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The Diffusion Limit for Reversible Jump Processes on Z^d with Ergodic Random Bond Conductivities

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Abstract. We consider a reversible jump process on $\mathbb{Z}^d$ whose jump rates themselves are random. We show mean square convergence of this process under diffusion scaling to a limiting Brownian motion with a certain diffusion matrix, characterizing effective conductivity.

0. Introduction

This paper contains a generalization of the well-known Theorem of Donsker (cf. Donsker [5]) to a pure jump process whose jump rates themselves contain a certain degree of randomness. This result can also be interpreted as giving the limiting Brownian motion and its diffusion matrix for a random walk in random environment under diffusion scaling.

Consider a probability space $(\Omega, \mathscr{F}, P)$. For $\omega \in \Omega$ fixed, let X_ω^ε be a pure jump process on the ε-spaced lattice $\varepsilon\mathbb{Z}^d$ with time structure governed by exponential waiting times with rate $\lambda^\varepsilon(x,\omega)$ at $x \in \varepsilon\mathbb{Z}^d$ and space structure given by the nearest neighbour jump probabilities, $p^\varepsilon_{i\pm}(x,\omega)$ being the probability under realisation ω to jump from x to $x \pm \varepsilon e_i$ at the next jump time, $1 \leqq i \leqq d$.

Let $a^\varepsilon_{i\pm}(x,\omega) = \lambda^\varepsilon(x,\omega) \cdot p^\varepsilon_{i\pm}(x,\omega)$ for all $\varepsilon > 0$, $x \in \varepsilon\mathbb{Z}^d$. Assume that

$$a^\varepsilon_{i\pm}(x,\omega) = a^1_{i\pm}\left(\frac{x}{\varepsilon}, \omega\right) =: a_{i\pm}\left(\frac{x}{\varepsilon}, \omega\right) \ \forall \varepsilon > 0,\ x \in \varepsilon\mathbb{Z}^d,\ \omega \in \Omega,\ 1 \leqq i \leqq d, \tag{0.1}$$

$$a_i(x,\omega) := a_{i+}(x,\omega) = a_{i-}(x+e_i,\omega) x \in \mathbb{Z}^d, 1 \leqq i \leqq d, \tag{0.2}$$

$$0 < A \leqq a_i(x,\omega) \leqq B < \infty \text{ for all } \forall x \in \mathbb{Z}^d, \omega \in \Omega, \tag{0.3}$$

$$a_i(x,\omega) \text{ is stationary and ergodic, } 1 \leqq i \leqq d. \tag{0.4}$$

A few words are now in order, considering these fundamental conditions. Equation (0.2) simply says that the process is reversible and that the "conductivity" $a^\varepsilon_i(x,\omega)$ is a "bond conductivity," i.e. independent of the direction in which the bond $(x, x+e_i)$ is used by the process. Equation (0.1) indicates intuitively that the configuration of bond conductivities $a^\varepsilon_{i\pm}(\omega)$ on $\varepsilon\mathbb{Z}^d$ is simply the configuration

$a_{i\pm}(\omega)$ on $\mathbb{Z}^d$ "looked at from a distance." Inequality (0.3) assumes the existence of uniform bounds A, B with $A > 0$. Let us now consider (0.4).

We may identify ω with the realisation of bond conductivities at ω: $\{a_i(x,\omega)\}_{x\in\mathbb{Z}^d} \sim \omega$. This enables us to define a "shift" τ_y on Ω for $y\in\mathbb{Z}^d$ by $a_i(x,\tau_y\omega) := a_i(x+y,\omega)$, i.e. $\tau_y\omega \sim \{a_i(x+y,\omega)\}_{x\in\mathbb{Z}^d}$. (0.4) contains the assumptions that the probability measure P on $(\Omega, \mathscr{F}, P)$ is stationary with respect to the shifts $\tau_y, y\in\mathbb{Z}^d$, and that the group $\{\tau_y\}_{y\in\mathbb{Z}^d}$ of shifts is ergodic for P, i.e. the only sets $E\in\mathscr{F}$ with $\tau_y E = E$ for all $y\in\mathbb{Z}^d$ are those with $P(E)=0$ or $P(E)=1$.

Our main result (Theorem 5) states that under these conditions, as $\varepsilon\to 0$, X^ε_ω converges to a Brownian motion X in a certain sense. Theorem 3 will furnish an expression for the diffusion matrix (q_{ij}) of the process X. In terms of physics Theorem 5 together with Theorem 3 can be seen as giving a formula of the "effective conductivity" for a conduction process on a lattice with random bond conductivities a_i.

If we consider a crystal with diffusion of atoms on interstitial positions what happens microscopically is in fact a jump process for the individual atoms with certain jump rates, determined by the potential barriers of the neighbouring lattice atoms. For details cf. Kittel [9]. For conduction phenomena on lattices cf. Kirkpatrick [8].

For the proof of Theorem 5 we work in suitable Hilbert spaces showing convergence of resolvents (Sect. 4), yielding semigroup convergence. Compactness (Sect. 2) of the family $\{X^\varepsilon_\omega, 1>\varepsilon>0, \omega\in\Omega\}$ is the crucial ingredient for proceeding to convergence in distribution.

This paper makes use of the approach developed by Papanicolaou and Varadhan in [13].

1. Some Properties of the Jump Processes $\mathbf{X}^\varepsilon_\omega$

Consider the cubic lattice $\varepsilon\mathbb{Z}^d$ with lattice constant ε and $a^\varepsilon_{i\pm}(x,\omega): \varepsilon\mathbb{Z}^d \times \Omega \to [A,B]$ satisfying (0.1) to (0.4), where $a_i(x,\omega)$ is the conductivity of the bond $(x, x+e_i)$ on the lattice $\mathbb{Z}^d$. Fix ω to consider the deterministic lattice first, and let

$$(\nabla^{\varepsilon-}_i f)(x) := \frac{1}{\varepsilon}[f(x-\varepsilon e_i) - f(x)],$$

$$(\nabla^{\varepsilon+}_i f)(x) := \frac{1}{\varepsilon}[f(x+\varepsilon e_i) - f(x)] \tag{1.1}$$

for a function f, square summable on $\varepsilon\mathbb{Z}^d$ or square integrable on $\mathbb{R}^d$, with e_i the unit vector in i-direction. It is not hard to verify that

$$\frac{\partial f(x,t)}{\partial t} = -\sum_{i=1}^d \nabla^{\varepsilon-}_i \left(a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla^{\varepsilon+}_i f(x,t)\right) =: \mathscr{L}^\varepsilon_\omega f(x,t) \tag{1.2}$$

is the diffusion equation on the lattice $\varepsilon\mathbb{Z}^d$ in the terminology introduced above with density $f(x)$ and conductivity $a_i(x/\varepsilon)$.

It is a standard result from the theory of Markov processes (e.g. Breiman [3]), that the operator $\mathscr{L}^\varepsilon_\omega$ is the infinitesimal generator of the pure jump process $X^\varepsilon_\omega(\bar{t})$

described above with scaled time $\bar{t} := \varepsilon^2 t$. Indeed, explosions are excluded, since we have a bound on the jump rate $\lambda^\varepsilon_\omega(x): \lambda^\varepsilon_\omega(x) = \sum_{i=1}^{d} \{a^\varepsilon_{i+}(x,\omega) + a^\varepsilon_{i-}(x,\omega)\} \leqq 2d \cdot B$ (by (0.3)).

Lemma 1. *$\mathscr{L}^\varepsilon_\omega$ is selfadjoint on the space of square integrable functions on $\varepsilon\mathbb{Z}^d$ with inner product $(f,g) := \sum_{x \in \mathbb{Z}^d} f(x)g(x)$.*

Proof. Observe that for $1 \leqq i \leqq d$

$$\sum_{x \in \varepsilon\mathbb{Z}^d} a_i\left(\frac{x}{\varepsilon} - e_i\right) f(x) g(x - \varepsilon e_i) = \sum_{x \in \varepsilon\mathbb{Z}^d} a_i\left(\frac{x}{\varepsilon}\right) f(x + \varepsilon e_i) g(x),$$

and

$$\sum_{x \in \varepsilon\mathbb{Z}^d} a_i\left(\frac{x}{\varepsilon} - e_i\right) g(x) f(x - \varepsilon e_i) = \sum_{x \in \varepsilon\mathbb{Z}^d} a_i\left(\frac{x}{\varepsilon}\right) g(x + \varepsilon e_i) f(x),$$

hence

$$\begin{aligned}
&(\mathscr{L}^\varepsilon_\omega f, g) \\
&= - \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{i=1}^{d} \frac{1}{\varepsilon^2} \Bigg[a_i\left(\frac{x}{\varepsilon} - e_i\right) \{ f(x - \varepsilon e_i + \varepsilon e_i) g(x) - f(x - \varepsilon e_i) g(x) \} \\
&\quad - a_i\left(\frac{x}{\varepsilon}\right) \{ f(x + \varepsilon e_i) g(x) - f(x) g(x) \} \Bigg] \\
&= - \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{i=1}^{d} \frac{1}{\varepsilon^2} \Bigg[a_i\left(\frac{x}{\varepsilon} - e_i\right) \{ g(x - \varepsilon e_i + \varepsilon e_i) \cdot f(x) - g(x - \varepsilon e_i) f(x) \} \\
&\quad - a_i\left(\frac{x}{\varepsilon}\right) \{ f(x) g(x + \varepsilon e_i) - f(x) g(x) \} \Bigg] = (f, \mathscr{L}^\varepsilon_\omega g). \quad \square
\end{aligned}$$

Therefore, the backwards and forwards equations for this process (which are satisfied by the transition probabilities $p_\varepsilon(y,t|x)$, cf. Breiman [3] and Chung [4]) read

$$\begin{aligned}
&\frac{\partial}{\partial t} p_\varepsilon(y,t|x) = [\mathscr{L}^\varepsilon_\omega p_\varepsilon(y,t|\cdot)](x) \quad \text{and} \\
&\frac{\partial}{\partial t} p_\varepsilon(y,t|x) = [\mathscr{L}^\varepsilon_\omega p_\varepsilon(\cdot,t|x)](y), \quad \text{respectively.}
\end{aligned} \tag{1.3}$$

Moreover with $\delta_x(z) = 1$ for $z = x$ and $\delta_x(z) = 0$ for $z \neq x$:

$$\begin{aligned}
\frac{\partial}{\partial t} p(y,t|x) &= (\delta_y(\cdot), \mathscr{L}^\varepsilon_\omega p(\cdot,t|x)) = (\delta_y(\cdot), \mathscr{L}^\varepsilon_\omega e^{t\mathscr{L}^\varepsilon_\omega} \delta_x(\cdot)) \\
&= (\delta_y(\cdot), e^{t\mathscr{L}^\varepsilon_\omega} \mathscr{L}^\varepsilon_\omega \delta_x(\cdot)) = (\mathscr{L}^\varepsilon_\omega e^{t\mathscr{L}^\varepsilon_\omega} \delta_y(\cdot), \delta_x(\cdot)) \\
&= (\mathscr{L}^\varepsilon_\omega p(\cdot,t|y), \delta_x(\cdot)) = \frac{\partial}{\partial t} p(x,t|y).
\end{aligned}$$

This being true for all t,x,y we can conclude that

$$p_\varepsilon(x,t|y) = p_\varepsilon(y,t|x),\ \forall t; x, y \in \varepsilon\mathbb{Z}^d. \tag{1.4}$$

Since there is at most one set of standard transition probabilities corresponding to $\mathscr{L}^\varepsilon_\omega$,

$$p_1(y,t|x) = p_\varepsilon(\varepsilon y, \varepsilon^2 t|\varepsilon x) \tag{1.5}$$

will follow from the following lemma, whose proof is straightforward.

Lemma 2. *If* $f_\varepsilon(x,t)$ *solves* $\partial f_\varepsilon/\partial t = \mathscr{L}^\varepsilon_\omega f_\varepsilon$, *then* $\tilde{f}(x,t) := f_\varepsilon(\varepsilon x, \varepsilon^2 t)$ *solves* $\partial\tilde{f}/\partial t = \mathscr{L}^\varepsilon_\omega \tilde{f}$.

The main result of this paper (Theorem 5) shows that under this type of contracting the bond lattice by ε and speeding up time by ε^{-2} the jump processes X^ε_ω approach a diffusion with matrix (q_{ij}) given by (3.17) below.

2. Relative Compactness

Relative compactness of the corresponding family plays an important role in most proofs of convergence of a family of stochastic processes (cf. Billingsley [2]). In our case we are dealing with measures $Q^\varepsilon_{x,\omega}$, respectively Q_x, belonging to the processes $X^\varepsilon_\omega(t)$ and $X(t)$ with generators $\mathscr{L}^\varepsilon_\omega$ and $\mathscr{L}$. These are measures on the set D, with $D := \{\zeta : [0,\infty) \to \mathbb{R}^d; \zeta(t) = \lim_{s\downarrow t} \zeta(s)$ and $\lim_{s\uparrow t} \zeta(s)$ exists for all $t\}$. This set of right-continuous functions with left limits contains the trajectories of our jump processes.

Let us recall some standard results (e.g. in Kurtz [10]): $C := C([0,\infty), \mathbb{R}^d)$, the set of "continuous paths," is a complete and separable metric space and so is D when furnished with a Skorokhod-type metric (cf. Kurtz [10]). Using the notation $q(x,y) := |x-y| \wedge 1$ on $x, y \in \mathbb{R}^d$ and for $\delta > 0$, $T > 0$

$$\omega'(\zeta,\delta,T) := \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} q(\zeta(s),\zeta(t)),$$

where $\{t_i\}$ is a partition on $[0,T]$ with $\min_i(t_i - t_{i-1}) > \delta$, we know that $K \subset D$ is relatively compact (i.e. cl(K) is compact), if for all $t\in\mathbb{Q}$, $t \geqq 0$, there is a compact set $\Gamma_t \subset \mathbb{R}^d$ such that

$$\zeta(t) \in \Gamma_t \quad \text{for all} \quad \zeta \in K, \tag{2.1}$$

and

$$\text{for all } T > 0: \lim_{\delta\to 0} \sup_{\zeta\in K} \omega'(\zeta,\delta,T) = 0. \tag{2.2}$$

Theorem (Prohorov). *Let* $\{P_\alpha\}_{\alpha\in A}$ *be a family of probability measures on* D *or* C. $\{P_\alpha\}_{\alpha\in A}$ *is relatively compact* iff *for all* $\varepsilon > 0$ *there is a compact set* K *with* $\inf_{\alpha\in A} P_\alpha(K) \geqq 1 - \varepsilon$.

For the description of processes of the type $X^\varepsilon_\omega(t)$ we can restrict our attention to trajectories in D, which have only isolated jumps of width ε. Such processes can be "smoothened" in a natural way: If the path ζ has a jump at t_n and the next jump at t_{n+1}, put

$$\tilde{\zeta}(t_n) := \zeta(t_n),$$

$$\tilde{\zeta}(t) := \zeta(t_n) + \frac{t - t_n}{t_{n+1} - t_n}(\zeta(t_{n+1}) - \zeta(t_n)) \quad \text{for} \quad t \in [t_n, t_{n+1}].$$

The corresponding process on the continuous paths will be denoted by $\tilde{X}^{\varepsilon}_{\omega(t)}$ (respectively its measure by $\tilde{Q}^{\varepsilon}_{\omega}$).

Let us recall that $S \subset C$ is relatively compact, if

$$\sup_{\zeta \in S} |\zeta(0)| < \infty, \tag{2.3}$$

$$\limsup_{\delta \downarrow 0} \sup_{\zeta \in S} \sup_{\substack{0 \leqq s \leqq t \leqq T \\ t - s \leqq \delta}} |\zeta(s) - \zeta(t)| = 0 \quad \text{for all } T, T < \infty. \tag{2.4}$$

From the previous compactness criterion for $K \subset D$ we can deduce the relative compactness of $\tilde{K} := \{\tilde{\zeta} \in C : \zeta \in K$ and ζ has only isolated jumps$\}$ in C.

Since $\lim_{\delta \to 0} \sup_{\zeta \in \tilde{K}} \omega'(\zeta, \delta, T) = 0$ is satisfied by (2.2), it suffices to prove

$$\lim_{\delta \to 0} \sup_{\zeta \in K} \omega'(\zeta, \delta, T) = 0 \Rightarrow \lim_{\delta \to 0} \sup_{\zeta \in K} \sup_{0 \leqq s \leqq t \leqq T} |\zeta_{(s)} - \zeta_{(t)}| = 0. \tag{2.5}$$

Choose δ_n such that $\sup_{\zeta \in \tilde{K}} \omega'(\zeta, \delta/2, T) < n^{-1}$ for all $\delta < \delta_n$, i.e. for all $\delta < \delta_n$ and $\zeta \in \tilde{K}$,

$$\inf_{\{t_i\}} \max_i \sup_{s, t \in [t_i, t_{i+1})} q(\zeta(s), \zeta(t)) < n^{-1}.$$

For δ, ζ fixed we can therefore find a partition $\{t_i\}$ (depending on ζ) with

$$\max_i \sup_{s, t \in [t_i, t_{i+1})} q(\zeta(s), \zeta(t)) < n^{-1}.$$

We now fix $s, t; 0 \leqq s \leqq t \leqq T$, with $|s - t| < \delta$: If there is an index i with $[s, t) \subset [t_i, t_{i+1})$, then $q(\zeta(s), \zeta(t)) < n^{-1}$. If such an index does not exist, we can certainly find an index i with

(a) $s \in [t_i, t_{i+1})$ and $t \in [t_{i+1}, t_{i+2})$, or

(b) $s \in [t_i, t_{i+1})$ and $t \in [t_{i+2}, t_{i+3})$, since $\min_i (t_{i+1} - t_i) > \delta/2$.

In case (a), we get $q(\zeta(t), \zeta(s)) \leqq q(\zeta(s), \zeta(t_{i+1})) + q(\zeta(t_{i+1}), \zeta(t)) < n^{-1} + n^{-1}$; in case (b) analogously

$$q(\zeta(s), \zeta(t)) \leqq q(\zeta(s), \zeta(t_{i+1})) + q(\zeta(t_{i+1}), \zeta(t_{i+2})) + q(\zeta(t_{i+2}), \zeta(t)) \leqq 3n^{-1},$$

i.e. we have $q(\zeta(s),\zeta(t)) \leqq 3n^{-1}$, for $|t-s| < \delta < \delta_n$, and

$$\sup_{\substack{0 \leqq s \leqq t \leqq t \\ |s-t| \leqq \delta}} q(\zeta(s),\zeta(t)) \leqq 3n^{-1}.$$

As this is true for all $\zeta \in \tilde{K}$, we have $\sup_{\zeta \in \tilde{K}} \sup_{|s-t| \leqq \delta} q(\zeta(s), \zeta(t)) < 3n^{-1}$ for all $\delta < \delta_n$, i.e. $\lim_{\delta \to 0} \sup_{\zeta \in \tilde{K}} \sup_{|s-r| < \delta} q(\zeta(s), \zeta(t)) = 0$, proving (2.5). Hence the closure $S := \mathrm{cl}(\tilde{K})$ is compact and $\tilde{Q}^{\varepsilon}_{\omega}(S) \geqq Q^{\varepsilon}_{\omega}(K)$.

We can then use Prohorov's Theorem to deduce the relative compactness of the family $\{\tilde{P}_\alpha\}_\alpha$ from the relative compactness of $\{P_\alpha\}_\alpha$, if $\{P_\alpha\}_\alpha$ is a family of jump processes. A compactness criterion appropriate to our situation is conveniently at hand:

Theorem 1 [Kurtz]. *Let $\{P_\alpha\}_{\alpha \in A}$ be a family of probability measures on D belonging to a family $\{X^\alpha\}_{\alpha \in A}$ of strong Markov processes. $\{P_\alpha\}_{\alpha \in A}$ is relatively compact if (2,6) and (2,7) hold:*

for all $T > 0$, $t \in \mathbb{Q}$, $0 \leqq t \leqq T$, $\eta > 0$,

$$\text{there is a compact set } \Gamma_t \subset \mathbb{R}^d \text{ with } \inf_{\alpha \in A} P_\alpha(\zeta(t) \in \Gamma_t) > 1 - \eta, \tag{2.6}$$

for all $T > 0$, $\delta > 0$, $\alpha \in A$ *there is a random variable* $Y_\alpha(\delta)$ *with* $\lim_{\delta \to 0} \sup_{\alpha \in A} E Y_\alpha(\delta) = 0$, and

$$E(Y_\alpha(\delta) | \mathscr{F}_t) \geqq E(q(X^\alpha(t+u), X^\alpha(t) | \mathscr{F}_t) \quad \text{a.s.}$$
$$\text{for all } 0 \leqq u \leqq \delta, t \leqq T. \tag{2.7}$$

The proof of this theorem can be obtained by a slight modification of the proof of Kurtz' original theorem in Kurtz [10].

Let us return now to our processes $X^{\varepsilon}_{\omega,x}$ with starting point $x \in \varepsilon\mathbb{Z}^d$ and corresponding measure $Q^{\varepsilon}_{\omega,x}$. We slightly extend our notion of $X^{\varepsilon}_{\omega,x}$ and $Q^{\varepsilon}_{\omega,x}$ in the sense that the starting point need not be a lattice point $x \in \varepsilon\mathbb{Z}^d$. The process may start at any $x \in \mathbb{R}^d$: we then simply identify $Q^{\varepsilon}_{x,\omega}(A)$ with $Q^{\varepsilon}_{\varepsilon[x/\varepsilon],\omega}(A - r)$ for $r := x - \varepsilon[x/\varepsilon]$, where $[y] := ([y_1], \ldots, [y_d])$, if $y = (y_1, \ldots, y_d)$, and $[y_i]$ is the largest integer not exceeding y_i. For $0 < M < \infty$, consider the set $A_M := \{(x,\omega,\varepsilon), x \in \mathbb{R}^d, |x| \leqq M, \omega \in \Omega, 0 < \varepsilon < 1\}$. We write $Q_\alpha := Q^{\varepsilon}_{\omega,x}$ for $x \in A_M$, and want to show that $\{Q_\alpha\}_{\alpha \in A_M}$ is a relatively compact family of measures. For this purpose we prove the following:

Theorem 2. *There is a constant C (independent of ε,ω) such that*

$$E(|X^{\varepsilon}_{\omega,0}(t)|) := E^{Q^{\varepsilon}_{\omega,0}}(|\zeta(t)|) \leqq C\sqrt{t}, \ \forall t. \tag{2.8}$$

Before starting the proof of Theorem 2, which will take up the rest of this section, we should convince ourselves that Theorem 2 is sufficient for the relative compactness of $\{Q_\alpha\}_{\alpha \in A_M}$. Since our processes are pure jump, they are also strong

Markov, and we try to apply Theorem 1. Note first that (assuming Theorem 2)

$$E[|X^{\varepsilon}_{\omega,0}(t+u)-X^{\varepsilon}_{\omega,0}(t)||X^{\varepsilon}_{\omega,0}(t)]=E[|X^{\varepsilon}_{\omega,X^{\varepsilon}_{\omega,0}(t)}(u)]$$
$$=E[X^{\varepsilon}_{\tau_{-X_{\omega,0(t)}}(\omega),0}(u)]\leqq C\sqrt{u}=:Y_\alpha(u), \tag{2.9}$$

where the first equality is just the Markov property and the second is the property of the shifts τ_y on Ω from Sect. 0. Note that ω in (2.9) is of course a parameter according to the use of the expectation operator E fixed in Theorem 1: $E\sim E^{Q^{\varepsilon}_{\omega,0}}$. With this choice of $Y_\alpha(u)$, condition (2.7) is certainly satisfied, by (2.9).

For the remaining condition (2.6) of Theorem 1 it is sufficient to show that one can find for any $t: 0<t<T$, $\eta>0$, a suitable $k(t)$ with $\sup_\alpha P(|X^\alpha_t|\geqq k(t))\leqq\eta$. But because of Theorem 1, we have $P(|X^\alpha_{(t)}|\geqq m)\leqq(1/m)E|X^\alpha_{(t)}|\leqq(1/m)C\sqrt{t}$, so put $k(t):=\eta^{-1}C\sqrt{t}$. This shows that condition (2.6) is satisfied, if we take $\Gamma_t:=\{x:|x|<M+k(t)\}$, for C is independent of ω and ε. Now Theorem 1 implies the relative compactness of $\{Q_\alpha\}_{\alpha\in A_M}$.

Proof of Theorem 2. Let us first observe that it is sufficient to prove (2.8) for $\varepsilon=1$ (in which case we write $X_{(t),\omega}$ for $X^1_{(t),\omega}$), since the fundamental solutions p_1 and p_ε of the Kolmogorov equations (1.3) have the scaling property $p_1(t,x,y)=p_\varepsilon(\varepsilon^2t,\varepsilon x,\varepsilon y)$ as was shown in (1.5). As p_1,p_ε suitably normed are also the transition densities of the Markov processes $X^1_\omega(t)$ and $X^\varepsilon_\omega(t)$, we get

$$E|X^{\varepsilon}_{\omega,0}(t)|=\sum_{y\in\mathbb{Z}^d}\varepsilon|y|\,p_\varepsilon(t,0,\varepsilon y)=\varepsilon\sum_{y\in\mathbb{Z}^d}|y|\,p_1\left(\frac{t}{\varepsilon^2},0,y\right)$$
$$=\varepsilon E\left(\left|X^1_{\omega,0}\left(\frac{t}{\varepsilon^2}\right)\right|\right)\leqq\varepsilon C\frac{\sqrt{t}}{\varepsilon^2}=C\sqrt{t},$$

assuming (2.8) for $\varepsilon=1$. The constant C will turn out to depend only on the dimension d and on A and B from (0.3). Hence we will drop the subscript ω in the sequel.

The following proof of (2.8) for $\varepsilon=1$ makes use of Nash's work on the "moment bound" in Nash [12]. We will bound the growth of $E|X_{(t)}|$ above by the growth of an entropy $S_{(t)}$ (Lemma 5) and bound $|EX_{(t)}|$ below by $ke^{S(t)/d}$ (Lemma 6). $S_{(t)}$ itself will be bounded below essentially by log t (Lemma 4). This way we will succeed in sandwiching $E|X_{(t)}|$ between two multiples of $t^{1/2}$, as we will see, for $t\geqq1$. For $t<1$ the result is trivial.

We start, however, with two technical lemmata

Lemma 1. *There is a constant $C(d)$ such that for any piecewise differentiable function $g\in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$, g continuous*

$$\int_{\mathbb{R}^d}|\nabla g|^2dx\geqq C(d)\left[\int_{\mathbb{R}^d}|g|dx\right]^{-4/d}\left[\int_{\mathbb{R}^d}|g|^2dx\right]^{1+2/d}. \tag{2.10}$$

Proof. (E. M. Stein, cf. Nash [12]). Consider the Fourier transform $\hat{g}$ of g,

$$\hat{g}(y)=(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot y}g(x)dx.$$

Recall that

$$\int_{\mathbb{R}^d}|\hat{g}(y)|^2dx=\int_{\mathbb{R}^d}|g(x)|^2dx.$$

Since $\partial g/\partial x_k$ has Fourier transform $y_k\hat{g}(y)$ (for which we need continuity, piecewise differentiability)

$$\begin{aligned}\int|\nabla g|^2dx&=\sum_{i=1}^{d}\int\left|\frac{\partial g}{\partial x_i}\right|^2dx=\sum_{i=1}^{d}\int|y_i|^2|\hat{g}(y)|^2dy\\&=\int|y|^2|\hat{g}(y)|^2dy.\end{aligned}\tag{2.11}$$

Since $|\hat{g}(y)|\leqq(2\pi)^{-d/2}\int|e^{ix,y}||g(x)|dx=(2\pi)^{-d/2}\int|g(x)|dx$, we get for $\rho>0$,

$$\int_{|y|\leqq\rho}|\hat{g}(y)|^2dy\leqq S_\rho[(2\pi)^{-d/2}\int|g|dx]^2,\tag{2.12}$$

where S_ρ is the volume of the d-sphere with radius ρ, $S_\rho=(\pi^{d/2}\rho^d/(d/2)!)$. On the other hand

$$\int_{|y|\geqq\rho}|\hat{g}(y)|^2dy\leqq\int_{|y|\geqq\rho}\left|\frac{y}{\rho}\right|^2|\hat{g}(y)|^2dy\leqq\rho^{-2}\int|\nabla g|^2dx\ \text{(by (2.6))}.\tag{2.13}$$

Now choose a ρ minimizing the sum of the two bounds in (2.12) and (2.13), to obtain a bound on $\int|\hat{g}|^2dy=\int|g|^2dx$ in terms of $\int|g|dx$ and $\int|\nabla g|^2dx$. Solved for $\int|\nabla g|^2dx$, this is

$$\begin{aligned}&\int|\nabla g|^2dx\\&\geqq(4\pi d/(d+2))\left(\left(\frac{d}{2}\right)!\Big/\left(1+\frac{d}{2}\right)\right]^{2/d}[\int|g|dx]^{-4/d}[\int|g|^2dx]^{1+2/d}.\ \square\end{aligned}$$

Lemma 2. *There is a continuous piecewise differentiable function* $g\in L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$, *such that for some constants* k_0,k_1

a) $\sum_{i=1}^{d}\sum_{x\in\mathbb{Z}^d}(\nabla^{i+}p(x,t))^2\geqq k_1\int|\nabla g|^2dx$,

b) $\int|g|dx\leqq k_0$,

c) $\int|g|^2dx\geqq\sum_{x\in\mathbb{Z}^d}p^2(x)$,

where we write $p(x,t)$ *or sometimes* $p(x)$ *for* $p(x,t,0,0)$, *the transition probability density of* $X_{(t)}$, *and where* k_0, k_1 *do not depend on* p.

Proof of Lemma 2. Considering the fact that the faces of unit cubes in dimension $(d+1)$ are unit cubes in dimension d, the step from dimension d to dimension

$(d+1)$ should be obvious from the following construction for $d=2$: Set $g(x):=4p(x)$ for $x \in \mathbb{Z}^2$;

$$g(x+\tfrac{1}{2}e_1+\tfrac{1}{2}e_2):=p(x)+p(x+e_1)+p(x+e_2)+p(x+e_1+e_2);$$

if y is on the line segment between $x \in \mathbb{Z}^2$ and $x+e_1$, let $g(y)$ be the linear interpolation of $g(x)$ and $g(x+e_1)$, i.e. $g(y)=g(x)+k(g(x+e_1)-g(x))$ for $y=x+ke_1$. Similarly for the other three edges of the unit square $\langle x, x+e_1, x+e_2, x+e_1+e_2 \rangle (=:C_2(x))$. For $y \in C_2(x)$, $y \neq x+\frac{1}{2}e_1+\frac{1}{2}e_2$, let $x_s(y)$ be the point on a side of $C_2(x)$ such that y is on the line segment from $x+\frac{1}{2}e_1+\frac{1}{2}e_2$ to $x_s(y)$. Let $g(y)$ be the linear interpolation of $g(x+\frac{1}{2}e_1+\frac{1}{2}e_2)$ and $g(x_s(y))$.

Continuity and piecewise differentiability of g are immediate; we are left with showing a), b), c) of Lemma 2.

to c): Set $\tilde{p}(y):=p(x)$ for $y \in [x-\frac{1}{2}e_1, x+\frac{1}{2}e_1] \times [x-\frac{1}{2}e_2, x+\frac{1}{2}e_2]$, $x \in \mathbb{Z}^2$. It suffices to show $g(y) \geqq \tilde{p}(y)$, $\forall y \in \mathbb{R}^2$. This is immediate for $y \in \partial C_2(x) \forall x \in \mathbb{Z}^2$, and hence for $y \in \operatorname{int}(C_2(x))$ in general.

to b): For $y \in C_2(x): g(y) \leqq 4\max\{p(x), p(x+e_1), \quad p(x+e_2), p(x+e_1+e_2)\}$. Since for any $x: p(x)$ can occur at most four times as such a maximum (namely for the four adjacent unit squares), we get $\int g(y)dy \leqq 4.4. \sum_{x \in \mathbb{Z}^2} p(x) = 16 =: k_0$.

to a): We compute $\nabla^1 g(y)$: Denote the triangle with vertices A, B, C by $\Delta(A, B, C)$. If $y \in \Delta(x, x+e_1, x+\frac{1}{2}e_1+\frac{1}{2}e_2)$, $y \in \Delta(x+e_2, x+\frac{1}{2}e_1+\frac{1}{2}e_2, x+e_1+e_2)$, then $\nabla^1 g(y) = 4\nabla^1 p(x)$, respectively $\nabla^1 g(y) = 4\nabla^1 p(x+e_2)$. If $y \in \Delta(x, x+\frac{1}{2}e_1+\frac{1}{2}e_2, x+e_2)$, respectively $y \in \Delta(x+e_1, x+e_1+e_2, x+\frac{1}{2}e_1+\frac{1}{2}e_2)$,

then

$$\nabla^1 p(y) = 2\left(p(x)+p(x+e_1)+p(x+e_2)+p(x+e_1+e_2) - \frac{4p(x)+4p(x+e_2)}{2}\right) = 2[\nabla^1 p(x) + \nabla^1 p(x+e_2)],$$

respectively

$$\nabla^1 g(y) = 2\left(\frac{4p(x+e_1)+4p(x+e_1+e_2)}{2} - (p(x)+p(x+e_1) + p(x+e_2)+p(x+e_1+e_2))\right) = 2[\nabla^1 p(x) + \nabla^1 p(x+e_2)].$$

Hence

$$\begin{aligned}\int (\nabla^1 g(y))^2 dy &= \sum_{x \in \mathbb{Z}^2} (\tfrac{1}{4}(4\nabla^1 p(x))^2 + \tfrac{1}{4}(4\nabla^1 p(x+e_2))^2 \\ &\quad + \tfrac{2}{4}(2[\nabla^1 p(x) + \nabla^1 p(x+e_2)])^2 \\ &= \frac{1}{4}\sum_{x \in \mathbb{Z}^2} \{16(\nabla^1 p(x))^2 + 16(\nabla^1 p(x+e_2))^2 \\ &\quad + 8[\nabla^1 p(x) + \nabla^1 p(x+e_2)]^2\}.\end{aligned}$$

But

$$\sum_{x\in\mathbb{Z}^2}[\nabla^1 p(x)+\nabla^1 p(x+e_2)]^2 \leqq \sum_{x\in\mathbb{Z}^2}\{(\nabla^1 p(x))^2+(\nabla^1 p(x+e_2))^2\}$$

$$+\sum_{x\in\mathbb{Z}^2}|\nabla^1 p(x)\nabla^1 p(x+e_2)|\cdot 2.$$

By Schwarz' inequality

$$\sum_{x\in\mathbb{Z}^2}|\nabla^1 p(x)\nabla^1 p(x+e_2)| \leqq \left\{\sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x))^2 \sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x+e_2))^2\right\}^{1/2}$$

$$=\sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x))^2.$$

Altogether then

$$\int(\nabla^1 g(y))^2 dy \leqq \frac{32}{4}\sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x))^2+\frac{32}{4}\sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x))^2$$

$$=16\sum_{x\in\mathbb{Z}^2}(\nabla^1 p(x))^2,$$

similarly $\int(\nabla^2 g(y))^2 dy \leqq 16\sum_{x\in\mathbb{Z}^2}(\nabla^2 p(x))^2$, so that

$$\frac{1}{16}\int(\nabla g(y))^2 dy \leqq \sum_{i=1}^{2}\sum_{x\in\mathbb{Z}^2}(\nabla^i p(x))^2. \quad \square$$

Using the previous lemmata we can bound $p(x,t)$ in terms of t:

Lemma 3. *There is a constant k_2, depending only on d, A, B, such that*

$$\forall x,t \quad p(x,t)\leqq k_2 t^{-d/2}.$$

Proof. Define $V_{(t)} := \sum_{x\in\mathbb{Z}^d} p^2(x,t)$,

$$-\frac{d}{dt}V_{(t)} = 2\sum_{x\in\mathbb{Z}^d} p(x,t)\frac{d}{dt}p(x,t) = 2\sum_{i=1}^{d}\sum_{x\in\mathbb{Z}^d} p(x,t)\nabla^{i-}(a_i(x)\nabla^{i+}p(x,t)), \tag{2.14}$$

by Kolmogorov's equation. Sum $\nabla^{1+}(A_m B_m) = A_m\nabla^{1+}B_m + B_{m+1}\nabla^{1+}A_m$ from $m=0$ to $m=q$

$$\sum_{m=1}^{q} a_m B_m = -\sum_{m=1}^{q-1} A_m b_{m+1} + A_q B_q - A_0 B_0, \tag{2.15}$$

where $\{a_m\}$, $\{b_m\}$ are given sequences and

$$A_m = A_0 + \sum_{k=1}^{m} a_k, B_m = B_0 + \sum_{k=1}^{m} b_k.$$

Now

$$\sum_{i=1}^{d} \sum_{x \in \mathbb{Z}^d} p(x,t)\nabla^{i-}(a_i(x)\nabla^{i+} p(x,t))$$

$$= \sum_{i=1}^{d} \sum_{x_1 \in \mathbb{Z}} \cdots \sum_{x_d \in \mathbb{Z}} p(x_1,\ldots,x_d)\nabla^{i-}(a_i(x_1,\ldots,x_d)\nabla^{i+} p(x_1,\ldots,x_d)).$$

Treat each summand of $\sum_{i=1}^{d}$ separately, say fix $i = 1$,

$$\sum_{x_1 \in \mathbb{Z}} \cdots \sum_{x_d \in \mathbb{Z}} p(x)\nabla^{1-}(a_1(x)\nabla^{1+} p(x))$$

$$= \sum_{x_2 \in \mathbb{Z}} \cdots \sum_{x_d \in \mathbb{Z}} \sum_{x_1 \in \mathbb{Z}} p(x)\nabla^{1-}(\cdots). \tag{2.16}$$

We will apply partial summation on the square bracketed part by identifying

$$A_m := p(m-N, x_2, \ldots, x_d), B_{m+1} := a_1(m-N)\nabla^{1+}p(m-N), m = 0,\ldots,q,$$

for N fixed:

$$\sum_{m=1}^{q} p(m-N,x_2,\ldots,x_d)\nabla^{1-}(a_i(m-N)\nabla^{1+} p(m-N))$$

$$= -\sum_{m=1}^{q} A_m b_{m+1} = \sum_{m=1}^{q} a_m B_m - A_q B_q + A_0 B_0 \text{ (by (2.15))}$$

$$= \sum_{m=1}^{q} \nabla^{1+} p(n-N-1)a_1(m-N-1)\nabla^{1+} p(m-N-1) - A_q B_q + A_0 B_0.$$

$$\sum_{x_1 \in \mathbb{Z}} p(x_1)\nabla^{1-}(a_1(x_1)\nabla^{1+} p(x_1))$$

$$= \lim_{N\to\infty} \lim_{q\to\infty} \sum_{m=1}^{q} p(m-N)\nabla^{1-}(a_1(m-N)\nabla^{1+} p(m-N))$$

$$= \lim_{N\to\infty} \lim_{q\to\infty} \sum_{m=1}^{q} \nabla^{1+} p(m-N-1)a_1(m-N-1)\nabla^{1+} p(m-N-1)$$

$$= \sum_{x_1 \in \mathbb{Z}} \nabla^{1+} p(x_1)a_1(x_1)\nabla^{1+} p(x_1),$$

since $\lim_{q\to\infty} p(q-N)\,[a_1(q-N-1)\nabla^{1+} p(q-N-1)] = 0 = \lim_{N\to\infty} p(-N) \times [a_1(-N-1)\nabla^{1+} p(-N-1)]$. Using this result for all i and putting back together the sums in (2.16), we can write (2.14) in the form

$$-\frac{d}{dt}V(t) = 2\sum_{i=1}^{d} \sum_{x \in \mathbb{Z}^d} \nabla^{i+} p(x,t)a_i(x)\nabla^{i+} p(x,t).$$

By uniform ellipticity (0.3), $\nabla^{i+}p(x,t)a_i(x)\nabla^{i+}p(x,t) \geqq |\nabla^{i+}p(x,t)|^2 A$, hence

$$-\frac{d}{dt}V(t) \geqq 2A \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} |\nabla^{i+}p|^2 \geqq 2Ak_1 \int |\nabla g|^2 dx, \tag{2.17}$$

for the function g of Lemma 2. Applying Lemma 1 to g

$$\int |\nabla g|^2 dx \geqq C_{(d)} k_0^{-4/d} [\int |g|^2 dx]^{1+2/d}.$$

Plugging this into (2.17) and using c of Lemma 2 yields

$$-\frac{d}{dt}V_{(t)} \geqq 2Ak_1 C_{(d)} k_0^{-4/d} \left[\sum_{x\in\mathbb{Z}^d} p^2(x)\right]^{1+2/d} = k' V_{(t)}^{1+2/d},$$

and

$$\frac{d}{dt}(V_{(t)}^{-2/d}) = -\frac{2}{d} V_{(t)}^{-2/d-1} \frac{d}{dt} V_{(t)} \geqq \frac{2}{d}k',$$

therefore

$$V_{(t)}^{-2/d} \geqq V_0^{-2/d} + \frac{2}{d}k't = 1 + \frac{2}{d}k't,$$

since $V_0 = \sum_{x\in\mathbb{Z}^d} p^2(x,0) = p^2(0,0) = 1$;

$$V_{(t)} \leqq \left(1 + \frac{2}{d}k't\right)^{-d/2} \leqq \left(\frac{2k't}{d}\right)^{-d/2}. \tag{2.18}$$

Finally by the Chapman–Kolmogorov identity $p(x,t) = \sum_{\bar{x}\in\mathbb{Z}^d} p(x,t,\bar{x},t/2) \times p(\bar{x},t/2,0,0)$, and by Schwarz' inequality

$$(p(x,t))^2 \leqq \sum_{\bar{x}\in\mathbb{Z}^d} p\left(x,t,\bar{x},\frac{t}{2}\right)^2 \sum_{\bar{x}\in\mathbb{Z}^d} p\left(\bar{x},\frac{t}{2},0,0\right)^2$$

$$\leqq \left[\left(\frac{2k'}{d}\cdot\frac{t}{2}\right)^{-d/2}\right]^2,$$

(by (2.18) and the reversibility $p^\omega(x,t,\bar{x},t/2) = p^\omega(\bar{x},t,x,t/2)$, together with the fact that $p^\omega(\bar{x},t,x,t/2) = p^{\tau_{-x}\omega}(\bar{x}-x,t/2,0,0)$, and (2.18) was independent of the particular ω. Put

$$k_2 = \left(\frac{k'}{2}\right)^{-d/2}, \quad \text{then} \quad p(x,t) \leqq k_2 t^{-d/2}. \quad \square$$

Now we can take up the program mentioned at the beginning of this proof of Theorem 1, and define the entropy $S_{(t)} = -\sum_{x\in\mathbb{Z}^d} p(x,t)\log p(x,t)$.

Lemma 4. *There is a constant k_3 with $S_{(t)} \geqq k_3 + (1/2)d\log t$, $\forall t$.*

Proof. $S_{(t)} \geqq \sum_{x \in \mathbb{Z}^d} p(x,t) \min_{x \in \mathbb{Z}^d}(-\log p(x,t))$

$$\geqq \sum_{x \in \mathbb{Z}^d} p(x,t)(-\log k_2 t^{-d/2}) = -\log k_2 t^{-d/2},$$

since $\sum_{x \in \mathbb{Z}^d} p(x,t) = 1$, hence $S_{(t)} \geqq -(\log k_2 + (-d/2)\log t) = k_3 + (d/2)\log t$, where $k_3 = -\log k_2$. □

Lemma 5. $\forall t \;\; 2dB(d/dt)S(t) \geqq [(d/dt)E|X_t|]^2$.

Proof. $E|X_{(t)}| = \sum_{x \in \mathbb{Z}^d} |x| p(x,t)$,

$$\begin{aligned}
\frac{d}{dt}E|X_{(t)}| &= \sum_{x \in \mathbb{Z}^d}\left[\frac{d}{dt}|x| \cdot p(x,t) + |x| \cdot \frac{d}{dt}p(x,t)\right] \\
&= \sum_{x \in \mathbb{Z}^d} |x| \frac{d}{dt}p(x,t) = -\sum_{i=1}^{d}\sum_{x \in \mathbb{Z}^d} |x| \nabla^{i-}(a_i(x)\nabla^{i+}p(x,t)) \\
&= -\sum_{i=1}^{d}\sum_{x \in \mathbb{Z}^d} \nabla^{i+}|x| a_i(x) \nabla^{i+}p(x,t),
\end{aligned}$$

by partial summation like in the proof of Lemma 3, observing that $p(x,t) = o(1/|x|)$ for $|x| \to \infty$, since the jump intensities are bounded, which takes care of the boundary term of partial summation, and

$$\left|\frac{d}{dt}E|X_{(t)}|\right| \leqq \sum_{i=1}^{d}\sum_{x \in \mathbb{Z}^d} |a_i(x)\nabla^{i+}p(x,t)|,$$

$$\text{since } |\nabla^{i+}|x|| \leqq 1. \tag{2.19}$$

Moreover, since $S_{(t)} = -\sum_{x \in \mathbb{Z}^d} p(x,t)\log p(x,t)$

$$\begin{aligned}
\frac{d}{dt}S_{(t)} &= -\sum_{x \in \mathbb{Z}^d}\left[\frac{d}{dt}p(x,t) \cdot \log p(x,t) + p(x,t)\frac{d}{dt}\log p(x,t)\right] \\
&= -\sum_{x \in \mathbb{Z}^d}\left[\frac{d}{dt}p(x,t) \cdot \log p(x,t) + p(x,t)\frac{1}{p(x,t)}\frac{d}{dt}p(x,t)\right] \\
&= -\sum_{x \in \mathbb{Z}^d}(1 + \log p(x,t))\frac{d}{dt}p(x,t) \\
&= \sum_{i=1}^{d}\sum_{x \in \mathbb{Z}^d}(1 + \log p(x,t))\nabla^{i-}(a_i \nabla^{i+}p(x,t)) \\
&= \sum_{i=1}^{d}\sum_{x \in \mathbb{Z}^d}\nabla^{i+}\log p(x,t) a_i \nabla^{i+}\log p(x,t)\frac{\nabla^{i+}p(x,t)}{\nabla^{i+}\log p(x,t)},
\end{aligned}$$

using summation by parts with $\lim_{p\to 0} p\log p = 0$ for the boundary term. We assume $\nabla^{i+} p(x,t) \neq 0$, for otherwise this summand would not contribute anyway.
Now

$$B\frac{dS}{dt} \geqq \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} |a_i \nabla^{i+} \log p(x,t)|^2 \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t)} \tag{2.20}$$

Since $p(x,t) < 1$,

$$0 \leqq \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t)} \leqq 1,$$

and by the mean value theorem

$$\frac{\nabla^{i+} \log p(x,t)}{\nabla^{i+} p(x,t)} = \frac{d}{dt} \log p \,\Big|_{p^*} = \frac{1}{p^*}$$

for some p^* between $p(x,t)$ and $p(x+e_i,t)$.
Therefore

$$\frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t)} = p^* \leqq \max\{p(x,t), p(x+e_i,t)\}$$

and,

$$K(p) := \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t)} \leqq \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} (p(x,t) + p(x+e_i,t))$$

$$= \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} p(x,t) + \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} p(x+e_i,t) = d + d = 2d.$$

Now, let us return to (2.20):

$$\frac{B}{K(p)} \frac{dS}{dt} \geqq \sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} |a_i \nabla^{i+} \log p(x,t)^2| |a_i \nabla^{i+}|^2 \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t) K(p)}. \tag{2.21}$$

Then

$$\sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t) K(p)} = 1 \quad \text{and} \quad 0 \leqq \frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t) K(p)} \leqq 1.$$

Hence we can consider

$$\frac{\nabla^{i+} p(x,t)}{\nabla^{i+} \log p(x,t) K(p)} =: \mu(\{(i,x)\}),$$

as a measure μ on $M := \{1,\dots,d\} \times \mathbb{Z}^d$ and apply Schwarz' inequality in the form $\int f^2 d\mu = \int 1^2 d\mu \cdot \int f^2 d\mu \geqq [\int f d\mu]^2$, (since $\mu(M) = 1$) on (2.21) to get

$$\frac{B}{K(p)} \frac{d}{dt} S_{(t)} \geqq \left[\sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} |a_i \nabla^{i+} \log p| \frac{|\nabla^{i+} p|}{|\nabla^{i+} \log p| K(p)} \right]^2$$

$$\geqq \left[\sum_{i=1}^{d} \sum_{x\in\mathbb{Z}^d} |a_i \nabla^{i+} p(x,t)| \right]^2 \frac{1}{(K(p))^2},$$

i.e.

$$2dB\frac{d}{dt}S \geqq \frac{K(p)^2}{K(p)}B\frac{d}{dt}S \geqq \left[\sum_{i=1}^{d}\sum_{x\in\mathbb{Z}^d}|a_i\nabla^{i+}p(x,t)|\right]^2$$

$$\geqq \left[\frac{d}{dt}E|X_t|\right]^2,$$

where we used (2.19) for the last inequality. □

For a function f on $\mathbb{Z}^d$, define $\bar{f}$ on $\mathbb{R}^d$ by $\bar{f}(x):=f(z)$ iff $x_i\in[z_i-\frac{1}{2},z_i+\frac{1}{2})$, $i=1,\ldots,d$; $x=(x_1,\ldots,x_d)$, $M_t:=\int dx|x|\bar{p}(x,t)$. Since $\overline{|x|}\geqq|x|-1$, $\forall x$,

$$E|X_t|=\sum_{x\in\mathbb{Z}^d}|x|p(x,t)=\int dx\overline{|x|}\bar{p}(x,t)\geqq\int dx(|x|-1)\bar{p}(x,t)$$

$$\geqq M_t-1. \tag{2.22}$$

Lemma 6. *There is a constant* $K>0$ *such that*

$$M_t\geqq Ke^{S(t)/d}.$$

Proof. Observe that for fixed λ: $\min_p(p\log p+\lambda p)=-e^{-\lambda-1}$, put $\lambda=a|x|+b$, $x\in\mathbb{R}^d$, where

$$a=\frac{d}{M_t}, e^{-b}=\left(\frac{e}{D_d}\right)a^d, \quad \text{with} \quad D_d:=\int_{\mathbb{R}^d}e^{-|x|}dx,$$

then $\bar{p}\log\bar{p}+(a|x|+b)\bar{p}\geqq-e^{-bd}e^{-a|x|}$, and $\int dx[\bar{p}\log\bar{p}+a|x|\bar{p}+b\bar{p}]\geqq -e^{-b-1}\int dxe^{-a|x|}$, that is $-S_{(t)}+aM_t+b\geqq-e^{-b-1}a^{-d}\int dxe^{-|x|}=-e^{-b-1}a^{-d}D_d$. Substitute on the right hand side for e^{-b} and on the left for a: $-S_{(t)}+d+b\geqq-1$, $d+b\geqq S_{(t)}-1$. Plug in for b

$$d-1+\log D_d-d\log a\geqq S_{(t)}-1, \qquad d+\log D_d-d[\log d-\log M_t]\geqq S_{(t)},$$

$$d\log M_t+d\geqq S_{(t)}+d\log d-\log D_d,$$

$$\log M_t+1\geqq\frac{S_{(t)}}{d}+\log d-\frac{\log D_d}{d},$$

$$\log M_t\geqq\frac{S_{(t)}}{d}+\log d-\log D_d^{1/d}-1,$$

$$M_t\geqq\exp\left(\frac{S_{(t)}}{d}\right)[d/D^{1/d}e]. \quad \square$$

Now we are in a position to conclude the proof of Theorem 2. Because of $E|X_0|=0$ and Lemma 5 we have

$$E|X_t|\leqq\int_0^t[2dB\frac{d}{dt}S_{(t)}]^{1/2}dt,$$

so that we get with Lemma 6 and (2.22)

$$Ke^{S(t)/d}-1\leqq M_t-1\leqq E|X_t|\leqq \int_0^t\left[2dB\frac{d}{dt}S_{(t)}\right]^{1/2}dt. \tag{2.23}$$

Define $R_{(t)}$ by $d\cdot R_{(t)}:=S_{(t)}-k_3-d/2\log t$, where k_3 is from Lemma 4, which says that $\forall t\, d\cdot R_{(t)}\geqq 0$. Then $d(d/dt)R=(d/dt)S-d/2t, (d/dt)S_{(t)}=d(d/dt)R_{(t)}+d/2t$. Substitute for S and $(d/dt)S$ in (2.23):

$$-1+Ke^{[R_{(t)}+k_3/d+(1/2)\log t]}\leqq E|X_t|\leqq (2d)^{1/2}(Bd)^{1/2}\int_0^1\left(\frac{1}{2t}+\frac{d}{dt}R\right)^{1/2}dt. \tag{2.24}$$

Use the inequality $(a+b)^{1/2}\leqq a^{1/2}+b/2a^{1/2}$ for $a>0$, $a+b>0$:

$$\int_0^t\left(\frac{1}{2t}+\frac{d}{dt}R\right)^{1/2}dt\leqq\int_0^t\left(\frac{1}{2t}\right)^{1/2}dt+\int_0^t\left(\frac{t}{2}\right)^{1/2}\frac{d}{dt}R\,dt$$

$$\leqq(2t)^{1/2}+R_t\left(\frac{t}{2}\right)^{1/2}-\int_0^t R_t/(8t)^{1/2}dt\leqq(2t)^{1/2}+R_t\left(\frac{t}{2}\right)^{1/2},\text{ since } R\geqq 0.$$

So we get from (2.24)

$$-1+Ke^{k_3/d}t^{1/2}e^{R_{(t)}}\leqq E|X_t|\leqq(2t)^{1/2}+R_t\left(\frac{t}{2}\right)^{1/2}$$

$$=t^{1/2}2^{1/2}[1+\tfrac{1}{2}R(t)],$$

$$Ke^{k_3/d}e^{R_{(t)}}\leqq 2^{1/2}[1+\tfrac{1}{2}R(t)]+t^{-1/2}$$

$$\leqq 2^{1/2}[1+\tfrac{1}{2}R(t)]+1. \tag{2.25}$$

for $t\geqq 1$.

Now, if $R(t)$ was unbounded as a function of t, then

$$e^{k_3/d}Ke^{R_{(t)}}\leqq(2^{1/2})[1+\tfrac{1}{2}R(t)]+1$$

could not hold, since $e^{R_{(t)}}$ grows much faster than $R(t)$, hence $R(t)\leqq B_0$ for all t, for some B_0, and consequently by (2.25) with $K':=2^{1/2}[1+1/2B_0]$:

$$E|X_t|\leqq K't^{1/2}\quad\text{for all}\quad t\geqq 1. \tag{2.26}$$

Take e.g. B_0 as the solution of

$$Ke^{k_3/d}e^{B_0}=2^{1/2}[1+\tfrac{1}{2}B_0]+1,$$

which depends only on the constants K, k_3, i.e. K' does not depend on ω.

We are left with bounding $E|X_t|$ for $t<1$, which is immediate, since $E|X_t|$ is bounded by the expected number of jumps of the process with highest jump intensity $2dB$ (cf (0.3)). Since its number of jumps before time t has Poisson-($2dBt$)-distribution, we get $E|X_t|\leqq 2dBt\leqq 2dB\sqrt{t}$ for $t<1$. Fusing this result with (2.26) to obtain $E|X_t|\leqq\max\{2dB,K'\}\sqrt{t}$, we have completed the proof of Theorem 2.

As a final remark concerning Theorem 2, I want to point to the fact that Rodolfo Figari, University of Naples, has recently proven another bond lattice version of

Nash's method in an unpublished paper, as I have just heard. Without using interpolation (Lemma 2) he got constants k_1, k_2 with $k_1(\sqrt{t}-\varepsilon) \leqq E|X^{\varepsilon}_{x,\omega}(t)-x| \leqq k_2(\sqrt{t}+\varepsilon)$ for all t,ω.

3. Effective Conductivity

In this section we will show the existence of an "effective conductivity matrix" (q_{ij}), which will serve as the diffusion matrix for the limiting Brownian motion of Sect. 5.

To develop a feeling for the theorem of this section, let us start with some heuristic remarks concerning the constructions of the effective conductivity q from the given conductivities a in the case of one dimension ($d=1$). Let us consider the lattice $\mathbb{Z}$ and the conducting bond $b(x)$ between x and $x+1$. The conductivity along $b(x)$ can be defined as the flux thru $b(x)$ under a potential of gradient 1. In order to construct some kind of effective conductivity on a possibly inhomogeneous lattice, the first problem arises in finding a potential on this lattice with "over-all gradient" 1. Obviously we have for a homogeneous lattice (i.e., $a(x)\equiv a$) an effective conductivity of a according to the previous definition, since the potential is trivial.

In our case of a stochastic inhomogeneous lattice we want to proceed analogously: We would like to put a potential $T(x,\omega)$ on the lattice with overall-unit-gradient and measure the average flux.

$$E(a(x)\nabla T(x,\omega)), \tag{3.1}$$

along a bond $b(x)$, where we assume an overall gradient condition in the sense of

$$\lim_{n\to\infty} E\left|\frac{T(x+n)-T(x-n)}{2n}\right| = 1. \tag{3.2}$$

Theorem 3 shows the existence of such a potential on $\mathbb{Z}^d$. It can be written in the form $x_k+\chi^k(x,\omega)$, where k denotes the coordinate-direction in which a unit-gradient potential is applied, and χ^k is some "correction" compensating for inhomogeneity and randomness of the lattice conductivities.

We start the rigorous part of this section with some remarks on the mathematical formalism of Theorem 3. Let B_d be the set of bonds in $\mathbb{Z}^d$, $\Omega := [A,B]^{B_d}$, $\mathscr{H} := L^2(\Omega,\ \mathscr{F},\ P)$, where $\mathscr{F}$ is generated by the cylinder sets whose images are balls in $\mathbb{R}^{d^2}$. Here $\omega\in\Omega$ is a configuration of conductivities $\{a_i(x,\omega)\}^{i\leqq d}_{x\in\mathbb{Z}^d}$. Recall from Sect. 0 that P was assumed to be invariant under the group $\{\tau_y\}_{y\in\mathbb{Z}^d}$ of shifts of the configuration. This will imply immediately that a function f on $\mathbb{Z}^d\times\Omega$ with $f(z,\omega) := \tilde{f}(\tau_{-z}\omega)$ for $\tilde{f}\in\mathscr{H}$ is stationary on $\mathbb{Z}^d$: Let T_x be the shift operator on $\mathbb{Z}^d$, i.e. $T_x(z)=z+x$:

$$f(T_x(z),\cdot) = f(z+x,\cdot) = \tilde{f}(\tau_{-z-x}(\,\cdot\,)) = f(z,\tau_{-x}\cdot) \doteq f(z,\cdot),$$

where $\doteq$ denotes equality in distribution.

Define $\nabla^{k+}\varphi$ for $\varphi\in\mathscr{H}$ by $\nabla^{k+}\varphi(\omega)=\varphi(\tau_{-e_k}\omega)-\varphi(\omega)$; $k=1,2,\dots,d$, and $\nabla^{i+}\chi$ for $\chi\in\mathbb{Z}^d\times\mathscr{H}$ by $\nabla^{i+}\chi(x,\omega)=\chi(x+e_i,\omega)-\chi(x,\omega)$, $i=1,2,\dots,d$. Define $a_i\in\mathscr{H}$ by $a_i(\omega) := a_i(0,\omega)$.

Theorem 3. *There are functions $\psi^k_i\in\mathscr{H}$, $i,k=1,\dots,d$, such that*

$$\sum_{i=1}^{d} \nabla^{i-}(a_i(\omega)(\delta_{ik} + \psi_i^k(\omega)) = 0, \text{ a.s. } [P], k = 1,\dots,d, \tag{3.3}$$

$$E(\psi_i^k(\omega)) = 0, \; i,k = 1,\dots,d, \tag{3.4}$$

$$\nabla^{i+}\psi_j^k = \nabla^{j+}\psi_i^k, \text{ a.s. } [P], i,k = 1,\dots,d. \tag{3.5}$$

Moreover, there are processes $\chi^k(x,\omega)$ *on* $\mathbb{Z}^d \times \Omega, k = 1,\dots,d$, *with* $\chi^k(0,\omega) = 0 \forall \omega \in \Omega$, *such that*

$$\nabla^{i+}\chi^k_{(x,\omega)} = \psi_i^k(\tau_{-x}\omega) = : \psi_i^k(x,\omega) \text{ a.s. } [P] : i,k = 1,\dots,d. \tag{3.6}$$

Extend χ^k *from* $\mathbb{Z}^d \times \Omega$ *to* $\mathbb{R}^d \times \Omega$, *such that* $\chi^k(x,\omega) := \chi([x],\omega)$, *where* $[x]$ *is the (unique) vector in* $\mathbb{Z}^d$ *with* $x \in \prod_{i=1}^{d} [[x], [x] + e_i)$.

$$\lim_{\varepsilon \to 0} E\left\{\left(\varepsilon\chi^k\left\{\frac{x}{\varepsilon}\right\}\right)^2\right\} = 0 \text{ for all } x \in \mathbb{R}^d, k = 1,\dots,d. \tag{3.7}$$

Part 1. Existence of a solution of (3.3)–(3.5). Here and in Sect. 4 the following lemma will be important.

Lax–Milgram Lemma [e.g. in Lions [11]]. *Let* $(H, (\,,\,))$ *and* $(V, ((\,,\,)))$ *be Hilbert spaces,* $V \subset H$ *dense,* $\|\varphi\| := ((\varphi,\varphi))^{1/2}$, *for* $\varphi \in V$, *let* $a(\psi,\varphi)$ *be a sesqui-linear form on* V *such that*

$$|a(\psi,\varphi)| \leqq \gamma \|\psi\| \, \|\varphi\| \text{ for some } \gamma > 0 \text{ and all } \psi,\varphi \in V, \tag{3.9}$$

$$a(\varphi,\varphi) \geqq c\|\varphi\|^2 \text{ for some } c > 0 \text{ and all } \varphi \in V. \tag{3.10}$$

For all $f \in H$, *the equation* $a(\psi,\varphi) = (f,\varphi), \forall \varphi \in V$, *has a unique solution* $\psi \in D(A)$, *where* $D(A) := \{\psi \in V : \varphi \to a(\psi,\varphi)$ *is continuous on* V *in the topology induced by* $H\}$.

In our case let $(H,(\,,\,)) = \mathscr{H} = L^2(dP)$ and let $(V,((\,,\,)))$ be $\mathscr{H}$ with inner product $((\psi,\varphi)) := \sum_{i=1}^{d} E(\nabla^{i+}\psi\nabla^{i+}\varphi) + E(\psi\varphi), (\psi,\varphi) := E(\psi\varphi)$. If we want to use this lemma for solving (3.3), we have to apply it to an equation of the form

$$\begin{aligned} a(\psi,\varphi) :&= \sum_{i=1}^{d} E(a_i(\omega)\nabla^{i+}\psi\nabla^{i+}\varphi) + \beta(\psi,\varphi) \\ &= (f,\varphi) \text{ for some fixed } \beta > 0. \end{aligned} \tag{3.11}$$

In this manner we can satisfy (3.10) with

$$\min\{A,\beta\}\|\varphi\|^2 \leqq |a(\varphi,\varphi)| = \left|\sum_{i=1}^{d} E(a_i(\omega)\nabla^{i+}\varphi\nabla^{i+}\varphi) + \beta(\varphi,\varphi)\right|,$$

and (3.9) with $\gamma := \max\{\beta,B\}$:

$$|a(\psi,\varphi)| = \left|\sum_{i=1}^{d} (a_i\nabla^{i+}\psi\nabla^{i+}\varphi) + \beta(\psi,\varphi)\right|$$

$$\leqq \gamma\left[\sum_{i=1}^{d}(|\nabla^{i+}\psi|,|\nabla^{i+}\varphi|)+(|\psi|,|\varphi|)\right]$$

$$\leqq \gamma\left\{\sum_{i=1}^{d}[E|\nabla^{i+}\psi|^2 E|\nabla^{i+}\varphi|^2]^{1/2}+[E|\psi|^2 E|\varphi|^2]\right\}$$

(by Schwarz in $(H,(\ ,\ ))$

$$\leqq \gamma\left\{\sum_{i=1}^{d}E|\nabla^{i+}\psi|^2+E|\varphi|^2\right)^{1/2}\left(\sum_{i=1}^{d}E|\nabla^{i+}\psi|^2+E|\varphi|^2\right)^{1/2}$$

(by Schwarz in $\mathbb{R}^{d+1}$)

$$=\gamma\|\psi\|\,\|\varphi\|.$$

Hence the Lax–Milgram Lemma can be applied in (3.11) with $f(\omega):=-\nabla^{k-}a_k(\omega)$ to get a unique $\chi^{k,\beta}\in\mathscr{H}$ solving $a(\psi,\varphi)=(-\nabla^{k-}a_k(\omega),\varphi)\ \forall\varphi\in\mathscr{H}$. Observe that for all $\psi,\varphi\in\mathscr{H}$, $i=1,\ldots,d$,

$$E(\psi\nabla^{i-}\varphi)=E[\psi(\varphi(\tau_{+e_i}\omega)-\varphi(\omega))]=E[\psi(\omega)\varphi(\tau_{e_i}\omega)-\psi(\omega)\varphi(\omega)]$$

$$=E[\psi[\tau_{-e_i}\omega)\varphi(\omega)-\psi(\omega)\varphi(\omega)]=E[(\nabla^{i+}\psi)\varphi],$$

hence

$$\sum_{i=1}^{d}E(a_i(\omega)(\delta_{ik}+\nabla^{i+}\chi^{k,\beta})\nabla^{i+}\varphi)+\beta E(\chi^{k,\beta}\varphi)=0,$$

$$\forall\varphi\in\mathscr{H}. \tag{3.12}$$

Now we want to let $\beta\to 0$ and hope that a limit of the solutions $\chi^{k,\beta}$ solves (3.3). For this argument we need

$$E\left[\sum_{j=1}^{d}(\nabla^{j+}\chi^{k,\beta})^2\right]\leqq c_1, \tag{3.13}$$

$$\beta E(\chi^{k,\beta})^2\leqq c_2, \tag{3.14}$$

where the constants c_1, c_2 do not depend on β. To see that these inequalities hold, substitute $\chi^{k,\beta}$ for φ in (3.12):

$$\sum_{i=1}^{d}E(a_i(\omega)(\delta_{ik}+\nabla^{i+}\chi^{k,\beta})\nabla^{i+}\chi^{k,\beta})+\beta E(\chi^{k,\beta})^2=0,\ \text{i.e.}$$

$$-E[a_k\nabla^{k+}\chi^{k,\beta}]=\sum_{i=1}^{d}E[a_i(\nabla^{i+}\chi^{k,\beta})^2]+E\beta(\chi^{k,\beta})^2$$

$$\geqq E[a_k(\nabla^{k+}\chi^{k,\beta})^2]+\beta E(\chi^{k,\beta})^2.$$

And therefore

$$B[E(\nabla^{k+}\chi^{k,\beta})^2]^{1/2}\geqq B|E(\nabla^{k+}\chi^{k,\beta})|\geqq|E(a_k\nabla^{k+}\chi^{k,\beta})|$$

$$\geqq A\sum_{i=1}^{d}E(\nabla^{i+}\chi^{k,\beta})^2+\beta E(\chi^{k,\beta})^2$$

$$\geqq AE(\nabla^{k+}\chi^{k,\beta})^2+\beta E(\chi^{k,\beta})^2. \tag{3.15}$$

by Schwarz' inequality and (1.3). Set $\alpha(\beta):=(E(\nabla^{k+}\chi^{k,\beta})^2)^{1/2}$ and $\gamma(\beta):=E(\chi^{k,\beta})^2$. Then (3.15) reads $B\alpha(\beta)\geqq A(\alpha(\beta))^2+\beta\gamma(\beta)$, $\alpha(\beta)$, $\gamma(\beta)\geqq 0$. This shows that $\alpha(\beta)$ is bounded (e.g. by B/A) and so is $\beta\gamma(\beta)$ (e.g. by B^2/A), which proves (3.14). Since the left hand side (3.15) is bounded by B^2/A, so is $A\sum_{i=1}^{d}E(\nabla^{i+}\chi^{k,\beta})^2$, proving (3.13).

Now, because of (3.13) there is some subsequence $\{\beta^{(1)}\}$ along which $\nabla^{1+}\chi^{k,\beta^{(1)}}\to\psi_1^k$ weakly in $\mathscr{H}$ for some $\psi_1^k\in\mathscr{H}$. Moreover, given a subsequence $\{\beta^{(i)}\}$, we can find by (3.13) a further subsequence $\{\beta^{(i+1)}\}\subset\{\beta^{(i)}\}$, along which $\nabla^{(i+1)+}\chi^{k,\beta^{(i+1)}}\to\psi_{i+1}^k$ weakly in $\mathscr{H}$ for some $\psi_{i+1}^k\in\mathscr{H}$. Therefore $\nabla^{i+}\chi^{k,\beta^{(d)}}\to\psi_i^k$ weakly in $\mathscr{H}$ for $i=1,\dots,d$. By (3.14) and Schwarz inequality

$$\beta E(\chi^{k,\beta}\varphi)\leqq\beta[E(\chi^{k,\beta})^2E\varphi^2]^{1/2}=[\beta c_2E\varphi^2]^{1/2},$$

so that (3.12) goes to (3.3) along the subsequence $\{\beta^{(d)}\}$.

Now let us check (3.4), (3.5):

$$E(\psi_j^k)=(\psi_j^k,1)=\lim_{\beta^{(d)}\to 0}(\nabla^{j+}\chi^{k,\beta^{(d)}},1)$$

$$=\lim_{\beta^{(d)}\to 0}E[\chi^{k,\beta^{(d)}}(\tau_{-e_j}\omega)-\chi^{k,\beta^{(d)}}(\omega)]=0,$$

since obviously $E(\varphi(\tau_{-e_j}(\omega)))=E(\varphi(\omega))$ for $\varphi\in\mathscr{H}$, using our remark on stationarity preceding Theorem 3.

Using the observation preceding (3.12) we have

$$E[\nabla^{j+}\chi^{k,\beta}\nabla^{i-}\varphi]=E(\chi^{k,\beta}\nabla^{j-}\nabla^{i-}\varphi)=E(\chi^{k,\beta}\nabla^{i-}\nabla^{j-}\varphi)$$
$$=E(\nabla^{i+}\chi^{k,\beta}\nabla^{j-}\varphi),$$

so that

$$E[(\nabla^{i-}\psi_j^k)\varphi]=E[\psi_j^k\nabla^{i-}\varphi]=\lim_{\beta'\to 0}E[\nabla^{j+}\chi^{k,\beta'}\nabla^{i-}\varphi]$$

$$=\lim_{\beta'\to 0}E(\nabla^{i+}\chi^{k,\beta'}\nabla^{j-}\varphi)=E[(\nabla^{j+}\psi_i^k)\varphi]$$

for all $\varphi\in\mathscr{H}$,

proving (3.5).

Part 2. Construction of χ^k. Let us define the shift operator T_x on $\mathscr{H}$ as follows: $T_xg(\omega):=g(\tau_{-x}\omega)$. $\{T_x\}_{x\in\mathbb{Z}^d}$ is a unitary group of operators on $\mathscr{H}$. Here T_x has the spectral representation $T_x=\int_{R^d}e^{i\lambda x}U(d\lambda)$, where $\{U(d\lambda)\}_\lambda$ is the corresponding family of spectral operators. Put

$$\chi^k(x,\omega):=\int_{\mathbb{R}^d}(e^{i\lambda x}-1)\frac{1}{|e^{i\lambda}-1|^2}\sum_{j=1}^{d}((e^{-i\lambda_j}-1)U(d\lambda)\psi_j^k(\omega)),$$

'where $|e^{i\lambda}-1|^2=\sum_{l=1}^{d}|e^{i\lambda_l}-1|^2$.

In the sequel we show first of all that χ^k is well-defined on $\mathbb{Z}^d \times \Omega$. Because of $\int U(d\lambda)\psi_j^k(\omega) = \psi_j^k(\omega)$ we need for this purpose simply an upper bound $S_{(x)}$ on the integrand:

$$\frac{|e^{i\lambda x}-1||e^{-i\lambda_j}-1|}{|e^{i\lambda}-1|^2} \leqq S(x), \text{ for all } x\in\mathbb{Z}^d, \quad j=1,\dots,d$$

implies $|\chi^k(x,\omega)| \leqq d\cdot S(x)\cdot|\psi_i^k|_{\mathscr{H}}$. In order to get a hand on $S(x)$, define $\rho:\mathbb{R}\to[0,\pi]$ such that $\rho(\lambda)=l$, if there is an l, $0\leqq l\leqq\pi$ and $k\in\mathbb{Z}$ with $\lambda=2\pi k+l$, or if there is an l, $0\leqq l<\pi$ and $k\in\mathbb{Z}$ with $\lambda=2\pi k-l$. For $\lambda_j\in\mathbb{R}$ we now have

$$\frac{\rho^{(\lambda_j)}}{2} \leqq |e^{\pm i\lambda j}-1| \leqq \rho(\lambda_j) \text{ for all } j=1,2,\dots,d,$$

and because of the triangle inequality

$$|e^{i\lambda x}-1| = |e^{i\lambda_1 x_1+\cdots+i\lambda_d x_d}-1| \leqq |e^{i\lambda_1 x_1}-1|+\cdots+|e^{i\lambda_d x_d}-1|,$$

and hence (since $\rho(\mu+\lambda)\leqq\rho(\mu)+\rho(\lambda)$) $|e^{i\lambda x}-1|\leqq\rho(\lambda_1 x_1)+\cdots+\rho(\lambda_d x_d)\leqq |x_1|\rho(\lambda_1)+\cdots+|x_d|\rho(\lambda_d)$. We redistribute indices if necessary, such that $\rho(\lambda_d)\geqq\rho(\lambda_i)>0$ for all $i\leqq d$. Then

$$\frac{|e^{i\lambda x}-1|\cdot|e^{-i\lambda_j}-1|}{|e^{i\lambda}-1|^2} \leqq \frac{\left(\sum_{i=1}^d |x_i|\rho(\lambda_i)\right)\rho(\lambda_j)}{\frac{1}{4}\sum_{i=1}^d \rho(\lambda_i)^2}$$

$$\leqq \frac{\sum_{i=1}^d |x_i|\rho(\lambda_d)^2}{\frac{1}{4}\rho(\lambda_d)^2} = 4\sum_{i=1}^d |x_i| =: S(x).$$

Hence $\rho(\lambda_d)\to 0$ does no harm.

We now turn to the properties of χ^k. Here $\chi^k(0,\infty)=0$ holds trivially for all $\omega\in\Omega$:

$$(\nabla^{i+}\chi^k,\varphi) = \int_{\mathbb{R}^d} (e^{i\lambda x}-1)\frac{1}{|e^{i\lambda}-1|^2}\sum_{j=1}^d ((e^{-i\lambda_j}-1)U(d\lambda)\psi_j^k,\nabla^{i-}\varphi)$$

$$= \int_{\mathbb{R}^d} (e^{i\lambda x}-1)\frac{1}{|e^{i\lambda}-1|^2}\sum_{j=1}^d (\psi_j^k,\nabla^{i-}(e^{i\lambda_j}-1)U(d\lambda)\varphi)$$

$$= \int_{\mathbb{R}^d} (e^{i\lambda x}-1)\frac{1}{|e^{i\lambda}-1|^2}\sum_{j=1}^d (\psi_i^k,\nabla^{j-}(e^{i\lambda_j}-1)U(d\lambda)\varphi)$$

$$= \left(\sum_{j=1}^d \nabla^{j+}\int_{\mathbb{R}^d} (e^{i\lambda x}-1)\frac{1}{|e^{i\lambda}-1|^2}(e^{-i\lambda_j}-1)U(d\lambda)\psi_i^k,\varphi\right)$$

$$= \left(\int_{\mathbb{R}^d} e^{i\lambda x}U(d\lambda)\psi_i^k,\varphi\right) = (T_x\psi_i^k,\varphi),$$

which implies (3.6). Note that we have made use of (3.5).

We show (3.7) for $x\in\mathbb{Z}^d$ first, with ε of the form $1/n$ (i.e. $x/\varepsilon\in\mathbb{Z}^d$):

$$E\left(\chi^k\left(\frac{x}{\varepsilon}\right)\right)^2 = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} |e^{i\lambda(x/\varepsilon)}-1|^2 \frac{(e^{-i\lambda_i}-1)(e^{i\lambda_j}-1)}{(|e^{i\lambda}-1|^2)^2}(U(d\lambda)\psi_i^k,\psi_j^k).$$

Put $\mu_j := |e^{-i\lambda_j}-1|$:

$$\sum_{i,j=1}^{d}(\mu_i\bar{\mu}_j U(d\lambda)\psi_i^k,\psi_j^k) = \left(\sum_{i=1}^{d}\mu_i U(d\lambda)\psi_i^k, \sum_{i=1}^{d}\mu_i U(d\lambda)\psi_i^k\right)$$
$$\leqq \left(\sum_{i=1}^{d}\mu_i^2\right)\left\|\sum_{i=1}^{d}U(d\lambda)\psi_i^k\right\|^2 \qquad \text{by Schwarz' inequality.}$$

Because of $\sum_{i=1}^{d}\mu_i^2 = |e^{i\lambda}-1|^2$, this implies

$$E\left(\varepsilon\chi^k\left(\frac{x}{\varepsilon}\right)\right)^2 \leqq \varepsilon^2\int_{R^d}\frac{|e^{i\lambda(x/\varepsilon)}-1|^2}{|e^{i\lambda}-1|^2}\sum_{j=1}^{d}(U(d\lambda)\psi_j^k, U(d\lambda)\psi_j^k).$$

As was shown above, the integrand is bounded independently of ε:

$$\frac{\varepsilon^2|e^{i\lambda_1(x_1/\varepsilon)+\dots+i\lambda_d(x_d/\varepsilon)}-1|^2}{|e^{i\lambda_j}-1|^2} \leqq \frac{4\varepsilon^2\sum_{j=1}^{d-1}\left|\frac{x_j}{\varepsilon}\right|^2\rho(\lambda_j)^2}{\rho(\lambda_j)^2} + \frac{4\varepsilon^2 d^2\left|\frac{x_d}{\varepsilon}\right|^2\rho(\lambda_d)^2}{\rho(\lambda_d)^2} \leqq 4d^2|x|^2,$$

where the indices have been redistributed if necessary, such that $|x_d|^2\rho(\lambda_d)^2 = \max_j\{|x_j|^2\rho(\lambda_j)^2\}$, and where we have used the inequality

$$|e^{i\lambda_1x_1+\dots+i\lambda_dx_d}-1|^2 \leqq (|x_1|\rho(\lambda_1)+\dots+|x_d|\rho(\lambda_d))^2$$
$$\leqq \sum_{j=1}^{d}|x_j|^2\rho(\lambda_j)^2 + d^2\max\{|x_i|^2\rho(\lambda_i)^2\}.$$

But then we can apply the Theorem on Bounded Convergence to proceed from

$$\lim_{\varepsilon\to 0}\varepsilon^2\frac{|e^{i\lambda(x/\varepsilon)}-1|^2}{|e^{i\lambda}-1|^2} = 0, \text{ for all } \lambda\notin 2\pi\mathbb{Z}^d, \quad \text{to}$$

$$\lim_{\varepsilon\to 0}E\left(\varepsilon\chi^k\left(\frac{x}{\varepsilon}\right)\right)^2 = \lim_{\varepsilon\to 0}\sum_{i,j=1}^{d}\sum_{\lambda\in 2\pi\mathbb{Z}^d}\varepsilon^2|e^{i\lambda(x/\varepsilon)}-1|^2 \cdot \frac{(e^{-i\lambda_i}-1)(e^{i\lambda_j}-1)}{(|e^{i\lambda}-1|^2)^2}(U(\{\lambda\})\psi_i^k,\psi_j^k). \tag{3.16}$$

For $\lambda \in 2\pi\mathbb{Z}^d$, however, $U\{\lambda\}\psi_i^k$ is T_x-invariant, for all $x \in \mathbb{Z}^d$:

$$T_x U(\{\lambda\})\psi_i^k = \int_{\mathbb{R}^d} e^{i\lambda x} U(d\lambda) U(\{\lambda\})\psi_i^k = U(\{\lambda\})\psi_i^k,$$

since $e^{i\lambda x} = 1$ and $U(d\lambda)U(\{\lambda\}) = U(\{\lambda\})$ for $\lambda \in d\lambda$, and $U(d\lambda)U(\{\lambda\}) = 0$ for $\lambda \notin d\lambda$, because U is a projection operator. Now, the only $\{T_x\}_{x\in\mathbb{Z}^d}$-invariant functions are the a.s.-constants, since the unitary group $\{T_x\}_{x\in\mathbb{Z}^d}$ was assumed to be ergodic in (0.4), hence $(U\{\lambda\}\psi_i^k, \psi_j^k) = U\{\lambda\}\psi_i^k(1, \psi_j^k) = U(\{\lambda\})\psi_i^k E(\psi_j^k) = 0$, so that $\lim_{\varepsilon\to 0} E(\varepsilon\chi^k(x/\varepsilon))^2 = 0$ for $x \in \mathbb{Z}^d$ and $\varepsilon \in \{1/n,\ n \in N\}$.

We now drop the conditions on x and ε: Recall that for $x \in \prod_{i=1}^{d} [[x], [x] + e_i]$: $\chi^k(x, \omega) = \chi^k([x], \omega)$. Assume first that $x = \alpha e_l$, $\alpha \in \mathbb{R}$, $l = 1, \ldots, d$,

$$\frac{1}{\alpha^2} \lim_{\varepsilon\to 0} E\left(\varepsilon\chi^k\left(\frac{\alpha e_l}{\varepsilon}\right)\right)^2 = \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{\alpha^2} E\left(\chi^k\left(\left[\frac{\alpha}{\varepsilon}\right] e_l\right)\right)^2 \leqq \lim_{\varepsilon\to 0} \left[\frac{\alpha}{\varepsilon}\right]^{-2} E\left(\chi^k\left(\left[\frac{\alpha}{\varepsilon}\right] e_l\right)\right)^2 = 0.$$

We now extend this result to $x \in \mathbb{R}^d$ in general. Set

$$\gamma_l\left(\frac{x}{\varepsilon}\right) := \sum_{m=0}^{[x_l/\varepsilon]-1} \psi_l^k\left(\left[\frac{x_1}{\varepsilon}\right], \ldots, \left[\frac{x_l - 1}{\varepsilon}\right], m, 0, \ldots, 0\right)$$

and

$$\tilde{\gamma}_l\left(\frac{x}{\varepsilon}\right) := \sum_{m=0}^{[x_l/\varepsilon]-1} \psi_l^k(0, \ldots, 0, m, 0, \ldots, 0) = \chi^k\left(\left[\frac{x_k}{\varepsilon}\right] e_l\right).$$

Because of the stationarity of the ψ_i^k, we see that γ_l and $\tilde{\gamma}_l$ have the same distribution, $l = 1, \ldots, d$, hence

$$E\left(\varepsilon\chi^k\left(\frac{x}{\varepsilon}\right)\right)^2 = E\left(\varepsilon\left(\sum_{i=1}^{d} \gamma_i\right)\right)^2 = E\left(\varepsilon^2 \sum_{i,j=1}^{d} \gamma_i\gamma_j\right)$$

$$\leqq \sum_{i,j=1}^{d} (E(\varepsilon\gamma_i)^2 E(\varepsilon\gamma_j)^2)^{1/2} = \sum_{i,j=1}^{d} (E(\varepsilon\tilde{\gamma}_i)^2 E(\varepsilon\tilde{\gamma}_j)^2)^{1/2}.$$

Since $\lim_{\varepsilon\to 0} E(\varepsilon\tilde{\gamma}_i(x/\varepsilon))^2 = 0$, $i = 1, \ldots, d$, as was shown above, this last sum will vanish in the limit, completing the proof of (3.7). □

In the beginning of this section we tried to develop some intuition how the "effective conductivities" q_{ij} should be defined and constructed. Theorem 3 gives us the necessary "correction potentials" χ^k, so that we can now define

$$q_{ij} := E(a_i(\omega)(\delta_{ij} + \nabla^{i+}\chi^j)), \quad i, j = 1, \ldots, d. \tag{3.17}$$

We conclude this section by proving some properties of (q_{ij}), which are more or less immediate from the definition. We will show that the matrix (q_{ij}) is symmetric and that for any eigenvalue q of (q_{ij})

$$A \leqq q \leqq \left(1 + \sum_{i,l=1}^{d} E(\psi_l^i)^2\right) B. \tag{3.18}$$

By (3.3) and the construction of $\chi^{k,\beta^{(d)}} \in \mathscr{H}$ in part 1 of Theorem 3 we get $\sum_{l=1}^{d} E(a_l(\omega)(\delta_{l_i} + \psi_l^i)\nabla^{l+}\chi^{k,\beta^{(d)}}) = 0$. Since $\forall l$, $\nabla^{l+}\chi^{k,\beta^{(d)}} \to \psi_l^k$ in $\mathscr{H}$ along $\{\beta^{(d)}\}$ weakly,

$$0 = \lim_{\beta^{(d)} \to 0} \sum_{l=1}^{d} E(a_l(\delta_{li} + \psi_l^i)\nabla^{l+}\chi^{k,\beta^{(d)}})$$

$$= \sum_{l=1}^{d} E(a_l(\delta_{li} + \psi_l^i)\psi_l^k),$$

hence by definition

$$q_{ki} = E(a_k(\delta_{ki} + \psi_k^i)) + E\left\{ \sum_{l=1}^{d} a_l(\delta_{li} + \psi_l^i)\psi_l^k \right\}$$

$$= E\left\{ a_k(\delta_{ki} + \psi_k^i) + a_k(\delta_{ki} + \psi_k^i)\psi_k^k + \sum_{l \neq k} a_l(\delta_{li} + \psi_l^i)\psi_l^k \right\}$$

$$= E\left\{ a_k(\delta_{ki} + \psi_k^i)(1 + \psi_k^k) + \sum_{l \neq k} a_l(\delta_{li} + \psi_l^i)\psi_l^k \right\},$$

so that

$$q_{ki} = E\left\{ \sum_{l=1}^{d} a_l(\delta_{li} + \psi_l^i)(\delta_{lk} + \psi_l^k) \right\}. \tag{3.19}$$

The symmetry of (q_{ki}) is immediate from this equation. Moreover by (3.19), we have for any $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\sum_{i,k=1}^{d} x_k q_{ki} x_i = \sum_{i,k,l=1}^{d} E(a_l(\delta_{li} + \psi_l^i)x_i(\delta_{lk} + \psi_l^k)x_k)$$

$$= \sum_{l=1}^{d} E\left(a_l \left(\sum_{j=1}^{d} (\delta_{lj} + \psi_l^j)x_j \right)^2 \right)$$

$$\geqq A \sum_{i,k,l=1}^{d} E(x_k x_i[\delta_{li}\delta_{lk} + \delta_{li}\psi_l^k + \delta_{lk}\psi_l^i + \psi_l^i\psi_l^k])$$

$$\geqq A \sum_{k=1}^{d} x_k^2 + A \sum_{i,k,l=1}^{d} x_k x_i [E(\delta_{li}\psi_l^k) + E(\delta_{lk}\psi_l^i) + E(\psi_l^i\psi_l^k)],$$

so that, since $E[\psi_l^k] = 0$, $\forall k, l$, and $\sum_{i,k=1}^{d} x_k x_i \psi_l^i \psi_l^k = \left(\sum_{j=1}^{d} x_j \psi_l^j \right)^2 \geqq 0$,

$$\sum_{i,k=1}^{d} x_k q_{ki} x_i \geqq A \sum_{i=1}^{d} x_i^2. \tag{3.20}$$

Similarly

$$\sum_{i,k=1}^{d} x_k q_{ki} x_i \leqq B \left(\sum_{k=1}^{d} x_k^2 + \sum_{i,k,l=1}^{d} x_k x_i E(\psi_l^i \psi_l^k) \right)$$

$$= B\left(\sum_{k=1}^{d} x_k^2 + \sum_{l=1}^{d} E\left(\sum_{j=1}^{d} x_j \psi_l^j\right)^2\right)$$

$$\leqq \left(1 + \sum_{k,l=1}^{d} E(\psi_l^k)^2\right) B \cdot \sum_{i=1}^{d} x_i^2, \tag{3.21}$$

by Schwarz' inequality. Inequalities (3.20) and (3.21) yield (3.18).

Being symmetric, (q_{ij}) is diagonalizable. If (q_{ij}) is diagonal, the upper bound in (3.18) can be slightly improved,

$$A \leqq q \leqq \left(1 + \max_{i \leqq d} \sum_{l=1}^{d} E(\psi_l^i)^2\right) B, \text{ if } (q_{ij}) \text{ is diagonal,} \tag{3.18'}$$

as can be seen by letting $x = e_i$, $i = 1, \ldots, d$, in the proof of (3.21).

4. Resolvent Convergence

Before introducing the Hilbert space framework for the formulation and proof of our Theorem 4 on strong resolvent convergence, let us consider for a moment the intuitive background of our approach, which is due to Papanicolaou and Varadhan (cf. [13]). As outlined in Sect. 1, we need the strong convergence of the semigroups $e^{\mathscr{L}_\omega^\varepsilon t} \to e^{\mathscr{L} t}$, where

$$\mathscr{L} = \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{(with } (q_{ij}) \text{ from (3.17)), and } \mathscr{L}_\omega^\varepsilon = -\sum_{i=1}^{d} \nabla_i^{\varepsilon -}\left(a_i\left(\frac{x}{\varepsilon}, \omega\right) \nabla_i^{\varepsilon +}\right)$$

are the generators of the corresponding jump, respectively diffusion processes. The convergence of semigroups will result from the convergence of resolvents:

$$\text{for } \alpha > 0, \quad (-\mathscr{L}_\omega^\varepsilon + \alpha)^{-1} \to (-\mathscr{L} + \alpha)^{-1}, \tag{4.1}$$

i.e. if f is a given function and $u^\varepsilon(\cdot, \omega) := (-\mathscr{L}_\omega^\varepsilon + \alpha)^{-1} f$, and $u(\cdot) := (-\mathscr{L} + \alpha)^{-1} f$, then we claim

$$u^\varepsilon(\cdot, \omega) \to u(\cdot) \quad \text{(in some sense).} \tag{4.2}$$

We use multiple scales for proving (4.2). This method will be indicated in a few words (for details cf. e.g. Bensoussan, Lions, Papanicolaou [1]): The idea is to expand u^ε as

$$u^\varepsilon(x, \omega) = u(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}, \omega\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}, \omega\right) + \ldots. \tag{4.3}$$

Plugging this into the equation for u^ε, collecting and equating coefficients of equal powers of ε, gives a sequence of equations for $u, u_1, u_2, \ldots$. The trick then is to set

$$u_1(x, y, \omega) := \sum_{i=1}^{d} \chi^k(y, \omega) \nabla_k^{\varepsilon +} u(x). \tag{4.4}$$

This will result in an equation for χ^k which is essentially (3.3), and is an equation characterizing u, of the form $(-\mathscr{L} + \alpha)u = f$. Can we hope for $\varepsilon^i u_i(x, x/\varepsilon, \omega) \to 0$, for $i \geqq 1, \varepsilon \to 0$, in some sense, or more directly:

Can we hope for $z^\varepsilon(x,\omega) := u^\varepsilon(x,\omega) - u - \sum_{k=1}^{d} \varepsilon\chi^k(x/\varepsilon,\omega)\nabla_k^{\varepsilon+} u(x)$ to vanish in some sense, as $\varepsilon \to 0$?

Now let us turn to making these ideas precise. Let H be the Hilbert space $H = L^2(\mathbb{R}^d; \mathscr{H})$ of square integrable functions on $\mathbb{R}^d$ with values in $\mathscr{H}$ and inner product $(f,g) := E\int dx\, fg$, $\|f\| := (f,f)^{1/2}$. Let H^1 be the subspace of H with square integrable distribution derivatives (cf. Richtmyer [14]) and inner product $((f,g)) := \sum_{i=1}^{d}\left(\frac{\partial}{\partial x_i}f, \frac{\partial}{\partial x_i}g\right) + (f,g)$; $\|f\|_1 := ((f,f))^{1/2}$. Let H_ε^1 be the Hilbert space consisting of the same functions as H, but with inner product $((f,g))_\varepsilon := \sum_{i=1}^{d} (\nabla_i^{\varepsilon+} f, \nabla_i^{\varepsilon+} g) + (f,g)$. Let $H_0 := L^2(\mathbb{R}^d, \mathbb{R})$ be the Hilbert space of square integrable real functions with inner product $(f,g)_0 := \int dx fg$. Let H_0^1 be the Hilbert space of functions in H_0 with square integrable distribution derivative and inner product $((f,g))_0 := \sum_{i=1}^{d}\left(\frac{\partial}{\partial x_i}f, \frac{\partial}{\partial x_i}g\right)_0 + (f,g)_0$. For $f \in H_0, \varepsilon > 0$, has $(-\mathscr{L}_\omega^\varepsilon u^\varepsilon + \alpha u^\varepsilon, \varphi) = (f,\varphi)$, $\forall \varphi \in H_\varepsilon^1$, a unique solution $u^\varepsilon \in H_\varepsilon^1$, as follows from the Lax–Milgram Lemma applied to the Hilbert spaces H_ε^1, H. It is sufficient for this matter to consider

$$\sum_{i=1}^{d} \int dx E\left(\nabla_i^{\varepsilon-}\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\psi\right)\right)\varphi + \alpha \int dx E\psi\varphi = E\int dx f(x)\cdot\varphi, \tag{4.5}$$

i.e. $a(\psi,\varphi) = (f,\varphi)$, where $a(\psi,\varphi) = \sum_{i=1}^{d} E\int dx a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\psi\nabla_i^{\varepsilon+}\varphi + \alpha E\int dx\psi\varphi$. This sesquilinear form on $H_\varepsilon, H_\varepsilon \subset H$, is of the same structure as (3.11) on V, $V \subset H$, hence satisfying (3.9), (3.10), so that the Lax–Milgram Lemma is applicable. Moreover

$$\alpha(u^\varepsilon,u^\varepsilon) \leqq |\alpha(u^\varepsilon,u^\varepsilon)| = |(f,u^\varepsilon)| \leqq (f,f)^{1/2}(u^\varepsilon,u^\varepsilon)^{1/2}, \text{ i.e.}$$

$$\|u^\varepsilon\| \leqq \alpha^{-1}\|f\|, \ \forall \varepsilon > 0. \tag{4.7}$$

Observe that even

$$\sum_{i=1}^{d} \|\nabla^{\varepsilon+}u^\varepsilon\|^2 \leqq \alpha\delta^{-1}\|f\|^2, \quad \text{where} \quad \delta := \min\{\alpha, A\}, \tag{4.8}$$

since $\delta\left(\sum_{l=1}^{d}(\nabla^{\varepsilon+}u^\varepsilon, \nabla^{\varepsilon+}u^\varepsilon) + (u^\varepsilon,u^\varepsilon)\right) = \delta((u^\varepsilon,u^\varepsilon))_\varepsilon \leqq |a(u^\varepsilon,u^\varepsilon)| \leqq \|f\|\cdot\|u^\varepsilon\|$, i.e. $\frac{1}{\|u^\varepsilon\|}\sum_{i=1}^{d}\|\nabla^{\varepsilon+}u^\varepsilon\|^2 + \|u^\varepsilon\| \leqq \delta^{-1}\|f\|$. By (4.7), then $\frac{1}{\alpha^{-1}\|f\|}\sum_{i=1}^{d}\|\nabla^{\varepsilon+}u^\varepsilon\|^2 \leqq \frac{1}{\|u^\varepsilon\|}\sum_{i=1}^{d}\|\nabla^{\varepsilon+}u^\varepsilon\|^2 \leqq \delta^{-1}\|f\|$. Now consider

$$(-\mathscr{L}u + \alpha u, \varphi)_0 = (f,\varphi)_0, \ \forall \varphi \in H_0^1. \tag{4.6}$$

This equation has a unique solution $u \in H_0^1$ for any $f \in H_0$, by the Lax–Milgram Lemma applied to H_0^1 and H_0. To see this we have to check the sesquilinear form

$a(\psi, \varphi) := (-\mathscr{L}\psi + \alpha\psi, \varphi)$ for (3.9) and (3.10). Inequality (3.10) is immediate from (3.20). To check (3.9) consider the matrix

$$(q_{ij})_\alpha := \begin{bmatrix} (q_{ij}) & & 0 \\ & & \vdots \\ & & 0 \\ 0 & 0\dots 0 & \alpha \end{bmatrix},$$

and let $\bar{f} := ((\partial/\partial x_1)f, \dots, (\partial/\partial x_d)f, f)$. Here $(q_{ij})_\alpha$ is real symmetric and its largest eigenvalue is $\overline{q_{\max}} = \max\{q_{\max}, \alpha\}$. Just as in (3.18), we get

$$\left| \sum_{i,j=1}^{d} q_{ij}\left(\frac{\partial}{\partial x_i} f, \frac{\partial}{\partial x_j} g\right) + \alpha(f,g) \right| = |(\bar{f}, (q_{ij})_\alpha \bar{g})|$$
$$\leqq \bar{q}_{\max} |\bar{f}|\cdot|\bar{g}| = \max\{q_{\max}, \alpha\} \, \|f\|_1 \cdot \|g\|_1, \text{ i.e. (3.9).}$$

Moreover, H_0^1 is dense in H_0, since C_0^∞ is, and $C_0 \subset H_0^1$. Hence the Lax–Milgram Lemma applies.

Theorem 4. *Let $f \in H_0$; $u \in H_0^1$, $u^\varepsilon \in H_\varepsilon^1$ be solutions of*

$$(-\mathscr{L}_\omega^\varepsilon u^\varepsilon + \alpha u^\varepsilon, \varphi) = (f, \varphi), \ \forall \varphi \in H_\varepsilon^1, \tag{4.5}$$

$$(-\mathscr{L} u + \alpha u, \varphi)_0 = (f, \varphi)_0 \ \forall \varphi \in H_0^1, \tag{4.6}$$

then $u^\varepsilon \to u$ strongly in H, as $\varepsilon \to 0$, i.e. $\|u^\varepsilon - u\| \to 0$.

Proof of Theorem 4. Observe that it is sufficient to give a proof for $f \in C_0^\infty$: We use the notation u_f^ε, respectively u_f, for $(-\mathscr{L}_\omega^\varepsilon + \alpha)^{-1} f$, respectively $(-\mathscr{L} + \alpha)^{-1} f$. Since C_0^∞ is dense in H_0^1, choose $\hat{f} \in C_0^\infty$ close enough to $f \in H_0^1$, so that the first and third summands in $\|u_f^\varepsilon - u_f\| \leqq \|u_f^\varepsilon - u_{\hat{f}}^\varepsilon\| + \|u_{\hat{f}}^\varepsilon - u_{\hat{f}}\| + \|u_{\hat{f}} - u_f\|$ can be made small uniformly in ε (cf. (4.7)), by continuity of $(-\mathscr{L} + \alpha)^{-1}$ and $(-\mathscr{L}_\omega^\varepsilon + \alpha)^{-1}$, since the left-hand-sides of (4.5) and (4.6) are sesquilinear forms. Now $f \in C_0^\infty$ implies $u \in \mathscr{S}$, the set of rapidly decreasing functions (cf. Richtmyer [14]).

The proof for $f \in C_0^\infty$ will proceed along a series of lemmata: Extend the function $\chi^k : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ of Sect. 3 to a function defined on $\Omega \times \mathbb{R}^d$ by $\chi^k(x, \omega) := \chi^k([x], \omega)$, where $[x]$ is defined to be the vector in $\mathbb{Z}^d$ satisfying $x \in \prod_{i=1}^{d} [[x], [x] + e_i)$. Define

$$z^\varepsilon(x, \omega) := u^\varepsilon(x, \omega) - u(x) - \sum_{k=1}^{d} \varepsilon \chi^k(x/\varepsilon, \omega) \nabla_k^{\varepsilon+} u(x).$$

Lemma 1. *There is a constant C_1 independent of ε such that $\|z^\varepsilon\| \leqq C_1$.*

Since
$$\|z^\varepsilon\| \geqq \left| \|u^\varepsilon - u\| - \left\| \sum_{k=1}^{d} \varepsilon \chi^k\left(\frac{x}{\varepsilon}, \omega\right) \nabla_k^{\varepsilon+} u \right\| \right|,$$

Lemma 2. $\lim_{\varepsilon \to 0} \left\| \sum_{k=1}^{d} \varepsilon \chi^k(x/\varepsilon, \omega) \nabla_k^{\varepsilon+} u \right\| = 0$, *and*

Lemma 3. $\lim_{\varepsilon\to 0}\|z^\varepsilon\| = 0$

will yield $\lim_{\varepsilon\to 0}\|u^\varepsilon - u\| = 0$, i.e. the claim of Theorem 4. For proving these lemmata, we need

Lemma 4. *There is a constant C_2 independent of ε, such that $\|\nabla_i^{\varepsilon+} z^\varepsilon\| \leqq C_2$, for all $i \leqq d$.*

Lemma 5. *There is a constant C_3 such that $E(\chi^k(x,\omega))^2 \leqq C_3(d+|x|)^2$, $\forall x\in\mathbb{R}^d$.*

Proof of Lemma 5. Consider the case $x\in\mathbb{Z}^d$ first. From the proof of (3.7) we recall that in this case

$$E(\chi^k(x))^2 \leqq \int_{\mathbb{R}^d} 4d^2|x|^2 \sum_{j=1}^{d} (U(d\lambda)\psi_j^k, U(d\lambda)\psi_j^k)$$

$$= 4d^2|x|^2 \sum_{j=1}^{d} (\psi_j^k,\psi_j^k) =: C_3|x|^2.$$

Extending χ^k from Z^d to $\mathbb{R}^d$ in the manner indicated prior to Lemma 1, we can only say that $E(\chi^k(x,\omega))^2 \leqq C_3(|x| + d_d)^2$, where d_d is the diagonal of the unit d-cube, $d_d \leqq d$. □

Lemma 6. *Let $u\in\mathcal{S}$, $\varphi\in L^2(\mathbb{R}^d)$, ψ a polynomial on $\mathbb{R}^d$. Then*

(i) $\lim_{\varepsilon\to 0}\int dx\psi(\nabla_k^{\varepsilon+}u)^2 = \int dx\psi\left(\frac{\partial}{\partial x_k}u\right)^2,$

(ii) $\lim_{\varepsilon\to 0}\int dx\varphi(\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u) = -\int dx\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}u\right)\varphi,$

(iii) $\lim_{\varepsilon\to 0}\int dx\psi(\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u)^2 = \int dx\psi\left(\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}u\right)^2$

(iv) $\lim_{\varepsilon\to 0}\int dx\psi(\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}\nabla_j^{\varepsilon+}u)^2 = \int dx\psi\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u\right)^2,$

for $i, j, k = 1,\ldots,d$.

Proof of Lemma 6.

Concerning (i): By the Mean Value Theorem $\nabla_k^{\varepsilon+}u(x) = (\partial/\partial x_k)u(x')$ for some $x'\in\mathbb{R}^d$ coinciding with x except in the k^{th} coordinate: $x = (x_1,\ldots,x_k,\ldots,x_d)$, $x' = (x_1,\ldots,x_k',\ldots,x_d)$ and $x_k'\in[x_k,x_k+\varepsilon]$. Define $g_k^\varepsilon(x) := \sup\{|(\partial/\partial x_k)u(x')| : x_k'\in[x_k,x_k+\varepsilon]\}$. Obviously $|\nabla_k^{\varepsilon+}u(x)| \leqq g_k^\varepsilon(x) \leqq g_k^1(x)$. Since $u\in\mathcal{S}$, g_k^1 goes fast enough to 0 as $|x|\to\infty$, so that $\int dx|\psi|(\nabla_k^{\varepsilon+}u)^2 \leqq \int|\psi|(g_k^1(x))^2 dx < \infty$, hence by bounded convergence and since $\nabla_k^{\varepsilon+}u \to (\partial/\partial x_k)u$ as $\varepsilon\to 0$, we get (i).

Concerning (ii): Since $\nabla_i^{\varepsilon+}u$ is differentiable: $-\nabla_k^{\varepsilon-}(\nabla_i^{\varepsilon+}u)_{(x)} = ((\partial/\partial x_k)$

$(\nabla_i^{\varepsilon+}u))_{(x')}$, for some $x'=(x_1,\ldots,x_k',\ldots,x_d)$ with $x_k'\in[x_k-\varepsilon,x_k]$, and $(\partial/\partial x_k)(\nabla_i^{\varepsilon+}u)_{(x)}=(1/\varepsilon)\,[(\partial/\partial x_k)u(x+\varepsilon e_i)-(\partial/\partial x_k)u(x)]=\nabla_i^{\varepsilon+}((\partial/\partial x_k)u(x))$. Since $(\partial/\partial x_k)u(x)$ is differentiable, we get altogether $-\nabla_k^{\varepsilon-}(\nabla_i^{\varepsilon+}u)_{(x)}=(\partial/\partial x_i)\times(\partial/\partial x_k)u(x'')$ for some $x''=(x_1',\ldots,x_{i-1}',x_i'',x_{i+1}',\ldots,x_d')$ with $x_i''\in[x_i',x_i'+\varepsilon]$. Since $x_i''\in\prod_{i=1}^{d}[x_i-2\varepsilon,x_i+2\varepsilon]$:

$$|\nabla_k^{\varepsilon-}(\nabla_i^{\varepsilon+}u)_{(x)}|\leqq\sup\left\{\left|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}u(y)\right|;\ y\in\prod_{i=1}^{d}[x_i-2\varepsilon,x_i+2\varepsilon]\right\}=:g_{ik}^{\varepsilon}.$$

For $\varepsilon\leqq 1$, $g_{ik}^{\varepsilon}\leqq g_{ik}^{1}$ and g_{ik}^{1} decreases rapidly enough for $|x|\to\infty$, so that

$$\int|\varphi(\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u)|dx\leqq\left[\int dx\varphi^2\int dx|g_{ik}^{1}|^2\right]^{1/2}<\infty, \tag{4.9}$$

$-\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u(x)=(\partial/\partial x_i)\,(\partial/\partial x_k)u(x'')$ for some $x''\in\prod_{i=1}^{d}[x_i-2\varepsilon,x_i+2\varepsilon]$, implies by continuity of derivatives (since $u\in\mathscr{S}$) that $\lim_{\varepsilon\to 0}(-\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u(x))=(\partial/\partial x_i)\,(\partial/\partial x_k)u(x)$. Hence by bounded convergence $\lim_{\varepsilon\to 0}\int dx\varphi(\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u)=-\int dx\varphi((\partial/\partial x_i)\,(\partial/\partial x_k)u)$.

Concerning (iii): Proof similar to (ii) except for $\int|\psi|(\nabla_k^{\varepsilon-}\nabla_i^{\varepsilon+}u)^2dx<\infty$ not by Schwarz' Inequality as in (4.9), but by using the fact that $g_{ik}^{1}(x)$ decays rapidly, as $|x|\to\infty$.

Concerning (iv): Proof similar to (iii) with g_{ik}^{ε} replaced by

$$g_{ikj}^{\varepsilon}:=\sup\left\{\left|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}u(y)\right|:y\in\prod_{i=1}^{d}[x_i-3\varepsilon,x_i+3\varepsilon]\right\}.\quad\square$$

Proof of Lemma 2.

$$\lim_{\varepsilon\to 0}E\left(\varepsilon\chi^k\left(\frac{x}{\varepsilon}\right)\right)^2=0\ \text{ for all } x\in\mathbb{R}^d \text{ by (3.7).} \tag{4.10}$$

Now, by Lemma 5 and the proof of Lemma 6(i),

$$E\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]^2(\nabla_k^{\varepsilon+}u(x))^2\leqq\varepsilon^2C_3\left(d+\left|\frac{x}{\varepsilon}\right|\right)^2\cdot(g_k^{\varepsilon}(x))^2$$

$$\leqq C_3(d+|x|)^2(g_k^{1}(x))^2\ \ \forall x,\ \text{for } \varepsilon\leqq 1;$$

and $\int dxC_3(d+|x|)^2\,(g_k^{1}(x))^2<\infty$. Therefore (4.10) implies by bounded convergence

$$\lim_{\varepsilon\to 0}\left\|\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+}u(x)\right\|\leqq\lim_{\varepsilon\to 0}\sum_{k=1}^{d}\left\|\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+}u(x)\right\|$$

$$=\lim_{\varepsilon\to 0}\sum_{k=1}^{d}\left[\int dx(\nabla_k^{\varepsilon+}u(x))^2E\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]^2\right]^{1/2}=0.\quad\square$$

Proof of Lemma 1. $\|z^{\varepsilon}\|\leqq\|u^{\varepsilon}\|+\|u\|+\left\|\sum_{k=1}^{d}\varepsilon\chi^k(x/\varepsilon,\omega)\nabla_k^{\varepsilon+}u\right\|$, $\|u^{\varepsilon}\|\leqq\alpha^{-1}\|f\|$

by (4.7), and $\left\| \sum_{k=1}^{d} \varepsilon\chi^k(x/\varepsilon,\omega)\nabla_k^{\varepsilon+}u \right\| \leqq C_4$ (independent of $\varepsilon \leqq 1$) by Lemma 2, hence we can choose $C_1 := \alpha^{-1}\|f\| + \|u\| + C_4$. □

Before we take up the crucial part of the proof of Theorem 4 (i.e. Lemma 3) one last technical point:

Proof of Lemma 4.

$$\|\nabla_i^{\varepsilon+} z^\varepsilon\| \leqq \|\nabla_i^{\varepsilon+} u^\varepsilon\| + \|\nabla_i^{\varepsilon+} u\| + \left\| \sum_{k=1}^{d} \varepsilon\nabla_i^{\varepsilon+}\left\{ \chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+} u \right\} \right\|.$$

Since $\|\nabla_i^{\varepsilon+} u^\varepsilon\| \leqq \alpha\cdot\delta^{-1}\|f\|^2$ by (4.8), and $\|\nabla_i^{\varepsilon+}u\| \leqq \|g_i^1\|$ as in the proof of Lemma 6, it is sufficient to show $\|\varepsilon\nabla_i^{\varepsilon+}\{\chi^k(x/\varepsilon,\omega)\nabla_k^{\varepsilon+}u\}\| \leqq C;\ i,k = 1,\ldots,d,$ for some constant C independent of ε. Use the following product rule

$$\nabla_i^{\varepsilon\pm}[\varphi(x)\psi(x)] = \varphi(x \pm \varepsilon e_i)\nabla_i^{\varepsilon\pm}\psi(x) + \psi(x)\nabla_i^{\varepsilon\pm}\varphi(x), \tag{4.11}$$

to get

$$\int dxE\left[\nabla_i^{\varepsilon+}\left(\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+}u(x) \right) \right]^2 = \int dxE\left[\varepsilon\chi^k\left(\frac{x+\varepsilon e_i}{\varepsilon},\omega\right) \cdot \nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u(x) + (\nabla_k^{\varepsilon+}u(x))\nabla_i^{\varepsilon+}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right) \right]^2$$

$$\leqq 3\int dxE\left\{ \left[\varepsilon\chi^k\left(\frac{x+\varepsilon e_i}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u(x) \right]^2 + \left[(\nabla_k^{\varepsilon+}u)(\nabla_k^{\varepsilon+}u)\nabla_i^{\varepsilon+}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right) \right]^2 \right\}. \tag{4.12}$$

Now, by Lemma 5

$$\int dxE\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon}+e_i,\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u \right]^2$$

$$\leqq \int dx\varepsilon^2 C_3\left(\left|\frac{x}{\varepsilon}+e_i\right| + d \right)^2 (\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u)^2$$

$$\leqq \int dxC_3(|x| + \varepsilon(1+d))^2(\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u)^2 \leqq \gamma_1 \ \text{(independent of } \varepsilon \leqq 1 \text{ by Lemma 6 iii).}$$

On the other hand $\varepsilon\nabla_i^{\varepsilon+}\chi^k(x/\varepsilon,\omega) = \chi^k(x/\varepsilon + e_i,\omega) - \chi^k(x/\varepsilon,\omega) = \psi(x/\varepsilon,\omega)$ a.s. (by (3.6)), where $\psi_i^k(y,\omega), y\in\mathbb{R}^d$, is the analogous extension from $\mathbb{Z}^d$ to $\mathbb{R}^d$ as in the case of χ^k, prior to Lemma 1. Since $\psi_i^k(x,\omega)$ is stationary, $E(\psi_i^k(x/\varepsilon,\omega))^2 = \gamma_2$, independent of x/ε. Therefore $\int dxE[\nabla_i^{\varepsilon+}\varepsilon\chi^k(x/\varepsilon,\omega)\cdot(\nabla_i^{\varepsilon+}u)]^2 = \int dx\gamma_2(\nabla_k^{\varepsilon+}u)^2 < \gamma_3$ independent of $\varepsilon \leqq 1$, by Lemma 6(i). Altogether $\int dxE[\nabla_i^{\varepsilon+}(\varepsilon\chi^k(x/\varepsilon,\omega)\nabla_k^{\varepsilon+}u(x))]^2 \leqq 3(\gamma_1+\gamma_3) =: C$. □

Proof of Lemma 3.

$$\alpha \| z^\varepsilon \|^2 \leqq \sum_{i=1}^{d} \left(a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+} z^\varepsilon, \nabla_i^{\varepsilon+} z^\varepsilon \right) + \alpha(z^\varepsilon, z^\varepsilon)$$

$$= \sum_{i=1}^{d} \left(\nabla_i^{\varepsilon-} a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+} z^\varepsilon, z^\varepsilon \right) + \alpha(z^\varepsilon, z^\varepsilon)$$

$$= ((-\mathscr{L}_\omega^\varepsilon + \alpha) z^\varepsilon, z^\varepsilon),$$

where the second step is justified by

$$E[\int dx \varphi(x) \nabla_i^{\varepsilon+} \psi(x)] = E \int dx \varphi(x) \frac{1}{\varepsilon}[\psi(x+\varepsilon e_i) - \psi(x)]$$

$$= \frac{1}{\varepsilon}[E\int dx \varphi(x-\varepsilon e_i)\psi(x) - E \int dx \varphi(x)\psi(x)]$$

$$= E \int dx (\nabla_i^{\varepsilon-}\psi(x))\varphi(x). \tag{4.13}$$

Hence it is sufficient to show $(-\mathscr{L}_\omega^\varepsilon z^\varepsilon + \alpha z^\varepsilon, z^\varepsilon) \to 0$, as $\varepsilon \to 0$. The first part of this proof will mainly consist of simplifying this limit up to (4.21). The second part beyond (4.21) contains the actual key of the proof in terms of the construction of the G_j^{ik}'s.

Take $\varphi \in H^1$, fix ω; then $\varphi(\omega) \in H_0^1$ and (4.6) implies

$$(-\mathscr{L}u + \alpha u, \varphi(\omega))_0 = (f, \varphi(\omega))_0, \ \forall \omega \forall \varphi \in H^1,$$

hence

$$(-\mathscr{L}u + \alpha u, \varphi) = E(-\mathscr{L}u + \alpha u, \varphi(\omega))_0 = E(f, \varphi(\omega))_0 = (f, \varphi), \ \forall \varphi \in H^1.$$

This identity together with (4.5) gives rise to the equation $((-\mathscr{L}_\omega^\varepsilon + \alpha)u^\varepsilon, \varphi) = (f, \varphi) = (-\mathscr{L}u + \alpha u, \varphi)$, for all $\varphi \in H^1$, $\varepsilon > 0$, hence

$$((-\mathscr{L}_\omega^\varepsilon + \alpha) z^\varepsilon, \varphi) = \left((-\mathscr{L}_\omega^\varepsilon + \alpha) \left[u^\varepsilon - u - \sum_{k=1}^{d} \varepsilon \chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+} u \right], \varphi \right)$$

$$= \left(-\sum_{i,k=1}^{d} \frac{\partial}{\partial x_i}\left[q_{ik} \frac{\partial}{\partial x_k} u \right], \varphi \right) + \alpha(u, \varphi)$$

$$- \left(\sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+} u \right], \varphi \right) - \alpha(u, \varphi)$$

$$- \left(\sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+} \left\{ \sum_{k=1}^{d} \varepsilon \chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+} u \right\} \right], \varphi \right)$$

$$- \alpha\left(\sum_{k=1}^{d} \varepsilon \chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+} u, \varphi \right). \tag{4.14}$$

Since (4.14) holds for all $\varphi \in H^1$, the corresponding identity for the left members of the inner products holds a.s. $[dP \times dx]$: Observe that $C_0^\infty \subset H^1$ and take as test functions e.g. a suitable $\varphi(x,\omega) := g(x) \in C_0^\infty$ to justify this statement.

But then we can as well take z^ε instead of φ in (4.14) by Lemma 1. Consider

$$F_\varepsilon := \left(\sum_{i,k=1}^{d} q_{ik} \nabla_i^{\varepsilon-} \nabla_k^{\varepsilon+} u + \sum_{i,k=1}^{d} q_{ik} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u, z^\varepsilon \right) + \alpha \left(\sum_{k=1}^{d} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_k^{\varepsilon+} u, z^\varepsilon \right),$$

$$|F_\varepsilon| \leqq \left(\left\| \sum_{i,k=1}^{d} q_{ik} \left(\nabla_i^{\varepsilon-} \nabla_k^{\varepsilon+} u + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u \right) \right\| + \alpha \left\| \sum_{k=1}^{d} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_k^{\varepsilon+} u \right\| \right) \cdot \|z^\varepsilon\|.$$

Now $\|z^\varepsilon\| \leqq C_1$ by Lemma 1, $\lim_{\varepsilon \to 0} \left\| \sum_{k=1}^{d} \varepsilon \chi^k(x/\varepsilon, \omega) \nabla_k^{\varepsilon+} u \right\| = 0$, since $|\nabla_k^{\varepsilon+} u|$ is bounded and by (4.10). Hence $\lim_{\varepsilon \to 0} |F_\varepsilon| = 0$ by an obvious analogue of Lemma 6(iii), and it is sufficient to show: $F_\varepsilon + ((-\mathscr{L}_\omega^\varepsilon + \alpha) z^\varepsilon, z^\varepsilon)$ vanishes as $\varepsilon \to 0$. With (4.14), we get

$$\begin{aligned} F_\varepsilon + ((-\mathscr{L}_\omega^\varepsilon + \alpha) z^\varepsilon, z^\varepsilon) = & \left(\sum_{i,k=1}^{d} q_{ik} \nabla_i^{\varepsilon-} \nabla_k^{\varepsilon+} u, z^\varepsilon \right) \\ & - \left(\sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i \left(\frac{x}{\varepsilon}, \omega \right) \nabla_i^{\varepsilon+} u \right], z^\varepsilon \right) \\ & - \left(\sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i \left(\frac{x}{\varepsilon}, \omega \right) \nabla_i^{\varepsilon+} \left\{ \sum_{k=1}^{d} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_k^{\varepsilon+} u \right\} \right], z^\varepsilon \right). \end{aligned} \tag{4.15}$$

We will now use the product rules (4.11) on some terms in (4.15).

i) First on the third term in (4.15):

$$\begin{aligned} & \sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i \left(\frac{x}{\varepsilon}, \omega \right) \nabla_i^{\varepsilon+} \left\{ \sum_{k=1}^{d} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_k^{\varepsilon+} u \right\} \right] \\ & = \sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i \left(\frac{x}{\varepsilon}, \omega \right) \sum_{k=1}^{d} \nabla_i^{\varepsilon+} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_k^{\varepsilon+} u(x + \varepsilon e_i) \right. \\ & \left. + a_i \left(\frac{x}{\varepsilon}, \omega \right) \sum_{k=1}^{d} \varepsilon \chi^k \left(\frac{x}{\varepsilon}, \omega \right) \nabla_i^{\varepsilon+} \nabla_k^{\varepsilon+} u \right]. \end{aligned}$$

ii) On the second term in (4.15):

$$\begin{aligned} \sum_{i=1}^{d} \nabla_i^{\varepsilon-} \left[a_i \left(\frac{x}{\varepsilon}, \omega \right) \nabla_i^{\varepsilon+} u \right] = \sum_{i,k=1}^{d} & \left\{ \left(\nabla_i^{\varepsilon-} a_i \left(\frac{x}{\varepsilon} \right) \right) \delta_{ik} \nabla_i^{\varepsilon+} u \right. \\ & \left. + a_i \left(\frac{x}{\varepsilon} - e_i \right) \delta_{ik} \nabla_i^{\varepsilon-} \nabla_i^{\varepsilon+} u \right\}. \end{aligned}$$

iii) Finally on the first summand in i):

$$\sum_{i=1}^{d} \nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d} \nabla_i^{\varepsilon+}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\varepsilon\right)\nabla_k^{\varepsilon+}u(x+\varepsilon e_i)\right]$$

$$= \sum_{i=1}^{d} \nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d} \varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\nabla_k^{\varepsilon+}u(x)$$

$$+ \sum_{i=1}^{d} a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d} \varepsilon\nabla_i^{\varepsilon+}\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i).$$

Making the corresponding replacements in (4.15) yields

$$F_\varepsilon + ((-\mathscr{L}_\omega^\varepsilon + \alpha)z^\varepsilon, z^\varepsilon) = \left(\sum_{i,k=1}^{d} q_{ik}\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u, z^\varepsilon\right)$$

$$-\left(\sum_{i,k=1}^{d}\left(\nabla_i^{\varepsilon-}a_i\left(\frac{x}{\varepsilon},\omega\right)\right)\delta_{ik}\nabla_i^{\varepsilon+}u(x), z^\varepsilon\right)$$

$$-\left(\sum_{i,k=1}^{d}\delta_{ik}a_i\left(\frac{x}{\varepsilon}-e_i\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x), z^\varepsilon\right)$$

$$-\left(\sum_{i=1}^{d}\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\nabla_i^{\varepsilon+}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\nabla_k^{\varepsilon+}u(x), z^\varepsilon\right)$$

$$-\left(\sum_{i=1}^{d}a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\nabla_i^{\varepsilon+}\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i), z^\varepsilon\right)$$

$$-\left(\sum_{k=1}^{d}\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u(x)\right], z^\varepsilon\right) \tag{4.16}$$

The second and fourth terms in (4.16) are combined to

$$-\left(\sum_{i,k=1}^{d}\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\left\{\delta_{ik}+\varepsilon\nabla_i^{\varepsilon+}\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right\}\right]\nabla_k^{\varepsilon+}u(x), z^\varepsilon\right). \tag{4.17}$$

In order to be able to combine the first, third and fifth terms in (4.16) we need a slight rearrangement. In the fifth term we first want to replace $\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i)$ by $\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)$. This can be done by adding $G_\varepsilon := \left(\sum_{i=1}^{d} a_i(x/\varepsilon,\omega)\sum_{k=1}^{d}\nabla_i^{\varepsilon+}[\varepsilon\chi^k(x/\varepsilon,\omega)]\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}(u(x)-u(x+\varepsilon e_i)), z^\varepsilon\right)$ on both sides of (4.16). Observe that $\lim_{\varepsilon\to 0} G_\varepsilon = 0$: Since $\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}(u(x)-u(x+\varepsilon e_i)) = -\varepsilon\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x)$,

$$|G_\varepsilon| = \left|\sum_{i,k=1}^{d}\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\varepsilon\nabla_i^{\varepsilon+}\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x), z^\varepsilon\right)\right|$$

$$\leqq \sum_{i,k=1}^{d} \left\| a_i\left(\frac{x}{\varepsilon},\omega\right)\varepsilon\nabla_i^{\varepsilon+}\left[\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x)\right\|\cdot\|z^\varepsilon\|$$

$$\leqq B\sum_{i,k=1}^{d}\left(\left\|\varepsilon\chi^k\left(\frac{x}{\varepsilon}+e_i,\omega\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x)\right\|+\left\|\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x)\right\|\right)\|z^\varepsilon\|.$$

Then $\lim_{\varepsilon\to 0}|G_\varepsilon|=0$, as a consequence, by Lemma 1 and (4.10) using $\|\varepsilon\chi^k((x+\varepsilon e_i)/\varepsilon,\omega)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x)\| = \|\varepsilon\chi^k(x/\varepsilon,\omega)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}\nabla_i^{\varepsilon+}u(x-\varepsilon e_i)\|$ and an obvious analogue of Lemma 2, (same proof, but g_k^1 replaced by $\bar{g}_{ijk}^1(x):=\sup\{|(\partial/\partial x_i)(\partial/\partial x_j)(\partial/\partial x_k)u(y)|;\ y\in\prod_{i=1}^{d}[x_i+6,\ x_i-6]\}$.)

The third term in (4.16) can be written as

$$\left(\sum_{i,k=1}^{d}\delta_{ik}a_i\left(\frac{x}{\varepsilon}-e_i\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u,z^\varepsilon\right)$$

$$=\sum_{i,k=1}^{d}\left(a_i\left(\frac{x}{\varepsilon}\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i),z^\varepsilon(x+\varepsilon e_i)\right)\delta_{ik}.$$

Let

$$K_\varepsilon:=\sum_{i,k=1}^{d}\delta_{ik}a_i\left(\frac{x}{\varepsilon}\right)[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)z^\varepsilon(x)-\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i)z^\varepsilon(x+\varepsilon e_i)],$$

$$\begin{aligned}
&[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)]z^\varepsilon(x)-[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i)]z^\varepsilon(x+\varepsilon e_i)\\
&\quad=[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)]z^\varepsilon(x)-[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)]z^\varepsilon(x+\varepsilon e_i)\\
&\qquad+[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)]z^\varepsilon(x+\varepsilon e_i)-[\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x+\varepsilon e_i)]z^\varepsilon(x+\varepsilon e_i)\\
&\quad=(-\varepsilon\nabla_i^{\varepsilon+}z^\varepsilon(x))\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)+\varepsilon z^\varepsilon(x+\varepsilon e_i)[\nabla_i^{\varepsilon+}\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u(x)].
\end{aligned}$$

Therefore

$$\begin{aligned}
|E\textstyle\int dxK_\varepsilon| &= |\varepsilon E\textstyle\int dx a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}z(x)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u|\\
&\quad+\left|\varepsilon E\textstyle\int dx z(x+\varepsilon e_i)a_i\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u\right|\\
&\leqq\varepsilon B(\|\nabla_i^{\varepsilon+}z(x)\|\cdot\|\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u\|+\|z(x+\varepsilon e_i)\|\cdot\|\nabla_i^{\varepsilon+}\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u\|)\\
&\leqq\varepsilon B(C_2\|\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u\|+C_1\|\nabla_i^{\varepsilon+}\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u\|),
\end{aligned}$$

by Lemma 1 and Lemma 4. These norms are bounded independently of $\varepsilon\leqq 1$, by Lemma 6 (iii), (iv), hence

$$\lim_{\varepsilon\to 0}E\textstyle\int dxK_\varepsilon=0. \tag{4.18}$$

Making use of the observations $\lim_{\varepsilon\to 0}|G_\varepsilon| = 0 = \lim_{\varepsilon\to 0} E\int dx K_\varepsilon$ and of the term (4.17), we conclude by combining the new third and fifth terms with the first term in (4.16), that we only have to show the vanishing of the following term, as $\varepsilon\to 0$:

$$\left(\sum_{k,i=1}^{d}\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\left\{\delta_{ik}+\varepsilon\nabla_i^{\varepsilon+}\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right\}-q_{ik}\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u,z^\varepsilon\right)$$

$$+\sum_{i,k=1}^{d}\left(\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\left\{\delta_{ik}+\varepsilon\nabla_i^{\varepsilon+}\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right\}\right]\nabla_k^{\varepsilon+}u(x),z^\varepsilon\right)$$

$$+\sum_{i=1}^{d}\left(\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_k^{\varepsilon+}\nabla^{\varepsilon+}u\right],z^\varepsilon\right). \tag{4.19}$$

Now consider the third summand of (4.19). Using (4.13),

$$\left|\sum_{i=1}^{d}\left(\nabla_i^{\varepsilon-}\left[a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u\right],z^\varepsilon\right)\right|$$

$$\leqq\sum_{i=1}^{d}\left|\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u,\nabla_i^{\varepsilon+}z^\varepsilon\right)\right|$$

$$\leqq\sum_{i=1}^{d}\left\|a_i\left(\frac{x}{\varepsilon},\omega\right)\sum_{k=1}^{d}\varepsilon\chi^k\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u\right\|\cdot\|\nabla_i^{\varepsilon+}z^\varepsilon\|, \tag{4.20}$$

with $\left\|a_i(x/\varepsilon,\omega)\sum_{k=1}^{d}\varepsilon\chi^k\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u\right\|\leqq B\left\|\sum_{k=1}^{d}\varepsilon\chi^k(x/\varepsilon,\omega)\nabla_i^{\varepsilon+}\nabla_k^{\varepsilon+}u\right\|$, which vanishes as $\varepsilon\to 0$ by an analogue of Lemma 2 (same proof except g_{ik}^1 replacing g_k^1). Using this result and Lemma 4 in (4.20) shows that the third summand of (4.19) goes to 0 in the limit.

Now consider $\int dx\sum_{k=1}^{d}\nabla_k^{\varepsilon+}u\sum_{i=1}^{d}E\{\nabla_i^{\varepsilon-}(a_i(x/\varepsilon,\omega)[\delta_{ik}+\varepsilon\nabla_i^{\varepsilon+}\chi^k(x/\varepsilon,\omega)])z^\varepsilon\}$, the second summand of (4.19), Fix $x:z^\varepsilon(x,\cdot)\in\mathscr{H}$. Recall that $\varepsilon\nabla_i^{\varepsilon+}\chi^k(x/\varepsilon,\omega)=\psi_i^k(x/\varepsilon,\omega)$, which is stationary by Theorem 3, and so is $a_i(x/\varepsilon,\omega)$, hence

$$\sum_{i=1}^{d}E\left\{\nabla_i^{\varepsilon-}\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\left[\delta_{ik}+\varepsilon\nabla_i^{\varepsilon+}\chi^k\left(\frac{x}{\varepsilon},\omega\right)\right]\right)z^\varepsilon(x,\omega)\right\}$$

$$=\frac{1}{\varepsilon}\sum_{i=1}^{d}E\{\nabla^{i-}(a_i(0,\tau_{-x/\varepsilon}\omega)[\delta_{ik}+\psi_i^k(0,\tau_{-x/\varepsilon}\omega)])$$

$$\cdot z^\varepsilon(x,\tau_{x/\varepsilon}\tau_{-x/\varepsilon}\omega)\}=0,$$

by (3.3) in the form $\frac{1}{\varepsilon}\sum_{i=1}^{d}E(\nabla^{i-}[a_i(\omega)(\delta_{ik}+\psi_i^k(\omega)]\varphi(\omega))=0$, with $\varphi(\omega):=z^\varepsilon(x,\tau_{x/\varepsilon}\omega)$. Hence the second summand of (4.19) is identically zero.

The proof of Lemma 3 is therefore reduced to showing that the first term in (4.19), i.e.

$$\sum_{i,k=1}^{d}\left(a_i\left(\frac{x}{\varepsilon},\omega\right)\left\{\delta_{ik}+\psi_i^k\left(\frac{x}{\varepsilon},\omega\right)\right\}-q_{ik}\right)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u,z^\varepsilon\Big) \tag{4.21}$$

vanishes as $\varepsilon\to 0$

We are now ready to enter the key portion of the proof.

Set $g^{ik}(x,\omega):=a_i(x,\omega)(\delta_{ik}+\psi_i^k(x,\omega))-q_{ik}$, $i,k=1,\dots,d$, where g^{ik} is stationary since a_i and ψ_i^k are. We want to show $\lim_{\varepsilon\to 0}\left(\sum_{i=1}^{d} g^{ik}(x/\varepsilon,\omega)\nabla_i^{\varepsilon-}\nabla_k^{\varepsilon+}u,z^\varepsilon\right)=0$. We define the shift operator T_x on H as $T_x g(y,\omega):=g(y+x,\omega)$. Here $\{T_x\}_{x\in\mathbb{Z}^d}$ is a unitary group of operators on H; with spectral representation $T_x=\int_{\mathbb{R}^d} e^{i\lambda x}U(d\lambda)$, where $\{U(d\lambda)\}_\lambda$ is the corresponding family of spectral projectors. Set

$$G_j^{ik}(x,\omega):=\int_{\mathbb{R}^d}(e^{i\lambda x}-1)\frac{(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega),\text{ for } x\in\mathbb{Z}^d. \tag{4.22}$$

where $|e^{i\lambda}-1|^2:=\sum_{l=1}^{d}|e^{i\lambda_l}-1|^2$. It is immediate that G_j^{ik} is well-defined, by the very same argument used for χ^k in Sect. 3.

The extension of G_j^{ik} on $\mathbb{Z}^d\times\Omega$ as usual to $\mathbb{R}^d\times\Omega$ has the following properties:

$$\sum_{j=1}^{d}\nabla_j G_j^{ik}(x,\omega)=g^{ik}(x,\omega)\qquad\text{for all } x\in\mathbb{R}^d, \tag{4.23i}$$

$$E(G_j^{ik}(x,\omega))^2\leqq C_3(d^2+|x|)^2\quad\text{for all } x\in\mathbb{R}^d, \tag{4.23ii}$$

$$E\left(\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\right)^2\to 0,\ as\ \varepsilon\to 0,\qquad\text{for all } x\in\mathbb{R}^d. \tag{4.23iii}$$

The proof of (4.23i) is straightforward:

$$\sum_{j=1}^{d}\nabla_j G_j^{ik}(x,\omega)=\sum_{j=1}^{d}\nabla_j\int_{\mathbb{R}^d}(e^{i\lambda x}-1)\frac{(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega)$$

$$=\sum_{j=1}^{d}\int_{\mathbb{R}^d}(e^{i\lambda(x+e_j)}-e^{i\lambda x})\frac{(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega)$$

$$=\sum_{j=1}^{d}\int_{\mathbb{R}^d}e^{i\lambda x}\frac{(e^{i\lambda_j}-1)(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega)$$

$$=\int_{\mathbb{R}^d}e^{i\lambda x}U(d\lambda)g^{ik}(0,\omega)=g^{ik}(x,\omega).$$

For the proof of (4.23ii), we recall from the proof of (3.7), that for $x\in\mathbb{Z}^d$

$$\frac{|e^{i\lambda x}-1|^2}{|e^{i\lambda}-1|^2}\leqq 4d^2|x|^2,\text{ hence for } x\in\mathbb{Z}^d,$$

$$E(G_j^{ik}(x,\omega))^2 = \int_{\mathbb{R}^d} \frac{|e^{i\lambda x}-1|^2|e^{i\lambda_j}-1|^2}{|e^{i\lambda}-1|^2|e^{i\lambda}-1|^2}(U(d\lambda)g^{ik}(0,\omega),g^{ik}(0,\omega))$$

$$\leqq \int_{\mathbb{R}^d} \frac{|e^{i\lambda x}-1|^2}{|e^{i\lambda}-1|^2}(U(d\lambda)g^{ik}(0,\omega),g^{ik}(0,\omega))$$

$$\leqq 4d^2|x|^2(g^{ik}(0,\omega),g^{ik}(0,\omega)).$$

Choosing $C_3 := (g^{ik}(0,\omega),g^{ik}(0,\omega))4\cdot d^2$, we get (4.23ii) as in the proof of Lemma 5 for $x\in\mathbb{R}^d$.

For the proof of (4.23iii) we start out with $x\in\mathbb{Z}^d$, as usual:

$$E\left(\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\right)^2 \leqq \int_{\mathbb{R}^d} \varepsilon^2\frac{|e^{i\lambda(x/\varepsilon)}-1|^2}{|e^{i\lambda}-1|^2}(U(d\lambda)g^{ik}(0,\omega),g^{ik}(0,\omega)).$$

The argument for $x\in\mathbb{Z}^d$ is completely analogous to the argument of $\lim_{\varepsilon\to 0} E(\varepsilon\chi^k(x/\varepsilon,\omega))^2 = 0$ in Sect. 3. For $x\in\mathbb{R}^d$ in general, we have to be a bit more careful. If $x\in\mathbb{R}e_l$, $l=1,\ldots,d$, the argument for χ^k can still be adopted. Now consider more general $x\in\mathbb{R}^d$; set

$$\eta_l\left(\frac{x}{\varepsilon}\right) := \sum_{m=0}^{[x_l/\varepsilon]-1} G_j^{ik}\left(\left[\frac{x_1}{\varepsilon}\right],\ldots,\left[\frac{x_{l-1}}{\varepsilon}\right],m+1,0,\ldots,0\right)$$

$$- G_j^{ik}\left(\left[\frac{x_1}{\varepsilon}\right],\ldots,\left[\frac{x_{l-1}}{\varepsilon}\right],m,0,\ldots,0\right),$$

$$\tilde{\eta}_l\left(\frac{x}{\varepsilon}\right) := \sum_{m=0}^{(x_l/\varepsilon]-1} (G_j^{ik}((m+1)e_l) - G_j^{ik}(me_l))$$

$$= \sum_{m=0}^{[x_l/\varepsilon]-1} \nabla_l G_j^{ik}(me_l).$$

The $\nabla_l G_j^{ik}$, however, is stationary, since

$$\nabla_l G_j^{ik}(x,\omega) = \int_{\mathbb{R}^d} e^{i\lambda x}\frac{(e^{i\lambda_l}-1)(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega),$$

and

$$\nabla_l G_j^{ik}(x+y,\omega)$$

$$= \int_{\mathbb{R}^d} e^{i\lambda(x+y)}\frac{(e^{i\lambda_l}-1)(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega)$$

$$= \int_{\mathbb{R}^d} e^{i\lambda' y}U(d\lambda')\int_{\mathbb{R}^d} e^{i\lambda x}\frac{(e^{i\lambda_l}-1)(e^{-i\lambda_j}-1)}{|e^{i\lambda}-1|^2}U(d\lambda)g^{ik}(0,\omega)$$

$$= T_y\nabla_l G_j^{ik}(x,\omega).$$

From the definitions of η_l and $\tilde{\eta}_l$ it is therefore immediate that η_l and $\tilde{\eta}_l$ are equally distributed, hence

$$E\left(\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\right)^2 = E\left(\varepsilon\left(\sum_{l=1}^{d}\eta_l\left(\frac{x}{\varepsilon}\right)\right)\right)^2 = E\left(\sum_{l,m=1}^{d}\varepsilon\eta_l\left(\frac{x}{\varepsilon}\right)\varepsilon\eta_m\left(\frac{x}{\varepsilon}\right)\right)$$

$$\leqq \sum_{l,m=1}^{d}\left(E\left(\varepsilon\eta_l\left(\frac{x}{\varepsilon}\right)\right)^2 E\left(\varepsilon\eta_m\left(\frac{x}{\varepsilon}\right)\right)^2\right)^{1/2}$$

$$= \sum_{l,m=1}^{d}\left(E\left(\varepsilon\tilde{\eta}_l\left(\frac{x}{\varepsilon}\right)\right)^2 E\left(\varepsilon\tilde{\eta}_m\left(\frac{x}{\varepsilon}\right)\right)^2\right)^{1/2}.$$

Since $G_j^{ik}(0)=0$, $\tilde{\eta}_l(x/\varepsilon)=G_j^{ik}((x_l/\varepsilon)e_l)$, and since we have already established the result for $x\in\mathbb{R}e_l$, $l=1,\ldots,d$, we have $\lim_{\varepsilon\to 0} E(\varepsilon\tilde{\eta}_l(x/\varepsilon))^2=0$, hence $E(\varepsilon G_j^{ik}(x/\varepsilon,\omega))^2\to 0$, as $\varepsilon\to 0$, by the inequality above, so that we have proven the last of the three properties of G_j^{ik}.

Now we use G_j^{ik} for the proof of (4.21) in the form

$$\lim_{\varepsilon\to 0}\left(\sum_{i=1}^{d} g^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u, z^\varepsilon\right)=0,\quad k=1,\ldots,d. \tag{4.21'}$$

By (4.23):

$$\left(\sum_{i=1}^{d} g^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u, z^\varepsilon\right)=\left(\varepsilon\sum_{i,j=1}^{d}\nabla_j^{\varepsilon +}G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u, z^\varepsilon\right)$$

$$=\varepsilon\int dx E\sum_{i,j=1}^{d}\nabla_j^{\varepsilon +}G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u z^\varepsilon$$

$$=\varepsilon\int dx E\sum_{i,j=1}^{d}G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_j^{\varepsilon -}[(\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u)z^\varepsilon]$$

$$=\varepsilon\int dx E\sum_{i,j=1}^{d}G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)[\nabla_j^{\varepsilon -}\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u(x)z^\varepsilon(x-\varepsilon e_i)+\nabla_j^{\varepsilon -}z^\varepsilon(x)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u(x)]$$

$$\leqq\sum_{i,j=1}^{d}\left\|\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_j^{\varepsilon -}\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u\right\|\|z^\varepsilon\|+\sum_{i,j=1}^{d}\left\|\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u\right\|\cdot\|\nabla_j^{\varepsilon -}z^\varepsilon\|$$

$$\leqq(C_1+C_2)\sum_{i,j=1}^{d}\left\{\left\|\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u\right\|+\left\|\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u\right\|\right\},$$

by (4.11), (4.13), Lemma 1 and Lemma 4.

Here $\nabla_j^{\varepsilon -}\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u$, respectively $\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u$, are bounded independently of ε by a function in $\mathscr{S}$ (cf. proof of Lemma 6), say by g_3, respectively g_2. Then by (4.23) for $\varepsilon\leqq 1$:

$$\left\|\varepsilon G_j^{ik}\left(\frac{x}{\varepsilon},\omega\right)\nabla_i^{\varepsilon -}\nabla_k^{\varepsilon +}u\right\|\leqq\left[\int dx C\varepsilon^2\left(d+\left|\frac{x}{\varepsilon}\right|\right)^2 g_2^2\right]^{1/2}$$

$$\leqq\left[\int dx C(d+|x|)^2 g_2^2\right]^{1/2}<\infty.$$

By dominated convergence and (4.24),

$$\lim_{\varepsilon \to 0} \| \varepsilon G_j^{ik}(x/\varepsilon) \nabla_i^{\varepsilon -} \nabla_k^{\varepsilon +} u \| \leqq [\int dx E(\varepsilon G_j^{ik}(x/\varepsilon,\omega))^2 (\nabla_i^{\varepsilon -} \nabla_k^{\varepsilon +} u)^2]^{1/2} = 0.$$

Similarly $\lim_{\varepsilon \to 0} \| \varepsilon G_j^{ik}(x/\varepsilon,\omega) \nabla_i^{\varepsilon -} \nabla_k^{\varepsilon +} u \| = 0$. This proves (4.21)′ and hence Lemma 3. □

Lemma 2 and Lemma 3 yield Theorem 4.

An obvious consequence is the uniqueness of the q_{ij}: We have just proven $u^\varepsilon \to u$ strongly in H, where we did not use uniqueness of χ^k. Since u^ε is formulated (as the solution of (4.5)) independently of χ^k, its limit u is independent of χ^k, hence q_{ij} (characterizing the limit u) is independent of χ^k, as long as χ^k satisfies the properties of Theorem 3.

5. Mean Square Convergence in Distribution

In this section we combine the results of sections 2 and 4 to prove the main theorem of this paper.

Theorem 5. *Let Q_x be the measure of a diffusion process starting at x with generator $\mathscr{L} = \sum_{i,j=1}^{d} q_{ij}(\partial/\partial x_i \partial x_j)$, with q_{ij} as in (3.17).*

Let $Q_{x,\omega}^\varepsilon$ be the measure of the jump process of Sect. 0, starting at x, with generator $\mathscr{L}_\omega^\varepsilon = -\sum_{i=1}^{d} \nabla_i^{\varepsilon -}(a_i(x/\varepsilon,\omega)\nabla_i^{\varepsilon +})$. Let $\tilde{Q}_{x,\omega}^\varepsilon$ be the measure of the corresponding smoothened process of Sect. 2

Let F be a bounded, continuous function on the space $C := C([0,\infty],\mathbb{R}^d)$.

Then, for any nonnegative function $\varphi, \varphi \in L^2(\mathbb{R}^d)$:

$$\lim_{\varepsilon \to 0} E | \int_{\mathbb{R}^d} dx \varphi(x) \int_C F(\zeta) \tilde{Q}_{x,\omega}^\varepsilon(d\zeta) - \int_{\mathbb{R}^d} dx \varphi(x) \int_C F(\zeta) Q_x(d\zeta)|^2 = 0. \tag{5.1}$$

Proof of Theorem 5. $\mathscr{L}_\omega^\varepsilon$ and $\mathscr{L}$ are generators of Markov processes. Let us denote their respective semigroups by $e^{t\mathscr{L}_\omega^\varepsilon}$ and $e^{t\mathscr{L}}$.

Lemma 1. $$e^{t\mathscr{L}_\omega^\varepsilon} f \to e^{t\mathscr{L}} f, \text{ as } \varepsilon \to 0, \text{ for all } f \in H_0, \tag{5.2}$$
strongly in H and uniformly in any finite interval $0 \leqq t \leqq T$.

Proof. Using the strong resolvent convergence of Theorem 4, (5.2) can be shown following Kato's proof of his well-known Theorem in Kato [6], p. 504. □

Since $\mathscr{L}$, $\mathscr{L}_\omega^\varepsilon$ are generators of Markov processes with measures $Q_{x,\omega}^\varepsilon$, Q_x, we have the respresentations $(e^{t\mathscr{L}_\omega^\varepsilon} f)(x) = E^{Q_{x,\omega}^\varepsilon} f(\zeta(t))$, $(e^{t\mathscr{L}} f)(x) = E^{Q_x} f(\zeta(t))$, for all $f \in H_0$, so that (5.2) can be written as

$$\lim_{\varepsilon \to 0} \sup_{0 \leqq t \leqq T} E \int dx | E^{Q_{x,\omega}^\varepsilon}(f(\zeta(t)) - E^{Q_x}(f(\zeta(t))|^2 = 0, \text{ for all } f \in H_0. \tag{5.3}$$

In order to show (5.1), start with taking M, $0 < M < \infty$;

$$E\left|\int_{\mathbb{R}^d} dx\varphi(x)\left[\int_C F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_C F(\zeta)Q_x(d\zeta)\right]\right|^2$$

$$\leqq 3E\left|\int_{|x|\leqq M} dx\varphi(x)\int_C F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{|x|\leqq M} dx\varphi(x)\int_C F(\zeta)Q_x(d\zeta)\right|^2$$

$$+3E\left|\int_{|x|\geqq M} dx\varphi(x)\int_C F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{|x|\geqq M} dx\varphi(x)\int_C F(\zeta)Q_x(d\zeta)\right|^2. \tag{5.4}$$

Since $\int_C F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)\leqq \|F\|_\infty \geqq \int_C F(\zeta)Q_x(d\zeta)$, we have

$$\left|\int_{|x|\geqq M} dx\varphi(x)\left\{\left|\int_C F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_C F(\zeta)Q_x(d\zeta)\right|\right\}\right|^2 \leqq 2\|F\|_\infty \int_{|x|\geqq M}\varphi^2 dx = :K(M),$$

which can be made arbitrarily small by choosing M large enough, since $\varphi\in L^2$. Hence it is sufficient to show that the first summand in (5.4) can be made small.

In Sect. 2 we have shown that for $\delta>0$ a relatively compact set $K_\delta\subset D$ can be found with $Q^{\varepsilon}_{x,\omega}(K_\delta)\geqq 1-\delta$, for all $\varepsilon, 0<\varepsilon<1$, $\omega\in\Omega$, $|x|\leqq M$, and that the corresponding set $S_\delta := \mathrm{cl}(\tilde{K}_\delta)$ is compact in C and satisfies $\tilde{Q}^{\varepsilon}_{x,\omega}(S_\delta)\geqq 1-\delta$, for all $\varepsilon, 0<\varepsilon<1$; $\omega\in\Omega$, $|x|\leqq M$. We can use S_δ now to reformulate and bound the first summand in (5.4)

$$E\left|\int_{|x|\leqq M} dx\varphi(x)\left(\int_{S_\delta} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)+\int_{S_\delta^c} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)\right)\right.$$

$$\left.-\int_{|x|\leqq M} dx\varphi(x)\left(\int_{S_\delta} F(\zeta)Q(d\zeta)+\int_{S_\delta^c} F(\zeta)Q(d\zeta)\right)\right|^2$$

$$\leqq 3E\left|\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta} F(\zeta)Q_x(d\zeta)\right|^2$$

$$+3E\left|\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta^c} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta^c} F(\zeta)Q_x(d\zeta)\right|^2. \tag{5.5}$$

Now, $\int_{S_\delta^c} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)$ and $\int_{S_\delta^c} F(\zeta)Q_x(d\zeta)$ are bounded by $\|F\|_\infty\cdot\delta$, so that the second summand in (5.5) is bounded by $3K(M)\cdot\delta$, which can be made small by choosing δ small.

So we are left with showing

$$\lim_{\varepsilon\to 0} E\left|\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{|x|\leqq M} dx\varphi(x)\int_{S_\delta} F(\zeta)Q_x(d\zeta)\right|^2=0,$$

or, making use of Schwarz' inequality and $\varphi\in L^2$, with showing

$$\lim_{\varepsilon\to 0} E\int dx\left|\int_{S_\delta} F(\zeta)\tilde{Q}^{\varepsilon}_{x,\omega}(d\zeta)-\int_{S_\delta} F(\zeta)Q_x(d\zeta)\right|^2=0. \tag{5.6}$$

The set of finite linear combinations of products of the form

$$f_1(\zeta(t_1))\dots(f_n(\zeta(t_n)), 0\leqq t_1\leqq\dots\leqq t_n<\infty, \text{ with } f_j\in C_0^\infty(\mathbb{R}^d) \tag{5.7}$$

is an algebra in $C(S_\delta)$, the set of continuous functions on S_δ, and moreover it contains the constant functions and separates points. Hence, since S_δ is compact, this algebra is dense in $C(S_\delta)$ by the Stone–Weierstraß-Theorem (cf. e.g. Kelley [7]).

It is therefore sufficient to show (5.6) for F of the form (5.7), i.e.

$$\lim_{\varepsilon\to 0} E\left|\int_{|x|\leqq M} dx\left\{\int_{S_\delta} f_1(\zeta(t_1))\dots f_n(\zeta(t_n))\tilde{Q}^\varepsilon_{x,\omega}(d\zeta) - \int_{S_\delta} f_1(\zeta(t_1))\dots f_n(\zeta(t_n))Q_x(d\zeta)\right\}\right|^2 = 0, \tag{5.8}$$

for all $n\in\mathbb{N}$. But for this purpose, we need only consider

$$\lim_{\varepsilon\to 0} E\int dx\left|\left\{\int_D f_1(\zeta(t_1))\dots f_n(\zeta(t_n))Q^\varepsilon_{x,\omega}(d\zeta) - \int_D f_1(\zeta(t_1))\dots f_n(\zeta(t_n))Q_x(d\zeta)\right\}\right|^2 = 0, \tag{5.9}$$

since all functions $f_i\in C_0^\infty$ and their derivatives are bounded, and since

$$|f_i(\zeta(t_i)) - f_i(\hat{\zeta}(t_i))| \leqq \varepsilon\cdot\max\{f'(x);\ x\in[\min(\zeta(t_i),\hat{\zeta}(t_i)),\max(\zeta(t_i),\hat{\zeta}(t_i))]\},$$

where $\hat{\zeta}$ is the smoothened path corresponding to ζ, and $|\zeta_{(t)} - \hat{\zeta}_{(t)}|\leqq\varepsilon$ by construction, for all t.

We begin the proof of (5.9) by considering the case $n=2$, $n=1$ is covered by (5.3), i.e. we show

$$\lim_{\varepsilon\to 0} E\int dx|E^{Q^\varepsilon_{x,\omega}} f_1(\zeta(t_1))f_2(\zeta(t_2))) - E^{Q_x}(f_1(\zeta(t_1))f_2(\zeta(t_2)))|^2 = 0, \tag{5.10}$$

$$\begin{aligned}
&E\int dx|E^{Q^\varepsilon_{x,\omega}}[f_1(\zeta(t_1))E^{Q^\varepsilon_{x,\omega}}(f_2(\zeta(t_2))|\mathscr{F}_{t_1})]\\
&\quad - E^{Q_x}[f_1(\zeta(t_1))E^{Q_x}(f_2(\zeta(t_2))|\mathscr{F}_{t_1})]|^2\\
&\leqq 3E\int dx|E^{Q^\varepsilon_{x,\omega}}[f_1(\zeta(t_1))e^{(t_2-t_1)\mathscr{L}^\varepsilon_\omega}f_2(\zeta(t_1))]\\
&\quad - E^{Q^\varepsilon_{x,\omega}}[f_1(\zeta(t_1))e^{(t_2-t_1)\mathscr{L}}f_2(\zeta(t_1))]|^2\\
&\quad + 3E\int dx|E^{Q^\varepsilon_{x,\omega}}[f_1(\zeta(t_1))e^{(t_2-t_1)\mathscr{L}}f_2(\zeta(t_1))]\\
&\quad - E^{Q_x}[f_1(\zeta(t_1))e^{(t_2-t_1)\mathscr{L}}f_2(\zeta(t_1))]|^2,
\end{aligned} \tag{5.11}$$

using the Markov property. The second summand vanishes as $\varepsilon\to 0$ by (5.3), with

$$f(x) := f_1(x)e^{(t_2-t_1)\mathscr{L}}f_2(x), f\in H_0,\ \text{since } f_1, f_2\in C_0^\infty(\mathbb{R}^d).$$

Now consider the first summand in (5.11).

$$\begin{aligned}
&E\int dx|E^{Q^\varepsilon_{x,\omega}}f_1(\zeta(t_1))[e^{(t_2-t_1)\mathscr{L}^\varepsilon_\omega}f_2(\zeta(t_1)) - e^{(t_2-t_1)\mathscr{L}}f_2(\zeta(t_1))])|^2\\
&\leqq E\int dx E^{Q^\varepsilon_{x,\omega}}[f_1^2(\zeta(t_1))]E^{Q^\varepsilon_{x,\omega}}[G^\varepsilon(\omega,\zeta(t_1))^2]\\
&\leqq \max_{x\in\mathbb{R}^d} f_1^2(x)E\int dx E^{Q^\varepsilon_{x,\omega}}[G^\varepsilon(\omega,\zeta(t_1))]^2,
\end{aligned} \tag{5.12}$$

by Schwarz' inequality, since $f_1 \in C_0^\infty(\mathbb{R}^d)$, where

$$G^\varepsilon(\omega,x) := [e^{(t_2-t_1)\mathscr{L}_\omega^\varepsilon} f_2(x) - e^{(t_2-t_1)\mathscr{L}} f_2(x)].$$

By (5.3) we have

$$\lim_{\varepsilon\to 0} E \int dx (G^\varepsilon(\omega,x))^2 = 0. \tag{5.13}$$

Let $p_\omega^\varepsilon(y,t|x)$ be the transition probability for the $Q_{x,\omega}^\varepsilon$-process at time t. Consider p_ω^ε as a density on $\mathbb{R}^d$ by our usual extension of functions from $\varepsilon\mathbb{Z}^d$ to $\mathbb{R}^d$. By self-adjointness (cf. (1.4)) $\int dx p_\omega^\varepsilon(y,t|x) = \int dx p_\omega^\varepsilon(x,t|y) = 1$.

Altogether then

$$\begin{aligned} E \int dx E^{Q_{x,\omega}^\varepsilon}(G^\varepsilon(\omega,\zeta(t_1))^2 &= E \int dx \int dy p_\omega^\varepsilon(y,t_1|x)(G^\varepsilon(\omega,y))^2 \\ &= E \int dy (G^\varepsilon(\omega,y))^2 \int dx p_\omega^\varepsilon(y,t_1|x) = E \int dy (G^\varepsilon(\omega,y))^2, \end{aligned}$$

which vanishes in the limit $\varepsilon \to 0$ by (5.13), and consequently the first summand (5.12) of (5.11) vanishes also, proving (5.10).

It is now obvious how we can conclude (5.9) for all $n \in \mathbb{N}$ by induction. This completes the proof of Theorem 5.

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