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Quantization and the Uniqueness of Invariant Structures

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Abstract. We determine and classify certain algebraic structures, defined on the space of all complex-valued polynomials in $2n$ real variables, which admit *affine* contact transformations as automorphisms. These are the structures which have the minimum symmetry necessary to define the basic linear and angular momentum observables of classical and quantum mechanics. The results relate to the so-called Dirac problem of finding an appropriate mathematical characterization of the canonical quantization procedure.

Introduction

Consider the space P of complex-valued polynomials in two real variables. The Poisson bracket operation

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

makes P into a complex Lie algebra. The *Dirac problem* asks if it is possible to derive from first principles a mapping θ from P to the algebra D of differential operators [with polynomial coefficients, acting on $L^2(\mathbb{R})$] which produces the correct spin zero quantization of classical mechanical systems [2, 3]. For definiteness, we take this to mean that θ should transform each function on $\mathbb{R}^2$ in the form of a Hamiltonian of a classical system

$$h(x,y) = \tfrac{1}{2}y^2 + V(x),$$

where V is a polynomial, into the Schrödinger operator on $L^2(\mathbb{R})$,

$$H = -\tfrac{1}{2}\Delta + V.$$

Here θ should certainly be a linear mapping, which hopefully would transform Poisson brackets into commutator brackets in the following sense:

$$\theta(f)\theta(g) - \theta(g)\theta(f) = \sqrt{-1}\,\theta(\{f,g\}). \tag{0.1}$$

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Assuming that θ is surjective, it is not hard to see from (0.1) alone that θ is injective, and thus the Lie algebras P and D would be isomorphic. This is known to be false [13, 6], or see Sects. 3 and 4 below.

Let q, p be the canonical coordinate functions: $q(x,y)=x$, $p(x,y)=y$. Another property one might hope to find is

$$\theta(q)=Q, \qquad \theta(p)=P, \tag{0.2}$$

where Q and P are, respectively, multiplication by x and $(1/\sqrt{-1})(d/dx)$. Again, (0.1) and (0.2) lead to a contradiction, and in fact Chernoff [2] and Joseph [6] have shown this to be the state of affairs even if one allows Q and P to act with finite multiplicity on vector-valued functions in $L^2(\mathbb{R})$. If one allows infinite multiplicity for P and Q, then it is possible to satisfy (0.1) and (0.2) [1]. Nevertheless, the relevance of that positive mathematical result to the problem originally considered by Dirac is rather dubious, because in the correct Schrödinger operator

$$\theta(h)=-\tfrac{1}{2}\Delta+V=\tfrac{1}{2}P^2+V(Q), \tag{0.3}$$

the pair (Q, P) is irreducible and therefore has multiplicity one, not infinity.

Our starting point has been to take seriously the fact that P and D fail to be isomorphic as Lie algebras, and we have consequently abandoned the hypothesis (0.1) altogether. We ask instead about the existence and uniqueness of non-classical structures defined on P, having the minimum symmetry necessary to define the basic linear and angular momentum observables. In both cases considered below (Lie structures in Sect. 2, algebra structures in Sect. 3), there is precisely one non-classical isomorphism class, and these considerations give rise to a naturally defined linear map θ having properties (0.2) and (0.3).

In this paper, all configuration spaces are flat (i.e., there are no constraints) and finite dimensional. While it is not clear how one might correctly formulate these results for systems with constraints, a significant part of the development can be generalized to the flat infinite-dimensional case appropriate for quantization problems involving nonlinear field equations [10], such as

$$\Box\phi+m^2\phi+g\phi^3=0 \tag{0.4}$$

in the realistic case of four-dimensional spacetime. Specifically, the algebras $P(\lambda)$ and $A(\lambda)$ of Sects. 3 and 4 have been constructed in this case, and the existence of Hilbert space representations of $A(\lambda)$ for imaginary λ has been established. These matters will be taken up elsewhere.

1. Bilinear Maps

Throughout this paper, (Σ, ω) will denote a symplectic vector space; that is, Σ is a finite-dimensional real vector space and ω is a distinguished bilinear mapping of Σ into the reals satisfying

(i) $$\omega(x,y)=-\omega(y,x),$$

(ii) $$\omega(x,\cdot)=0 \Rightarrow x=0.$$

There is no loss of generality if one considers Σ to be the direct sum $E \oplus E'$ of a real "configuration space" E with its dual E', and ω to be given by $\omega((x,f),(y,g)) = f(y) - g(x)$. The simplest example is of course $\Sigma = \mathbb{R}^2$, with

$$\omega((x_1, y_1), (x_2, y_2)) = x_2 y_1 - x_1 y_2 .$$

Now $C^\infty(\Sigma)$ (or merely C^∞ if there is no chance for confusion) will denote the algebra of all *complex*-valued smooth functions on Σ. A *polynomial* is an element in the complex subalgebra of C^∞ generated by the constants and the real linear functionals $u \in \Sigma'$. Here P will denote the complex vector space of all polynomials. By a known polarization argument, one can see that every polynomial f is a complex linear combination of elementary polynomials $f(x) = u(x)^n$, $n = 0, 1, \ldots, u \in \Sigma'$, [4].

The purpose of this section is to determine all bilinear maps of P into itself which are invariant under the action of *affine* contact transformations. The principal result is that such a mapping is a unique infinite linear combination of basic ones $\{\cdot, \cdot\}_p$, $p = 0, 1, \ldots$, which are higher order analogues of Poisson brackets (for odd p) and of functional multiplication (for even p).

We begin by recalling the definition of Poisson brackets. Let f be a *real*-valued smooth function. For each $x \in \Sigma$ we obtain a linear functional on Σ by

$$y \mapsto \frac{d}{dt} f(x + ty)|_{t=0} .$$

Since ω is nondegenerate, there is a unique vector $Df(x) \in \Sigma$ satisfying the condition

$$\omega(Df(x), y) = \frac{d}{dt} f(x + ty)|_{t=0} ,$$

for all $y \in \Sigma$. Thus $x \mapsto Df(x)$ is a nonlinear smooth mapping of Σ into itself. If f is a polynomial of degree n, then Df is a polynomial mapping of degree $n-1$.

For real-valued $f, g \in C^\infty$, the Poisson bracket is defined to be the smooth function

$$\{f, g\}(x) = \omega(Df(x), Dg(x)) .$$

For complex f and g, $\{f, g\}$ is defined by bilinearity

$$\{f_1 + if_2, g_1 + ig_2\} = \{f_1, g_1\} - \{f_2, g_2\} + i(\{f_1, g_2\} + \{f_2, g_1\}) .$$

If f and g are polynomials of respective degrees m and n, then $\{f, g\}$ is a polynomial of degree at most $m+n-2$. Note also that if f and g are linear functionals on Σ, then $\{f, g\}$ is a *constant* whose value is $\omega(\tilde{f}, \tilde{g})$, where for example $\tilde{f}$ is the element of Σ defined by $\omega(\tilde{f}, x) = f(x)$, $x \in \Sigma$.

We now define a sequence $\{\cdot, \cdot\}_p$ of bilinear maps of C^∞ into itself as follows. Put $\{f, g\}_0 = f \cdot g$, $\{f, g\}_1 = \{f, g\}$. In order to define $\{\cdot, \cdot\}_p$ for $p \geqq 2$ we require some preliminaries. Fix p and let Σ^p denote the *symmetric* tensor product of p copies of Σ. There is a bilinear form ω^p on Σ^p which is determined uniquely by the condition

$$\omega^p(x^{(p)}, y^{(p)}) = \omega(x, y)^p , \qquad x, y \in \Sigma ,$$

where $z^{(p)}$ denotes the elementary tensor $z \otimes z \otimes \ldots \otimes z \in \Sigma^p$. Since ω is nondegenerate, ω^p is nondegenerate for every $p \geqq 1$. ω^p is symmetric for even values of p and antisymmetric for the rest.

Now choose a *real*-valued f in C^∞ and fix $x \in \Sigma$. We may define a symplectic p^{th} order derivative $D^p f(x) \in \Sigma^p$ by the requirement that

$$\omega^p(D^p f(x), y^{(p)}) = \frac{d^p}{dt^p} f(x+ty)|_{t=0} \tag{1.1}$$

should hold for all $y \in \Sigma$. In more detail, the function

$$y \mapsto \frac{d^p}{dt^p} f(x+ty)|_{t=0}$$

is a real-valued homogeneous polynomial of degree p on Σ and therefore defines a linear functional on Σ^p via

$$\sum_{j=1}^{n} a_j y_j^{(p)} \mapsto \sum_{j=1}^{n} a_j \frac{d^p}{dt^p} f(x+ty_j)|_{t=0},$$

$n=1,2,\ldots, a_1,\ldots,a_n \in \mathbb{R}$, $y_1,\ldots,y_n \in \Sigma$. The existence and uniqueness of $D^p f(x)$ in Eq. (1.1) now follow from the fact that ω^p is nondegenerate and $\{y^{(p)} : y \in \Sigma\}$ spans Σ^p.

Here $D^p f$ is a smooth mapping of Σ into Σ^p and, if f is a polynomial of degree n, then $D^p f$ vanishes for $p>n$ and is a polynomial of degree $n-p$ if $p \leqq n$. For f, g real-valued functions in C^∞, we put $\{f,g\}_p(x) = \omega^p(D^p f(x), D^p g(x))$. As before, the definition of $\{f,g\}_p$ is extended to complex-valued f and g by bilinearity over $\mathbb{C}$. If f and g are polynomials of degree $m \geqq p$ and $n \geqq p$, respectively, then $\{f,g\}_p$ is a polynomial of degree at most $m+n-2p$; if p is larger than the *minimum* of the degrees of f and g, then $\{f,g\}_p = 0$. Here $\{\cdot,\cdot\}_p$ is symmetric for even p and antisymmetric for odd p.

A *contact transformation* is a self-diffeomorphism of Σ which preserves the two-form associated with ω. It is well-known [1] that every contact transformation ϕ leaves the Poisson bracket invariant in the sense that $\{f\circ\phi, g\circ\phi\} = \{f,g\}\circ\phi$, for all $f, g \in C^\infty$. While this is false for the higher order brackets $\{\cdot,\cdot\}_p$, $p \geqq 2$, these structures are invariant under a large enough subgroup of contact transformations to enable one to define the basic linear and angular momentum observables of classical mechanics. An *affine contact transformation* is a contact transformation that preserves the affine structure of the vector space Σ. The most general affine contact transformation has the form $\phi(x) = Ax + x_0$, where $x_0 \in \Sigma$ and A belongs to the symplectic group $\mathrm{sp}(\Sigma)$, i.e., A is a linear automorphism of Σ such that $\omega(Ax, Ay) = \omega(x,y)$ for all x, y. The set $AC(\Sigma)$ of all affine contact transformations has the structure of a semidirect product of Lie groups $AC(\Sigma) = \Sigma \circledS \mathrm{sp}(\Sigma)$, where Σ denotes the additive group of the vector space structure on Σ, and $AC(\Sigma)$ is of course itself a Lie group.

Proposition 1.2. *For each $p \geqq 0$ and $\phi \in AC(\Sigma)$, we have*

$$\{f\circ\phi, g\circ\phi\}_p = \{f,g\}_p \circ \phi,$$

for each $f, g \in C^\infty$.

Proof. It suffices to verify the formula for real-valued f and g. For $p=0$, the assertion is simply that the map $f \mapsto f\circ\phi$ preserves the multiplicative structure of C^∞. So fix $p \geqq 1$.

Because of the semidirect product structure of $AC(\Sigma)$, it suffices to verify $\{f\circ\phi, g\circ\phi\}_p=\{f,g\}_p\circ\phi$ for maps ϕ which are either translations or belong to $\mathrm{sp}(\Sigma)$. Suppose first that $\phi(x)=x+x_0$, $x_0\in\Sigma$. Then

$$\{f,g\}_p(x+x_0)=\omega^p(D^pf(x+x_0), D^pg(x+x_0)),$$

and it suffices to notice that $D^pf(x+x_0)=D^p(f\circ\phi)(x)$, a fact that follows immediately from the definition of D^p:

$$\begin{aligned}\omega^p(D^p(f\circ\phi)(x), y^{(p)})&=\frac{d^p}{dt^p}f\circ\phi(x+ty)|_{t=0}\\ &=\frac{d^p}{dt^p}f(x+x_0+(y)|_{t=0}=\omega^p(D^pf(x+x_0), y^{(p)}),\end{aligned}$$

for all $x, y\in\Sigma$.

Now assume ϕ belong to $\mathrm{sp}(\Sigma)$. There is a unique representation Γ_p of $\mathrm{sp}(\Sigma)$ on the vector space Σ^p, defined on elementary tensors by $\Gamma_p(A)x^{(p)}=(Ax)^{(p)}$, for $x\in\Sigma$, $A\in\mathrm{sp}(\Sigma)$. Note first that for each $A\in\mathrm{sp}(\Sigma)$, we have $\omega^p(\Gamma_p(A)\xi, \Gamma_p(A)\eta)=\omega^p(\xi,\eta)$, for all $\xi,\eta\in\Sigma^p$. Indeed, it suffices to check this on elementary tensors $\xi=x^{(p)}$, $\eta=y^{(p)}$, and then we have

$$\omega^p(\Gamma_p(A)x^{(p)}, \Gamma_p(A)y^{(p)})=\omega^p((Ax)^{(p)}, (Ay)^{(p)})=\omega(Ax, Ay)^p=\omega(x,y)^p=\omega^p(x^{(p)}, y^{(p)}),$$

because A leaves ω invariant.

Next we claim that $D^p(f\circ A)(x)=\Gamma_p(A)^{-1}D^pf(Ax)$, for each $x\in\Sigma$. Indeed, for each $y\in\Sigma$ we can write

$$\begin{aligned}\omega^p(D^p(f\circ A)(x), y^{(p)})&=\frac{d^p}{dt^p}(f\circ A)(x+ty)|_{t=0}=\frac{d^p}{dt^p}f(Ax+tAy)|_{t=0}\\ &=\omega^p(D^pf(Ax), (Ay)^{(p)})=\omega^p(D^pf(Ax), \Gamma_p(A)y^{(p)})\\ &=\omega^p(\Gamma_p(A)^{-1}D_pf(Ax), y^{(p)}),\end{aligned}$$

proving the claim.

Finally, we have the required identity

$$\begin{aligned}\{f\circ A, g\circ A\}_p(x)&=\omega^p(D^p(f\circ A)(x), D^p(g\circ A)(x))\\ &=\omega^p(\Gamma_p(A)^{-1}D^pf(Ax), \Gamma_p(A)^{-1}D^pg(Ax))\\ &=\omega^p(D^pf(Ax), D^pg(Ax))=\{f,g\}_p(Ax). \quad \square\end{aligned}$$

Let $a_0, a_1, a_2, \ldots$ be an arbitrary sequence of complex numbers. Since for any two fixed polynomials f, g the brackets $\{f,g\}_p$ vanish for large p, the infinite series

$$\sum_{p=0}^{\infty} a_p\{f,g\}_p$$

is only finitely nonzero and it therefore defines a bilinear mapping $[f,g]$ of $P\times P$ into P. The preceding proposition implies that this bilinear map is invariant under all affine contact transformations. Conversely, we have

Theorem 1.3. *Every bilinear mapping $[\cdot,\cdot]$ of P into itself satisfying*

$$[f\circ\phi, g\circ\phi]=[f,g]\circ\phi,$$

for all affine contact transformations ϕ *has the form*

$$[f,g]=\sum_{p=0}^{\infty} a_p\{f,g\}_p$$

for a unique sequence of complex numbers $a_0, a_1, \ldots$.

Thus, the brackets $\{\cdot,\cdot\}_p$ form in a natural sense a linear basis for the vector space of all bilinear maps of polynomials which are invariant under the action of $AC(\Sigma)$.

Before giving the proof, we require a result from classical invariant theory [12, Theorem 6.1A] in a form appropriate for our purposes. For the reader's convenience, we sketch a proof.

Lemma 1. *Let* $\phi:\Sigma'\times\Sigma'\to\mathbb{R}$ *be a real-valued polynomial in two vector variables which is invariant under the action of the symplectic group:*

$$\phi(f\circ A, g\circ A)=\phi(f,g), \qquad f,g\in\Sigma',$$

for all $A\in\mathrm{sp}(\Sigma)$. *Then* ϕ *has the form*

$$\phi(f,g)=\sum_{k=0}^{n} a_k\{f,g\}^k,$$

for some $a_0,\ldots,a_n\in\mathbb{R}$, $n\geqq 0$.

Proof. ϕ decomposes uniquely into a finite sum

$$\phi(f,g)=\sum_{p,q\geqq 0}\phi_{pq}(f,g),$$

where $\phi_{pq}:\Sigma'\times\Sigma'\to\mathbb{R}$ is a homogeneous polynomial of degree (p,q) in the sense that $\phi_{pq}(sf,tg)=s^p t^q\phi_{pq}(f,g)$, for scalars s,t. Thus it suffices to show that $\phi_{pq}=0$ if $p\neq q$, and that $\phi_{pp}(f,g)=\lambda_p\{f,g\}^p$, for some $\lambda_p\in\mathbb{R}$.

Recall first the representations Γ_p, $p=0,1,2,\ldots$ of $\mathrm{sp}(\Sigma)$ on Σ^p defined in the proof of 1.2 (for $p=0$, Σ^0 is taken as $\mathbb{R}$ and Γ_0 is the trivial representation). It is classical that each of these representations is irreducible in the sense that it has no nontrivial invariant subspace. This implies that if $L:\Sigma^p\to\Sigma^q$ is a nonzero linear transformation which intertwines Γ_p and Γ_q,

$$L\Gamma_p(A)=\Gamma_q(A)L, \qquad A\in\mathrm{sp}(\Sigma), \tag{1.4}$$

then $p=q$ and L is a scalar multiple of the identity. Indeed, since $L\neq 0$ and Γ_p and Γ_q are irreducible, (1.4) implies that L is one-to-one and onto. Since Σ^p and Σ^q have different dimensions if $p\neq q$, we must have $p=q$; and since L commutes with the irreducible set of operators $\Gamma_p(\mathrm{sp}(\Sigma))$, Burnside's theorem [5, p. 276] implies that L is a scalar.

Now fix p,q such that $\phi_{pq}\neq 0$. Define a polynomial $\psi_{pq}:\Sigma\times\Sigma\to\mathbb{R}$ by $\psi_{pq}(x,y)=\phi_{pq}(\tilde{x},\tilde{y})$, where $x\mapsto\tilde{x}$ is the linear isomorphism of Σ onto Σ' defined by the condition $\tilde{x}(y)=\omega(x,y)$, $x,y\in\Sigma$. We have $\tilde{x}\circ A=(A^{-1}x)^{\sim}$ for each symplectic automorphism A of Σ, and hence ψ_{pq} is a nonzero homogeneous polynomial of degree (p,q) which satisfies $\psi_{pq}(Ax,Ay)=\psi_{pq}(x,y)$. We have to show that $p=q$ and $\psi_{pp}(\mathrm{x},\mathrm{y})=\lambda\omega(\mathrm{x},\mathrm{y})^p$ for some scalar λ.

Now for each $x \in \Sigma$, there is a unique vector x' in Σ^q such that $\omega^q(x', y^{(q)}) = \psi_{pq}(x, y)$, for all $y \in \Sigma$ [because ω^q is nondegenerate and $y \mapsto \psi_{pq}(x, y)$ is a homogeneous polynomial in y of degree q]. Since for each fixed $y, x \mapsto \psi_{pq}(x, y)$ is a homogeneous polynomial of degree p, there is a unique linear operator $L: \Sigma^p \to \Sigma^q$ satisfying $L(x^{(p)}) = x'$ for each $x \in \Sigma$, i.e., $\omega^q(L(x^{(p)}), y^{(q)}) = \psi_{pq}(x, y)$, $x, y \in \Sigma$. Here L is nonzero because $\psi_{pq} \neq 0$, and we have $L\Gamma_p(A) = \Gamma_q(A)L$ because

$$\begin{aligned}\omega^q(L\Gamma_p(A)x^{(p)}, y^{(q)}) &= \omega^q(L((Ax)^{(p)}), y^{(q)}) \\ &= \psi_{pq}(Ax, y) = \psi_{pq}(x, A^{-1}y) \\ &= \omega^q(Lx^{(p)}, \Gamma_q(A^{-1})y^{(q)}) = \omega^q(\Gamma_q(A)L(x^{(p)}), y^{(q)}),\end{aligned}$$

for all $x, y \in \Sigma$ and because the elementary tensors span Σ^p and Σ^q. The preceding remarks now imply that $p = q$ and $L = \lambda$, $\lambda \in \mathbb{R}$. Hence,

$$\psi_{pq}(x, y) = \psi_{pp}(x, y) = \omega^p(\lambda x^{(p)}, y^{(p)}) = \lambda \omega^p(x^{(p)}, y^{(p)}) = \lambda \omega(x, y)^p,$$

as required. □

We shall also require a convenient formula for $\{f, g\}_p$ when f and g have a particular form. Let $F: \mathbb{R} \to \mathbb{C}$ be a complex-valued smooth function of a real variable. For every linear functional $u \in \Sigma'$, we can form the composite function $F(u) \in C^\infty(\Sigma)$, $F(u): x \mapsto F(u(x))$.

Lemma 2. *Let $u, v \in \Sigma'$ and $F, G \in C^\infty(\mathbb{R})$. Then we have*

$$\{F(u), G(v)\}_p = \{u, v\}^p F^{(p)}(u) G^{(p)}(v),$$

where $F^{(p)}$, $G^{(p)}$ denote the p^{th} derivatives of F, G.

Proof. To avoid confusion between the notation for p^{th} derivatives and elementary tensors in Σ^p, in the proof to follow we will write F^p for the p^{th} derivative of F and reserve $x^{(p)}$ for elementary tensors $x \otimes \ldots \otimes x$ in Σ^p.

Using bilinearity, the proof of Lemma 2 reduces to the case where both F and G are real-valued. We claim first that one has the following variant of the chain rule:

$$D^p(F(u))(x) = F^p(u(x))\tilde{u}^{(p)}, \qquad x \in \Sigma,$$

where for $u \in \Sigma'$, $\tilde{u}$ is the vector in Σ defined by the condition $\omega(\tilde{u}, z) = u(z)$, $z \in \Sigma$. Indeed, for each $x, y \in \Sigma$ we have

$$\begin{aligned}\frac{d^p}{dt^p} F(u(x + ty))|_{t=0} &= \frac{d^p}{dt^p} F(u(x) + tu(y))|_{t=0} = F^p(u(x))u(y)^p \\ &= F^p(u(x))\omega^p(\tilde{u}^{(p)}, y^{(p)}) = \omega^p(F^p(\tilde{u}(x))u^{(p)}, y^{(p)}),\end{aligned}$$

and the formula follows from the definition of $D^p h$ for $h \in C^\infty(\Sigma)$.

Noting that Du is the constant function $Du(x) = \tilde{u}$, we obtain

$$\begin{aligned}\{F(u), G(v)\}_p(x) &= \omega^p(F^p(u(x))\tilde{u}^{(p)}, G^p v(x))\tilde{v}^{(p)}) \\ &= \omega^p(\tilde{u}^{(p)}, \tilde{v}^{(p)}) F^p(u(x)) G^p(v(x)) \\ &= \omega(\tilde{u}, \tilde{v})^p F^p(u(x)) G^p(v(x)) \\ &= \{u, v\}^p F^p(u(x)) G^p(v(x)). \quad \square\end{aligned}$$

Proof of Theorem 1.3 (Existence). Let $[\cdot,\cdot]$ be a bilinear map of P having the stated invariance property. For each $p,q\geqq 0$, define a function $\phi_{pq}:\Sigma'\times\Sigma'\to\mathbb{R}$ by

$$\phi_{pq}(u,v)=\frac{1}{p!q!}[u^p,v^q](0).$$

Note that ϕ_{pq} is a homogeneous polynomial of degree (p,q) in its two variables and, for each A in the symplectic group of Σ, we have

$$\begin{aligned}\phi_{pq}(u\circ A,v\circ A)&=\frac{1}{p!q!}[u^p\circ A,v^q\circ A](0)\\&=\frac{1}{p!q!}[u^p,v^q](A0)=\phi_{pq}(u,v).\end{aligned}$$

By applying Lemma 1 separately to the real and imaginary parts of ϕ_{pq}, we conclude that $\phi_{pq}=0$ if $p\neq q$, and that ϕ_{pp} has the form $\phi_{pp}(u,v)=a_p\{u,v\}^p$, for some $a_p\in\mathbb{C}$. This defines the sequence $a_0,a_1,\ldots$.

Now choose $u,v\in\Sigma'$ and fix $m,n=0,1,2,\ldots$. Using the binomial theorem and translation invariance of $[\cdot,\cdot]$, we write

$$\begin{aligned}[u^m,v^n](x)&=[(u+u(x)1)^m,(v+v(x)1)^n](0)\\&=\sum_{p,q=0}^{m,n}\binom{m}{p}\binom{n}{q}[u^p,v^q](0)u(x)^{m-p}v(x)^{n-p}.\end{aligned}$$

Thus we have

$$\begin{aligned}[u^m,v^n]&=\sum_{p,q=0}^{m,n}\phi_{pq}(u,v)\frac{m!n!}{(m-p)!(n-q)!}u^{m-p}v^{n-q}\\&=\sum_{p=0}^{n}a_p\{u,v\}^p\frac{m!}{(m-p)!}u^{m-p}\frac{n!}{(n-q)!}v^{n-q}\\&=\sum_{p=0}^{n}a_p\{u^m,v^n\}_p,\end{aligned}$$

where the last equality uses the formula from Lemma 2. Notice that since $\{u^m,v^n\}_p=0$ for all $p>n$, we can rewrite this formula as

$$[u^m,v^n]=\sum_{p=0}^{\infty}a_p\{u^m,v^n\}_p,$$

holding simultaneously for all $u,v\in\Sigma'$ and all nonnegative integers m,n. Since polynomials of the form u^n, $u\in\Sigma'$, $n\geqq 0$, span P, we are done.

Proof of Uniqueness. Let $a_0,a_1,\ldots$ be a sequence of complex numbers such that

$$\sum_{p=0}^{\infty}a_p\{f,g\}_p=0$$

for all $f,g\in P$. We have to show that $a_p=0$ for all p. Taking $f=g=1$ yields $a_0=0$.

Assume $a_p = 0$ for all $p \leqq n$. Then for every pair $u, v \in \Sigma'$, we have $\{u^{n+1}, v^{n+1}\}_p = 0$ for all $p > n+1$, and

$$\{u^{n+1}, v^{n+1}\}_{n+1} = (n+1)!^2 \{u, v\}^{n+1}$$

by Lemma 2 above. So if we choose u and v so that $\{u, v\} = 1$, then we obtain

$$(n+1)!^2 a_{n+1} = [u^{n+1}, v^{n+1}] = 0.$$

Hence $a_{n+1} = 0$, and the proof is completed by induction. □

2. Lie Structures

In this paper we use the term *Lie algebra* to denote a complex vector space E endowed with a bilinear mapping $[\cdot, \cdot]$ of E into itself which satisfies $[x, y] = -[y, x]$, together with the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

It is well known that the Poisson bracket operation defines a Lie algebra structure on $C^\infty(\Sigma)$, [1]. This is false for the higher order brackets $\{\cdot, \cdot\}_p$, $p \geqq 2$. Indeed, for even values of p the bracket $\{\cdot, \cdot\}_p$ is not even antisymmetric, and for odd values of p the Jacobi identity fails. In order to obtain a new Lie bracket on P which is invariant under the action of $AC(\Sigma)$, it is necessary to form an infinite linear combination of the $\{\cdot, \cdot\}_p$ in such a way that the combined defects from the Jacobi identity cancel out. The purpose of this section is to show that there is one way, and essentially only one way, to accomplish this.

We begin by considering bilinear maps which are *finite* linear combinations of the basic brackets $\{\cdot, \cdot\}_p$. Let ϕ be a polynomial in a complex variable z: $\phi(z) = a_0 + a_1 z + \ldots + a_N z^N$. We can define a bilinear operation $[\cdot, \cdot]$ on smooth functions $f, g \in C^\infty$ by

$$[f, g] = \sum_{p=0}^{N} a_p \{f, g\}_p.$$

We shall use the notation

$$\langle f, g, h \rangle_\phi = [f, [g, h]_\phi]_\phi + [g, [h, f]_\phi]_\phi + [h, [f, g]_\phi]_\phi \tag{2.0}$$

for triples of functions f, g, h in C^∞, and we will consider along with ϕ the three-variable polynomial $\Phi: \mathbb{C}^3 \to \mathbb{C}$ defined by

$$\Phi(x, y, z) = \phi(z)\phi(x+y) + \phi(-y)\phi(z-x) + \phi(x)\phi(-y-z).$$

Proposition 2.1. *For every $u, v, w \in \Sigma'$ we have*

$$\langle e^u, e^v, e^w \rangle_\phi = \Phi(\{u, v\}, \{u, w\}, \{v, w\}) e^{u+v+w}.$$

Proof. By Lemma 2 of the preceding section we have

$$\begin{aligned} [e^v, e^w]_\phi &= \sum_p a_p \{e^v, e^w\}_p \\ &= \sum a_p \{v, w\}^p e^{v+w} = \phi(\{v, w\}) e^{v+w}. \end{aligned}$$

Hence,

$$\begin{aligned}[e^u, [e^v, e^w]_\phi]_\phi &= \phi(\{v, w\})\,[e^u, e^{v+w}]_\phi \\ &= \phi(\{v, w\})\phi(\{u, v+w\})e^{u+v+w} \\ &= \phi(\{v, w\})\phi(\{u, v\}+\{u, w\})e^{u+v+w}.\end{aligned}$$

The required equation follows by permuting u, v, w cyclically in this formula and adding the three expressions. □

For each $u \geqq 0$, let P_n denote the space of all polynomials $f \in P$ having degree at most n. We will say that $\langle\cdot,\cdot,\cdot\rangle_\phi$ *vanishes on* P_n if $\langle f, g, h\rangle_\phi = 0$, for all f, g, h in P_n. We have to connect this property with the polynomial Φ in a way that is independent of the degree of ϕ. For that, let

$$\Phi(x, y, z) = \sum_{p,q,r \geqq 0} A_{pqr} x^p y^q z^r$$

be the (finite) power series expansion of Φ. Then we have:

Proposition 2.2. *Let ϕ be a polynomial and let n be a positive integer.*

(i) *If $\langle\cdot,\cdot,\cdot\rangle_\phi$ vanishes on P_n, then $A_{pqr} = 0$ for all p, q, r satisfying $p+q+r \leqq \frac{n}{2}$.*

(ii) *If $A_{pqr} = 0$ for all p, q, r satisfying $p+q+r \leqq \frac{3n}{2}$, then $\langle\cdot,\cdot,\cdot\rangle_\phi$ vanishes on P_n.*

Proof. (i) We write $e^z = p(z) + z^{n+1}q(z)$, where

$$p(z) = \sum_{k=0}^{n} \frac{1}{k!} z^k,$$

and $q(z)$ is the entire function defined by the above formula. Fix three linear functionals $u, v, w \in \Sigma'$. Using trilinearity of $\langle\cdot,\cdot,\cdot\rangle_\phi$, we can write $\langle e^u, e^v, e^z\rangle_\phi$ as a sum of eight terms

$$\langle e^u, e^v, e^w\rangle_\phi = \langle p(u), p(v), p(w)\rangle_\phi + \sum_{j=1}^{7} \langle f_j(u), g_j(v), h_j(w)\rangle_\phi,$$

where, for each j, at least one of the entire functions f_j, g_j, h_j has a zero of order $n+1$ at the origin. Since $p(u), p(v), p(w)$ all belong to P_n, the first term vanishes, and hence for each real number t we can write $\langle e^{tu}, e^{tv}, e^{tw}\rangle_\phi$ in the form $t^{n+1}h(t)$, where $h: \mathbb{R} \to P$ is a P-valued analytic function which, of course, depends on u, v, w. Using Proposition 2.1, we conclude that the complex-valued polynomial in t,

$$t \in \mathbb{R} \mapsto \Phi(t^2\{u, v\}, t^2\{u, w\}, t^2\{v, w\}), \tag{2.3}$$

vanishes at $t = 0$ at least to order $n+1$.

Let (x, y, z) denote the (fixed) triple of real numbers $(\{u, v\}, \{u, w\}, \{v, w\})$. Using the power series expansion for Φ in the function of t defined by (2.3) we conclude that

$$\sum_{v=0}^{\infty} t^{2v} \sum_{p+q+r=v} A_{pqr} x^p y^q z^r = \Phi(t^2 x, t^2 y, t^2 z)$$

vanishes at $t=0$ at least to order $n+1$. It follows that

$$\sum_{p+q+r=\nu} A_{pqr} x^p y^q z^r = 0, \tag{2.4}$$

for every nonnegative integer ν satisfying $\nu \leqq \frac{n+1}{2}$, and every triple (x, y, z) of real numbers arising from a triple $u, v, w \in \Sigma'$ as above. Since the image of the mapping

$$(u, v, w) \in \Sigma' \times \Sigma' \times \Sigma' \mapsto (x, y, z) \in \mathbb{R}^3$$

has nonvoid interior (for example, if we choose any pair $f, g \in \Sigma'$ such that $\{f, g\} = 1$, then the image of all triples of the form $(af, bg, c(f+g))$, $a, b, c \in \mathbb{R}$, contains the set $\{(ab, ac, -bc) : a, b, c \in \mathbb{R}\}$, which has nontrivial interior), it follows that the homogeneous polynomial on the left in (2.4) must vanish *identically* in x, y, z. We conclude that $A_{pqr} = 0$ for all p, q, r satisfying $p+q+r \leqq \frac{n}{2} \leqq \frac{n+1}{2}$, as required.

(ii) For each integer $\nu \geqq 0$, define a function $M_\nu : \Sigma' \times \Sigma' \times \Sigma' \to P$ by

$$M_\nu(u, v, w) = \sum_{i+j+k=\nu} \frac{1}{i!j!k!} \langle u^i, v^j, w^k \rangle_\phi .$$

Here M_ν is a polynomial in (u, v, w), and notice that for every triple i, j, k of nonnegative integers satisfying $i+j+k=\nu$, we have

$$\langle u^i, v^j, w^k \rangle_\phi = \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} M_\nu(xu, yv, zw)|_{(0,0,0)} . \tag{2.5}$$

We claim that $M_\nu = 0$ for all $\nu \leqq 3n$. To see this, fix u, v, w in Σ'. Letting p be the complex polynomial,

$$p(z) = \sum_{k=u}^{3n} \frac{z^k}{k!},$$

we can write $e^z = p(z) + z^{3n+1} q(z)$, where q is an entire function. Using trilinearity, we may expand $\langle e^{tu}, e^{tv}, e^{tw} \rangle_\phi$ to obtain

$$\langle e^{tu}, e^{tv}, e^{tw} \rangle_\phi = \langle p(tu), p(tv), p(tw) \rangle_\phi + R(t),$$

where the remainder $R(t)$ is a sum of seven terms, each of which vanishes at $t=0$ at least to order t^{3n+1}. Write

$$\begin{aligned} \langle p(tu), p(tv), p(tw) \rangle_\phi &= \sum_{i,j,k=0}^{3n} \frac{t^{i+j+k}}{i!j!k!} \langle u^i, v^j, w^k \rangle_\phi \\ &= \sum_{\nu=0}^{9n} t^\nu M_\nu(u, v, w), \end{aligned}$$

where M_ν is as defined in the preceding paragraph. Utilizing Proposition 2.1, we have

$$\sum_{\nu=0}^{9n} t^\nu M_\nu(u, v, w) + R(t) = \Phi(t^2 x, t^2 y, t^2 z) e^{tu+tv+tw}, \tag{2.6}$$

where we have taken $x = \{u, v\}$, $y = \{u, w\}$, $z = \{v, w\}$.

Notice that the right side of (2.6) vanishes at $t=0$ to order at least t^{3n+1}. Indeed, since $A_{pqr}=0$ for $2p+2q+2r \leqq 3n$ by hypothesis, the lowest power of t that can appear in

$$\Phi(t^2x, t^2y, t^2z) = \Sigma A_{pqr} t^{2p+2q+2r} x^p y^q z^r$$

is t^{3n+1}, and the assertion follows. Thus,

$$\sum_{\nu=0}^{9n} t^\nu M_\nu(u, v, w) = \Phi(t^2x, t^2y, t^2z) e^{tu+tv+tw} - R(t)$$

vanishes to order t^{3n+1} at $t=0$, and hence $M_\nu(u, v, w)=0$ for all $\nu \leqq 3n$, which is the original claim we wanted to prove.

To complete the proof, we have to show that $\langle f, g, h\rangle_\phi = 0$ for all f, g, h in P_n. Since P_n is spanned by elements of the form u^i, where $u \in \Sigma'$ and $i=0, 1, \ldots, n$, it suffices to show that $\langle u^i, v^j, w^k\rangle_\phi = 0$ for all $u, v, w \in \Sigma'$, $i, j, k = 0, 1, \ldots, n$. So fix these six quantities and let $\nu = i+j+k$. Since $\nu \leqq 3n$, the preceding paragraphs imply that $M_\nu = 0$, and so Eq. (2.5) provides the conclusion. □

We now show that Proposition 2.2 can be extended to the case of bilinear maps which are infinite linear combinations of the basic brackets $\{\cdot, \cdot\}_p$. Let ϕ be an arbitrary formal power series with complex coefficients

$$\phi(z) = \sum_{p=0}^{\infty} a_p z^p,$$

and let $[\cdot, \cdot]$ be the associated bilinear mapping of polynomials

$$[f, g]_\phi = \sum_{p=0}^{\infty} a_p \{f, g\}_p.$$

We may form the power series $\Phi(x, y, z)$ much as we did before when ϕ was a polynomial,

$$\Phi(x, y, z) = \phi(z)\phi(x+y) + \phi(-y)\phi(z-x) + \phi(x)\phi(-y-z), \tag{2.7}$$

but now Φ must be interpreted as an element of the algebra of all formal power series in three variables:

$$\Phi(x, y, z) = \sum_{p, q, r=0}^{\infty} A_{pqr} x^p y^q z^r,$$

where the coefficients A_{pqr} are uniquely determined by the coefficients of ϕ via (2.7). Let $\langle \cdot, \cdot, \cdot\rangle_\phi$ be the trilinear mapping of *polynomials* defined by the formula (2.0).

Proposition 2.8. *Let n be a positive integer and let ϕ be a formal power series.*

(i) *If $\langle \cdot, \cdot, \cdot\rangle_\phi$ vanishes on P_n, then $A_{pqr}=0$ for all p, q, r satisfying $p+q+r \leqq \frac{n}{2}$.*

(ii) *If $A_{pqr}=0$ for all p, q, r satisfying $p+q+r \leqq \frac{3n}{2}$, then $\langle \cdot, \cdot, \cdot\rangle$ vanishes on P_n.*

Proof. For each $n \geqq 1$, let ϕ_n be n^{th} order polynomial obtained by truncating ϕ, $\phi_n(z) = a_0 + a_1 z + \ldots + a_n z^n$. We claim first that if $f, g, h \in P_n$, then

$$[f, [g, h]_\phi]_\phi = [f, [g, h]_{\phi_n}]_{\phi_n},$$

and

$$[[f, g]_\phi, h]_\phi = [[f, g]_{\phi_n}, h]_{\phi_n}.$$

Indeed, since $\{f, g\}_p$ vanishes when p is larger than the minimum of the degrees of f and g, we have $\{f, \{g, h\}_p\}_q = 0$, $f, g, h \in P_n$, whenever either one of the indices p, q exceeds n. It follows that

$$\begin{aligned} [f, [g, h]_\phi]_\phi &= \sum_{p,q=0}^{\infty} a_p a_q \{f, \{g, h\}_p\}_q \\ &= \sum_{p,q=0}^{n} a_p a_q \{f, \{g, h\}_p\}_q = [f, [g, h]_{\phi_n}]_{\phi_n}. \end{aligned}$$

The proof of the second formula is similar.

It follows that the restrictions of $\langle \cdot, \cdot, \cdot \rangle_\phi$ and $\langle \cdot, \cdot, \cdot \rangle_{\phi_\nu}$ to $P_n \times P_n \times P_n$ agree whenever $\nu \geqq n$.

To prove (i), fix $n \geqq 1$. For each $\nu \geqq n$, we form ϕ_ν and the polynomial Φ_ν obtained from it via (2.7). Let

$$\Phi_\nu(x, y, z) = \sum_{p,q,r=0}^{\infty} A_{pqr}(\ ) x^p y^q z^r$$

be its finite series expansion. Now it is clear from the definition of the Φ's in terms of the ϕ's that, for fixed p, q, r, we will have $A_{pqr}(\nu) = A_{pqr}$ for large enough values of ν (which will, of course, depend on p, q, r). Thus, assuming that $\langle \cdot, \cdot, \cdot \rangle_\phi$ vanishes on P_n, we see from the preceding comments that $\langle \cdot, \cdot . \cdot \rangle_{\phi_\nu}$ vanishes on P_n for every $\nu \geqq n$. Proposition 2.2 implies that $A_{pqr}(\nu) = 0$ for every triple p, q, r satisfying $p + q + r \leqq \frac{n}{2}$ and for every $\nu \geqq n$. Since there are only a finite number of such triples, we may choose ν so large that $A_{pqr}(\nu) = A_{pqr}$ for all such p, q, r, hence the conclusion of (i).

Assertion (ii) follows along similar lines. Fix $n \geqq 1$ and assume that $A_{pqr} = 0$ whenever $p + q + r \leqq \frac{3n}{2}$. Again, since there are only a finite number of such triples p, q, r, we may find $\nu \geqq n$ so that

$$A_{pqr}(\nu) = A_{pqr} = 0$$

for all p, q, r in the stated region. Applying 2.2(i) to the polynomial ϕ_ν, we conclude that $\langle \cdot, \cdot, \cdot \rangle_{\phi_\nu}$ vanishes on P_n; and since $\langle \cdot, \cdot, \cdot \rangle_\phi$ and $\langle \cdot, \cdot, \cdot \rangle_{\phi_\nu}$ agree on $P_n \times P_n \times P_n$ whenever $\nu \geqq n$, the proof is complete. $\square$

Let $\alpha_0, \alpha_1, \ldots$ be the numbers defined by the power series expansion

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \alpha_n x^n = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - + \ldots.$$

For every complex number λ, define

$$[f,g]_\lambda = \sum_{n=0}^{\infty} \alpha_n \lambda^n \{f,g\}_{n+1}$$
$$= \{f,g\} - \frac{\lambda^2}{3!}\{f,g\}_3 + \frac{\lambda^4}{5!}\{f,g\}_5 - + \dots,$$

for polynomials f, g. The results of the preceding section imply that $[\cdot,\cdot]_\lambda$ is a bilinear map of P into itself which is invariant under the action of all affine contact transformations. This bracket is antisymmetric because $\{\cdot,\cdot\}_p$ is antisymmetric for odd values of p. Notice also that, for fixed f and g, the function $\lambda \mapsto [f,g]_\lambda$ is a P-valued polynomial in the complex variable λ, which coincides with the Poisson bracket $\{f,g\}$ when $\lambda=0$. We are now in position to show that $[\cdot,\cdot]_\lambda$ is a Lie bracket for all complex λ.

Theorem 2.9. $[\cdot,\cdot]_\lambda$ *satisfies the Jacobi identity for every* $\lambda \in \mathbb{C}$.

Proof. We may assume $\lambda \neq 0$. Fix λ, and let ϕ be the power series

$$\phi(z) = \sin(\lambda z) = \lambda z - \frac{(\lambda z)^3}{3!} + \frac{(\lambda z)^5}{5!} - + \dots.$$

We can write $[f,g]_\lambda = \lambda^{-1}[f,g]_\phi$, and therefore it suffices to show that $[\cdot,\cdot]_\phi$ satisfies the Jacobi identity. By Proposition 2.8, this is equivalent to the condition that

$$\Phi(x,y,z) = \phi(z)\phi(x+y) + \phi(-y)\phi(z-x) + \phi(x)\phi(-y-z)$$

should vanish in the ring of formal power series in three variables. But the standard trigonometric identities $\sin(-A) = -\sin A$, and

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

imply the formal identities

$$\begin{aligned} \phi(-z) &= -\phi(z), \\ \phi(z_1+z_2) &= \phi(z_1)\psi(z_2) + \psi(z_1)\phi(z_2), \end{aligned} \tag{2.10}$$

where ψ is the power series

$$\psi(z) = \cos(\lambda z) = 1 - \frac{\lambda^2 z^2}{z!} + \frac{\lambda^4 z^4}{4!} - + \dots,$$

and a routine substitution of (2.10) in the expression for $\Phi(x,y,z)$ implies that $\Phi = 0$. $\square$

Remark. Paul Chernoff has pointed out that this bracket is essentially the same as the Lie bracket introduced by J. E. Moyal using quite a different method [8]. The definition of Moyal brackets involves operator methods and the Fourier transform; consequently it becomes a formal expression when applied to polynomials. But the formalism is essentially equivalent to the above definition of $[\cdot,\cdot]_\lambda$, and with care can be reformulated rigorously.

We show now that the brackets $[\cdot,\cdot]_\lambda$ are essentially the only ones possible.

Theorem 2.11. *Let* $[\cdot,\cdot]$ *be a Lie bracket on the space* P *which is invariant under all affine contact transformations. Then* $[\cdot,\cdot]$ *has the form* $[f,g]=\alpha[f,g]_\lambda$ *for some pair of complex numbers* α,λ.

Proof. The results of Sect. 1 imply that $[\cdot,\cdot]$ has the form

$$[f,g]=\sum_{p=0}^{\infty} a_p\{f,g\}_p$$

for a unique sequence of complex numbers a_p. Since $[f,g]+[g,f]=0$ and $\{g,f\}_p=(-1)^p\{f,g\}_p$, we have

$$\sum_{p=0}^{\infty} a_p(1+(-1)^p)\{f,g\}_p=0.$$

By the uniqueness assertion of Theorem 1.3, we conclude that $a_p=0$ for even values of p.

It follows that the power series

$$\phi(z)=\sum_{p=0}^{\infty} a_p z^p$$

satisfies the formal identity $\phi(-z)=-\phi(z)$. Moreover, by Proposition 2.8(i) we may conclude that ϕ satisfies the formal identity

$$\phi(z)\phi(x+y)+\phi(-y)\phi(z-x)+\phi(x)\phi(-y-z)=0 \tag{2.12}$$

in the ring of formal power series in x,y,z.

The remainder of the proof simply consists of showing that the only formal power series solutions of (2.12) satisfying $\phi(-z)=-\phi(z)$ are of the form

$$\phi(z)=A\sin(\lambda z)=A\left(\lambda z-\frac{(\lambda z)^3}{3!}+\frac{(\lambda z)^5}{5!}-+\dots\right),$$

for some A,λ in $\mathbb{C}$. Note that this implies the conclusion of the theorem by setting $\alpha=A\lambda$.

We may clearly assume that $\phi \neq 0$. Using $\phi(-z)=-\phi(z)$, we rewrite (2.12) in the form

$$\phi(z)\phi(x+y)=\phi(y)\phi(z-x)+\phi(x)\phi(y+z).$$

We may formally differentiate this once with respect to y and twice with respect to z, and then set $y=0$ and $z=x$ to obtain

$$\phi''(x)\phi'(x)=\phi'(0)\phi''(0)+\phi(x)\phi'''(x).$$

Since $\phi''(0)=2a_2=0$, we have

$$\phi''(x)\phi'(x)-\phi(x)\phi'''(x)=0.$$

We will make use of the following result from the elementary theory of formal power series

Lemma. *Let* ϕ,ψ *be formal power series in a single variable such that* $\phi'\psi-\phi\psi'=0$. *Then* ϕ *and* ψ *are proportional.*

Applying this to the pair ϕ, ϕ'' and using the fact that $\phi \neq 0$, we conclude that there is a complex number λ such that $\phi''(x) = -\lambda^2\phi(x)$. By a familiar argument, the only power series solutions of the formal differential equation $\phi'' + \lambda^2\phi = 0$, are of the form $\phi(z) = A\sin(\lambda z) + B\cos(\lambda z)$. Since ϕ satisfies $\phi(-z) = -\phi(z)$, we conclude that $B = 0$. $\square$

We now take up the question of equivalence among the various Lie structures described in Theorem 2.11. By an *isomorphism* of two Lie algebras $\mathscr{A}_1, \mathscr{A}_2$ we mean the usual complex linear isomorphism $\phi : \mathscr{A}_1 \to \mathscr{A}_2$ which satisfies $[\theta(x), \theta(y)]_2 = \theta([x, y]_1)$ for all $x, y \in \mathscr{A}_1$. Note that if $\mathscr{A}$ is any Lie algebra and α is a nonzero complex number, we can define a new Lie bracket $[\cdot, \cdot]'$ on $\mathscr{A}$ by $[x, y]' = \alpha[x, y]$. The resulting Lie algebra $\mathscr{A}'$ is however isomorphic to the original one via the mapping $\theta : \mathscr{A} \to \mathscr{A}'$ given by $\theta(x) = \alpha^{-1}x$.

For every complex number λ, let $P(\lambda)$ denote the Lie algebra of all complex polynomials on Σ, relative to the bracket

$$[f, g] = \{f, g\} - \frac{\lambda^2}{3!}\{f, g\}_3 + \frac{\lambda^4}{5!}\{f, g\}_5 - + \dots .$$

By (2.11) and the preceding remarks, every nontrivial $AC(\Sigma)$-invariant Lie structure on P gives rise to a Lie algebra isomorphic to $P(\lambda)$ for some complex number λ.

Theorem 2.13. *The Lie algebras $P(\lambda)$ are mutually isomorphic for all nonzero λ.*

The proof of 2.13 uses a universal property of polynomials for which we lack an appropriate reference. We state this result as a lemma and sketch the proof, a routine application of the lore of tensor products.

Lemma. *Let E be a complex vector space, let $f_0 \in E$ and, for each $n \geqq 1$, let $f_n : \Sigma' \to E$ be a homogeneous polynomial mapping of real vector spaces of degree n. Then there is a unique linear map $F : P(\Sigma) \to E$ satisfying*

$$F(1) = f_0,$$

$$F(u^n) = f_n(u),$$

for every $u \in \Sigma'$, $u \geqq 1$.

Sketch of Proof. Let Q_n denote the space of all complex-valued homogeneous polynomials of degree n in $P(\Sigma)$. Then we have a direct sum decomposition $P(\Sigma) = Q_0 + Q_1 + \dots$, and so it suffices to show that for each $n \geqq 1$ there is a unique linear map $L_n : Q_n \to E$ satisfying $L_n(u^n) = f_n(u)$, $u \in \Sigma'$.

Now Q_n is the complexification of the space $\operatorname{Re} Q_n$ of all real-valued polynomials in Q_n. Moreover, if $(\Sigma')^n$ denotes the symmetric tensor product of n copies of Σ', then there is a natural real-linear isomorphism α of $(\Sigma')^n$ onto $\operatorname{Re} Q_n$ satisfying $\alpha(u^{(n)}) = u^n$, $u \in \Sigma'$ [the existence of α follows from the familiar universal property of tensor products, and injectivity of α is equivalent to the fact that the natural pairing of $(\Sigma')^n$ with Σ^n identifies $(\Sigma')^n$ with the dual of Σ^n]. It follows that Q_n is naturally isomorphic to the complexification of $(\Sigma')^n$, and the assertion now follows from the universal property of tensor products. $\square$

Proof of 2.13. Fix $\lambda \neq 0$. We will exhibit a Lie isomorphism of $P(\lambda)$ onto $P(1)$.

Let μ be any nonzero complex number. By the lemma, there is a unique linear mapping θ of P onto itself such that $\theta(1)=1$, and $\theta(u^n)=\mu^n u^n$, $u\in\Sigma'$, $n\geqq 1$. The same lemma implies that θ is a linear automorphism whose inverse is given by $\theta^{-1}(1)=1$, $\theta^{-1}(u^n)=\lambda^{-n}u^n$.

Let Q_n denote the space of all homogeneous polynomials of degree n in P. Then Q_n is spanned by $\{u^n : u\in\Sigma'\}$, and hence $\theta(f)=\mu^n f$ for all $f\in Q_n$. It follows that $\theta(u^p v^q)=\mu^{p+q}u^p v^q$ for all nonnegative integers p,q and all $u,v\in\Sigma'$.

We claim that $\{\theta(f),\theta(g)\}_p=\mu^{2p}\theta(\{f,g\}_p)$. Indeed, for all $u,v\in\Sigma'$ and $m,n\geqq 0$, we have

$$\{\theta(u^m),\theta(v^n)\}_p=\{\mu^m u^m,\mu^n v^n\}_p=\mu^{m+n}\{u^m,v^n\}_p .$$

On the other hand, using Lemma 2 of the preceding section, we can write

$$\{u^m,v^n\}_p=C_{mnp}\{u,v\}^p u^{m-p}v^{n-p}$$

for appropriate coefficients C_{mnp}, and hence

$$\begin{aligned}\mu^{2p}\theta(\{u^m,v^n\}_p)&=\mu^{2p+m+n-2p}C_{mnp}\{u,v\}^p u^{m-p}v^{n-p}\\&=\mu^{m+n}\{\theta(u^m),\theta(v^n)\}_p=\{\theta(u^m),\theta(v^n)\}_p .\end{aligned}$$

The claim follows because P is spanned by $\{u^m : m\geqq 0,\ u\in\Sigma'\}$.

Now choose μ to be a square root of λ. Then we have

$$\begin{aligned}[\theta(f),\theta(g)]_1&=\sum_{p=0}^{\infty}\alpha_p\{\theta(f),\theta(g)\}_{p+1}\\&=\Sigma\alpha_p\lambda^{p+1}\theta(\{f,g\}_p)=\lambda\theta([f,g]_\lambda) .\end{aligned}$$

Thus $\lambda^{-1}\theta$ is the required Lie isomorphism of $P(\lambda)$ onto $P(1)$. □

The only question remaining is whether or not $P(1)$ is isomorphic to $P(0)$. The answer is no. A theorem of Wollenberg [13] implies that the Poisson algebra $P(0)$ admits Lie derivations which are outer, whereas Joseph [6] has shown that every Lie derivation of a certain Lie algebra, which is isomorphic to $P(1)$ by the discussion of the following sections, is inner.

3. Associative Algebra Structures

We indicate in this section how the methods of Sect. 2 allow one to determine all associative algebra structures on P which admit the group $AC(\Sigma)$ as automorphisms. Again, there are exactly two isomorphism classes, the usual commutative algebra structure determined by pointwise multiplication and a new algebra which, in Sect. 4, we show is the complex algebra generated by the canonical commutation relations for an appropriate number of degrees of freedom. By a *multiplication* in P we mean a bilinear mapping $[\cdot,\cdot]:P\times P\to P$ which is *associative*, $[f,[g,h]]=[[f,g],h]$, and is *nontrivial* in the sense that $[f,g]\neq 0$ for at least one pair f,g of polynomials. The multiplication is called *invariant* if $[f\circ\phi,g\circ\phi]=[f,g]\circ\phi$ for every affine contact transformation ϕ. Of course, the usual pointwise multiplication of functions defines an invariant multiplication which is also *commutative.*

Let $\phi(z)=a_0+a_1z+\dots$ be a formal power series in a single variable and let $[\cdot,\cdot]_\phi$ be its associated bilinear map,

$$[f,g]_\phi=\sum_{p=0}^{\infty} a_p\{f,g\}_p.$$

To measure the defect of $[\cdot,\cdot]_\phi$ from the associative law we define a trilinear form

$$\langle f,g,h\rangle_\phi=[[f,g]_\phi,h]_\phi-[f,[g,h]_\phi]_\phi, \tag{3.0}$$

and a three-variable formal power series $\Phi(x,y,z)=\phi(x)\phi(y+z)-\phi(z)\phi(x+y)$.

In the special case where ϕ is a polynomial, then of course so is Φ, and we may define $[f,g]_\phi$ and $\langle f,g,h\rangle_\phi$ for arbitrary smooth functions $f,g,h\in C^\infty(\Sigma)$ as in Sect. 2.

Proposition 3.1. *If ϕ is a polynomial then, for every $u,v,w\in\Sigma'$, we have*

$$\langle e^u,e^v,e^w\rangle_\phi=\Phi(\{u,v\},\{u,w\},\{v,w\})e^{u+v+w}.$$

The proof of 3.1 is a trivial variation of the proof of Proposition 2.1, and we omit it. In general, let

$$\Phi(x,y,z)=\sum_{p,q,r=0}^{\infty} A_{pqr}x^py^qz^r$$

be the power series expansion of Φ.

Proposition 3.2. *Let n be a positive integer and let ϕ be a formal power series.*

(i) *If $\langle\cdot,\cdot,\cdot\rangle_\phi$ vanishes on P_n, then $A_{pqr}=0$ for all p,q,r satisfying $p+q+r\leqq\frac{n}{2}$.*

(ii) *If $A_{pqr}=0$ for all p,q,r satisfying $p+q+r\leqq\frac{3n}{2}$, then $\langle\cdot,\cdot,\cdot\rangle_\phi$ vanishes on P_n.*

Proof. Once we are given Proposition 3.1, the arguments of Propositions 2.2 and 2.8 can be repeated verbatim to establish (i) and (ii). □

As an immediate consequence, we have

Corollary. *Let $\phi\neq0$ be a formal power series. In order that $[\cdot,\cdot]_\phi$ should define a multiplication on P, it is necessary and sufficient that ϕ should satisfy the formal equation*

$$\phi(x)\phi(y+z)=\phi(z)\phi(x+y). \tag{3.3}$$

It is a simple matter to verify that the most general solution ϕ of (3.3) is given by

$$\phi(z)=Ae^{\lambda z}, \tag{3.4}$$

where A and λ are complex numbers. Indeed, assuming ϕ satisfies (3.3), we may formally differentiate (3.3) with respect to y and set $y=0$ to obtain $\phi(x)\phi'(z)=\phi(z)\phi'(x)$. This implies that $\phi'(z)=\lambda\phi(z)$ for some complex constant λ, and (3.4) follows. That (3.4) implies (3.3) is apparent. Thus we may conclude as in Sect. 2

Theorem 3.5. *The most general invariant multiplication in P is given by*

$$[f,g]=A\sum_{p=0}^{\infty}\frac{\lambda^p}{p!}\{f,g\}_p,$$

where A,λ are complex numbers with $A\neq 0$.

We now determine the isomorphism classes of these algebras as λ and A vary.

In general, if $\mathscr{A}$ is any complex associative algebra and A is a nonzero complex number, we can define a new associative multiplication on $\mathscr{A}$ by $[x,y]=Ax\cdot y$, thereby obtaining a new algebra $\mathscr{A}'$. The map $\theta(x)=A^{-1}x$ is an isomorphism of $\mathscr{A}$ onto $\mathscr{A}'$.

Thus we need only consider multiplications on P of the form

$$[f,g]=\sum_{p=0}^{\infty}\frac{\lambda^p}{p!}\{f,g\}_p,$$

where $\lambda\in\mathbb{C}$. Let $\mathscr{A}(\lambda)$ denote the corresponding algebra. Note that $\mathscr{A}(0)$ has the usual commutative algebra structure of pointwise multiplication.

Theorem 3.6. *The algebras $\mathscr{A}(\lambda)$ are mutually isomorphic for all nonzero λ. $\mathscr{A}(1)$ is not isomorphic to $\mathscr{A}(0)$.*

Proof. Let $\mu\in\mathbb{C}$ satisfy $\mu^2=\lambda$ and let $\theta: P\to P$ be the unique linear automorphism satisfying $\theta(1)=1$, $\theta(u^n)=\mu^n u^n$, $u\in\Sigma'$, $n\geqq 1$, as in the proof of 2.12. The proof of 2.12 also shows that $\{\theta(f),\theta(g)\}_p=\lambda^p\theta(\{f,g\}_p)$. It follows that

$$\begin{aligned}\sum_{p=0}^{\infty}\frac{1}{p!}\{\theta(f),\theta(g)\}_p&=\sum\frac{\lambda^p}{p!}\theta(\{f,g\}_p)\\&=\theta\left(\sum\frac{\lambda^p}{p!}\{f,g\}_p\right).\end{aligned}$$

So if we consider θ as a linear map of $\mathscr{A}(\lambda)$ to $\mathscr{A}(1)$, then θ is in fact an isomorphism of complex algebras.

Certainly, $\mathscr{A}(1)$ cannot be isomorphic to $\mathscr{A}(0)$ because $\mathscr{A}(0)$ is commutative while $\mathscr{A}(1)$ is not. □

We first want to relate the algebras $\mathscr{A}(\lambda)$ to the Lie algebra $P(1)$ of the preceding section. We may select any nonzero λ we like, and it is convenient to take $\lambda=\sqrt{-1}$. We will also write $f*g$ for the multiplication in $\mathscr{A}(\sqrt{-1})$:

$$f*g=\sum_{p=0}^{\infty}\frac{i^p}{p!}\{f,g\}_p.$$

It follows that

$$\begin{aligned}f*g-g*f&=2i\left(\{f,g\}-\frac{1}{3!}\{f,g\}_3+\frac{1}{3!}\{f,g\}_5-+\dots\right)\\&=2i[f,g]_1,\end{aligned}$$

where $[\cdot,\cdot]$ denotes the Lie bracket operation in $P(1)$. We may conclude:

Proposition 3.8. *The mapping* $f \mapsto \frac{1}{2i} f$ *is a Lie isomorphism of* $P(1)$ *onto the Lie algebra* $\mathscr{A}(\sqrt{-1})$ *relative to the commutator bracket operation.*

Notice that the constant function 1 is a multiplicative unit for $\mathscr{A}(\sqrt{-1})$. Letting $\bar{f}$ denote the complex conjugate of the polynomial f, we may also see (cf. 3.9 below) that $f \mapsto \bar{f}$ induces a $*$-operation in $\mathscr{A}(\sqrt{-1})$; that is to say $\overline{f * g} = \bar{g} * \bar{f}$, for all $f, g \in P$. *We conclude that* $\mathscr{A}(\sqrt{-1})$ *is a unital* $*$*-algebra.* More generally, let us say that a multiplication $[\cdot, \cdot]$ on P is *self-adjoint* if $\overline{[f,g]} = [\bar{g}, \bar{f}]$ for all $f, g \in P$.

Proposition 3.9. *The self-adjoint invariant multiplications on P are precisely those of the form*

$$[f,g] = A \sum_{p=0}^{\infty} \frac{\lambda^p}{p!} \{f,g\}_p,$$

where $A \neq 0$ *is real and* λ *is pure imaginary.*

Proof. It suffices to verify that

$$f, g \mapsto A \sum_{p=0}^{\infty} \frac{\lambda^p}{p!} \{f,g\}_p$$

defines a self-adjoint multiplication if, and only if, A is real and λ is imaginary. This verification is a straightforward application of the relations $\overline{\{f,g\}_p} = \{\bar{f}, \bar{g}\}_p$ in the above formula defining the multiplication. □

It is of interest to consider the $*$-algebras defined by Proposition 3.9 as A and λ vary. By the remark following Theorem 3.5, the algebra determined by the pair (A, λ) (with $A \neq 0$ real and λ imaginary) is $*$-isomorphic to the algebra determined by the pair $(1, \lambda)$. Thus we need only consider algebras of the form $\mathscr{A}(\alpha\sqrt{-1})$, where α is real and nonzero. We will see presently that these are mutually $*$-isomorphic, but in the proof, the cases $\alpha > 0$ and $\alpha < 0$ must be handled separately.

Lemma 1. $\mathscr{A}(\alpha\sqrt{-1})$ *is* $*$*-isomorphic (respectively* $*$*-anti-isomorphic) to* $\mathscr{A}(\sqrt{-1})$ *if* $\alpha > 0$ *(respectively* $\alpha < 0$*).*

Remark. Theorem 3.6 implies that $\mathscr{A}(\alpha\sqrt{-1})$ is isomorphic to $\mathscr{A}(\sqrt{-1})$ for all nonzero real α, but the reader should note that when α is negative it is not possible to produce a $*$-preserving isomorphism by the method of Theorem 3.6. See Lemma 2 below.

Proof of Lemma 1. If $\alpha > 0$, then we may choose $\mu = \sqrt{\alpha}$ and define $\theta: P \to P$ as in the proof of 3.6: $\theta(u^n) = \mu^n u^n$, $u \in \Sigma'$, $n \geqq 0$. As in the proof of 3.6, we have $\{\theta(f), \theta(g)\}_p = \alpha^p \theta(\{f,g\}_p)$, and it follows that θ is an algebra isomorphism of $\mathscr{A}(\alpha\sqrt{-1})$ onto $\mathscr{A}(\sqrt{-1})$. We need only check that θ is a self-adjoint linear mapping. But every real-valued polynomial is a real-linear combination of monomials of the form u^n, $u \in \Sigma'$, $n \geqq 0$, and hence θ maps real polynomials to real polynomials. It follows that $\theta(\bar{f}) = \overline{\theta(f)}$, as required.

Now assume $\alpha<0$. Choose μ to be the positive real number $\mu=\sqrt{-\alpha}$ and let θ be defined by the same formula as in the preceding paragraph. We have

$$\begin{aligned}\{\theta(f),\theta(g)\}_p&=\mu^{2p}\theta(\{f,g\}_p)\\&=(-\alpha)^p\theta(\{f,g\}_p)\\&=\alpha^p\theta(\{g,f\}_p),\end{aligned}$$

where the last equality results from $\{f,g\}_p=(-1)^p\{g,f\}_p$. Thus

$$\sum_{p=0}^{\infty}\frac{i^p}{p!}\{\theta(f),\theta(g)\}_p=\theta\left(\sum_{p=0}^{\infty}\frac{(i)^p}{p!}\{g,f\}_p\right).$$

This shows that θ is an algebraic anti-isomorphism of $\mathscr{A}(\alpha\sqrt{-1})$ onto $\mathscr{A}(\sqrt{-1})$, and it only remains to observe that, by an argument in the preceding paragraph, θ is a self-adjoint linear mapping. □

By a *reversal* of a symplectic vector space (Σ,ω) we mean a linear automorphism τ of Σ satisfying

$$\begin{aligned}&\text{(i)} && \tau^2=1\,,\\&\text{(ii)} && \omega(\tau x,\tau y)=-\omega(x,y)\,.\end{aligned}\tag{3.10}$$

The easiest way to see that reversals exist is to realize Σ as the direct sum $E\oplus E'$ of a vector space E with its dual, where ω is given by $\omega((q_1,p_1),(q_2,p_2))=p_1(q_2)-p_2(q_1)$, $q_i\in E$, $p_i\in E'$. In this case we can simply put $\tau(q,p)=(q,-p)$. This terminology is intended to suggest the operation of reversing the time sense of the flow of a classical mechanical system. We content ourselves with that brief remark, since an adequate discussion of this interpretation would involve a substantial digression from the development of this paper. In any case, reversals give rise to anti-automorphisms of the algebras $\mathscr{A}(\lambda)$ in the following way.

Lemma 2. *Let τ be a reversal of (Σ,ω) and let $\lambda\in\mathbb{C}$. Then $\theta(f)=f\circ\tau$ is an anti-automorphism of $\mathscr{A}(\lambda)$ satisfying*

$$\begin{aligned}&\text{(i)} && \theta^2=\underline{\mathrm{id}}\,,\\&\text{(ii)} && \theta(\bar f)=\overline{\theta(f)}\,,\qquad f\in P\,.\end{aligned}$$

Proof. The map θ is clearly a linear automorphism of P which satisfies (i) and (ii), and we need only prove that θ reverses the multiplication in $\mathscr{A}(\lambda)$. By taking appropriate linear combinations, it is enough to verify that $\{f\circ\tau,g\circ\tau\}_p=\{g,f\}_p\circ\tau$ for each $f,g\in P$, $p=0,1,2,\ldots$.

Now as in the proof of Proposition 1.2, there is a unique linear operator $\Gamma_p(\tau)$ on Σ^p satisfying $\Gamma_p(\tau)x^{(p)}=(\tau x)^{(p)}$, $x\in\Sigma$. Moreover, we have $\omega^p(\Gamma_p(\tau)\xi,\Gamma_p(\tau)\eta)=(-1)^p\omega^p(\xi,\eta)$ for all $\xi,\eta\in\Sigma^p$, a fact easily seen by taking ξ,η to be elementary tensors $x^{(p)},y^{(p)}$ and using the fact that the action of τ changes the sign of ω. It follows that $\omega^p(\xi,\Gamma_p(\tau)\eta)=(-1)^p\omega^p(\Gamma_p(\tau^{-1})\xi,\eta)$. Thus we may argue as in the proof of 1.2 to conclude that

$$D^p(f\circ\tau)(x)=(-1)^p\Gamma_p(\tau^{-1})D^pf(\tau x)=(-1)^p\Gamma_p(\tau)D^pf(\tau x)\,.$$

Finally, we arrive at the desired conclusion by writing

$$\begin{aligned}\{f\circ\tau, g\circ\tau\}_p(x) &= \omega^p(D^p(f\circ\tau)(x), D^p(g\circ\tau)(x))\\ &= \omega^p(\Gamma_p(\tau)D^pf(\tau x), \Gamma_p(\tau)D^pg(\tau x))\\ &= (-1)^p\omega^p(\Gamma_p(\tau^{-1})\Gamma_p(\tau)D^pf(\tau x), D^pg(\tau x))\\ &= (-1)^p\{f,g\}_p(\tau x)\\ &= \{g,f\}_p(\tau x). \quad \square\end{aligned}$$

Theorem 3.11. *For every real $\alpha\neq 0$, $\mathscr{A}(\alpha\sqrt{-1})$ is $*$-isomorphic to $\mathscr{A}(\sqrt{-1})$.*

Proof. If $\alpha>0$ there is nothing more to prove because of Lemma 1. If $\alpha<0$, Lemma 1 implies that $\mathscr{A}(\alpha\sqrt{-1})$ is $*$-anti-isomorphic to $\mathscr{A}(\sqrt{-1})$, and Lemma 2 implies $\mathscr{A}(\sqrt{-1})$ is $*$-anti-isomorphic to itself. Hence $\mathscr{A}(\alpha\sqrt{-1})$ is $*$-isomorphic to $\mathscr{A}(\sqrt{-1})$. $\square$

The algebras $\mathscr{A}(\lambda)$ contain many abelian subalgebras in which the multiplication reduces to ordinary multiplication. The following result, which we collect here for use later in Sect. 4, summarizes the situation.

Proposition 3.12. *Let $u_1,\dots,u_n\in\Sigma'$ be such that $\{u_i,u_j\}=0$ for all i,j, and let $\mathscr{U}$ denote the space of all polynomials of the form $F(u_1,u_2,\dots,u_n)$, where F is a complex-valued polynomial in n real variables.*

Then $\mathscr{U}$ is a subalgebra of $\mathscr{A}(\lambda)$, for every complex λ, in which multiplication in $\mathscr{A}(\lambda)$ coincides with ordinary multiplication.

Proof. Let F, G be two complex-valued polynomials in n real variables. We have to show that the product of $F(u_1,\dots,u_n)$ and $G(u_1,\dots,u_n)$ in $\mathscr{A}(\lambda)$ coincides with $F(u_1,\dots,u_n)G(u_1,\dots,u_n)$.

Now every complex-valued polynomial in n real variables is a (complex) linear combination of polynomials of the form

$$(x_1,\dots,x_n)\mapsto(a_1x_1+\dots+a_nx_n)^k, \tag{3.13}$$

where $k\geqq 0$ and $a_1,\dots,a_n$ are real numbers. Thus we need only prove the assertion when F and G are of the form (3.13) (for perhaps different sets of k, $a_1,\dots,a_n$, of course). This reduces to considering the product of u^k and v^l, where k,l are nonnegative integers and u,v both belong to the (real) linear span of $\{u_1,\dots,u_n\}$. But for such u,v we clearly have $\{u,v\}=0$, because $\{u_i,u_j\}=0$ for all i,j. By Lemma 2 of Sect. 1 we have

$$\{u^k,v^l\}_p=\begin{cases}u^kv^l, & \text{if } p=0\\ 0, & \text{if } p\geqq 1,\end{cases}$$

and by taking linear combinations we see that

$$\sum_{p=0}^{\infty}\frac{\lambda^p}{p!}\{u^k,v^l\}_p=u^kv^l,$$

which is the required formula. $\square$

4. Representations and Quantization

Let $\mathscr{A}$ be a complex unital associative algebra which is generated by elements $P_1, \dots, P_n, Q_1, \dots, Q_n$ satisfying the *canonical commutation relations*

$$\begin{aligned} &P_iP_j - P_jP_i = Q_iQ_j - Q_jQ_i = 0, \\ &Q_iP_j - P_jQ_i = \sqrt{-1}\delta_{ij}1. \end{aligned} \tag{4.1}$$

Using these relations, one can see that the elements of the form

$$e_{\alpha\beta} = Q_1^{\alpha_1} \dots Q_n^{\alpha_n} P_1^{\beta_1} \dots P_n^{\beta_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices of nonnegative integers, constitute a linear basis for $\mathscr{A}$. The product of $e_{\alpha\beta}$ with $e_{\gamma\delta}$ is a linear combination of the e's, where the coefficients are universal constants depending only on (α, β), (γ, δ), and the indices of summation.

It follows from these observations that if P_i', Q_j' are elements of a complex algebra $\mathscr{A}'$ which satisfy (4.1), then there is a unique homomorphism of complex algebras $\theta: \mathscr{A} \to \mathscr{A}'$ such that $\theta(Q_j) = Q_j'$, $\theta(P_j) = P_j'$. This universal property implies that the algebra $\mathscr{A}$ is unique up to isomorphism, and that it contains no nontrivial two-sided ideals. We denote this simple algebra by $\mathscr{C}_n$. By a slight variation of these considerations, one can see that there is a natural $*$-operation on $\mathscr{C}_n$, which is the unique complex algebra involution obtained by requiring the Q's and P's to be self-adjoint: $Q_j^* = Q_j$, $P_j^* = P_j$, $1 \leqq j \leqq n$.

Now let (Σ, ω) be a $2n$-dimensional symplectic vector space. By a system of *canonical coordinates* we mean a set of linear functionals $q_1, \dots, q_n, p_1, \dots, p_n \in \Sigma'$ which satisfy the conditions $\{p_i, p_j\} = \{q_i, q_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$. Such a set forms a linear basis for Σ', and if $\{q_1', \dots, q_n', p_1', \dots, p_n'\}$ is another such system, then there is a unique symplectic automorphism $A \in \mathrm{sp}(\Sigma)$ such that $q_j' = q_j \circ A$, $p_j' = p_j \circ A$, $j = 1, 2, \dots, n$. The existence of canonical coordinates is clear if one realizes Σ as a direct sum $E \oplus E'$, where $\omega((x, f), (y, g)) = f(y) - g(x)$, $x, y \in E$, $f, g \in E'$. In this case, we can choose any linear basis $e_1, \dots, e_n$ for E, choose $e_1', \dots, e_n'$ to be the dual basis for E', and define $q_j(x, f) = f(e_j)$, $p_j(x, f) = -e_j'(x)$.

We can now easily see that the algebras $\mathscr{A}(\lambda)$, $\lambda \neq 0$, of Sect. 3 are isomorphic to $\mathscr{C}_n$. It is convenient here to work with the value $\lambda = \frac{\sqrt{-1}}{2}$, and we will write $f * g$ for the multiplication in $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$:

$$f * g = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{i}{2}\right)^p \{f, g\}_p.$$

Notice that this convention conflicts with the notation used in Sect. 3 for the multiplication in $\mathscr{A}(\sqrt{-1})$, but no problems will arise because we have no further use for the algebra $\mathscr{A}(\sqrt{-1})$.

Theorem 4.2. *There is a unique $*$-isomorphism $\theta: \mathscr{A}\left(\frac{\sqrt{-1}}{2}\right) \to \mathscr{C}_n$ determined by $\theta(p_j) = P_j$, $\theta(q_j) = Q_j$, $j = 1, 2, \dots, n$.*

Proof. Note first that $q_1, \dots, q_n, p_1, \dots, p_n$ satisfy the canonical commutation relations 4.1 relative to the algebra structure of $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$. Indeed, if $f, g \in \Sigma'$ are any two linear functionals, then

$$f * g = \sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{i}{2}\right)^p \{f, g\}_p = f \cdot g + \frac{\sqrt{-1}}{2}\{f, g\}.$$

Hence $f * g - g * f = \sqrt{-1}\{f, g\}$, and the assertion follows from the definition of canonical coordinate systems.

Let $\mathscr{A}_0$ be the complex subalgebra of $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$ generated by $q_1, \dots, q_n, p_1, \dots, p_n$. Every element $u \in \Sigma'$ is a (unique) real linear combination of $q_1, \dots, p_n$, and hence $\Sigma' \subseteq \mathscr{A}_0$. Now Proposition 3.12 implies that, for each $u \in \Sigma'$ and $n \geqq 1$, we have

$$\underbrace{u * u * \dots * u}_{n} = u^n.$$

Thus $\mathscr{A}_0$ contains all polynomials of the form u^n, $n \geqq 1$, and it contains the constant 1 because

$$1 = \frac{1}{\sqrt{-1}}(q_1 * p_1 - p_1 * q_1).$$

Since these elements span P as a complex vector space, we conclude that $\mathscr{A}_0 = \mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$.

By the universal property of the CCR's discussed above, it follows that there is a unique isomorphism θ of complex algebras satisfying $\theta(q_j) = Q_j$, $\theta(p_j) = P_j$. Finally, by definition of the involution in $\mathscr{C}_n$, it must correspond through θ to the involution in $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$, simply because $q_1, \dots, q_n, p_1, \dots, p_n$ are self-adjoint elements of $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$ which satisfy (4.1). □

Remark. Theorem 4.2 has a coordinate-free reformulation which is of interest. Let $\mathscr{C}$ be any unital complex associative $*$-algebra, and let $\theta_0 : \Sigma' \to \mathscr{C}$ be a real-linear mapping of the dual of Σ into the self-adjoint elements of $\mathscr{C}$ which satisfies $\theta_0(u)\theta_0(v) - \theta_0(v)\theta_0(u) = \sqrt{-1}\{u, v\}1$. *Then θ_0 extends uniquely to a $*$-monomorphism of $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$ into $\mathscr{C}$.* The proof is a trivial variation of the proof of 4.3.

As for the Lie algebras $P(\lambda)$ of Sect. 2, we conclude

Corollary. *For each $\lambda \neq 0$, $P(\lambda)$ is isomorphic to the Lie algebra $\mathscr{C}_n$ relative to the commutator bracket operation*

$$[x, y] = xy - yx.$$

Theorem 4.2 implies that the $*$-algebraic structure defined on $\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)$ is the correct structure for quantum mechanics, but the question remains as to whether or not the map θ actually represents the process called "canonical quantization" [7]. We will show now that it does.

Let $p_1, \dots, p_n, q_1, \dots, q_n$ be a canonical coordinate system for (Σ, ω). Let $h: \Sigma \to \mathbb{R}$ be a polynomial which is in the form of the Hamiltonian of a classical system relative to the coordinates $\{q_i, p_j\}$; this is to say that h has the form

$$h(x) = \frac{1}{2} \sum_{i=1}^{n} p_i(x)^2 + V(q_1(x), \dots, q_n(x)),$$

where $V: \mathbb{R}^n \to \mathbb{R}$ is a polynomial representing the potential energy of the interaction forces. The corresponding quantum mechanical Hamiltonian is the Schrödinger operator on $L^2(\mathbb{R}^n)$ given by $H = -\frac{1}{2}\Delta + V$, where Δ is the Laplacian in n variables and V is the multiplication operator

$$f(x_1, \dots, x_n) \mapsto V(x_1, \dots, x_n) f(x_1, \dots, x_n).$$

Quantization is this more or less ad hoc procedure of passing from h to H.

Let $D \subsetneq L^2(\mathbb{R}^n)$ be the dense space of all complex-valued smooth functions of compact support, and let $\mathscr{C}_n$ be the algebra of all differential operators in n variables having complex-valued polynomial coefficients. We may consider $\mathscr{C}_n$ as an algebra of operators acting on D. For each $L \in \mathscr{C}_n$ there is a unique adjoint $L^* \in \mathscr{C}_n$ defined by $\langle Lf, g \rangle = \langle f, L^* g \rangle$, $f, g \in D$, where $\langle \cdot, \cdot \rangle$ is the complex inner product in $L^2(\mathbb{R}^n)$. Thus, $\mathscr{C}_n$ is a $*$-algebra of unbounded operators. If we define P_j, Q_j in the usual way, $P_j f = \dfrac{1}{\sqrt{-1}} \dfrac{\partial f}{\partial x_j}$, $Q_j f(x_1, \dots, x_n) = x_j f(x_1, \dots, x_n)$, then $\{P_i, Q_j\}$ generates $\mathscr{C}_n$ as a complex algebra and satisfies the canonical commutation relations. Moreover, the set of operators $\mathscr{C}_n$ is irreducible in an appropriate sense for $*$-algebras of unbounded operators ([11, Eqs. (3)–(8)] and [9]).

Now let $\theta: \mathscr{A}\left(\dfrac{\sqrt{-1}}{2}\right) \to \mathscr{C}_n$ be the $*$-isomorphism defined by $\theta(p_i) = P_i$, $\theta(q_i) = Q_i$. By 3.12, θ is multiplicative on polynomials which are functions of $p_1, \dots, p_n$ alone, and hence $\theta(\frac{1}{2}\sum p_j^2) = \frac{1}{2}\sum P_j^2 = -\frac{1}{2}\Delta$. For the same reason, $\theta(V(q_1, \dots, q_n)) = V(Q_1, \dots, Q_n)$, and so by linearity it follows that θ carries the classical Hamiltonian h to the correct Schrödinger operator H.

These remarks show that, from a purely mathematical point of view, one may consider (or define) quantization to be nothing more than *an irreducible $*$-representation of the algebra of functions* $\mathscr{A}\left(\dfrac{\sqrt{-1}}{2}\right)$.

The essential ingredient in the preceding discussion is the fact that if $u_1, \dots, u_n$ are linear functionals on Σ such that $\{u_i, u_j\} = 0$ for all i and j, then the operators $X_j = \theta(u_j)$ mutually commute, and

$$\theta(f(u_1, \dots, u_n)) = f(X_1, \dots, X_n) \tag{4.3}$$

for any n-variate polynomial f. It is less simple to calculate quantities such as $\theta(u^m v^n)$ when u and v are linear functionals whose Poisson bracket is not zero. In order to quantize such polynomials, one must appeal to the special case of the formula (4.3) for $n = 1$.

To illustrate this, consider the symplectic space $\Sigma=\mathbb{R}^2$, $\omega((x,y),(x',y'))=yx'-xy'$, together with the canonical coordinate system $q(x,y)=y$, $p(x,y)=x$. Let P,Q be the usual operators on $L^2(\mathbb{R})$, $P=\frac{1}{\sqrt{-1}}\frac{d}{dx}$, $Q=$ *multiplication by* x, and let $\theta:\mathscr{A}\left(\frac{\sqrt{-1}}{2}\right)\to\mathscr{C}_1$ be defined by $\theta(p)=P$, $\theta(q)=Q$. Let us quantize a homogeneous cubic polynomial such as $f(x,y)=x^2y$. We first express f as a monomial in p and q, $f=p^2q$. Noting that $3p^2q$ is the coefficient of st^2 in the formal expansion of $(sq+tp)^3$, it follows that $3\theta(f)=3\theta(p^2q)$ is the coefficient of st^2 in the expansion of $\theta((sq+tp)^3)=(sQ+tP)^3$. From this we obtain

$$\begin{aligned}\theta(f)&=\tfrac{1}{3}(QP^2+PQP+P^2Q)\\&=QP^2-\sqrt{-1}P=-x\frac{d^2}{dx^2}-\frac{d}{dx}.\end{aligned}$$

This method has an obvious generalization in which, for the case where Σ has dimension $2n$, one quantizes homogeneous polynomials of degree k by working with the formal expansion of

$$(s_1q_1+\ldots+s_nq_n+t_1p_1+\ldots+t_np_n)^k$$

in an entirely similar way.

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