^2, \pm\sigma^3, \ldots$, that these are all simple with the possible exception of 0, and that the elements of the basis listed above constitute the leading terms of the corresponding eigenvectors. For suppose λ is in the spectrum of T but not in the above set. Since T is compact, λ is an eigenvalue. Let E be the projection onto the generalized eigenspace of λ and $I-E$ the complementary projection. All the polynomial eigenvectors must lie in $(I-E)\mathscr{B}$. Therefore $\mathscr{P}\subset(I-E)\mathscr{B}$. But that contradicts the fact that $\mathscr{P}$ is dense in $\mathscr{B}$. A similar argument proves that every non-zero eigenvalue is simple.

Note that the eigenvectors

$$b_1=(1,-\sigma),$$

$$b_2=(-\sigma^4+\sigma z, -\sigma^4+z),$$

$$b_3=((2\sigma-1)+2\sigma z+\sigma z^2,(2\sigma-1)-2\sigma z+z^2),$$

are complementary to the subspace $\mathscr{B}_T\subset\mathscr{B}$. In fact b_1, b_2, and b_3 violate (a″), (b″), and (c″), respectively. Thus $\mathscr{B}_T$ is the closure of the space generated by the remaining eigenvectors of T, and the spectrum of $T|\mathscr{B}_T$ is equal to

$$\left\{0, -\frac{1}{\sigma^2}, \pm\sigma, \pm\sigma^2, \pm\sigma^3, \ldots\right\}.$$

Let $\mathscr{B}_T^\sigma$ be the linear subspace of $\mathscr{B}_T$ consisting of those $(X, Y)\in\mathscr{B}_T$ such that

$$\int_{-\sigma^2}^{0} X(t)dt+\int_{0}^{\sigma} Y(t)dt=0.$$

This subspace is invariant under T as can be seen by direct calculation or from the following lemma.

Lemma 3.1. *$\mathscr{B}_T^\sigma$ is the tangent space to $\varrho=\sigma$ at (f_0, g_0).*

Proof (cf. [2, 5]). If $f=f_0+X$ and $g=g_0+Y$, then let k be the unique homeomorphism of $\mathbb{R}$ satisfying $k(x+1)=k(x)+1$, $k|[-\sigma^2,0)=f$ and $k|[0,\sigma)=g+1$. Let $Z:[-\sigma^2,\sigma]\to\mathbb{R}$ be such that $Z=X$ on $[-\sigma^2,0)$ and $Z=Y$ on $[0,\sigma)$. Also let $k_0(x)=x+\sigma$. Then for sufficiently small $\|Z\|$, $\|Z\|_1\leqq K\|Z\|$ because of (3), where $\|Z\|_1$ denotes the sup norm of the first derivative of Z. As in [5, p. 284] we combine

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} Z\circ k_0^i(x)-\int_{T'} Z(x)dx\right|\leqq\|Z\|\varepsilon$$

and

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} Z\circ k^i(x)-\frac{1}{n}\sum_{i=0}^{n-1} Z\circ k_0^i(x)\right|\leqq\|Z\|_1\delta,$$

where δ is an upper bound on the quantities $\|k^i-k_0^i\|$, $i=1,2,\ldots,n$. For a given $\varepsilon>0$ and for $\|Z\|$ sufficiently small, this gives the following inequality, first for a specific $n = n_0$ and thus for all sufficiently large n:

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} Z \circ k^i(x) - \int_{T'} Z(x) dx \right| \leqq c \| Z \| \varepsilon .$$

That is,

$$\left. \frac{d}{dt} \right|_{t=0} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} t Z \circ k^i \right\} = \int_{T'} Z(x) dx .$$

This completes the proof, as

$$\varrho(\phi) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\sigma + Z) \circ k^i = \sigma + \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} Z \circ k^i . \quad \square$$

Clearly the eigenvector $(X, Y) = (1, 1)$ is complementary to R_T^σ, so we have the following theorem.

Theorem 3.1. *T leaves invariant the decomposition*

$$\mathscr{B}_T = \mathbb{R}(1,1) \oplus \mathscr{B}_T^\sigma .$$

The spectral radius of $T | \mathscr{B}_T^\sigma$ is σ.

3.2. Proof of Theorem 2

We saw in Sect. 3.1 that the eigenvectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are complementary to the subspace $\mathscr{B}_T \subset \mathscr{B}$. We let $\mathscr{B}_\perp$ denote the space spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, so that $\mathscr{B} = \mathscr{B}_\perp \oplus \mathscr{B}_T$.

Define $A : (-\varepsilon_0, \varepsilon_0) \times \mathscr{U} \subset (-\varepsilon_0, \varepsilon_0) \times \mathscr{B}_T \times \mathscr{B}_\perp \to \mathbb{R}^3$ as follows: $A = (A_1, A_2, A_3)$, where A_1, A_2, and A_3 are given by conditions (a′), (b′), (c′) of Sect. 2.1,

$$A_1(\varepsilon, f, g) = f(0) - g(0) - 1 ,$$

$$A_2(\varepsilon, f, g) = f(g(0)^{(\varepsilon)}) - g(f(0)^{(\varepsilon)}) ,$$

$$A_3(\varepsilon, f, g) = \left. \frac{d}{dz} \right|_{z=0} [f(g(z)^\varepsilon) - g(f(z)^{(\varepsilon)})] .$$

A calculation similar to that in Sect. 2.3 shows that A is C^∞.

We seek to solve $A(\varepsilon, u + f_0 + \delta u, v + g_0 + \delta v) = 0$ for (ε, u, v) on some neighbourhood of $(0,0,0)$ in $(-\varepsilon_0, \varepsilon_0) \times \mathscr{B}_T$ and with $(\delta u, \delta v) = (\delta u(\varepsilon, u, v), \delta v(\varepsilon, u, v)) \in \mathscr{B}_\perp$, so that $\delta u(0) = \delta v(0) = 0$. In fact the existence of the functions δu and δu and δv follows directly from the Implicit Function Theorem if

$$S = d_{(f,g)} A(0, f_0, g_0) | \mathscr{B}_\perp : \mathscr{B}_\perp \to \mathbb{R}^3$$

is surjective, and this is the case because

$$S(b_1) = (\sigma + 1, 0, 0) ,$$

$$S(b_2) = (0, -\sigma^2 - 1, 0) ,$$

$$S(b_3) = (0, 0, -\sigma(\sigma^2 + 1)) .$$

It also follows that for some $\varepsilon_1>0$ and for some neighbourhood $\mathscr{W}$ of $(0,0)$ in $\mathscr{B}_T$,

$$\Lambda:(\varepsilon,u,v)\to(\varepsilon,u+f_0+\delta u(\varepsilon,u,v),v+g_0+\delta v(\varepsilon,u,v))$$

is a diffeomorphism of $(-\varepsilon_1,\varepsilon_1)\times\mathscr{W}$ onto the C^∞ submanifold $\mathscr{M}=\Lambda^{-1}(0)\cap((-\varepsilon_1,\varepsilon_1)\times\mathscr{W}\times\mathscr{B})$. Clearly there exists a neighbourhood $\mathscr{W}_0$ of 0 in $\mathscr{W}$ and $\varepsilon_2>0$ such that if $\mathscr{M}_0=\Lambda((-\varepsilon_2,\varepsilon_2)\times\mathscr{W}_0)$, then $\mathscr{T}(\mathscr{M}_0)\subset\mathscr{M}$ and (f,g) satisfies (e′) for all $(\varepsilon,f,g)\in\mathscr{M}_0$. Henceforth we consider $\mathscr{M}_0$ as the domain of $\mathscr{T}$, and we let $\mathscr{M}_0^\varepsilon=\Lambda(\{\varepsilon\}\times\mathscr{W}_0)$. We will use $\mathscr{N}_0^\varepsilon$ to denote the corresponding subset of $\mathscr{N}^\varepsilon$.

Let $\mathscr{S}:(-\varepsilon_2,\varepsilon_2)\times\mathscr{W}_0\to(-\varepsilon_2,\varepsilon_2)\times\mathscr{W}$ be defined by $\mathscr{S}=\Lambda\circ\mathscr{T}\circ\Lambda^{-1}$, and let $\mathscr{S}_\varepsilon:\mathscr{W}_0\to\mathscr{W}$ be defined by $\mathscr{S}(\varepsilon,u,v)=(\varepsilon,\mathscr{S}_\varepsilon(u,v))$. Then $\mathscr{S}_\varepsilon$ represents $\mathscr{T}_\varepsilon$ in this coordinate system.

Now, by Theorem 3.1, $(0,0)$ is a hyperbolic fixed point of $\mathscr{S}_0$. Therefore, for small $|\varepsilon|$, $\mathscr{S}_\varepsilon$ has a unique hyperbolic fixed point $(u_\varepsilon,v_\varepsilon)$ near $(0,0)$ and this has C^∞ dependence on ε. Moreover, $d\mathscr{S}_\varepsilon(u_\varepsilon,v_\varepsilon)$ is a C^∞ function of ε. Using Proposition 3.1 and the Implicit Function Theorem, each non-zero eigenvalue has C^∞ dependence upon small ε.

Remark. Numerical studies by Siggia [12] clearly indicate that $\delta_\varepsilon=\delta_0+O(\varepsilon^2)$, but we have not been able to prove this. However, we can obtain an expression for $(A,B)=\frac{d}{d\varepsilon}(f_\varepsilon,g_\varepsilon)$ at $\varepsilon=0$: By differentiating $\xi_\varepsilon(z)=-(1/a_\varepsilon)\eta_\varepsilon(-a_\varepsilon z)$ and $\eta_\varepsilon(z)=-(1/a_\varepsilon)\eta_\varepsilon(\xi_\varepsilon(-a_\varepsilon z))$, where $a_\varepsilon=\eta_\varepsilon(\xi_\varepsilon(0))-\eta_\varepsilon(0)$, one gets

$$A(z)=-B(\sigma)-\frac{1}{\sigma}B(-\sigma z)+U(z),$$

$$B(z)=\sigma B(\sigma)-\frac{1}{\sigma}A(-\sigma z)-\frac{1}{\sigma}B(-\sigma z+\sigma)+V(z),$$

where

$$U(z)=-\sigma\log\sigma+z\log\sigma,$$

and

$$V(z)=-\sigma\log\sigma+2z\log\sigma+(z-1)\log(1-z).$$

That is, $(A,B)=(U,V)+\hat{T}(A,B)$, where the spectrum of $\hat{T}$ is equal to

$$\left\{0,-\frac{1}{\sigma^2},\frac{1}{\sigma},-1,\pm\sigma,\pm\sigma^2,\ldots\right\}.$$

Therefore $\hat{T}-I$ is invertible on $\mathscr{B}$, and we get $(A,B)=(\hat{T}-I)^{-1}(U,V)$. If we let U_n and V_n be the polynomial expansions of U and V up to degree n, we get

$$(A,B)=\lim_{n\to\infty}(\hat{T}-I)^{-1}(U_n,V_n).$$

Since $\hat{T}-I$ maps $\mathscr{P}_n$ to itself isomorphically, this gives an approximation of $\frac{d}{d\varepsilon}(f_\varepsilon,g_\varepsilon)|_{\varepsilon=0}$ by polynomials. Presumably, this expression is related to that obtained formally by Kadanoff in [4] and [8].

4. Stable and Unstable Manifolds and the Proofs of Theorems 3–5

We gave the definition of stable and unstable manifolds in Sect. 2.2. We assume that ε is sufficiently small for Theorem 2 to hold. Thus $\mathcal{S}_\varepsilon$, the local coordinate representation of $\mathcal{T}_\varepsilon$, is defined on a neighbourhood $\mathcal{W}_0$ of 0 in $\mathcal{B}_T$, and has a unique fixed point $(u_\varepsilon, v_\varepsilon)=l_\varepsilon$ in $\mathcal{W}_0$. The derivative $d\mathcal{S}_\varepsilon(l_\varepsilon)$ has a simple eigenvalue outside the unit circle with the rest of its spectrum inside.

The existence and properties of stable and unstable manifolds, complicated somewhat by the non-invertibility of $\mathcal{S}_\varepsilon$, follows from Sect. 5 in [7]. From this work we obtain the following facts: For ε and $\mathcal{W}_0$ sufficiently small the set

$$W_\varepsilon^s = \bigcap_{i=0}^{\infty} \mathcal{S}_\varepsilon^{-1} \mathcal{W}_0$$

is a smooth connected submanifold of $\mathcal{W}_0$ of codimension one. The tangent space to W_ε^s at l_ε is the spectral subspace of $d\mathcal{S}_\varepsilon(l_\varepsilon)$ corresponding to that part of the spectrum which is contained inside the unit circle. Also, $\mathcal{S}_\varepsilon W_\varepsilon^s \subset W_\varepsilon^s$ and $\bigcap_{j \geqq 0} \mathcal{S}_\varepsilon^j W_\varepsilon^s = \{l_\varepsilon\}$.

Similarly, a smooth local unstable manifold is defined as follows: Inductively define $\mathcal{W}_{j+1} = (\mathcal{S}_\varepsilon \mathcal{W}_j) \cap \mathcal{W}_0$, $j=0,1,2,\ldots$. Then

$$W_\varepsilon^u = \bigcap_{j=0}^{\infty} \mathcal{W}_j$$

is a smooth connected one-dimensional submanifold of $\mathcal{W}_0$ tangent at l_ε to the eigenspace of $d\mathcal{S}_\varepsilon(l_\varepsilon)$ corresponding to the eigenvalue of largest modulus. It satisfies $\mathcal{S}_\varepsilon W_\varepsilon^u \supset W_\varepsilon^u$, and for any $l \in W_\varepsilon^u$ and any $j=1,2,\ldots$, there is a unique $l_j \in W_\varepsilon^u$ such that $\mathcal{S}_\varepsilon^j l_j = l$. Moreover, this sequence (l_j) converges to the fixed point l_ε. Thus we see that W_ε^s and W_ε^u are respectively stable and unstable manifolds in the sense of Sect. 2.2. Furthermore, W_ε^s is unique (for each value of ε) in the sense that if W^s is another stable manifold of $\mathcal{S}_\varepsilon$ at l_ε, then the connected component of l_ε in $W^s \cap \mathcal{W}_0$ is contained in W_ε^s. The W_ε^u is unique in the same sense. Finally, the manifolds W_ε^s and W_ε^u depend continuously on ε in the C^r-sense, for any r.

Using the coordinate map Λ, we now obtain local stable and unstable manifolds of $\mathcal{T}_\varepsilon$ at $\zeta_\varepsilon = (\xi_\varepsilon, \eta_\varepsilon)$ in $\mathcal{N}_0^\varepsilon$. We shall also refer to them as W_ε^s and W_ε^u.

Though the mapping $\mathcal{T}_0$ is not globally defined on $\mathcal{N}$, it is clearly well-defined along the line

$$W^u = \{(\xi, \eta) : \xi = \xi_0 + \lambda, \eta = \eta_0 + \lambda, \lambda \in \mathbb{R}\}.$$

We claim that W^u is a global unstable manifold for $\mathcal{T}_0$. To prove this we only need to show that $W^u = \bigcup_{n=0}^{\infty} \mathcal{T}_0^n(W_\varepsilon^u)$. From the uniqueness of W_ε^u it follows that $W_\varepsilon^u \subset W^u$. The equality $W^u = \bigcup_{n=0}^{\infty} \mathcal{T}_0^n(W_\varepsilon^u)$ follows immediately from the fact that $\mathcal{T}_0(z+\lambda, z+\lambda-1) = \left(z+\frac{1}{\lambda}-1, z+\frac{1}{\lambda}-2\right)$, $\lambda \in (0,1)$.

4.1. Proof of Theorem 3

The only part of Theorem 3 that is not yet obvious is the identification of W_ε^s with the set of pairs (ξ, η) with $\varrho(\xi, \eta) = \sigma$.

There is a group action $R_\theta : \mathscr{B}_T \to \mathscr{B}_T$, $\theta \in \mathbb{R}$, which at a function pair $(u,v) \in \mathscr{B}_T$ is given by $R_\theta(u,v) = (u+\theta, v+\theta)$. When $\varepsilon = 0$ the orbit $\{\Lambda(0, R_\theta(0,0)) : \theta \in \mathbb{R}\} = \{R_\theta(f_0, g_0) : \theta \in \mathbb{R}\}$ equals the line $W^u(f_0, g_0)$. When $\varepsilon \neq 0$ the curve in $\mathscr{M}^\varepsilon$ corresponding to an orbit $\{R_\theta(u,v) : \theta \in R,\ (u,v) \in \mathscr{W}_0\}$, is given by $\Lambda(\varepsilon, R_\theta(u,v))$, θ varying over an open real interval. Whenever the value of ε is fixed by the context we shall use $\varrho(u,v)$ to refer to the rotation number $\varrho(f,g)$, where $(\varepsilon, f, g) = \Lambda(\varepsilon, u, v)$. It is clear that for $\varepsilon = 0$, $\varrho(R_\theta(0,0))$ is a monotonically increasing function of θ. We claim corresponding results for $\varepsilon \neq 0$.

Proposition 4.1.1. *For $(\varepsilon, R_\theta(u,v)) \in (-\varepsilon_0, \varepsilon_0) \times \mathscr{W}_0$ the rotation number $\varrho(R_\theta(u,v))$ is an increasing function of θ.*

Proposition 4.1.2. *If for a given $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and a given $(u,v) \in \mathscr{W}_0$ the rotation number $\varrho(u,v)$ is irrational, then $\varrho(R_\theta(u,v))$ is strictly increasing at $\theta = 0$.*

We let ϕ_θ be the circle mapping corresponding to $R_\theta(u,v)$ (for the fixed value of ε) and we let k_θ be the lift of ϕ_θ to $\mathbb{R}$. We will assume $\theta \geqq 0$ throughout this section. Both propositions follow from the inequality

$$k_\theta \geqq k_0 + C\theta, \tag{10}$$

which we will establish below. Here C is a positive constant.

Proof of Proposition 4.1.1 (cf. III, 1.3 in [5]). Inequality (10) implies

$$\varrho(k_\theta) = \lim_{n\to\infty} \frac{k_\theta^n - \mathrm{Id}}{n} \geqq \lim_{n\to\infty} \frac{k_0^n - \mathrm{Id}}{n} = \varrho(k_0). \quad \square$$

Proof of Proposition 4.1.2. We have from Proposition III.4.1.1 in [5] that if $\theta > 0$ then $\varrho(k_0 + C\theta) > \varrho(k_0)$. Therefore, using (10) again,

$$\varrho(k_\theta) = \lim_{n\to\infty} \frac{k_\theta^n - \mathrm{Id}}{n} \geqq \lim_{n\to\infty} \frac{(k_0 + C\theta)^n - \mathrm{Id}}{n} = \varrho(k_0 + C\theta) > \varrho(k_0). \quad \square$$

It remains to prove (10). For convenience we will write $R_\theta(u,v) = (R_\theta u, R_\theta v)$. Since the functions $\delta u(\varepsilon, u, v)$ and $\delta v(\varepsilon, u, v)$ defined by $\Lambda(\varepsilon, u, v) = (\varepsilon, u + f_0 + \delta u, v + g_0 + \delta v)$ are C^∞ in (ε, u, v), therefore the mapping $(-\varepsilon_0, \varepsilon_0) \times \mathscr{W}_0 \times \bar{\Omega}_1 \to \mathbb{C}$ given by $((\varepsilon, u, v), z) \to [d(\delta u)(\varepsilon, u, v) \cdot (0,1,1)](z)$ is continuous. Now $[d(\delta u)(0,0,0) \cdot (0,1,1)](z) = 0$ for all $z \in \bar{\Omega}_1$. A similar result holds for $d(\delta v)$. Since $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are compact, it follows that for every $\delta_1 > 0$ there is a neighbourhood $(-\varepsilon_0, \varepsilon_0) \times \mathscr{W}_0$ (perhaps smaller than the original) of $(0,0,0)$ on which

$$\| d(\delta u)(\varepsilon, u, v) \cdot (0,1,1) \| < \delta_1,$$

and

$$\| d(\delta v)(\varepsilon, u, v) \cdot (0,1,1) \| < \delta_1. \tag{11}$$

Now suppose that for a certain $\theta\in\mathbb{R}$ we have $(\varepsilon, R_t u, R_t v)\in(-\varepsilon_0,\varepsilon_0)\times\mathscr{W}_0$ for all t between 0 and θ. Then for $z\in\bar{\Omega}_1\cap R$ we have by (11):

$$R_\theta u(z)+\delta u(\varepsilon, R_\theta u, R_\theta v)(z)\geqq u(z)+\delta u(\varepsilon, u, v)+(1-\delta_1)\theta\,. \tag{12}$$

Thus, if $\Lambda(\varepsilon, R_\theta u, R_\theta v)=(\varepsilon, f_\theta, g_\theta)$, then

$$f_\theta(z)\geqq f_0(z)+(1-\delta_1)\theta\,. \tag{13}$$

Similarly,

$$g_\theta(z)\geqq g_0(z)+(1-\delta_1)\theta\,. \tag{14}$$

If as usual we let $\xi_\theta(z)=f_\theta(z|z|^\varepsilon)$ and $\eta_\theta(z)=g_\theta(z|z|^\varepsilon)$, then the same inequalities also hold for ξ_θ and η_θ.

Unfortunately it does not follow that $k_\theta\geqq k_0+(1-\delta)\theta$. However, keeping in mind that $k=k(f,g)$ is given by

$$k(x)=\begin{cases}\xi(x-n)+n & \text{if}\quad x\in[\eta(0)+n,n], n\in\mathbb{Z}\\ \eta(x-n)+n+1 & \text{if}\quad x\in[n,\xi(0)+n]\,,\end{cases}$$

we define for small $\varrho>0$:

$$k_\varrho(x)=\begin{cases}\xi(x-n)+n & \text{if}\quad x\in[\eta(0)+\varrho+n,n]\,,\\ \eta(x-n)+n & \text{if}\quad x\in[n,\xi(0)+\varrho+n]\,.\end{cases}$$

It is easy to see that then

$$|k(x)-k_\varrho(x)|\leqq|\xi'(\eta(0))-\eta'(\xi(0))|\cdot\varrho+O(\varrho^2)\,.$$

Since the mapping $(\varepsilon,u,v)\to\xi'(\eta(0))-\eta'(\xi(0))$ is continuous and has the value 0 at $(0,0,0)$, therefore for every $\delta_2>0$ there exists a neighbourhood $(-\varepsilon_0,\varepsilon_0)\times\mathscr{W}_0$ of $(0,0,0)$ (again smaller than the original in general), such that for $(\varepsilon,u,v)\in(-\varepsilon_0,\varepsilon_0)\times\mathscr{W}_0$, $|\xi'(\eta(0))-\eta'(\xi(0))|<\delta_2$. But then

$$|k(x)-k_\varrho(x)|\leqq\delta_2\varrho+O(\varrho^2)\,. \tag{15}$$

We let $\varrho=\eta_\theta(0)-\eta_0(0)$ now. Then (13) and (14) imply that $k_\theta\geqq(k_0)_\varrho+(1-\delta_1)\theta$. Combining this with (15) we get

$$\begin{aligned}k_\theta&\geqq k_0+(1-\delta_1)\theta-\delta_2|\varrho|-O(\varrho^2)\\&\geqq k_0+(1-\delta_1)\theta-\delta_2(1-\delta_1)\theta-O(\theta^2)\,.\end{aligned}$$

Inequality (10) follows immediately.

Proof of Theorem 3. First suppose $(u,v)\in W_\varepsilon^s\cap\mathscr{W}_0$ and suppose $\alpha=\varrho(u,v)$. Then $0<\varrho(\mathscr{S}_\varepsilon^n(u,v))<1$ for all $n\geqq0$. But $\varrho(\mathscr{S}_\varepsilon^n(u,v))=\tau^n(\alpha)$, where $\tau(\alpha)=\alpha^{-1}-1$. Consequently, $\alpha=1/(1+1/(1+\ldots=\sigma$.

To prove the converse we suppose $(u,v)\in\mathscr{W}_0$ and $\varrho(u,v)=\sigma$. Let $(\hat{u},\hat{v})$ be the point on W_ε^s that lies on the same leaf of the R_θ-foliation as does (u,v). By Proposition 4.1.2, we have $(u,v)=(\hat{u},\hat{v})$. This completes the proof of Theorem 3.

4.2. *Proof of Theorem 4*

As usual, if $(f,g)\in\mathscr{M}_0^\varepsilon$ for some ε, we let ξ, η and ϕ denote the following functions: $\xi(z)=f(z|z|^\varepsilon)$, $\eta(u)=g(z|z|^\varepsilon)$ and $\phi=\phi(\xi,\eta)$. Suppose $(\hat{f},\hat{g})\in\mathscr{M}_0^\varepsilon$ is such that 0 is periodic under the associated map $\hat{\phi}=\phi(\hat{\xi},\hat{\eta})$ on the circle. Say $\hat{\phi}^n(0)=0$. As always we identify the circle with the pair of intervals $[\hat{\eta}(0),0]\cup[0,\hat{\xi}(0)]$ with $\hat{\eta}(0)$ and $\hat{\xi}(0)$ identified. For $(f,g)\in\mathcal{M}_0^\varepsilon$ we now define this set of functions ϕ_k, $k=1,2,\ldots,n$, depending on f and g: For $k>2$ we let $\phi_k=\xi$ if $\hat{\phi}^{k-1}(0)\in[\hat{\eta}(0),0]$ and $\phi_k=\eta$ if $\hat{\phi}^{k-1}(0)\in[0,\hat{\xi}(0)]$. We choose $\phi_1=\phi_2=\xi$.

Since for any k the composition $\phi_k\circ\phi_{k-1}\circ\ldots\circ\phi_1(0)$ is continuous in (f,g), there is an open neighbourhood $\mathcal{W}_1$ of $(\hat{f},\hat{g})$ in $\mathcal{M}_0^\varepsilon$ such that for any $(f,g)\in\mathcal{W}_1$ the composition $\phi_n\circ\phi_{n-1}\circ\ldots\circ\phi_1(0)$ is well-defined. Let $(\varepsilon,f,g)=\Lambda(\varepsilon,u,v)$, where $(u,v)\in\mathcal{B}_T$ and where Λ is the coordinate map defined in Sect. 3.2. Also let $u_\theta=u+\theta$, $v_\theta=v+\theta$, $(\varepsilon,f_\theta,g_\theta)=\Lambda(\varepsilon,u_\theta,v_\theta)$, $\xi_\theta(z)=f_\theta(z|z|^\varepsilon)$, $\eta_\theta(z)=g_\theta(z|z|^\varepsilon)$, and $\phi_{i,\theta}=\xi_\theta$ or η_θ according as $\hat{\phi}_i=\hat{\xi}$ or $\hat{\eta}_\theta$. Finally let $\phi_\theta=\phi(\xi_\theta,\eta_\theta)$. Then for $(f,g)\in\mathcal{W}_1$ and for θ sufficiently small, $\phi_\theta^n(0)=\phi_{n,\theta}\circ\phi_{n-1,\theta}\circ\ldots\circ\phi_{1,\theta}(0)$.

We want to prove that there is a neighbourhood $(-\varepsilon_3,\varepsilon_3)\times\mathcal{W}_2'$ of $(0,0,0)$ in $\mathbb{R}\times\mathcal{B}_T$ such that if $\mathcal{W}_2=\Lambda(\{\varepsilon\}\times\mathcal{W}_2')$, $\varepsilon\in(-\varepsilon_3,\varepsilon_3)$, then the set $\{\phi^n(0)=0\}$ meets $\mathcal{W}_2$ in a C^∞ manifold transverse to the R_θ-foliation (f_θ,g_θ). To do this we assume $(\hat{f},\hat{g})\in\mathcal{W}_1\subset\mathcal{W}_2$ and $(f,g)\in\mathcal{W}_1$, and prove that $\frac{d}{d\theta}\phi_\theta^n(0)|_{\theta=0}>0$. But

$$\begin{aligned}\frac{d}{d\theta}\phi_\theta^n(0)|_{\theta=0}&=\frac{d}{d\theta}(\phi_{n,\theta}\circ\phi_{n-1,\theta}\circ\ldots\circ\phi_{1,\theta}(0))|_{\theta=0}\\&=\left(\frac{d}{d\theta}\phi_{n,\theta}\right)(\phi^{n-1}(0))+\phi_n'(\phi^{n-1}(0))\left(\frac{d}{d\theta}\phi_{n-1,\theta}\right)(\phi^{n-2}(0))+\ldots\\&\quad+\phi_n'(\phi^{n-1}(0))\phi_{n-1}'(\phi^{n-2}(0))\ldots\phi_2'(\phi(0))\left(\frac{d}{d\theta}\phi_{1,\theta}\right)(0)|_{\theta=0}\,.\end{aligned}$$

Thus it suffices to show that if ε_3 and $\mathcal{W}_2'$ are sufficiently small, then $\frac{d}{d\theta}\xi_\theta(z)|_{\theta=0}>0$ and $\frac{d}{d\theta}\eta_\theta(z)|_{\theta=0}>0$. But when $(\varepsilon,f,g)=(0,f_0,g_0)$, $\frac{d}{d\theta}\xi_\theta(z)=1$ and $\frac{d}{d\theta}\eta_\theta(z)=1$. Since $f_\theta(z|z|^\varepsilon)$ and $g_\theta(z|z|^\varepsilon)$ are continuous in $(\theta,\varepsilon,f,g,z)$, it follows that ε_3 and $\mathcal{W}_2'$ may be chosen with the required properties. This proves that for sufficiently large n the set $\{(f,g)\in\mathcal{M}_0^\varepsilon:\phi^{q_{n+1}}(0)=0,\ \varrho(\phi)=q_n/q_{n+1}\}$ meets $\mathcal{W}_2$ in a C^∞ manifold. Therefore, by identification, $(\xi_\varepsilon,\eta_\varepsilon)$ has a neighbourhood $\mathcal{V}_2$ which meets $\mathcal{P}(q_n/q_{n+1},\varepsilon)$ in a C_∞ submanifold of $\mathcal{N}_0^\varepsilon$.

To prove that $\mathcal{P}(q_n/q_{n+1},\varepsilon)$ is transverse to W_ε^u, we note that by standard stable manifold theory, for some $\varepsilon_3>0$ and $\delta>0$ there exists a C^∞ mapping $C:(-\varepsilon_3,\varepsilon_3)\times(-\delta,\delta)\to\mathcal{M}_0$, such that $C(\varepsilon,\cdot)$ is an embedding of $(-\delta,\delta)$ into $\mathcal{M}_0^\varepsilon$ whose image is an unstable manifold of $(f_\varepsilon,g_\varepsilon)$ and such that $C(0,\lambda)=(f_0+\lambda,g_0+\lambda)$. Now $\frac{d}{d\lambda}\xi(C(0,\lambda))\equiv1$ and $\frac{d}{d\lambda}\eta(C(0,\lambda))\equiv1$. Therefore, by continuity, if $|\varepsilon|$ is sufficiently small, then $\frac{d}{d\lambda}\xi(C(0,\lambda))>1-\delta$ and $\frac{d}{d\lambda}\eta(C(0,\lambda))>1-\delta$ for some $\delta>0$. Using these facts and a calculation similar to that used above, one can show that $\frac{d}{d\lambda}\phi^{q_{n+1}}(0)>0$ at a point of $\mathcal{P}(q_n/q_{n+1},\varepsilon)\cap W_\varepsilon^u$. This proves that $\mathcal{P}(q_n/q_{n+1},\varepsilon)$ is transverse to the unstable manifold of $(\xi_\varepsilon,\eta_\varepsilon)$. □

4.4. Proof of Theorem 5

The proof of this theorem is an almost immediate corollary of the following linearization result of Collet et al. (see [3, Theorem 6.3]). Let $\mathscr{X}_1$ denote the open unit ball in $\mathscr{B}_T^{\sigma}$ and assume $|\varepsilon|<\varepsilon_2$.

Proposition 4.4. *There exists a C^1 diffeomorphism of $\mathscr{X}_1\times(-1,1)$ onto an open neighbourhood $\mathscr{V}\subset\mathscr{N}_0^{\varepsilon}$ of $(\xi_\varepsilon,\eta_\varepsilon)=\zeta_\varepsilon$ such that in this coordinate system*

(i) $(0,0)$ *represents* ζ_ε;

(ii) $\mathscr{X}_1\times\{0\}$ *represents* $W_\varepsilon^s\cap\mathscr{V}$;

(iii) $\{0\}\times(-1,1)$ *represents* $W_\varepsilon^u\cap\mathscr{V}$;

(iv) $\mathscr{T}_\varepsilon(X,Y)=(M(X,Y),\delta_\varepsilon Y)$, *where* $M(0,Y)=0$ *and* $\|d_xM(X,Y)\|\leqq\tau<1$ *for all* $(X,Y)\in\mathscr{X}_1\times(-1,1)$;

(v) *For some* $m\in\mathbb{N}$ *and for some* $y_m\in(0,1)$ *the set* $\{y=y_m\}$ *represents* $\mathscr{P}(q_{m-1}/q_m,\varepsilon)\cap\mathscr{V}$.

Since $\varrho(\mathscr{T}(\xi,\eta))=(1/\varrho(\xi,\eta))-1$, we have $\mathscr{T}_\varepsilon\mathscr{P}(q_{n-1}/q_n,\varepsilon)\subset\mathscr{P}(q_{n-2}/q_{n-1},\varepsilon)$. Thus, for every $n\geqq m$ the intersection of $\mathscr{P}(q_{n-1}/q_n,\varepsilon)$ with $\mathscr{V}$ is given in the above coordinate system by the equation $y=\delta_\varepsilon^{-(n-m)}y_m$. In particular, if the C^1 curve $\gamma(\mu)=(\xi(\mu),\eta(\mu))\in\mathscr{V}$ crosses W_ε^s transversely at $\mu=\mu_\infty$, then for $|\mu-\mu_\infty|$ sufficiently small and n sufficiently large, the intersection $(\xi(\mu_n),\eta(\mu_n))$ of $\mathscr{P}(q_{n-1}/q_n,\varepsilon)$ and γ is uniquely defined. By the mean value theorem on the function $y(\xi(\mu),\eta(\mu))$, we have

$$\frac{\delta_\varepsilon^{-(n-m)}}{\mu_n-\mu_\infty}=\frac{y(\xi(\mu_n),\eta(\mu_n))}{\mu_n-\mu_\infty}=\frac{dy}{d\mu}(\theta_n),$$

where $\theta_n\to\mu_\infty$. Thus,

$$\delta_\varepsilon^n(\mu_\infty-\mu_n)=\delta_\varepsilon^m\Big/\left(-\frac{dy}{d\mu}(\theta_n)\right).$$

Therefore,

$$\lim\delta_\varepsilon^n(\mu_\infty-\mu_n)=\delta_\varepsilon^m\Big/\left(-\frac{dy}{d\mu}(\mu_\infty)\right).$$

To prove the last assertion of the theorem we note that by Theorem 3,

$$\lim_{n\to\infty}\beta_n^n(k^{q_n}(0)-q_{n-1})=\xi_\varepsilon(0),$$

where β_n is the geometric mean of the first n scale changes $\alpha_0,\ldots,\alpha_{n-1}$, $\alpha_i=-1/(\eta_n\xi_n(0)-\eta_n(0))$, where $(\xi_n,\eta_n)=\mathscr{T}^n(\xi,\eta)$. Since η_n and ξ_n converge uniformly we get $\lim\limits_{n\to\infty}\alpha_i=\alpha_\varepsilon$, where $\alpha_\varepsilon=-1/(\eta_\varepsilon\xi_\varepsilon(0)-\eta_\varepsilon(0))=\xi_\varepsilon(0)/(\xi_\varepsilon(0)-1)$. From this it follows immediately that $\lim\limits_{n\to\infty}(\beta_n^n/\alpha_\varepsilon^n)=1$. □

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