A Property of the Structure Constants
of Finite Dimensional Compact Simple Lie Algebras

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Abstract. We consider products of structure constants of a finite-dimensional compact simple Lie algebra, in which all indices except a few are contracted in pairs. We prove that such a product is zero if only one index is free, is proportional to the Cartan-Killing tensor if two indices are free and is proportional to a structure constant itself if three indices are free. For SU(n), $n \geq 3$ we also consider products of usual d (related to the anti-commutator) and structure constants f. The results for one and two free indices are still valid. For three free indices the product is proportional to either an f or a d according to whether the number of f's in the product is odd or even.

1. Introduction

In quantum chromodynamics (QCD), the various fields carry colour and the interaction vertices depend explicitly on quantities connected with the colour group SU(3)$_c$, which is gauged to produce the interactions. In particular, the colour dependence of the bare triple gluon and ghost-gluon vertices is given by the structure constants c_{ijk} of the colour group (forgetting for the moment about the contravariant or covariant character of the indices). The question then arises about the colour dependence of the vertices in higher loop orders. Is it still given by c_{ijk} as above, or are there additional terms with a different dependence on the colour indices? This question is important from the viewpoint of having a renormalizable theory. The point is that unless the counterterms required for these vertices in higher orders depend only on c_{ijk}, such theories will not be renormalizable in the usual way. Fortunately, using charge conjugation invariance, one can show [1] that the proper triple gluon vertex depends solely on c_{ijk}. However, this proof does not work for the ghost-gluon vertex. Instead one can use the Slavnov-Taylor identity [2] connecting the two vertices to show [1] that in two momentum subtraction schemes [3] the divergent part of the ghost-gluon vertex depends only on c_{ijk}, though finite parts with a different colour dependence are not ruled out. Much before this proof was found, one was led to speculate that there may be a

mathematical property of the structure constants themselves which would ensure that the proper ghost-gluon vertex is always proportional to c_{ijk}.

In any higher order contribution to the ghost-gluon vertex, the colour dependent part will factor out and will be a combination of the structure constants with three free colour indices. Moreover, since any higher order diagram is constructed from the elementary vertices connected by propagators, and since the ghost fields do not couple directly to quarks, the dummy indices will be contracted pairwise. Thus, the general form of the colour dependence expected is

$$T_{ijk} = c_{i\ldots} c_{j\ldots} c_{k\ldots} \cdots c_{\ldots} c_{\ldots}. \tag{1.1}$$

The dots represent dummy indices which are contracted pairwise only. Our considerations include the T_{ijk} in which ij, ik or jk may belong to the same c. Elementary counting tells us that the number of c's in Eq. (1.1) must be odd. In words: a combination of an odd number of c's with three free indices is a number times a single c, provided the dummy indices in the combination are contracted pairwise.

Apart from the original motivation from QCD, it is interesting in its own right to prove that all such T_{ijk} are proportional to c_{ijk}. We prove this not only for SU(3) but for all compact simple Lie algebras. We also prove that contracted tensor products of c's with one and two free indices are, respectively, zero and proportional to the Cartan-Killing form. Two proofs are given; one pleasing to the mathematically minded (Sects. 2–4) and the other which might appeal to the physicists (Sect. 5). In the special case of SU(n), $n \geq 3$ these results are extended using the first proof to the contracted tensor products containing f_{ijk} (proportional to c_{ijk}) and d_{ijk}. Some numerical results for SU(3) are given in Sect. 6.

2. Invariance of the Structure Constants under the Adjoint Representation

In this section we recall some well known results which will be relevant for us.

Let $u, v, \ldots$ be the elements of a finite-dimensional Lie Algebra $\mathscr{A}$ and $[u, v]$ their anti-symmetric Lie product. The Jacobi identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \tag{2.1}$$

tells us that the adjoint operator $\varDelta_u$ defined for every v in $\mathscr{A}$ by

$$\varDelta_u v = [u, v], \tag{2.2}$$

gives a representation of the algebra $\mathscr{A}$ into itself: the so-called adjoint representation. The Cartan-Killing form is defined by

$$g(u, v) = \mathrm{Tr}(\varDelta_u \varDelta_v). \tag{2.3}$$

Let $\mathscr{G}$ be the simply connected Lie group associated with $\mathscr{A}$; u in $\mathscr{A}$ corresponds to $\exp u$ in $\mathscr{G}$. The adjoint representation $\varDelta \equiv \varDelta(\mathscr{G})$ of $\mathscr{G}$ is defined by the formal power series

$$\exp \varDelta_u = \sum_{n=0}^{\infty} \frac{1}{n!} (\varDelta_u)^n. \tag{2.4}$$

It is known [1] that the Lie product is invariant under the adjoint representation Δ, i.e. for every u, v and w in $\mathscr{A}$

$$[\exp(\Delta_u)v, \exp(\Delta_u)w] = \exp(\Delta_u)[v, w]. \qquad (2.5)$$

With respect to any basis (e_i) of $\mathscr{A}$, the structure constants $c_{ij}{}^k$ are defined by

$$[e_i, e_j] = c_{ij}{}^k e_k. \qquad (2.6)$$

Here and in what follows, summation over repeated indices will be understood. The matrix elements of Δ_{e_i} are thus equal to the structure constants and the Cartan-Killing form reads

$$g_{ij} \equiv g(e_i, e_j) = c_{ik}{}^l c_{jl}{}^k. \qquad (2.7)$$

From now on we will be concerned only with the semi-simple Lie algebras over the complex field. They are characterized by a non-degenerate Cartan-Killing form. In these algebras one can lower or raise the indices at will using the non-singular metric tensor (g_{ij}) and its inverse (g^{ij}). Tensorial sequences which are related by the above procedure are said to be equivalent and an equivalent class is called a tensor, the components of which are the tensorial sequences with a definite tensorial (contravariant or covariant) character. For instance, $(c_{ij}{}^k)$ are particular components of a tensor $c^{(3)}$ of order three. Its completely covariant components (c_{ijk}) are known to be totally anti-symmetric in the three indices. This is an intrinsic property of the structure constant tensor $c^{(3)}$. Similarly, (g_{ij}) are the covariant components of a tensor $g^{(2)}$ of order two which is symmetric.

Under a change of basis

$$e_i' = e_j M_i^j, \qquad (2.8)$$

where (M_i^j) is any non-singular matrix, the structure constants and the metric tensor transform as tensorial sequences of order three and two respectively. For example,

$$c_{ijk}' = M_i^l M_j^m M_k^n c_{lmn}, \qquad c_{ij}'{}^k = M_i^l M_j^m (M^{-1})_n^k c_{lm}{}^n, \qquad (2.9)$$

$$g_{ij}' = M_i^l M_j^m g_{lm}. \qquad (2.10)$$

We now restrict ourselves to changes of basis in $\mathscr{A}$ generated by the adjoint representation Δ,

$$e_i' = e_j (\exp \Delta_u)_i^j \qquad (2.11)$$

for any u in $\mathscr{A}$. It follows from Eq. (2.5) that the structure constants form an invariant tensorial sequence, i.e.

$$c_{ij}'{}^k = (\exp \Delta_u)_i^l (\exp \Delta_u)_j^m (\exp -\Delta_u)_n^k c_{lm}{}^n = c_{ij}{}^k. \qquad (2.12)$$

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1   Expanding the exponential series in the left hand side of Eq. (2.5), and using induction on n and the Jacobi identity (2.1) to prove $\sum\limits_{\substack{p,q=0 \\ p+q=n}}^{\infty} \dfrac{n!}{p!q!} [(\Delta_u)^p v, (\Delta_u)^q w] = (\Delta_u)^n [v, w]$, one gets the result

Since the metric tensor (g_{ij}) is given in terms of the structure constants [Eq. (2.7)], it also forms an invariant tensorial sequence, i.e.

$$g'_{ij} = (\exp \varDelta_u)^l_i (\exp \varDelta_u)^m_j g_{lm} = g_{ij}. \tag{2.13}$$

Hence $c^{(3)}$ and $g^{(2)}$ are two invariant tensors under the adjoint representation $\varDelta$. This property plays a central role in proving our main results.

3. Contracted Tensor Products of Structure Constant Tensors

Let

$$T^{(n)} = c^{(3)} \otimes c^{(3)} \otimes \dots \otimes c^{(3)} \tag{3.1}$$

be a contracted tensor product of p tensors $c^{(3)}$ in which n indices are free. Elementary counting implies that $n + p$ is even. For example, $g^{(2)}$ is itself a $T^{(2)}$ tensor, and for $p = 5$:

$$T_i = c_{ijk} c^{jl}{}_m c^r{}_{nq} c^k{}_r{}^m c_l{}^{qn} \tag{3.2}$$

will represent the components of a $T^{(1)}$.

We now prove that for a finite-dimensional compact simple Lie algebra $\mathscr{A}$ one has:

$$T^{(1)} = 0, \quad T^{(2)} = \lambda g^{(2)}, \quad T^{(3)} = \mu c^{(3)}, \tag{3.3}$$

where λ and μ are constants (possibly zero) which depend upon the tensor considered.

The components of $T^{(n)}$ are a tensorial sequence of order n transforming according to the nth power of the adjoint representation $\varDelta$. Since from Sect. 2 the tensor $c^{(3)}$ is invariant under $\varDelta$, the same holds true for any $T^{(n)}$.

For compact Lie groups any finite-dimensional representation is either irreducible or completely reducible. Therefore, the invariant tensor $T^{(n)}$, as any other invariant, transforms according to the unit representation of $\mathscr{G}$ (Wigner-Eckart theorem). Hence, to find what $T^{(n)}$ can be, we have to answer the question: how many times does the unit representation occur in the nth power of $\varDelta$?

From now on we consider only compact simple Lie groups. For these groups it is known that $\varDelta$ is irreducible and different from the unit representation. Also that the unit representation occurs once in $D_1 \otimes D_2^*$ if and only if the two irreducible representations D_1 and D_2 are equivalent [4]. Moreover, $\varDelta$ is real [5, 6], $\varDelta \approx \varDelta^*$. Therefore, the unit representation occurs:

  (i) zero times in $\varDelta$,
  (ii) once in $\varDelta \otimes \varDelta$, and
  (iii) as many times in $\varDelta \otimes \varDelta \otimes \varDelta$ as $\varDelta$ occurs in $\varDelta \otimes \varDelta$.

It is verified in the appendix that $\varDelta$ occurs twice in $\varDelta \otimes \varDelta$ for the algebra A_n (isomorphic to the algebra of $SU(n+1)$), $n \geq 2$, and once for every other compact simple Lie algebra including the five exceptional ones.

Thus:

  (i) $T^{(1)} = 0$.
  (ii) All $T^{(2)}$ are proportional to one of them, say the Cartan-Killing tensor $g^{(2)}$.

(iii)   For all compact simple Lie algebras, except A_n, $n \geq 2$, all $T^{(3)}$ are proportional to one of them, say the structure constant tensor $c^{(3)}$ itself. This ends the proof of Eqs. (3.3) for these algebras.

(iv)   For A_n, $n \geq 2$, one needs further arguments to show that $T^{(3)}$ is proportional to $c^{(3)}$, since there are two linearly independent third order invariant tensors under Δ. This is done in the next section.

4. The Special Case of SU(n), n ≥ 3

The Lie algebra of $SU(n)$ is isomorphic to the $(n^2 - 1)$-dimensional Lie algebra $\mathscr{A}$ over the real field of $n \times n$ traceless hermitian matrices, the Lie product being defined by $-i$ times the commutator. Without loss of generality, we choose the commonly used [7] basis (V_i) of $\mathscr{A}$ normalized such that:

$$\mathrm{Tr}(V_i V_j) = 2\delta_{ij}. \tag{4.1}$$

The commutator and the anti-commutator of two V's are then given by:

$$[V_i, V_j] = 2if_{ijk}V_k, \tag{4.2}$$

$$\{V_i, V_j\} = \frac{4}{n}\delta_{ij}\mathbf{1} + 2d_{ijk}V_k. \tag{4.3}$$

The structure constants f_{ijk} are related to the c's considered previously by

$$f_{ijk} = \frac{1}{2}c_{ij}{}^{k} = -\frac{1}{8n}c_{ijk}. \tag{4.4}$$

The f_{ijk} and the d_{ijk} are, respectively, completely anti-symmetric and symmetric in i, j, and k. Both of them are real.

The adjoint representation of $SU(n)$ corresponds to changes of basis in $\mathscr{A}$ defined for all U in $SU(n)$ by:

$$V_i' = UV_iU^{-1}. \tag{4.5}$$

Thus, the tensorial sequences (f_{ijk}) and (d_{ijk}) are seen to be invariant under the adjoint representation. They are the components of the two linearly independent third order tensors $f^{(3)}$ and $d^{(3)}$. Consequently, any invariant tensor $T^{(3)}$ is a linear combination of $f^{(3)}$ and $d^{(3)}$.

To distinguish between $f^{(3)}$ and $d^{(3)}$ we examine the law of transformation of these tensors under the change of basis in $\mathscr{A}$ defined by the complex conjugation of the basis matrices V_i. Since f_{ijk} and d_{ijk} are real, taking the complex conjugate of Eqs. (4.2) and (4.3) one finds that $f^{(3)}$ changes sign while $d^{(3)}$ remains invariant:

$$V_i \to V_i^* \;\Rightarrow\; f^{(3)} \to -f^{(3)}, \qquad d^{(3)} \to d^{(3)}. \tag{4.6}$$

It follows that $T^{(3)}$ defined as the contracted tensor product of an odd number of tensors $f^{(3)}$ changes sign under complex conjugation of the V_i, and therefore is a multiple of $f^{(3)}$. This completes the proof of Eq. (3.3).

More generally, let $T_{qp}^{(n)}$ be a tensor of order n defined as a contracted tensor product of q tensors $d^{(3)}$ and p tensors $f^{(3)}$. Elementary counting implies that
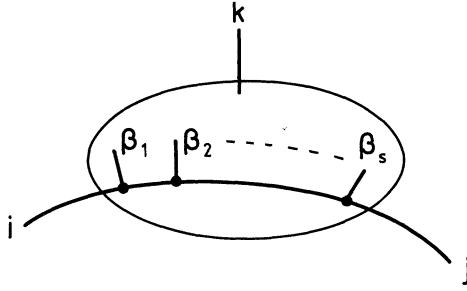
Fig. 1. A path starting at line i and ending in line j associated with the expression for T_{ijk} given in Eq. (5.1)

$n+p+q$ is even. This tensor $T_{qp}^{(n)}$ is invariant under the adjoint representation of $SU(n)$, and following the same arguments as above yields for all q and p:

$$T_{qp}^{(1)} = 0, \qquad T_{qp}^{(2)} = \lambda g^{(2)}, \tag{4.7}$$

$$T_{qp}^{(3)} = \begin{cases} \mu d^{(3)} & \text{if } q \text{ is odd and } p \text{ even} \\ \nu f^{(3)} & \text{if } q \text{ is even and } p \text{ odd}, \end{cases} \tag{4.8}$$

where λ, μ, and ν are constants (possibly zero) depending upon the tensor considered.

5. Second Proof of the Nature of $T^{(n)}$ for $n \leq 3$

This proof is based on the matrix representation of the adjoint operator. Let us denote by Δ_i the matrix of Δ_{e_i} with respect to a basis (e_i) of $\mathscr{A}$. Without loss of generality, for compact simple Lie algebras we choose (g_{ij}) to be proportional to the unit matrix. Then $c_{ijk} = (\Delta_i)_{kj} = (\Delta_j)_{ik} = (\Delta_k)_{ji}$ follows from the complete anti-symmetry of the c_{ijk}. This is used below to express products of c's in terms of matrix products.

A product of any number of c's with certain contracted indices can be represented by a trivalent graph. Apart from a sign, a c is denoted by a point and its indices by lines radiating from it, a line joining two points corresponds to a contracted index. Thus T_{ijk} is represented by a trivalent graph with three outgoing lines marked i, j, and k. It is sufficient to consider connected graphs. One can then always trace a path starting at line i and ending in line j, see Fig. 1. Therefore, T_{ijk} can always be written as:

$$T_{ijk} = M_{ij} c_{\alpha_1 \alpha_2 \alpha_3} \cdots c_{\alpha_{r-2} \alpha_{r-1} \alpha_r}, \tag{5.1}$$

where M_{ij} is the (i,j) matrix element of the matrix

$$M = \Delta_{\beta_1} \Delta_{\beta_2} \cdots \Delta_{\beta_s}. \tag{5.2}$$

By definition all the indices α and β, except k which is one of them, are pairwise contracted. Since the graph is connected, there is at least one index in the set $\{\beta_1, \ldots, \beta_s\}$, call it β, which occurs in the set $\{\alpha_1, \ldots, \alpha_r\}$. Denote the unique structure constant which carries the index β in Eq. (5.1) by $c_{\alpha \alpha' \beta}$. Now using the

obvious matrix relation

$$c_{\alpha\alpha'\beta}\Delta_\beta = \Delta_\alpha\Delta_{\alpha'} - \Delta_{\alpha'}\Delta_\alpha \qquad (5.3)$$

again and again, we can absorb all the c's in Eq. (5.1). Then T_{ijk} can be written as the (i,j) matrix element of a linear combination of matrices

$$T_{ijk} = (M' + M'' + \ldots)_{ij}, \qquad (5.4)$$

where M', M'', ... are, up to a sign, matrices of the type defined by Eq. (5.2). This time Δ_k occurs in each of them.

Given Eq. (5.4), it is sufficient to show that any such matrix

$$M_k = \Delta_{\beta_1}\ldots\Delta_{\beta_m}\Delta_k\Delta_{\beta_{m+2}}\ldots\Delta_{\beta_p}, \qquad (5.5)$$

p odd, all indices β except k being contracted in pairs, is proportional to Δ_k. Using

$$[X, Y_1 Y_2 \ldots Y_p] = [X, Y_1]Y_2\ldots Y_p + Y_1[X, Y_2]\ldots Y_p + \ldots + Y_1 Y_2 \ldots [X, Y_p], \quad (5.6)$$

the commutator $[\Delta_i, M_j]$ can be written as a sum of terms. One of them is:

$$\Delta_{\beta_1}\ldots[\Delta_i, \Delta_j]\ldots\Delta_{\beta_p} = c_{ijk}\Delta_{\beta_1}\ldots\Delta_k\ldots\Delta_{\beta_p} = c_{ijk}M_k, \qquad (5.7)$$

while the other terms are paired to have a zero sum. Indeed, for any contracted index β which occurs twice in the right hand side of Eq. (5.5), there are two terms

$$\ldots[\Delta_i, \Delta_\beta]\ldots\Delta_\beta\ldots + \ldots\Delta_\beta\ldots[\Delta_i, \Delta_\beta]\ldots = c_{i\beta\gamma}(\ldots\Delta_\gamma\ldots\Delta_\beta\ldots + \ldots\Delta_\beta\ldots\Delta_\gamma\ldots), \quad (5.8)$$

which sum to zero owing to the anti-symmetry of $c_{i\beta\gamma}$. Thus one finds:

$$[\Delta_i, M_j] = c_{ijk}M_k. \qquad (5.9)$$

The number of linearly independent solutions M_i of this equation is the number of times the adjoint representation Δ occurs in the reduction of $\Delta \otimes \Delta$. The arguments of Sects. 3 and 4 above then complete the proof that M_k is proportional to Δ_k.

It then follows that

$$T_{ijk} = \mu c_{ijk}, \qquad (5.10)$$

from which one can easily deduce the results for one and two free indices, i.e.

$$T_i = T_{ijk}\delta_{jk} = \mu c_{ijj} = 0, \qquad (5.11)$$

$$T_{ij} = T_{ikl}c_{jlk} = \lambda\delta_{ij}. \qquad (5.12)$$

6. Some Numerical Results for T_{ijk} in the Case of SU(3)

Instead of Δ_i, we use the proportional matrix F_i, the (k,j) matrix element of which is f_{ijk}. In the graphical representation of T_{ijk} with at least three f's there always exists a closed path, see Fig. 2. Then, the same arguments as in Sect. 5 allow us to write any T_{ijk} as a linear combination of traces of products of F's. In the compact notation of indicating F_i by the index i it carries, we write:

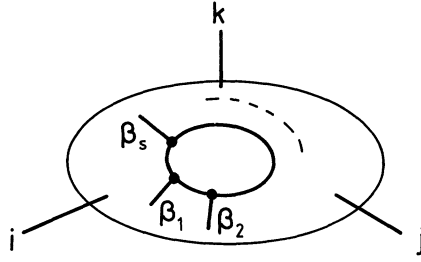$$\text{Tr}(F_i\ldots F_j\ldots F_k\ldots) \equiv \text{Tr}(i\ldots j\ldots k\ldots). \qquad (6.1)$$

Fig. 2. A closed path which allows us to write any T_{ijk} as $\mathrm{Tr}(\varDelta_{\beta_1}\varDelta_{\beta_2}...\varDelta_{\beta_s})$ times a product of c's

Table 1. Traces containing products of nine F's for $SU(3)$. Each is proportional to f_{ijk}. The constant of proportionality is given in the second column

Tr $(i\ j\ k\ l\ m\ n\ l\ m\ n)$	0	
Tr $(i\ j\ l\ k\ m\ n\ l\ m\ n)$	0	
Tr $(i\ j\ l\ m\ k\ n\ l\ m\ n)$	27/4	
Tr $(i\ j\ l\ m\ k\ n\ m\ l\ n)$	189/16	
Tr $(i\ l\ j\ m\ k\ n\ l\ m\ n)$	9/4	
Tr $(i\ l\ j\ m\ k\ n\ m\ l\ n)$	225/16	
Tr $(i\ j\ l\ m\ n\ k\ l\ m\ n)$	27/4	
Tr $(i\ j\ l\ m\ n\ k\ l\ n\ m)$	27/4	
Tr $(i\ j\ l\ m\ n\ k\ m\ n\ l\ )$	0	
Tr $(i\ l\ j\ m\ n\ k\ l\ m\ n)$	9/2	
Tr $(i\ l\ j\ m\ n\ k\ l\ n\ m)$	$-$ 9/4	
Tr $(i\ l\ j\ m\ n\ k\ m\ l\ n)$	9/4	
Tr $(i\ l\ m\ j\ l\ n\ k\ m\ n)$	$-$ 9/2	
Tr $(i\ l\ m\ j\ n\ l\ k\ m\ n)$	0	

For products of two and three f's the results are well-known

$$f_{ikl}f_{jlk} = -3\delta_{ij} = \mathrm{Tr}(ij),\tag{6.2}$$

$$f_{ilm}f_{jmn}f_{knl} = \tfrac{3}{2}f_{ijk} = -\mathrm{Tr}(ijk).\tag{6.3}$$

All the products of five f's can be reduced to a product of three or less f's by using Eqs. (6.2) and (6.3).

All the products of seven f's can be expressed in terms of products of a fewer number of f's and

$$\mathrm{Tr}(ijlmklm) = 0,\tag{6.4}$$

$$\mathrm{Tr}(iljmklm) = 36 f_{ijk}.\tag{6.5}$$

On a systematic study of the traces of nine f's we find that one can express them all in terms of a set of fourteen traces and products of a fewer number of f's. The values of these fourteen traces are given in Table 1. Traces containing seven and nine f's were calculated on a computor using the explicit numerical values of the f_{ijk}.

Appendix – Number of Times the Adjoint Representation Δ Occurs in $\Delta \otimes \Delta$

As we could not find a convenient reference, we worked out the reduction of $\Delta \otimes \Delta$ into its irreducible components. We list the result for all compact simple Lie algebras. An irreducible representation is characterised by its highest weight $\mathbf{M} = \sum_i \lambda_i \pi_i$ expressed in terms of the fundamental dominant weights π_i. In labelling these π_i we have adopted the notation of Cartan (see for example, Refs. [5] and [6])

A_1:
$$2\pi_1 \otimes 2\pi_1 = 4\pi_1 \oplus 2\pi_1 \oplus 0,$$
dimensions $\quad 3 \times 3 = 5 + 3 + 1$.

A_2:
$$(\pi_1 + \pi_2) \otimes (\pi_1 + \pi_2)$$
$$= (2\pi_1 + 2\pi_2) \oplus 3\pi_1 \oplus 3\pi_2 \oplus (\pi_1 + \pi_2) \oplus (\pi_1 + \pi_2) \oplus 0,$$
dimensions $\quad 8 \times 8 = 27 + 10 + 10 + 8 + 8 + 1$.

$A_n, n \geq 3, (D_3 = A_3)$:
$$(\pi_1 + \pi_n) \otimes (\pi_1 + \pi_n)$$
$$= (2\pi_1 + 2\pi_n) \oplus (2\pi_1 + \pi_{n-1}) \oplus (\pi_2 + 2\pi_n) \oplus (\pi_2 + \pi_{n-1})$$
$$\oplus (\pi_1 + \pi_n) \oplus (\pi_1 + \pi_n) \oplus 0,$$
dimensions $\quad n(n+2) \times n(n+2)$
$$= \tfrac{1}{4} n(n+1)^2(n+4) + \tfrac{1}{4} n(n-1)(n+2)(n+3)$$
$$+ \tfrac{1}{4} n(n-1)(n+2)(n+3) + \tfrac{1}{4}(n+1)^2(n^2-4)$$
$$+ n(n+2) + n(n+2) + 1.$$
$$(A_n = \mathrm{SU}(n+1), n \geq 1).$$

B_3:
$$\pi_3 \otimes \pi_3 = 2\pi_3 \oplus (2\pi_1 + \pi_2) \oplus 2\pi_2 \oplus 2\pi_1 \oplus \pi_3 \oplus 0,$$
dimensions $\quad 21 \times 21 = 168 + 189 + 27 + 35 + 21 + 1$.

B_4:
$$\pi_3 \otimes \pi_3 = 2\pi_3 \oplus (\pi_2 + \pi_4) \oplus 2\pi_2 \oplus 2\pi_1 \oplus \pi_3 \oplus 0,$$
dimensions $\quad 36 \times 36 = 495 + 594 + 44 + 126 + 36 + 1$.

$B_n, n \geq 5$:
$$\pi_3 \otimes \pi_3 = 2\pi_3 \oplus (\pi_2 + \pi_4) \oplus 2\pi_2 \oplus \pi_5 \oplus \pi_3 \oplus 0,$$
dimensions $\quad n(2n+1) \times n(2n+1)$
$$= \tfrac{1}{3}(n^2-1)(2n+1)(2n+3) + \tfrac{1}{2}n(n-1)(2n+1)(2n+3)$$
$$+ n(2n+3) + \tfrac{1}{6}n(n-1)(2n-1)(2n+1) + n(2n+1) + 1.$$

$C_n, n \geq 2, (B_2 = C_2)$:
$$2\pi_1 \otimes 2\pi_1 = 4\pi_1 \oplus (2\pi_1 + \pi_2) \oplus 2\pi_2 \oplus 2\pi_1 \oplus \pi_2 \oplus 0,$$
dimensions $\quad n(2n+1) \times n(2n+1)$
$$= \tfrac{1}{6}n(n+1)(2n+1)(2n+3) + \tfrac{1}{2}n(n-1)(2n+1)(2n+3)$$
$$+ \tfrac{1}{3}n(n-1)(2n-1)(2n+3) + n(2n+1) + (n-1)(2n+1) + 1.$$

D_4:
$$\pi_4 \otimes \pi_4 = 2\pi_4 \oplus (\pi_1 + \pi_2 + \pi_3) \oplus 2\pi_3 \oplus 2\pi_1 \oplus \pi_4 \oplus 0,$$
dimensions $\quad 28 \times 28 = 300 + 350 + 35 + 70 + 28 + 1$.

D_5:
$$\pi_4 \otimes \pi_4 = 2\pi_4 \oplus (\pi_3 + \pi_5) \oplus 2\pi_3 \oplus (\pi_1 + \pi_2) \oplus \pi_4 \oplus 0,$$
dimensions $\quad 45 \times 45 = 770 + 945 + 54 + 210 + 45 + 1$.

$D_n, n \geqq 6$:

$$\pi_4 \otimes \pi_4 = 2\pi_4 \oplus (\pi_3 + \pi_5) \oplus 2\pi_3 \oplus \pi_6 \oplus \pi_4 \oplus 0,$$

dimensions $\quad n(2n-1) \times n(2n-1)$

$$= \tfrac{1}{3}n(n+1)(2n+1)(2n-3) + \tfrac{1}{2}n(n+1)(2n-1)(2n-3)$$
$$+ (n+1)(2n-1) + \tfrac{1}{6}n(n-1)(2n-1)(2n-3) + n(2n-1) + 1.$$

E_6:

$$\pi_2 \otimes \pi_2 = 2\pi_2 \oplus \pi_6 \oplus (\pi_1 + \pi_3) \oplus \pi_2 \oplus 0,$$

dimensions $\quad 78 \times 78 = 2430 + 2925 + 650 + 78 + 1.$

E_7:

$$\pi_1 \otimes \pi_1 = 2\pi_1 \oplus \pi_5 \oplus \pi_4 \oplus \pi_1 \oplus 0,$$

dimensions $\quad 133 \times 133 = 7371 + 8645 + 1539 + 133 + 1.$

E_8:

$$\pi_1 \otimes \pi_1 = 2\pi_1 \oplus \pi_3 \oplus \pi_2 \oplus \pi_1 \oplus 0,$$

dimensions $\quad 248 \times 248 = 27000 + 30380 + 3875 + 248 + 1.$

F_4:

$$\pi_2 \otimes \pi_2 = 2\pi_2 \oplus \pi_4 \oplus 2\pi_1 \oplus \pi_2 \oplus 0,$$

dimensions $\quad 52 \times 52 = 1053 + 1274 + 324 + 52 + 1.$

G_2:

$$\pi_2 \otimes \pi_2 = 2\pi_2 \oplus 3\pi_1 \oplus 2\pi_1 \oplus \pi_2 \oplus 0,$$

dimensions $\quad 14 \times 14 = 77 + 77 + 27 + 14 + 1.$

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