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The Scattering of Certain Yang–Mills Fields*

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Abstract. The Yang–Mills fields considered by us in an earlier paper are asymptotically non-interacting. Also any free field is an incoming field for some Yang–Mills field.

Introduction

Interest in the classical solutions of the Yang–Mills equations in Minkowski space has grown in recent years. The definitive global existence theorem (solution of the Cauchy problem) has been found by Eardley and Moncrief [3], following the local theorem of Segal [8]. Other existence results and methods appear in [6], [4] and [1]. However, the scattering problem has so far been left untouched. Christodoulou's transform method can be used to derive some decay properties. In [5] we showed how the conformal invariance directly implies certain asymptotic properties of the fields, in particular, local decay of the energy.

In Sect. 1 of this paper we use these asymptotic properties, and the special properties of the class of solutions discussed in [6], to prove that these solutions α are asymptotically free fields in the energy norm. In Sect. 3 we show that any free field α_- of our special type is the incoming field of some α. In Sect. 2 we derive some explicit pointwise bounds needed in the proof.

Our class of solutions is defined by a condition of the Polyakov-t'Hooft type for the gauge group SU(2). We emphasize that there is no restriction on the size of the solutions we consider. Specifically, the gauge potentials have the form

$$A^k = \alpha(r,t)v^k \quad (k=1,2,3), \quad A^0 = 0.$$

where t is time, $x\in\mathbb{R}^3$, $r=|x|$ and v^k are certain vectors (see [6]). Such a field belongs to both the temporal and Coulomb gauges. The field equations reduce to a single scalar wave equation (see (1) below). Although this equation appears rather innocuous, its asymptotic analysis is surprisingly non-trivial due to the singularity at the origin. The free fields we consider are simply solutions of the Yang–Mills

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equations with vanishing coupling constant; that is, they satisfy the same equation but linearized around zero.

Thus physically interesting phenomena such as solitons appear to be absent from the pure Yang–Mills equations. It seems that a Higgs mechanism is necessary to make them appear.

1. Asymptotic Freedom of the Yang–Mills Fields

We consider the solutions of the real scalar equation

$$\alpha_{tt} - \alpha_{rr} - \frac{2}{r}\alpha_r + \frac{2}{r^2}\alpha - \frac{3}{r}\alpha^2 + \alpha^3 = 0. \tag{1}$$

Denote the energy density by

$$e = e(r,t) = \frac{1}{2}\alpha_t^2 + \frac{1}{2}\alpha_r^2 + \frac{1}{4}\alpha^2\left(\frac{2}{r} - \alpha\right)^2,$$

and the inversional density by

$$I = I(r,t) = \frac{1}{2}(t^2 + r^2)(\alpha_t^2 + \alpha_r^2) + 2tr\alpha_t\alpha_r + 2t\alpha_t\alpha - \alpha^2 + \frac{1}{4}(t^2 + r^2)\alpha^2\left(\frac{2}{r} - \alpha\right)^2.$$

Furthermore we denote $x \in \mathbb{R}^3, r = |x|$ and $dx = 4\pi r^2 dr$. Let

$$\|\alpha\|_*^2 = \int\left(\alpha_t^2 + \alpha_r^2 + \frac{2}{r^2}\alpha^2\right)dx \tag{2}$$

at any time t, where the integration is over all of space $\mathbb{R}^3$. Let $\mathscr{H}$ be the Hilbert space of Cauchy data provided with this norm. That is, $\mathscr{H} = \tilde{H}_r^1 \oplus L^2$ in the notation of [6].

Theorem 1. (a) *If $\int e\,dx < \infty$ when $t = 0$, then there is a unique solution $\alpha = \alpha(r,t)$ of (1) with given Cauchy data at $t = 0$ such that $\alpha \in C(\mathbb{R}; \mathscr{H})$, the energy $\int e\,dx$ is independent of time and*

(i) $B^2 = \int\int \alpha^2\left(\frac{2}{r} - \alpha\right)^2 \frac{1}{r}dx\,dt < \infty$

(integral taken over all space-time).

(b) *In case $\int(1 + r^2)e\,dx < \infty$ when $t = 0$, we have*

(ii) $\int I(r,t)dx \leqq \int I(r,0)dx < \infty$.

Furthermore, $\int(1 + r^2)e\,dx < \infty$ for all time and

(iii) $\int_{r<\theta t} e\,dx + \int \alpha^2\left(\frac{2}{r} - \alpha\right)^2 dx = 0\left(\frac{1}{t^2}\right)$

for large t, for any $\theta < 1$.

Proof. See [6]. For smooth α the last statements follow from (ii) because I can be rewritten as

$$4I = (t + r)^2\left(\alpha_t + \alpha_r + \frac{\alpha}{r}\right)^2 + (t - r)^2\left(\alpha_t - \alpha_r - \frac{\alpha}{r}\right)^2$$

$$+(t^2+r^2)\alpha^2\left(\frac{2}{r}-\alpha\right)^2-2r^{-2}[(t^2+r^2)r\alpha^2]_r,$$

where the last term drops out upon integration. Hence

$$\int_{r<\theta t}\left[\alpha_t^2+\left(\alpha_r+\frac{\alpha}{r}\right)^2\right]dx=0(t^{-2}).$$

But for $R\leqq\theta t$

$$\int_{r<R}\left(\alpha_r+\frac{\alpha}{r}\right)^2dx=\int_{r<R}\left[\alpha_r^2+\frac{1}{r^2}(r\alpha^2)_r\right]dx,$$

whence

$$\int_{r<R}\alpha_r^2dx+4\pi R\alpha^2(R,t)=0(t^{-2}). \tag{3}$$

By a *free solution* we mean a solution of the equation

$$L\beta\equiv\beta_{tt}-\beta_{rr}-\frac{2}{r}\beta_r+\frac{2}{r^2}\beta=0, \tag{4}$$

which is equation (1) with the nonlinear terms dropped. The main result of this section is

Theorem 2. *Given a solution α of (1) with $\int(1+r^2)e\,dx<\infty$. Then there exist a unique pair of free solutions α_- and α_+ such that*

$$\|\alpha-\alpha_\pm\|_*\to 0 \text{ as } t\to\pm\infty.$$

Lemma 1. *As $t\to\infty$ we have*

(a) $\sup_{r\leqq\theta t} r\alpha^2=0(t^{-2})$ *for* $\theta<1$,

(b) $\int\alpha^2dx=0(\ln t)$,

(c) $\int\frac{\alpha^2}{r^2}\,dx=0(t^{-2}\ln t)$,

(d) $\sup_{0\leqq r\leqq\infty} r\alpha^2=0(t^{-1}(\ln t)^{1/2})$.

Proof. (a) is immediate from (3). Next we rewrite I again as

$$I=\frac{1}{2}(t\alpha_t+r\alpha_r+\alpha)^2+\frac{1}{2}(r\alpha_t+t\alpha_r)^2+\frac{1}{4}(t^2+r^2)\alpha^2\left(\frac{2}{r}-\alpha\right)^2$$
$$+t\alpha_t\alpha-\frac{1}{2r^2}[r^3\alpha^2]_r.$$

In this expression the first three terms are non-negative and the last term integrates to zero. So by Theorem 1(b) $t\int\alpha_t\alpha dx<c$. This implies (b). By (b), $\int_{r>\theta t}\frac{\alpha^2}{r^2}dx=$

$0(t^{-2}\ln t)$. Next we write

$$\alpha^2 = (r\alpha^2)_r - 2r\alpha_r\alpha, \tag{*}$$

whence

$$\int_0^R \frac{\alpha^2}{r^2}dx = 4\pi r\alpha^2|_0^R - 2\int_{r<R}\alpha_r\frac{\alpha}{r}dx$$
$$\leqq 4\pi R\alpha^2(R,t) + 2\int_{r<R}\alpha_r^2\,dx + \frac{1}{2}\int_{r\leq R}\frac{\alpha^2}{r^2}dx.$$

The last term is moved to the left side. The other terms are $0(t^{-2})$ by (a) and (3), if $R \leqq \theta t$. This proves (c). Integrating (*) once again, we have

$$r\alpha^2 = \int\frac{\alpha^2}{r^2}dx + 2\int\alpha_r\frac{\alpha}{r}dx$$
$$\leqq \int\frac{\alpha^2}{r^2}dx + 2\left(\int\alpha_r^2\,dx\right)^{1/2}\left(\int\frac{\alpha^2}{r^2}dx\right)^{1/2}$$
$$= 0(t^{-2}\ln t + [t^{-2}\ln t]^{1/2})$$

which proves (d). Q.E.D

Lemma 2.

$$\int\left\{\int\left[\frac{3}{r}\alpha^2 - \alpha^3\right]^2 dx\right\}^{1/2} dt < \infty.$$

Proof. We write

$$\frac{3}{r}\alpha^2 - \alpha^3 = \frac{3}{2}\alpha^2\left(\frac{2}{r} - \alpha\right) + \frac{1}{2}\alpha^3,$$

and estimate the L^2 norm of each term on the right separately. First,

$$\int\alpha^6 dx \leqq \sup r^2\alpha^4\cdot\int\frac{\alpha^2}{r^2}dx$$
$$= 0(t^{-4}\ln^2 t),$$

by Lemma 1 (c) and (d). Second,

$$A^2(t) \equiv \int_{r<t/2}\alpha^4\left(\frac{2}{r} - \alpha\right)^2 dx$$
$$\leqq \int\alpha^2\left(\frac{2}{r} - \alpha\right)^2\frac{1}{r}dx\cdot\sup_{r<t/2} r\alpha^2,$$

so that by Lemma 1(a)

$$\int_1^\infty A(t)dt \leqq c\int_1^\infty\left[\int\alpha^2\left(\frac{2}{r} - \alpha\right)^2\frac{1}{r}dx\right]^{1/2}\frac{dt}{t} \leqq cB < \infty.$$

Finally,

$$\int_{r>t/2} \alpha^4\left(\frac{2}{r}-\alpha\right)^2 dx \leqq \int \alpha^2\left(\frac{2}{r}-\alpha\right)^2 dx \cdot \sup_{r>t/2} \frac{1}{r} \cdot \sup r\alpha^2$$
$$= 0(t^{-4}\ln^{1/2} t)$$

by Lemma 1(d) and Theorem 1(b). Q.E.D

Proof of Theorem 2. Let $R(t)$ denote the Riemann operator for the free equation (4). This means that the solution β of (4) with the initial data $\beta(r,0)\equiv 0, \beta_t(r,0)=\gamma(r)$ is written as $\beta(r,t)=[R(t)\gamma](r)$. If we multiply (4) by β_t, we get the energy identity

$$\int\left(\beta_t^2+\beta_r^2+\frac{2}{r^2}\beta^2\right)dx=\int \gamma^2 dx$$

for all t. Thus $R(t)$ is bounded from $L^2(\mathbb{R}^3)$ into $\tilde{H}_r^1(\mathbb{R}^3)$ and dR/dt from $L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. Let $\alpha(r,t)$ be the given solution of (1). Let $\alpha_0(r,t)$ be the free solution with the same Cauchy data at time $t=0$ as $\alpha(r,t)$ has. Then we have the integral form of (1)

$$\alpha(t)=\alpha_0(t)+\int_0^t R(t-s)\left[\frac{3}{r}\alpha^2(s)-\alpha^3(s)\right]ds.$$

Now we *define*

$$\alpha_+(t)=\alpha_0(t)+\int_0^\infty R(t-s)\left[\frac{3}{r}\alpha^2(s)-\alpha^3(s)\right]ds.$$

By Lemma 2 this integral converges in the norm of $\mathscr{H}$ and

$$\|\alpha_+(t)-\alpha(t)\|_* \leqq \int_t^\infty \left\|\frac{3}{r}\alpha^2(s)-\alpha^3(s)\right\|_{L^2} ds$$

tends to zero as $t\to\infty$. It is clear that, being a linear combination of free solutions, it is itself a free solution. Q.E.D.

2. The Representation and L^∞-Estimates

In Sect. 3 we will demonstrate the existence of the free-to-perturbed wave operators. For that purpose we need L^∞-estimates on the solution α of (1). Inverting the linear wave operator does not suffice for this purpose, because of the singularity in the term $\frac{2}{r^2}\alpha$. Therefore we will first find a fundamental solution for the operator L, given by (4).

Lemma 3. *A classical solution α of the problem*

$$L\alpha = F(r,t) \qquad \text{in } \{r>0, t>0\}$$
$$\alpha(r,0)=\alpha_t(r,0)=0, \tag{5}$$

is represented by

$$\alpha(r,t)=\frac{\omega_4}{3\omega_5 r^2}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau} K(\rho,r,t-\tau)F(\rho,\tau)d\rho d\tau, \tag{6}$$

where the kernel K is given by

$$K(\rho,r,t)=\rho^2+r^2-t^2, \tag{7}$$

and where ω_n *denotes the area of the unit sphere in* $\mathbb{R}^n$. *Thus, in the notation of the proof of Theorem 2,*

$$\alpha(t)=\int_0^t R(t-\tau)F(\tau)d\tau.$$

Proof. Introduce the change of variables

$$\alpha=rv \tag{8}$$

in (5). The Cauchy problem (5) then becomes

$$v_{tt}-\frac{4}{r}v_r-v_{rr}=\frac{1}{r}F(r,t)\equiv\tilde{F}(r,t). \tag{9}$$

We recognize (9) as the nonhomogeneous five-dimensional radial wave equation, whose solution, as is well-known (cf. [2]), can be represented by

$$v(r,t)=\frac{1}{3!}\int_0^t d\tau\frac{\partial^3}{\partial t^3}\int_0^{t-\tau}[(t-\tau)^2-\rho^2]\rho Q(x,\rho,\tau)d\rho, \tag{10}$$

where $Q(x,\rho,\tau)=(1/\omega_5)\int_{|\omega|=1}\tilde{F}(x+\rho\omega,\tau)d\omega$.

When we carry out explicitly the differentiation indicated in (10), we get the result

$$6v(r,t)=\int_0^t\left[6(t-\tau)Q(x,t-\tau,\tau)+2(t-\tau)^2\frac{\partial Q}{\partial\rho}(x,t-\tau,\tau)\right]d\tau. \tag{11}$$

Now we compute Q explicitly, using the fact that F (and hence $\tilde{F}$) is radial:

$$Q(x,\rho,\tau)=\frac{\omega_4}{\omega_5}\int_0^\pi\tilde{F}((|x|^2+\rho^2+2\rho|x|\cos\phi)^{1/2},\tau)\sin^3\phi d\phi. \tag{12}$$

Making the change of variable

$$\lambda=(|x|^2+\rho^2+2\rho|x|\cos\phi)^{1/2},$$

we see that (12) is the same as

$$Q(x,\rho,\tau)=\frac{\omega_4}{4\omega_5\rho^3|x|^3}\int_{|\rho-|x||}^{\rho+|x|}\lambda[\lambda^2-(\rho-|x|)^2][(\rho+|x|)^2-\lambda^2]\tilde{F}(\lambda,\tau)d\lambda. \tag{12$'$}$$

Next, we calculate $\partial Q/\partial\rho$ using (12)$'$; the result is

$$\frac{\partial Q}{\partial\rho}=\frac{-3}{\rho}Q+\frac{\omega_4}{\omega_5\rho^2|x|^3}\int_{|\rho-|x||}^{\rho+|x|}\lambda[\lambda^2-\rho^2+|x|^2]\tilde{F}(\lambda,\tau)d\lambda.$$

Substitution of this in the formula (11) for v gives

$$v(r,t)=\frac{\omega_4}{3\omega_5|x|^3}\int_0^t \int_{||x|-t+\tau|}^{|x|+t-\tau} \lambda K(\lambda,|x|,t-\tau)\tilde{F}(\lambda,\tau)d\lambda d\tau.$$

Now $r\equiv|x|$. When we replace v by α/r here, and recall that $\tilde{F}(r,t)=(1/r)F(r,t)$, we obtain the statement of the lemma.

Corollary. *The solution α_0 of the linear problem*

$$L\alpha_0=0;\ \alpha_0(r,0)=f(r),\ \partial_t\alpha_0(r,0)=g(r) \tag{13}$$

is represented by

$$\alpha_0(r,t)=\frac{\omega_4}{3\omega_5 r^2}\int_{|r-t|}^{r+t} K(\rho,r,t)g(\rho)d\rho+\frac{\omega_4}{3\omega_5 r^2}\frac{\partial}{\partial t}\int_{|r-t|}^{r+t} K(\rho,r,t)f(\rho)d\rho. \tag{14}$$

We now make the following standing hypotheses on the Cauchy data f,g in (13):

$$(1+r^2)g(r)\in L^2(\mathbb{R}^1_+),\ r^2g\in L^\infty(\mathbb{R}^1_+);$$
$$rf(r)\in L^\infty(\mathbb{R}^1_+),\ r^2f'\in L^\infty(\mathbb{R}^1_+),\ rf'\in L^1(\mathbb{R}^1_+)\cap L^2(\mathbb{R}^1_+). \tag{H$_0$}$$

In making the L^∞-estimates, we will need the following result, which is simply a calculus computation.

Lemma 4.

$$\int_{t-r}^{t+r}\rho^{-2}K^2(\rho,r,t)d\rho=\frac{8}{3}r^3.$$

Define

$$I_g=r^{-3/2}\int_{|r-t|}^{r+t} K(\rho,r,t)g(\rho)d\rho, \tag{15}$$

$$I_f=r^{-3/2}\frac{\partial}{\partial t}\int_{|r-t|}^{r+t} K(\rho,r,t)f(\rho)d\rho. \tag{16}$$

Thus $\alpha_0=(c/\sqrt{r})(I_g+I_f)$.

Lemma 5. (Estimates on I_g). Assume (H_0). Then

(a) $|I_g|\leqq ct^{-1/2}\left(\int_{|r-t|}^{r+t}\rho^3g^2(\rho)d\rho\right)^{1/2}$ for all r,t

(b) $|I_g|\leqq c\left(\int_{|t-r|}^{t+r}\rho^2g^2d\rho\right)^{1/2}$ for all r,t

(c) $|I_g|\leqq ct^{-1}\left(\int_{t-r}^{t+r}\rho^4g^2d\rho\right)^{1/2}$ for $r\leqq\frac{t}{2}$

(d) $|I_g|\leqq ct^{-1}r^{1/2}\|\rho^2g\|_{L^\infty}$ for $r\leqq\frac{t}{2}$

Proof. Call $a=|r-t|, b=r+t$. We may assume that $t>0$. We first establish (a) above in the case $r>t/2$. For, splitting the kernel, we clearly have

$$
\begin{aligned}
|I_g| &\leqq r^{-3/2}\int_a^b \rho^2|g|\,d\rho + r^{-3/2}|r^2-t^2|\int_a^b |g|\,d\rho \\
&\leqq r^{-3/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2}\left(\int_a^b \rho\, d\rho\right)^{1/2} \\
&\quad + r^{-3/2}|r^2-t^2|\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2}\left(\int_a^b \rho^{-3} d\rho\right)^{1/2} \\
&\leqq c r^{-3/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2}\left[r^{1/2}t^{1/2} + |r^2-t^2|\cdot\frac{r^{1/2}t^{1/2}}{|r^2-t^2|}\right] \\
&\leqq c t^{1/2} r^{-1}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2} \leqq c t^{-1/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2},
\end{aligned}
$$

since $r>t/2$ by assumption. In order to prove (a) in the case $r<t/2$, write

$$
\begin{aligned}
|I_g| &\leqq r^{-3/2}\int_a^b |\rho^{-3/2}K|\rho^{3/2}|g|\,d\rho \\
&\leqq r^{-3/2}\left(\int_a^b \rho^{-3}K^2 d\rho\right)^{1/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2}
\end{aligned}
$$

But for $r<t/2$, $a=|t-r|=t-r\geqq t/2$. Thus

$$
\begin{aligned}
|I_g| &\leqq c r^{-3/2} t^{-1/2}\left(\int_a^b \rho^{-2}K^2 d\rho\right)^{1/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2} \\
&\leqq c t^{-1/2}\left(\int_a^b \rho^3 g^2 d\rho\right)^{1/2},
\end{aligned}
$$

in view of Lemma 4. This proves (a).

To establish (b) we write

$$
\begin{aligned}
|I_g| &= \left| r^{-3/2}\int_a^b \left(\frac{K}{\rho}\right)\rho g\, d\rho\right| \\
&\leqq r^{-3/2}\left(\int_a^b \rho^{-2}K^2 d\rho\right)^{1/2}\left(\int_a^b \rho^2 g^2 d\rho\right)^{1/2} \\
&\leqq c\left(\int_a^b \rho^2 g^2 d\rho\right)^{1/2}
\end{aligned}
$$

by Lemma 4. This proves (b).

Note that (a) implies (c) trivially, since $r\leqq t/2$ is assumed in (c). Clearly (c) implies (d), which concludes the proof of Lemma 5.

Lemma 5 contains the basic estimates which we need to solve the nonlinear problem. However, we digress to estimate the integral I_f, appearing in (16), to complete our study of the free equation.

Lemma 6. (*Estimates on* I_f).

(a) $\sup_{r\geqq 0}|I_f|\leqq c(1+t)^{-1/2}$ *for all* $t\geqq 0$

(b) $\sup_{r\leqq 1} r^{-1/2}|I_f|\leqq c(1+t)^{-1}$ *for all* $t\geqq 0$.

Proof. We compute I_f explicitly, with the result

$$I_f=2r^{-1/2}[(r+t)f(r+t)+|r-t|f(|r-t|)] - 2tr^{-3/2}\int_{|t-r|}^{t+r} f(\lambda)d\lambda. \tag{17}$$

The last integral is, after integration by parts,

$$\int_{|t-r|}^{t+r} f(\lambda)d\lambda=(t+r)f(t+r)-|t-r|f(|t-r|)-\int_{|t-r|}^{t+r}\lambda f'(\lambda)d\lambda.$$

Using this in (17), we obtain

$$I_f=2r^{-3/2}[(r^2-t^2)f(t+r)+|r^2-t^2|f(|t-r|)] + 2tr^{-3/2}\int_{|t-r|}^{t+r}\lambda f'(\lambda)d\lambda. \tag{18}$$

In view of (H_0), it is clear that $I_f=0(t^{-1/2})$ for large t, provided $r>t/2$. On the set $r<t/2$, we have from (18)

$$I_f=2r^{-3/2}(t^2-r^2)[f(t-r)-f(t+r)]+2tr^{-3/2}\int_{t-r}^{t+r}\lambda f'(\lambda)d\lambda.$$

The first term here is equal to

$$-2(t^2-r^2)r^{-3/2}\int_{t-r}^{t+r} f'(\lambda)d\lambda.$$

Hence I_f can be written as

$$I_f=2r^{-3/2}\int_{t-r}^{t+r}[t\lambda+r^2-t^2]f'(\lambda)d\lambda,$$

and so

$$\begin{aligned}|I_f|&\leqq 2r^{1/2}\int_{t-r}^{t+r}\frac{1}{\lambda}\cdot\lambda|f'|d\lambda+2tr^{-3/2}\int_{t-r}^{t+r}\frac{|\lambda-t|}{\lambda^2}\cdot\lambda^2|f'|d\lambda\\ &\leqq 2r^{1/2}(t-r)^{-1}\|\lambda f'\|_{L^1(\mathbb{R}^1)}+2tr^{-3/2}\cdot\frac{r}{(t-r)^2}\int_{t-r}^{t+r}\lambda^2|f'|d\lambda\\ &\leqq cr^{1/2}t^{-1}\|\lambda f'\|_{L^1}+ct^{-1}r^{-1/2}\cdot\|\lambda^2 f'\|_{L^\infty}\cdot 2r\\ &\leqq cr^{1/2}t^{-1}.\end{aligned}$$

This estimate completes the proof of (a) on the set $r < t/2$, and establishes (b) simultaneously.

Corollary. *Let α_0 be a solution to the linear Cauchy problem (13) whose data satisfy (H_0). Then we have the following estimates:*

$$\text{(a)} \quad \sup_{r \geqq 0} r^{1/2} |\alpha_0(r,t)| = 0(t^{-1/2}) \qquad \text{as } t \to \infty.$$

$$\text{(b)} \quad \sup_{r \leqq 1} |\alpha_0(r,t)| = 0(t^{-1}) \qquad \text{as } t \to \infty.$$

Proof. The first estimate is a consequence of Lemma (5a) and Lemma (6a). The second conclusion follows immediately from Lemma (5d) and (6b).

We now turn to the L^∞-estimates for the nonlinear problem. Let $\alpha_-(r,t)$ be a free solution ($L\alpha_- = 0$) with data given at time $t = -\infty$ which satisfy (H_0). In Sect. 3 (to follow) we will convert (1) to integral form by inverting L, and will show that there exists a solution $\alpha(r,t)$ to the integral equation

$$\alpha = \alpha_- + \frac{\omega_4}{3\omega_5 r^2} \int_{-\infty}^{t} \int_{a}^{b} K(\rho, r, t-\tau) \left(\frac{3\alpha^2}{\rho} - \alpha^3 \right) d\rho\, d\tau, \tag{19}$$

where $a = |r - t + \tau|$, $b = r + t - \tau$, and K is given by (7). We denote by $\mathscr{R}\alpha$ the operator mapping α into the right-hand side of (19). Let $T < 0$. We will show that $\mathscr{R}\alpha$ is a contraction for sufficiently large $|T|$. The resulting solution α will be asymptotic to the given α_- in the energy norm $\|\cdot\|_*$ as $t \to -\infty$. For our present purposes we define a norm by

$$\|\alpha\|_0 = \sup_{-\infty < t \leqq T} \left\{ |t|^{1/2} \|r^{1/2}\alpha(r,t)\|_{L^\infty(\mathbb{R}^3)} + \frac{|t|}{|\ln^{1/2}|t||} \left\| \frac{\alpha(r,t)}{r} \right\|_{L^2(\mathbb{R}^3)} \right\} \tag{20}$$

Lemma 7. (A priori global L^∞-estimate). *If $\|\alpha\|_0 < \infty$ on an interval $(-\infty, T]$, then the following estimate holds*:

$$\|r^{1/2}(\mathscr{R}\alpha - \alpha_-)\|_\infty < c(\|\alpha\|_0^2 + \|\alpha\|_0^3) |\ln^{1/2}|t|| \, |t|^{-1}$$

for all $t \in (-\infty, T)$.

Proof. From (19) we clearly have

$$|r^{1/2}(\mathscr{R}\alpha - \alpha_-)| \leqq c \int_{-\infty}^{t} (|I'| + |I''|) d\tau,$$

where

$$I' = r^{-3/2} \int_a^b K \cdot \frac{\alpha^2}{\rho} d\rho,$$

$$I'' = r^{-3/2} \int_a^b K \cdot \alpha^3 d\rho.$$

Note that these integrals are of precisely the same form as those treated in Lemma 5 (i.e. I_g there). Hence, by Lemma (5a),

$$\begin{aligned} |I'| &\leqq c(t-\tau)^{-1/2} \left(\int_a^b \rho^3 \cdot \frac{\alpha^4}{\rho^2} d\rho \right)^{1/2} \\ &\leqq c(t-\tau)^{-1/2} \| r^{1/2} \alpha(\tau) \|_\infty \left(\int_a^b \alpha^2 d\rho \right)^{1/2} \\ &\leqq c(t-\tau)^{-1/2} |\ln^{1/2} |\tau|| \cdot |\tau|^{-3/2} \| \alpha \|_0^2 . \end{aligned}$$

It follows that

$$\begin{aligned} \int_{-\infty}^{t} |I'| d\tau &\leqq c \| \alpha \|_0^2 \int_{-\infty}^{t} (t-\tau)^{-1/2} |\ln^{1/2} |\tau|| \cdot |\tau|^{-3/2} d\tau \\ &\leqq c \| \alpha \|_0^2 |t|^{-1} |\ln^{1/2} |t||, \end{aligned}$$

which is the required estimate for this term. To treat I'', apply Lemma (5b):

$$\begin{aligned} |I''| \leqq c \left(\int_a^b \rho^2 \alpha^6 d\rho \right)^{1/2} &\leqq c \| r^{1/2} \alpha(\tau) \|_\infty^2 \left(\int_a^b \alpha^2 d\rho \right)^{1/2} \\ &\leqq c \| \alpha \|_0^3 \tau^{-2} |\ln^{1/2} |\tau||. \end{aligned}$$

Thus

$$\begin{aligned} \int_{-\infty}^{t} |I''| d\tau &\leqq c \| \alpha \|_0^3 \int_{-\infty}^{t} \frac{|\ln^{1/2} |\tau|| d\tau}{\tau^2} \\ &\leqq c \| \alpha \|_0^3 \frac{|\ln^{1/2} |t||}{|t|}, \end{aligned}$$

as desired.

Lemma 8. (A priori local L^∞-estimates). *Let* $\| \alpha \|_0 < \infty$ *on an interval* $(-\infty, T]$. *Then the following estimates hold:*

(a) $r^{1/2} |\mathscr{R}\alpha - \alpha_-| \leqq c r^{1/4} (\|\alpha\|_0^2 + \|\alpha\|_0^3) |\ln^{1/4} |t|| |t|^{-1}$ *for* $r \leqq 1, t \in (-\infty, T]$.

(b) *For* $r \leqq 1$, $t \in (-\infty, T]$, $|\mathscr{R}\alpha - \alpha_-| \leqq c(\|\alpha\|_0^2 + \|\alpha\|_0^3) |t|^{-1} \times$
$\left[1 + \left| \ln \left(\frac{2r}{2r + |t|} \right) \right| \right]$.

Proof. We split up the integral operator (19) as

$$r^{1/2}(\mathscr{R}\alpha-\alpha_-)=\frac{c}{r^{3/2}}\left[\int_{-\infty}^{t-2r}+\int_{t-2r}^{t}\right]\int_a^b K\left(\frac{3\alpha^2}{\rho}-\alpha^3\right)d\rho d\tau$$

$$\equiv E_1+E_2.$$

Consider E_2. By applying Lemma (5a) as above, we find

$$|E_2|\leqq c\int_{t-2r}^{t}(t-\tau)^{-1/2}\left[\left(\int_a^b\rho^3\cdot\frac{\alpha^4}{\rho^2}d\rho\right)^{1/2}+\left(\int_a^b\rho^3\alpha^6 d\rho\right)^{1/2}\right]d\tau$$

$$\leqq c\int_{t-2r}^{t}(t-\tau)^{-1/2}\left[|\ln^{1/2}|\tau||\ \|\alpha\|_0^2|\tau|^{-3/2}+\|r^{1/2}\alpha(\tau)\|_\infty^3\left(\int_a^b d\rho\right)^{1/2}\right]d\tau.$$

Here we have used on the first term the estimate for I' in Lemma 7. Therefore

$$|E_2|\leqq c\|\alpha\|_0^2|\ln^{1/2}|t||\ |t|^{-3/2}r^{1/2}$$

$$+c\|\alpha\|_0^3 r^{1/2}\int_{t-2r}^{t}|\tau|^{-3/2}(t-\tau)^{-1/2}d\tau,$$

since $r\leqq 1$. Hence we find

$$|E_2|\leqq c(\|\alpha\|_0^2+\|\alpha\|_0^3)r^{1/2}|\ln^{1/2}|t||\ |t|^{-3/2}\tag{21}$$

for $r\leqq 1$.

To handle E_1, we apply Lemma (5c):

$$|E_1|\leqq c\int_{-\infty}^{t-2r}(t-\tau)^{-1}\left[\left(\int_a^b\rho^4\cdot\frac{\alpha^4}{\rho^2}d\rho\right)^{1/2}+\left(\int_a^b\rho^4\cdot\alpha^6 d\rho\right)^{1/2}\right]d\tau.\tag{22}$$

The second term here can be bounded as

$$\left(\int_a^b\rho^4\alpha^6 d\rho\right)^{1/2}\leqq\|r^{1/2}\alpha(\tau)\|_\infty^3\left(\int_a^b\rho d\rho\right)^{1/2}$$

$$\leqq c\|\alpha\|_0^3|\tau|^{-3/2}r^{1/2}(t-\tau)^{1/2}.\tag{23}$$

The first term we bound as

$$\left(\int_a^b\rho^2\alpha^4 d\rho\right)^{1/2}\leqq\|r^{1/2}\alpha(\tau)\|_\infty^{3/2}\left(\int_a^b\alpha^2 d\rho\right)^{1/4}\left(\int_a^b\rho d\rho\right)^{1/4}$$

$$\leqq c\|\alpha\|_0^2|\ln^{1/4}|\tau||\,|\tau|^{-5/4}r^{1/4}(t-\tau)^{1/4}.\tag{24}$$

Using (23), (24) in the estimate (22) for E_1, we find

$$|E_1|\leqq c\int_{-\infty}^{t-2r}(t-\tau)^{-1}[\|\alpha\|_0^2|\ln^{1/4}|\tau||\,|\tau|^{-5/4}r^{1/4}(t-\tau)^{1/4}$$

$$+\|\alpha\|_0^3|\tau|^{-3/2}r^{1/2}(t-\tau)^{1/2}]d\tau$$

$$\leqq cr^{1/4}(\|\alpha\|_0^2+\|\alpha\|_0^3)|\ln^{1/4}|t||\,|t|^{-1},$$

since $r \leqq 1$. This and the estimate (21) for E_2 establish part (a) of Lemma 8.

It remains to prove part (b). Notice that the estimate (21) for E_2 is stronger than that claimed in part (b) of Lemma 8. Therefore we need only estimate E_1. Since the second integral in (22) is dominated by $c\|\alpha\|_0^3 r^{1/2}|t|^{-1}$, as follows from (23), we can write

$$|E_1| \leqq c(\|\alpha\|_0^2 + \|\alpha\|_0^3)r^{1/2}|t|^{-1} + c\int_{-\infty}^{t-2r}(t-\tau)^{-1}\left(\int_a^b \rho^2\alpha^4 d\rho\right)^{1/2} d\tau.$$

The last integral is less than

$$\int_{-\infty}^{t-2r}(t-\tau)^{-1}\|r^{1/2}\alpha(\tau)\|_\infty^2\left(\int_a^b d\rho\right)^{1/2} d\tau \leqq c\|\alpha\|_0^2 r^{1/2}\int_{-\infty}^{t-2r}\frac{d\tau}{|\tau|(t-\tau)}$$
$$\leqq c\|\alpha\|_0^2 r^{1/2}|t|^{-1}\left|\ln\left(\frac{2r}{2r+|t|}\right)\right|,$$

and this establishes part (b).

3. The Free-to-Perturbed Wave Operators

Let $\alpha_-(r,t)$ be a free solution ($L\alpha_- = 0$) with data given at time $t = -\infty$ which satisfy (H_0). Let $-\infty < T < 0$, and let

$$a = |r - t + \tau|, b = r + t - \tau,$$
$$K(\rho, r, t) = \rho^2 + r^2 - t^2.$$

We wish to show that there exists a unique solution $\alpha(r,t)$ of the nonlinear equation

$$\alpha_{tt} - \frac{2}{r}\alpha_r - \alpha_{rr} + \frac{2}{r^2}\alpha - \frac{3\alpha^2}{r} + \alpha^3 = 0, \qquad (1)$$

such that

$$\|\alpha(t) - \alpha_-(t)\|_* \to 0 \text{ as } t \to -\infty.$$

Here the *energy norm* $\|\cdot\|_*$ is given by (2).

The existence of such an α will be achieved by first showing that the integral equation,

$$\alpha(t) = \alpha_-(t) + \frac{\omega_4}{3\omega_5 r^2}\int_{-\infty}^{t}\int_a^b K(\rho, r, t-\tau)\left(\frac{3\alpha^2}{\rho} - \alpha^3\right)d\rho\, d\tau \equiv \mathscr{R}\alpha, \qquad (19)$$

has a unique solution on the interval $-\infty < t \leqq T$, provided $|T|$ is sufficiently large. This in turn follows from our showing below that the operator $\mathscr{R}$ on the right-hand side of (19) is a contraction for large enough $|T|$. The solution so obtained will be shown to exist for all times $t \in \mathbb{R}$, as follows from positivity of the energy.

Define a norm on functions $\alpha(r,t)$ $(0 \leqq r < \infty, -\infty < t \leqq T)$ by

$$|||\alpha||| = \sup_{-\infty < t \leqq T}\left\{\sup_r |\sqrt{r|t|}\,|\alpha| + \sup_{r\leqq 1}\left|\frac{t}{\ln t}\frac{\alpha}{\ln\left(\frac{2r}{2r+|t|}\right)}\right|\right.$$

$$+\frac{|t|}{\sqrt{\ln|t|}}\left\|\frac{\alpha}{r}\right\|_2+\|\alpha\|_*+\|\alpha\|_2+|t|^{2/3}\|r^{-1/3}\alpha\|_3\Big\}. \qquad (25)$$

All spatial norms here are to be taken over $\mathbb{R}^3$. It follows from the corollary to Lemma 6 and from estimates to be made below that $|||\alpha|||<\infty$ for a free solution with data satisfying (H_0).

Theorem 3. *Let $\alpha_-(r,t)$ be a solution of the free equation $L\alpha_-=0$, whose data satisfy (H_0). Then there exists a unique solution α of the perturbed equation*

$$L\alpha-\frac{3\alpha^2}{r}+\alpha^3=0,$$

such that $\|\alpha(t)-\alpha_-(t)\|_*\to 0$ as $t\to-\infty$. Moreover, this perturbed solution enjoys the following properties:

(i) $\alpha\in C^0(\mathbb{R},\tilde{H}_r^1)$; $\alpha_t\in C^0(\mathbb{R},L_r^2)$,

(ii) *the total energy is conserved:*

$$\int_{\mathbb{R}^3}\left[\tfrac{1}{2}\alpha_t^2+\tfrac{1}{2}|\nabla\alpha|^2+\tfrac{1}{4}\alpha^2\left(\alpha-\frac{2}{r}\right)^2\right]dx=\tfrac{1}{2}\|\alpha_-\|_*^2=\text{constant},$$

(iii) $|||\alpha|||<\infty$.

Proof. Consider the integral equation (19) on the interval $(-\infty,T)$. We will prove that there exists a solution α of (19) for $|T|$ large, and such a solution is certainly a weak solution of (1). Notice that $\|\alpha\|_0\leqq|||\alpha|||$ and that the L^∞-estimates for $\mathscr{R}\alpha-\alpha_-$ in the first two terms of $|||\cdot|||$ (cf. (25)) have already been given in Lemmas 7 and 8b).

Lemma 9. *Let $|||\alpha|||<\infty$ on an interval $(-\infty,T]$. Then there exists a constant c such that*

$$\sup_{-\infty<t\leqq T}\|\mathscr{R}\alpha(t)-\alpha_-(t)\|_*\leqq c|T|^{-1/2}|\ln^{1/2}|T||(|||\alpha|||^2+|||\alpha|||^3). \qquad (26)$$

Proof. First we note that, given a solution u of $Lu=f$, where $f\in L^1_{\text{loc}}(\mathbb{R},L_r^2)$, with zero Cauchy data, we have

$$\|u(t)\|_*\leqq c\int_0^t\|f(\tau)\|_2\,d\tau, \qquad (27)$$

as follows from the proof of Theorem 2.

Hence, by applying (27) to (19), we get

$$\|\mathscr{R}\alpha(t)-\alpha_-(t)\|_*\leqq c\int_{-\infty}^t\left\|\left(\frac{3\alpha^2}{r}-\alpha^3\right)\right\|_2d\tau. \qquad (28)$$

Now $\|\alpha^3\|_2\leqq\|r^{1/2}\alpha\|_\infty^2\|\alpha/r\|_2\leqq c|||\alpha|||^3|\tau|^{-2}|\ln^{1/2}|\tau||$, and

$$\left\|\frac{\alpha^2}{r}\right\|_{L^2(r>1)}\leqq\|r^{1/2}\alpha\|_\infty\left\|\frac{\alpha}{r}\right\|_2\sup_{r\geqq 1}r^{-1/2}\leqq c|||\alpha|||^2|\ln^{1/2}|\tau|||\tau|^{-3/2}.$$

The estimate for this term near the origin is made by using Lemma 8b):

$$\left\|\frac{\alpha^2}{r}\right\|_{L^2(r\leqq 1)} = c\left(\int_0^1 \alpha^4\,dr\right)^{1/2}$$

$$= c\left(\int_0^1 \frac{\tau^4\alpha^4}{\ln^4|\tau|\ln^4\left(\frac{2r}{2r+|\tau|}\right)}\cdot\frac{\ln^4|\tau|\cdot\ln^4\left(\frac{2r}{2r+|\tau|}\right)dr}{\tau^4}\right)^{1/2}$$

$$\leqq c|||\alpha|||^2\frac{\ln^2|\tau|}{\tau^2}\left(\int_0^1 \ln^4\left(\frac{2r}{2r+|\tau|}\right)dr\right)^{1/2} \leqq |||\alpha|||^2\ln^4|\tau|\cdot\tau^{-2}$$

Using these estimates in (28) and integrating in τ, we obtain (26).

To complete the estimation of $|||\mathscr{R}\alpha-\alpha_-|||$, we need the following L^p-estimates:

Lemma 10. *Let β be a solution of $L\beta=f$ in $\{t>0\}$ having zero Cauchy data. Then*

(i) $\|r^{-1/3}\beta\|_{L^3(\mathbb{R}^3)} \leqq c\int_0^t (t-\tau)^{-2/3}\|r^{1/3}f(\tau)\|_{L^{3/2}(\mathbb{R}^3)}d\tau;$

(ii) $\|\beta\|_{L^2(\mathbb{R}^3)} \leqq c\int_0^t \|r^{2/5}f(\tau)\|_{L^{10/7}(\mathbb{R}^3)}d\tau.$

Proof. As in Sect. 2, Eq. (8), introduce the change of variables $\beta=rv$ into the equation $L\beta=f$. The equation for v becomes

$$v_{tt}-\frac{4}{r}v_r-v_{rr}=r^{-1}f,$$

and the operator on the left-hand side here is again the five-dimensional wave operator. Then as a special case of the estimates proved in [7], we conclude

$$\|v(t)\|_{L^3(\mathbb{R}^5)} \leqq c\int_0^t (t-\tau)^{-2/3}\|r^{-1}f(\tau)\|_{L^{3/2}(\mathbb{R}^5)}d\tau$$

and

$$\|v(t)\|_{L^2(\mathbb{R}^5)} \leqq c\int_0^t \|r^{-1}f(\tau)\|_{L^{10/7}(\mathbb{R}^5)}d\tau.$$

Since $\beta\equiv rv$, we then have

$$\|v(t)\|_{L^3(\mathbb{R}^5)} = \|r^{-1/3}\beta(t)\|_{L^3(\mathbb{R}^3)},$$

and

$$\|v(t)\|_{L^2(\mathbb{R}^5)} = \|\beta(t)\|_{L^2(\mathbb{R}^3)},$$

and these identifications complete the proof.

Lemma 11. *Let $|||\alpha|||<\infty$ on an interval $(-\infty,T]$. Then there exists a constant c such* *that*

$$\| r^{-1/3}(\mathscr{R}\alpha - \alpha_-)(t) \|_3 \leqq c(|||\alpha|||^2 + |||\alpha|||^3)|t|^{-1} \quad \text{for } -\infty < t \leqq T. \tag{29}$$

Proof. Applying i) of Lemma 10 to (19), we find

$$\| r^{-1/3}(\mathscr{R}\alpha - \alpha_-)(t) \|_3 \leqq c \int_{-\infty}^{t} (t-\tau)^{-2/3} \left\| \left(\frac{-3\alpha^2}{r} + \alpha^3 \right) r^{1/3}(\tau) \right\|_{3/2} d\tau.$$

Now $\| r^{1/3}\alpha^2/r \|_{3/2} = \| r^{-1/3}\alpha \|_3^2 \leqq c|\tau|^{-4/3} |||\alpha|||$, and

$$\begin{aligned} \| r^{1/3}\alpha^3 \|_{3/2} &\leqq \| r^{1/2}\alpha \|_\infty^{4/3} \| \alpha \|_2^{2/3} \| r^{-1/3}\alpha \|_3 \\ &\leqq c|||\alpha|||^3 |\tau|^{-4/3}. \end{aligned}$$

Putting these estimates together, we obtain

$$\| r^{-1/3}(\mathscr{R}\alpha - \alpha_-)(t) \|_3 \leqq c(|||\alpha|||^2 + |||\alpha|||^3) \int_{-\infty}^{t} (t-\tau)^{-2/3} |\tau|^{-4/3} d\tau,$$

and this proves the Lemma.

Lemma 12. *Let $|||\alpha||| < \infty$ on an interval $(-\infty, T]$. Then there exists a constant c such that*

$$\| (\mathscr{R}\alpha - \alpha_-)(t) \|_2 \leqq c(|||\alpha|||^2 + |||\alpha|||^3)|t|^{-1/5} \quad \text{for } -\infty < t \leqq T. \tag{30}$$

Proof. Applying ii) of Lemma 10 to (19), we find

$$\| (\mathscr{R}\alpha - \alpha_-)(t) \|_2 \leqq c \int_{-\infty}^{t} \left\| r^{2/5} \left(\frac{-3\alpha^2}{r} + \alpha^3 \right)(\tau) \right\|_{10/7} d\tau. \tag{31}$$

Now

$$\left\| r^{2/5} \cdot \frac{\alpha^2}{r} \right\|_{10/7} \leqq \| \alpha \|_2^{1/5} \| r^{-1/3}\alpha \|_3^{9/5} \leqq c|||\alpha|||^2 |\tau|^{-6/5},$$

and

$$\begin{aligned} \| r^{2/5}\alpha^3 \|_{10/7} &\leqq \| r^{1/2}\alpha \|_\infty^{4/3} \| \alpha \|_2^{13/15} \| r^{-1/3}\alpha \|_3^{4/5} \\ &\leqq c|||\alpha|||^3 |\tau|^{-6/5}. \end{aligned}$$

Putting these estimates into (31), we obtain the result of the lemma.

There is only one more term in the norm $|||\mathscr{R}\alpha - \alpha_-|||$ to be estimated. This is the content of

Lemma 13. *Let $|||\alpha||| < \infty$ on an interval $(-\infty, T]$. Then there exists a constant c such that*

$$\| r^{-1}(\mathscr{R}\alpha - \alpha_-)(t) \|_{L^2(\mathbb{R}^3)} \leqq c|||\alpha|||^2 + |||\alpha|||^3)|\ln^{1/4}|t|| |t|^{-1} \tag{32}$$

for $t \in (-\infty, T]$.

Proof. This result follows from already derived bounds. First, let

$$J_1 = \int_{r \geqq 1} r^{-2} |\mathscr{R}\alpha - \alpha_-|^2 \, dx.$$

Then

$$J_1 \leqq \| r^{-1/3}(\mathcal{R}\alpha - \alpha_-)(t)\|_3^2 \left(\int_{r \geqq 1} r^{-4} dx \right)^{1/3}$$
$$\leqq c(|||\alpha|||^2 + |||\alpha|||^3)^2 t^{-2},$$

where we have used (29) from Lemma 11. Hence

$$\| r^{-1}(\mathcal{R}\alpha - \alpha_-)(t)\|_{L^2(r \geqq 1)} \leqq c(|||\alpha|||^2 + |||\alpha|||^3)|t|^{-1}, \tag{33}$$

and this is stronger than the desired estimate (32). For the estimate on the complement $\{r < 1\}$, we have

$$|J_2| \equiv \int_{r<1} \frac{(\mathcal{R}\alpha - \alpha_-)^2}{r^2} d^3x = \int_{r<1} \frac{r|\mathcal{R}\alpha - \alpha_-|^2}{r^3} d^3x.$$

Now bound $r|\mathcal{R}\alpha - \alpha_-|^2$ using a) of Lemma 8:

$$|J_2| \leqq c \int_{r<1} r^{-3} \cdot r^{1/2}(|||\alpha|||^2 + |||\alpha|||^3)^2 |\ln^{1/2}|t|| t^{-2} d^3x$$
$$\leqq c(|||\alpha|||^2 + |||\alpha|||^3)^2 |\ln^{1/2}|t|| t^{-2} \int_0^1 r^{-1/2} dr.$$

Therefore

$$\| r^{-1}(\mathcal{R}\alpha - \alpha_-)(t)\|_{L^2(r \leqq 1)} \leqq c(|||\alpha|||^2 + |||\alpha|||^3) \ln^{1/4}|t|| |t|^{-1}.$$

This establishes Lemma 13.

If we now examine the results of Lemmas 7–13, we conclude the following:

Lemma 14. *Let* $|||\alpha||| < \infty$ *on an interval* $(-\infty, T]$. *Then there exists a constant* c *such that*

$$|||\mathcal{R}\alpha - \alpha_-||| \leqq c|\ln^{-1/4}|T||(|||\alpha|||^2 + |||\alpha|||^3). \tag{34}$$

It is now clear from the methods of Lemmas 9–13 and the corollary to Lemma 6 that $|||\alpha_-||| < \infty$ for a free solution α_- with data satisfying (H_0). Inequality (34) shows that the operator $\alpha \to \mathcal{R}\alpha$ given by (19) is a contraction for $|T|$ sufficiently large. Thus there exists a time T and a unique solution $\alpha = \alpha(r, t)$ of the integral equation (19) for $-\infty < t < T < 0$. By construction and the definition (25) of the norm $|||\cdot|||$ we see that

$$\alpha \in C((-\infty, T); \tilde{H}_r^1), \text{ and}$$
$$\alpha_t \in C((-\infty, T); L_r^2).$$

This solution α can be continued for all times $-\infty < t < \infty$. This follows exactly as in [6] (from the positive definite nature of the energy density e). By (28), we have, since $\mathcal{R}\alpha = \alpha$,

$$\|(\alpha - \alpha_-)(t)\|_* \leqq c \int_{-\infty}^{t} \left\| \left(\frac{3\alpha^2}{r} - \alpha^3 \right)(\tau) \right\|_2 d\tau, \tag{35}$$

and the result of Lemma 9 shows that this integral tends to zero as $t \to -\infty$.

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