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On Zamolodchikov's Solution of the Tetrahedron Equations

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Abstract. The tetrahedron equations arise in field theory as the condition for the S-matrix in $2+1$-dimensions to be factorizable, and in statistical mechanics as the condition that the transfer matrices of three-dimensional models commute. Zamolodchikov has proposed what appear (from numerical evidence and special cases) to be non-trivial particular solutions of these equations, but has not fully verified them. Here it is proved that they are indeed solutions.

1. Introduction

A number of two-dimensional models in statistical mechanics have been exactly solved [1–4] by using the "star-triangle equations" (or simply "triangle equations") [5–7], which are generalizations of the star-triangle relation of the Ising model [8, 9]. These equations are the conditions for two row-to-row transfer matrices to commute.

Alternatively, these models can be put into field-theoretic form by considering the transfer matrix that adds a single face to the lattice [10], and regarding this as an S-matrix. The star-triangle relations then become the condition for the S-matrix to factorize [11].

These equations can be generalized to three-dimensional models in statistical mechanics, corresponding to a $1+2$-dimensional field theory. Unfortunately, the resulting "tetrahedron" equations are immensely more complicated, the main problem being that there are 2^{14} individual equations to satisfy for an Ising-type model, as against 2^6 in two-dimensions. Symmetries reduce this number somewhat, but there are still apparently many more equations than unknowns and until recently there was little reason to suppose that the equations permitted any interesting solutions at all.

However, by what appears to be an extraordinary feat of intuition, Zamolodchikov [12, 13] has written down particular possible solutions and has shown that they satisfy some of the tetrahedron equations in various limiting cases. Extensive numerical tests have also been made by V. Bajanov and Yu.

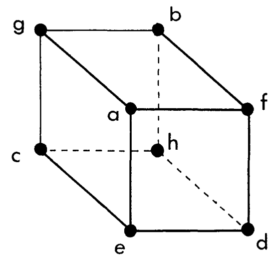

Fig. 1. Arrangement of the spins $a, \ldots, h$ on the corner sites of a cube

Stroganov, but a complete algebraic proof has hitherto not been obtained. Here (in Sects. 3–5) I give the required proof, which depends very much on classical nineteenth (and eighteenth) century mathematics, in particular on spherical trigonometry.

Let me refer to Zamolodchikov's two papers [12] and [13] as ZI and ZII, respectively. In ZI he considers the "static limit," when his solution simplifies considerably. Some of the Boltzmann weights then vanish, and the others satisfy various symmetry and anti-symmetry relations. In Sect. 6 I show that it is the only solution of the tetrahedron equations with these zero elements, symmetries and anti-symmetries.

From the statistical mechanical point of view, the anti-symmetry relations are rather unsatisfactory as they require that some of the Boltzmann weights be negative. We should like to replace them by strict symmetry relations, but unfortunately the tetrahedron equations then no longer admit a solution.

2. Interactions-Round-a-Cube Model and the Tetrahedron Equations

Just as the two-dimensional star-triangle relations can be obtained in convenient generality for an "Interactions-Round-a-Face" (IRF) model [7, 10], so can the three-dimensional tetrahedron relations be obtained for an "Interactions-Round-a-Cube" (IRC) model.

Consider a simple cubic lattice $\mathscr{L}$ of N sites. At each site i there is a "spin" σ_i, free to take some set of values. Each cube of the lattice has eight corner sites: let the spins thereon be $a, b, \ldots, h$, arranged as in Fig. 1, and allow all possible interactions between them. Then the Boltzmann weight of the cube will be some function of $a, b, \ldots, h$: let us write it (omitting commas) as $W(a|efg|bcd|h)$. The partition function is

$$Z = \sum \prod W(\sigma_i|\sigma_m \sigma_n \sigma_p|\sigma_j \sigma_k \sigma_l|\sigma_q), \tag{2.1}$$

where the product is over all N cubes of the lattice; for each cube $i, j, \ldots, p$ are the eight corner sites; the summation is over all values of all the N spins.

Let T be the layer-to-layer transfer matrix of the model. It depends on W, so can be written as $T(W)$. Consider another model, with a different weight function W'. Then, as has been shown by Jaekel and Maillard [14], the two-dimensional argument [7, 10] can be generalized to establish that $T(W)$ and $T(W')$ commute if

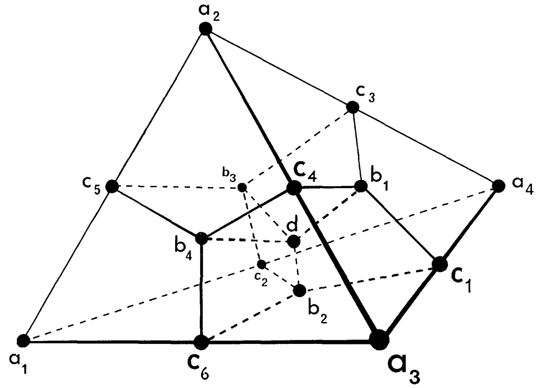

Fig. 2. The graph (a rhombic dodecahedron) whose partition function is the left hand side of (2.2). Some edges are shown by broken lines: this is merely to help visualisation

there exist two other weight functions W'' and W''' such that

$$\begin{aligned}&\sum_d W(a_4|c_2c_1c_3|b_1b_3b_2|d)\,W'(c_1|b_2a_3b_1|c_4dc_6|b_4)\\&W''(b_1|dc_4c_3|a_2b_3b_4|c_5)\,W'''(d|b_2b_4b_3|c_5c_2c_6|a_1)\\&\quad=\sum_b W'''(b_1|c_1c_4c_3|a_2a_4a_3|d)\,W''(c_1|b_2a_3a_4|dc_2c_6|a_1)\\&W'(a_4|c_2dc_3|a_2b_3a_1|c_5)\,W(d|a_1a_3a_2|c_4c_5c_6|b_4)\end{aligned} \tag{2.2}$$

for all values of the 14 spins $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, \ldots, c_6$. I shall refer to these 14 spins as "external," and to d as the "internal" spin.

We can think of each side of (2.2) as the partition function of four skewed cubes joined together, with a common interior spin d. This graph is a rhombic dodecahedron. For the left hand side of (2.2), the graph can be drawn (by distorting the angles) as in Fig. 2; for the right hand side, the centre spin d is to be connected to $a_1, \ldots, a_4$, instead of $b_1, \ldots, b_4$. In either case the lattice is bipartite: $a_1, \ldots, a_4$, $b_1, \ldots, b_4$ lie on one sub-lattice; $c_1, \ldots, c_6$, d on the other.

Now suppose that each spin σ_i can only take values $+1$ and -1, and that W is unchanged by negating all its eight arguments, i.e.

$$W(-a|-e,-f,-g|-b,-c,-d|-h) = W(a|efg|bcd|h)\,. \tag{2.3}$$

The Eq. (2.2) are then the tetrahedron equations used by Zamolodchikov. To see this, work with the duals $\mathscr{L}_D$ of the lattices discussed above. The cube shown in Fig. 1 is then replaced by three planes intersecting at a point, dividing three-dimensional space into eight volumes associated with the spins $a, \ldots, h$. Two parallel cross-sections of this diagram (one above the point of intersection, the other below) are shown in Fig. 3.

Adjacent spins are separated by faces of $\mathscr{L}_D$ (shown as lines in Fig. 3). Colour each face white if the spins on either side of it are equal, black if they are different. Then by letting the spins $a, \ldots, h$ take all possible values, one obtains the allowed colourings shown in Eq. (6.1) of ZI, a typical example being shown in Fig. 4.

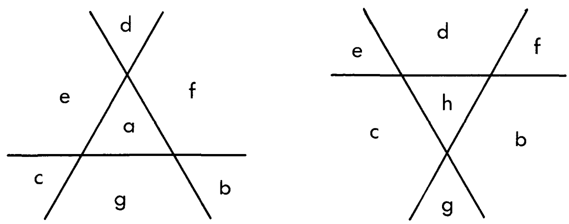

Fig. 3. Two cross-sections through the dual graph of Fig. 1: the spins $a, \ldots, h$ are here associated with volumes

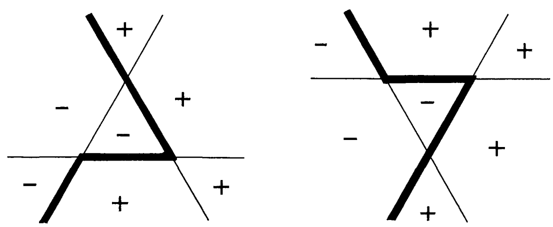

Fig. 4. A typical set of values of $a, \ldots, h$, and Zamolodchikov's corresponding plaquette colouring (heavy lines denote "black," lighter lines "white")

Conversely, any allowed colouring corresponds to just two sets of values of the spins $a, \ldots, h$, one set being obtained from the other by negating all of them. From (1.3), this negation leaves W unchanged, so the Boltzmann weight of the configuration is uniquely determined by the colouring of the faces. We can therefore replace the function W of the eight spins by a function S of the colours of the 12 faces. If we write "white" and "black" simply as "+" and "−", then the colour on the face between two spins c and g is simply the product cg of the spins. We can therefore explicitly define S by

$$\overset{cg,ae,df,bh}{\underset{bf,ag,ce,dh}{S}}{}_{de,af,bg,ch} = W(a|efg|bcd|h). \tag{2.4}$$

Define S', S'', S''' similarly, with W replaced repsectively by W', W'', W'''. Then (2.2) becomes precisely Eq. (3.9) of ZI, except only that in Zamolodchikov's notation S, S', S'', S''' become

$$S(z^{(23)}, z^{(12)}, z^{(13)}), \qquad S(z^{(24)}, z^{(12)}, z^{(14)}),$$
$$S(z^{(34)}, z^{(13)}, z^{(14)}), \qquad S(z^{(34)}, z^{(23)}, z^{(24)}),$$

respectively.

3. Zamolodchikov's Model

Zamolodchikov's model has the "black-white" symmetry property that S is unchanged by reversing the colours of all 12 faces. This means that W not only satisfies (2.3), but has the stronger sub-lattice symmetry properties

$$\begin{aligned} W(-a|efg|-b,-c,-d|h) &= W(a|-e,-f,-g|bcd|-h) \\ &= W(a|efg|bcd|h). \end{aligned} \tag{3.1}$$

Table 1. Values of the function W ; the spin products λ, μ, ν, are defined by (3.3), the parameters $P_0, \ldots, R_3$ by (3.13)

λ	μ	ν	$W(a\|efg\|bcd\|h)$
+	+	+	$P_0 - abcd\, Q_0$
−	+	+	R_1
+	−	+	R_2
+	+	−	R_3
+	−	−	$ab\, P_1 + cd\, Q_1$
−	+	−	$ac\, P_2 + bd\, Q_2$
−	−	+	$ad\, P_3 + bc\, Q_3$
−	−	−	R_0

To express the function W in terms of Zamolodchikov's matrix elements $\sigma, S, a, \ldots, K, H, V$, we simply note that $W(a|efg|bcd|h)$ corresponds to the spins as arranged in Fig. 3, translate each set of spin values to a face colouring, and look up the corresponding matrix element in Eq. (6.1) of ZI (using if necessary the black-white colour symmetry). For instance, using Fig. 4, considering also the effect of negating h:

$$W(-|-++|+-+|-)=K(\theta_1,\theta_2),$$
$$W(-|-++|+-+|+)=H(\theta_2,\theta_1). \tag{3.2}$$

Doing this, using Eqs. (2.2) and (4.9) of ZII, we find that W is "almost determined" by the values of the three spin products

$$\lambda = abeh, \qquad \mu = acfh, \qquad \nu = adgh. \tag{3.3}$$

More precisely, W has the values given in Table 1, where $P_0, \ldots, P_3, Q_0, \ldots, Q_3$, and $R_0, \ldots, R_3$ are constants, independent of the spins $a, \ldots, h$. The functions W', W'', W''' are also given by Table 1, but with different values of $P_0, \ldots R_3$.

Spin Symmetries

Before specifying these constants, it is worth considering the symmetries of W and the tetrahedron equations (2.2). Since (2.2) has to be valid for all values of all the 14 spins $a_1, a_2, \ldots c_6$, it consists of 2^{14} individual equations: a dauntingly large number! The spin reversal symmetry (2.3) helps: it reduces the number to 2^{13}. The stronger symmetry (3.1), together with the fact that the graph in Fig. 2 is bi-partite (the a_i and b_i lie on one sub-lattice, the c_i and d on the other), implies that (2.2) is unchanged not only by reversing all spins, but also by reversing all those on one sub-lattice. This reduces the number of distinct equations in (2.2) to 2^{12}.

This is still a very large number, but fortunately when we use the specific form of W given in Table 1 we find some dramatic simplifications. If W (and W', W'', W''') depended only on λ, μ, ν, then it would be true that (2.2) involved the 14 external spins $a_1, \ldots, c_6$ only via the 10 products a_1b_1, a_2b_2, a_3b_3, a_4b_4, $a_3a_4c_1$, $a_1a_4c_2$, $a_2a_4c_3$, $a_2a_3c_4$, $a_1a_2c_5$, $a_1a_3c_6$. Further, negating either the first four of these products, or the last six, would merely be equivalent to negating d. This would mean that (2.2) depended only on eight combinations of the external spin,s e.g.

$$\begin{array}{cccc} a_1a_2b_1b_2\,, & a_1a_3b_1b_3\,, & a_1a_4b_1b_4\,, & a_1a_3c_1c_2\,, \\ a_2a_3c_1c_3\,, & a_2a_4c_1c_4\,, & a_1a_4c_1c_6\,, & a_1a_2a_3a_4c_1c_5\,. \end{array} \tag{3.4}$$

Hence there would only be $2^8=256$ distinct equations. In fact W depends not only on λ, μ, v: from Table 1 it has the form

$$W(a|efg|bcd|h)=\tau F(\lambda,\mu,v,abcd)\,, \tag{3.5}$$

where F is a function of four spin products and τ is a sign factor, equal to either 1, ab, ac or ad. Also, the terms involving $abcd$ have $\lambda\mu v=1$, i.e. $abcd=efgh$. It follows that $abcd$ in (3.5) can be replaced by $efgh$ if desired.

Using the form (3.5) (and corresponding forms for W', W'', W''') in (2.2), the eight $abcd$ (or $efgh$) arguments that occur can be taken to be $b_1b_2b_3a_4$, $b_1b_2a_3b_4$, $b_1a_2b_3b_4$, $a_1b_2b_3b_4$, $b_1a_2a_3a_4$, $a_1b_2a_3a_4$, $a_1a_2b_3a_4$, $a_1a_2a_3b_4$. The ratios of these depend only on the first three products in (3.4), so they are determined by the products (3.4), apart from a single overall sign factor. This means that each of the previously mentioned 2^8 equations is in fact a pair of equations, one being obtained from the other by negating (say) a_1, a_2, a_3, a_4. Equivalently, one equation can be obtained from the other by negating R_0, Q_0, Q_1, Q_2, Q_3 in Table 1, for all four functions W, W', W'', W'''.

It still remains to examine the contributions to (2.2) of the τ sign factor in (3.5). I have done this (aided by a computer) for each of the 2^8 pairs of equations. In every case it is true that the multiplied contributions also depend only on $b_1b_2b_3a_4$ and the eight spin products in (3.4) [apart possibly from an overall sign factor multiplying both sides of (2.2)].

Thus there are just $2\times 2^8=512$ distinct equations: a great reduction on the original 2^{14}!

There are still further simplifications: the function W has the "diagonal reversal" property:

$$W(a|efg|bcd|h)=W(h|bcd|efg|a)\,, \tag{3.6}$$

and similarly for W', W'', W'''. It follows that the two sides of (2.2) are interchanged by the transformation

$$\begin{array}{c} a_i \leftrightarrow b_i\,, \quad i=1,\dots,4\,, \\ c_1 \leftrightarrow c_5\,, \quad c_2 \leftrightarrow c_4\,, \quad c_3 \leftrightarrow c_6\,. \end{array} \tag{3.7}$$

This means that 64 of the 512 equations are satisfied identically, the right hand side being the same as the left hand side. The remaining 448 occur in pairs of type $B=A$ and $A=B$, so there are only 224 distinct equations remaining. These can conveniently be regarded as 112 pairs, one being obtained from the other by negating R_0, Q_0, ..., Q_3 for all four functions W, W', W'', and W'''.

Set

$$X_j=P_j+Q_j\,, \qquad Y_j=P_j-Q_j \tag{3.8}$$

for $j=0,1,2,3$, and take W' to be also given by Table 1 and (3.8), but with P_j, Q_j, R_j, X_j, Y_j replaced by P'_j, Q'_j, R'_j, X'_j, Y'_j. Similarly for W'', W'''. Then two typical equations [obtained from (2.2) by taking all the external spins positive except a_1, and except a_1 and a_2] are

$$Y_0 Y_0' Y_0'' R_0''' + R_0 R_2' R_1'' X_0''' = Y_0''' R_0'' R_3' R_1 + R_0''' Y_1'' Y_1' Y_1 , \tag{3.9}$$

$$Y_0 Y_0' R_1'' R_0''' + R_0 R_2' X_0'' X_0''' = R_1''' R_0'' X_2' X_2 + Y_1''' Y_1'' R_0' R_2 . \tag{3.10}$$

Two other equations can be obtained immediately from these by negating $R_0, \ldots, R_0'''$ and $Q_j, \ldots, Q_j'''$, i.e. by negating each R_0 and replacing every X by a Y, and every Y by an X.

The Constants $P_0, \ldots, R_3$

Now let us return to the procedure described before (3.2), so as to obtain the constants $P_0, \ldots, R_3$ in Table 1, and hence the X_j, Y_j in (3.8), from Eqs. (2.2) and (4.9) of ZII. To do this we need various functions of the angles ϕ_1, ϕ_2, ϕ_3 of a spherical triangle, namely the spherical excesses

$$\begin{aligned} 2\alpha_0 &= \phi_1 + \phi_2 + \phi_3 - \pi , \\ 2\alpha_1 &= \pi + \phi_1 - \phi_2 - \phi_3 , \\ 2\alpha_2 &= \pi + \phi_2 - \phi_3 - \phi_1 , \\ 2\alpha_3 &= \pi + \phi_3 - \phi_1 - \phi_2 , \end{aligned} \tag{3.11}$$

and the quantities

$$t_i = [\tan(\alpha_i/2)]^{1/2} , \qquad s_i = [\sin(\alpha_i/2)]^{1/2} , \qquad c_i = [\cos(\alpha_i/2)]^{1/2} , \tag{3.12}$$

for $i = 0, 1, 2, 3$. We then find that $P_0, \ldots, R_3$ are given by

$$\begin{aligned} P_0 &= 1 , \qquad Q_0 = t_0 t_1 t_2 t_3 , \qquad R_0 = s_0/(c_1 c_2 c_3) , \\ P_i &= t_j t_k , \qquad Q_i = t_0 t_i , \qquad R_i = s_i/(c_0 c_j c_k) , \end{aligned} \tag{3.13}$$

for all permutations (i, j, k) of $(1, 2, 3)$. (Here ϕ_1, ϕ_2, ϕ_3 are Zamolodchikov's angles $\theta_2, \theta_1, \theta_3$.)

It is convenient to regard Table 1 and Eqs. (3.11)–(3.13) as defining W as a function of the angles ϕ_1, ϕ_2, ϕ_3, as well as of the spins $a, \ldots, h$. We can write it as

$$W[\phi_1, \phi_2, \phi_3 ; a|efg|bcd|h] , \tag{3.14}$$

or, if the explicit spin dependence is not required, as $W[\phi_1, \phi_2, \phi_3]$. The other weights W', W'', W''' are also given by this function, but with different values of the arguments ϕ_1, ϕ_2, ϕ_3. Zamolodchikov's assertion is that (2.2) is satisfied if W, W', W'', W''' therein are given by

$$\begin{aligned} W &= W[\theta_2, \theta_1, \theta_3] , \qquad W' = W[\pi - \theta_6, \theta_1, \pi - \theta_4] , \\ W'' &= W[\theta_5, \pi - \theta_3, \pi - \theta_4] , \qquad W''' = W[\theta_5, \theta_2, \theta_6] , \end{aligned} \tag{3.15}$$

where $\theta_1, \ldots, \theta_6$ are the six angles of a spherical quadrilateral, as shown in Fig. 5, and equivalently in Fig. 7 of ZII. These angles are not independent: they necessarily satisfy the relation (3.2) of ZII.

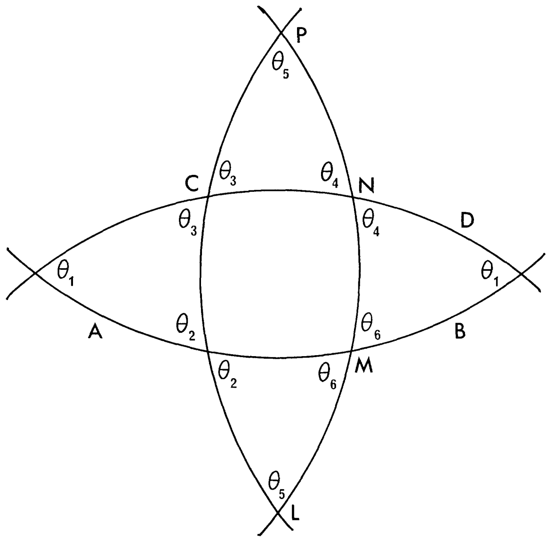

Fig. 5. A segment of a spherical quadrilateral, showing its associated four triangles, with interior angles $\theta_1, \ldots, \theta_6$

The parameters $P_i', P_i'', P_i''', Q_i', \ldots, R_3'''$ are of course given by adding primes to P, Q, R in (3.13) and substituting the appropriate values of ϕ_1, ϕ_2, ϕ_3 into (3.11), and thence into (3.12) and (3.13). For instance, the double-primed parameters $P_0'', \ldots, R_3''$ are obtained by taking ϕ_1, ϕ_2, ϕ_3 to be θ_5, $\pi - \theta_3$, $\pi - \theta_4$.

4. Angle Symmetries

We want to verify Zamolodchikov's assertion that (2.2) is satisfied by (3.15). Fortunately we do not have to prove each of the 224 equations individually. We can regard each of them as an identity, to be verified for all values of $\theta_1, \ldots, \theta_6$ satisfying the spherical quadrilateral constraint (3.2) of ZII. It turns out that many of these identities are simple corollaries of one another.

Q Negation

We can regard $\theta_1, \theta_2, \theta_3$ as determined by $\theta_4, \theta_5, \theta_6$ and the arc lengths LM, MN in Fig. 5. These parameters can be varied so as to shift the line AB in Fig. 5 upwards through the point C. The $(\theta_1, \theta_2, \theta_3)$ triangle first shrinks to a point, and then reappears in an inverted configuration, as shown in Fig. 6. This gives a new spherical quadrilateral, with angles θ_1, θ_2, θ_3, $\pi - \theta_4$, $\pi - \theta_5$, $\pi - \theta_6$.

Since $\theta_1 + \theta_2 + \theta_3 - \pi$ is the area of the $(\theta_1, \theta_2, \theta_3)$ triangle [15], it cannot become negative during this process: it has a double zero when the triangle shrinks to a point. This means that $[\tan(\theta_1 + \theta_2 + \theta_3 - \pi)/4]^{1/2}$ has a simple zero, so is negated when analytically continued from Fig. 5 to Fig. 6. The same is true of $[\sin(\theta_1 + \theta_2 + \theta_3 - \pi)/4]^{1/2}$. All other square roots retain their original positive sign.

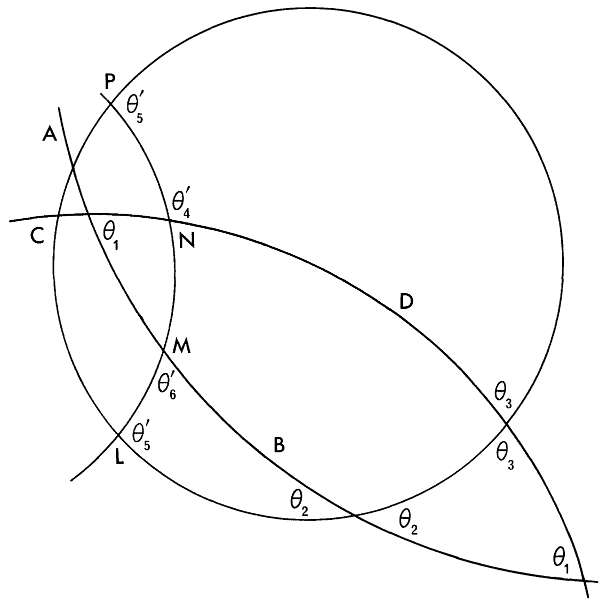

Fig. 6. The spherical quadrilateral of Fig. 5, after the line AB has been shifted through C. Here θ'_j denotes the supplement $\pi-\theta_j$ of the angle θ_j

It follows that for each of our equations, we can obtain another by the following procedure:

Use (3.15) and (3.11)–(3.13) to write the equation explicitly in terms of $\theta_1, \ldots \theta_6$; negate $[\tan(\theta_1+\theta_2+\theta_3-\pi)/4]^{1/2}$ and $[\sin(\theta_1+\theta_2+\theta_3-\pi)/4]^{1/2}$ throughout; replace $\theta_4, \theta_5, \theta_6$ by $\pi-\theta_4$, $\pi-\theta_5$, $\pi-\theta_6$.

Let us call this procedure P_{123}, and define P_{146}, P_{345}, P_{256} similarly. (Each corresponds to shrinking one of the triangles in Fig. 5 through a point.) By itself, each such procedure gives an equation which is not in our original set, but if we perform all four sequentially, then the result is to return $\theta_1, \ldots, \theta_6$ to their original values, having negated R_0 and Q_j for $j=0,1,2,3$ and for all four functions W, W', W'', W'''. (This corresponds to inverting each S matrix.) This is the (R_0, Q)-negation pair symmetry discussed between (3.5) and (3.11).

The 224 equations therefore occur in 112 pairs, each equation of a pair being a corollary of the other.

Negation of θ_1

Another way to analytically continue $\theta_1, \ldots, \theta_6$ is to allow the great circles AB and CD in Fig. 5 to first become coincident and then cross one another. The result (after vertically mirror inverting) is to replace $\theta_1, \ldots, \theta_6$ in Fig. 5 by $-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$, $\pi-\theta_4$, θ_5, $\pi-\theta_6$, respectively; i.e. to negate θ_1 and supplement $\theta_2, \theta_3, \theta_4, \theta_6$. In this process the spherical excesses (triangle areas)

$$\begin{aligned} &\theta_1+\theta_2+\theta_3-\pi, \quad \pi+\theta_1-\theta_2-\theta_3, \\ &\theta_1+\theta_4+\theta_6-\pi, \quad \pi+\theta_1-\theta_4-\theta_6, \end{aligned} \tag{4.1}$$

all pass through zero and become negative, their ratios remaining positive. Thus for each such spherical excess E in (4.1), $[\tan(E/4)]^{1/2}$ and $[\sin(E/4)]^{1/2}$ should be replaced by $i[\tan(-E/4)]^{1/2}$ and $i[\sin(-E/4)]^{1/2}$, and this should be done before transforming $\theta_1, \ldots, \theta_6$.

This procedure can be simplified by following it by procedures P_{123} and P_{146}, the effect of which is to restore $\theta_2, \theta_3, \theta_4, \theta_6$ to their original values. (It also keeps us within our original set of equations.) The result is to change W by replacing the parameters $(P_0, P_1, P_2, P_3; Q_0, Q_1, Q_2, Q_3; R_0, R_1, R_2, R_3)$ by $\xi(P_2, iQ_3, P_0, iQ_1; Q_2, -iP_3, Q_0, -iP_1; -iR_2, R_3, iR_0, R_1)$, the common factor ξ being P_2^{-1}. The same changes are made in W' (all the parameters being primed), but the functions W'' and W''' are unaltered.

For instance, under this transformation (3.9) becomes (after cancelling the common ξ factors)

$$Y_2 Y_2' Y_0'' R_0''' + R_2 R_0' R_1'' X_0''' = Y_0''' R_0'' R_1' R_3 - R_0''' Y_1'' X_3' X_3, \tag{4.2}$$

which is another of our 112 equations [obtainable from (2.2) by taking b_2, c_2, $c_6 = -1$, all other external spins $= +1$]. Applying the transformation to any of the equations gives another equation of the set (not obtainable from the first by Q negation), and repeating it gives back the original equation. The 224 equations can therefore now be grouped in 56 sets of 4, any three equations of a set being corollaries of the fourth.

Permutation Symmetry

The weight function (3.14) has various symmetry properties, in particular

$$W(\phi_i, \phi_j, \phi_k; a|e_i e_j e_k|b_i b_j b_k|h) = \text{unchanged by permuting } i, j, k, \tag{4.3}$$

and

$$\begin{aligned} W[\phi_1, \phi_2, \phi_3; a|efg|bcd|h] &= W[\pi-\phi_1, \pi-\phi_3, \phi_2; f|dba|geh|c] \\ &= W[\phi_1, \pi-\phi_3, \pi-\phi_2; b|hfg|acd|e] \\ &= W[\phi_1, \phi_2, \phi_3; h|bcd|efg|a]. \end{aligned} \tag{4.4}$$

Using these, (2.2) can be put into the more obviously symmetric form

$$\begin{aligned} &\sum_d W[\theta_2, \theta_3, \theta_1; a_4|c_2 c_3 c_1|b_1 b_2 b_3|d]\, W[\theta_1, \theta_6, \theta_4; a_3|c_1 c_6 c_4|b_4 b_1 b_2|d] \\ &\quad \cdot W[\theta_4, \theta_3, \theta_5; a_2|c_4 c_3 c_5|b_3 b_4 b_1|d]\, W[\theta_5, \theta_6, \theta_2; a_1|c_5 c_6 c_2|b_2 b_3 b_4|d] \\ &= \sum_d W[\theta_5, \theta_6, \theta_2; b_1|c_1 c_3 c_4|a_2 a_3 a_4|d]\, W[\theta_4, \theta_3, \theta_5; b_2|c_2 c_6 c_1|a_3 a_4 a_1|d] \\ &\quad \cdot W[\theta_1, \theta_6, \theta_4; b_3|c_5 c_3 c_2|a_4 a_1 a_2|d]\, W[\theta_2, \theta_3, \theta_1; b_4|c_4 c_6 c_5|a_1 a_2 a_3|d]. \end{aligned} \tag{4.5}$$

For some purposes it is convenient to replace the angles θ_r and the spins c_r by ψ_{ij} and e_{ij}, where

$$\begin{aligned} &\theta_1 = \psi_{34}, \quad \theta_2 = \psi_{14}, \quad \theta_3 = \psi_{24}, \\ &\theta_4 = \psi_{23}, \quad \theta_5 = \psi_{12}, \quad \theta_6 = \psi_{13}, \\ &c_1 = e_{34}, \quad c_2 = e_{14}, \quad c_3 = e_{24}, \\ &c_4 = e_{23}, \quad c_5 = e_{12}, \quad c_6 = e_{13}. \end{aligned} \tag{4.6}$$

Making these substitutions into (4.5), all indices are integers between 1 and 4, and it is easy to check by using (4.3) that the equation is unchanged by permuting the integers 1, 2, 3, 4. Further, any such permutation is equivalent to merely re-drawing Fig. 5, by extending the quadrilateral to another part of the spherical surface and possibly rotating or reflecting it. Thus the permutation takes one set of quadrilateral angles $\theta_1, \dots \theta_6$ to another.

There are 24 such permutations (we can think of them as the permutations of the four vertices of the tetrahedron in Fig. 2, $\theta_1, \dots, \theta_6$ being associated with the edges). Each takes one of the 224 equations to itself or another, so this is a very strong symmetry. In fact, when use is also made of the Q-negation and θ_1-negation symmetries, it turns out that all the 224 equations are corollaries of just two archetypical equations, which we can take to be Eqs. (3.9) and (3.10) above. Thus our original 2^{14} equations have finally reduced to two!

[We cannot get down to just one equation with the above transformations: they all take R-parameters to R-parameters, and X or Y-parameters to X or Y-parameters. Since each term in (3.9) has an odd number of each, and each term in (3.10) has an even number, one equation cannot be transformed to the other. The 224 equations fall into two distinct classes: an "odd" class with 128 members, and an "even" class with 96.]

5. Proof of the two Archetypal Identities

To verify that Zamolodchikov's solution does indeed satisfy the tetrahedron equations, it remains only to prove that (3.9) and (3.10) are satisfied for all sets $(\theta_1, \dots, \theta_6)$ of spherical quadrilateral angles.

First let us follow Zamolodchikov's notation [Eqs. (4.9) and (2.2) of ZII] and define functions σ, S, a, U, ω, V of ϕ_1, ϕ_2, ϕ_3 by

$$\begin{aligned} \sigma(\phi_1, \phi_2, \phi_3) &= 1 + t_0 t_1 t_2 t_3 , \\ S(\phi_1, \phi_2, \phi_3) &= 1 - t_0 t_1 t_2 t_3 , \\ a(\phi_1, \phi_2, \phi_3) &= s_0/(c_1 c_2 c_3) , \\ U(\phi_1, \phi_2 | \phi_3) &= s_3/(c_0 c_1 c_2) , \\ \omega(\phi_1, \phi_2 | \phi_3) &= t_0 t_3 + t_1 t_2 , \\ V(\phi_1, \phi_2 | \phi_3) &= t_0 t_3 - t_1 t_2 , \end{aligned} \tag{5.1}$$

where t_i, s_i, c_i are defined in terms of ϕ_1, ϕ_2, ϕ_3 by (3.12) and (3.11). (The functions σ, S, a are symmetric in ϕ_1, ϕ_2, ϕ_3; U, ω, V are symmetric only in ϕ_1 and ϕ_2.)

The 48 parameters X_j, Y_j, $R_j, \dots, R_j'''$ are given by (3.11)–(3.13) and (3.8), with arguments ϕ_1, ϕ_2, ϕ_3 determined by (3.15). They are equal to particular values of the functions $\sigma, \dots, V$ (possibly negated). For instance, setting ϕ_1, ϕ_2, ϕ_3 to be θ_5, $\pi - \theta_3$, $\pi - \theta_4$, we can verify that $Y_1'' = -V(\theta_3, \theta_4 | \theta_5)$. Doing this for all the parameters in (3.9) and (3.10), these equations can be written more explicitly as

$$\begin{aligned} &S(\theta_1,\theta_2,\theta_3)\,S(\theta_1,\theta_4,\theta_6)\,S(\theta_3,\theta_4,\theta_5)\,\mathfrak{a}(\theta_2,\theta_5,\theta_6)\\ &\quad+\mathfrak{a}(\theta_1,\theta_2,\theta_3)\,\mathfrak{a}(\theta_1,\theta_4,\theta_6)\,\mathfrak{a}(\theta_3,\theta_4,\theta_5)\,\sigma(\theta_2,\theta_5,\theta_6)\\ &=U(\theta_1,\theta_3|\theta_2)\,U(\theta_1,\theta_4|\theta_6)\,U(\theta_3,\theta_4|\theta_5)\,S(\theta_2,\theta_5,\theta_6)\\ &\quad+V(\theta_1,\theta_3|\theta_2)\,V(\theta_1,\theta_4|\theta_6)\,V(\theta_3,\theta_4|\theta_5)\,\mathfrak{a}(\theta_2,\theta_5,\theta_6)\,, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &S(\theta_1,\theta_2,\theta_3)\,S(\theta_1,\theta_4,\theta_6)\,\mathfrak{a}(\theta_3,\theta_4,\theta_5)\,\mathfrak{a}(\theta_2,\theta_5,\theta_6)\\ &\quad+\mathfrak{a}(\theta_1,\theta_2,\theta_3)\,\mathfrak{a}(\theta_1,\theta_4,\theta_6)\,\sigma(\theta_3,\theta_4,\theta_5)\,\sigma(\theta_2,\theta_5,\theta_6)\\ &=\omega(\theta_2,\theta_3|\theta_1)\,\omega(\theta_4,\theta_6|\theta_1)\,U(\theta_3,\theta_4|\theta_5)\,U(\theta_2,\theta_6|\theta_5)\\ &\quad+U(\theta_2,\theta_3|\theta_1)\,U(\theta_4,\theta_6|\theta_1)\,V(\theta_3,\theta_4|\theta_5)\,V(\theta_2,\theta_6|\theta_5)\,. \end{aligned} \tag{5.3}$$

[Equation (5.2) is precisely Eq. (3.1) of ZII.]

Simplification of $\sigma, \ldots, V$

An irritating feature of these equations is the proliferation of square roots that enter via (3.12). We can remove these by introducing the lengths of the sides of the spherical triangles, as well as their angles.

To do this, we need some basic formulae of spherical trigonometry, which are given by Todhunter and Leathem [15], here referred to as TL. Let A, B, C be the interior angles of a spherical triangle, and a, b, c the lengths of the corresponding (opposite) sides. Let

$$E=A+B+C-\pi\,, \tag{5.4}$$

$$s=\tfrac{1}{2}(a+b+c) \tag{5.5}$$

(E is the "spherical excess" of the triangle, $2s$ is the perimeter). Then, from (5.1), (3.12), and (3.11),

$$\sigma(A,B,C)=1+\left[\tan\frac{E}{4}\tan\frac{2A-E}{4}\tan\frac{2B-E}{4}\tan\frac{2C-E}{4}\right]^{1/2}. \tag{5.6}$$

The square-root expression in (5.6) is the "Lhuilierian" of the triangle (Sect. 137 of TL) and is equal to $\tan(E/4)\cot(s/2)$. It follows at once that

$$\begin{aligned} \sigma(A,B,C)&=1+\tan(E/4)/\cot(s/2)\\ &=\sin[(2s+E)/4]/\cos\frac{E}{4}\sin\frac{s}{2}\,. \end{aligned} \tag{5.7}$$

Similarly,

$$S(A,B,C)=\sin[(2s-E)/4]/\cos\frac{E}{4}\sin\frac{s}{2}\,. \tag{5.8}$$

As is shown in Sect. 28 of TL, any relation in spherical trigonometry remains true if the angles are changed into the supplements of the corresponding sides, and vice-versa. Applying this duality principle to Eq. (32) of Sect. 139 of TL, we obtain

$$\sin^2\frac{1}{2}s=\frac{\sin\frac{E}{4}\cos\frac{2A-E}{4}\cos\frac{2B-E}{4}\cos\frac{2C-E}{4}}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}\,, \tag{5.9}$$

and from this and (5.1), (3.11), and (3.12) we can verify that

$$\mathcal{a}(A,B,C)=\frac{\sin(E/4)}{\sin\frac{s}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]^{1/2}}. \tag{5.10}$$

Using Cagnoli's theorem (Sect. 132 of TL), we can establish that

$$\sin\tfrac{1}{2}E\sin(C-\tfrac{1}{2}E)=\sin^2\frac{c}{2}\sin A\sin B, \tag{5.11}$$

while from (5.1), (3.11), and (3.12),

$$\frac{U(A,B|C)}{\mathcal{a}(A,B|C)}=\left[\frac{\sin(C-\frac{1}{2}E)}{\sin\frac{1}{2}E}\right]^{1/2}. \tag{5.12}$$

Eliminating $\sin(C-\frac{1}{2}E)$ between (5.11) and (5.12), then using (5.10), we obtain

$$U(A,B|C)=\frac{\sin\frac{c}{2}\left[\cos\frac{A}{2}\cos\frac{B}{2}\right]^{1/2}}{\cos\frac{E}{4}\sin\frac{s}{2}\left[\sin\frac{C}{2}\right]^{1/2}}. \tag{5.13}$$

Taking the duals of Eqs. (34) and (35) of Sect. 140 of TL, then dividing by (5.9), gives the formulae

$$\frac{\sin\frac{1}{2}(s-c)}{\sin\frac{1}{2}s}=\left\{\tan\frac{2A-E}{4}\tan\frac{2B-E}{4}\tan\frac{A}{2}\tan\frac{B}{2}\right\}^{1/2}, \tag{5.14}$$

$$\frac{\cos\frac{1}{2}(s-c)}{\sin\frac{1}{2}s}=\left\{\cot\frac{E}{4}\tan\frac{2C-E}{4}\tan\frac{A}{2}\tan\frac{B}{2}\right\}^{1/2}. \tag{5.15}$$

Using these, it follows from (5.1), (3.11), and (3.12) that

$$\omega(A,B|C)=\frac{\tan\frac{E}{4}\cos\frac{s-c}{2}+\sin\frac{s-c}{2}}{\sin\frac{s}{2}\left[\tan\frac{A}{2}\tan\frac{B}{2}\right]^{1/2}}$$

$$=\frac{\sin[(E+2s-2c)/4]}{\cos\frac{E}{4}\sin\frac{s}{2}\left[\tan\frac{A}{2}\tan\frac{B}{2}\right]^{1/2}}. \tag{5.16}$$

Similarly,

$$V(A,B|C)=\frac{\sin[(E-2s+2c)/4]}{\cos\frac{E}{4}\sin\frac{s}{2}\left[\tan\frac{A}{2}\tan\frac{B}{2}\right]^{1/2}}. \tag{5.17}$$

For the purpose of verifying (5.2) and (5.3), these expressions for σ, S, $\mathcal{a}$, U, ω, V are more convenient than the original definitions (5.1). They contain square roots only of multiplicative functions of individual angles, and these cancel out of Eqs. (5.2) and (5.3).

To use these forms, we need a notation for the lengths of the sides of the four triangles $(\theta_1, \theta_3, \theta_2)$, $(\theta_4, \theta_1, \theta_6)$, $(\theta_3, \theta_4, \theta_5)$, $(\theta_2, \theta_6, \theta_5)$ in Fig. 5. If θ_i and θ_j are two interior angles of a triangle, let $r_{ij}(=r_{ji})$ be the length of the side joining them. Thus $r_{65}=r_{56}=LM$, $r_{64}=MN$ and $r_{45}=NP$. Since every great circle has length 2π, and any two great circles bisect each other, the length LP (along either circle) is π. The lengths r_{ij} therefore satisfy the four relations

$$r_{56}+r_{64}+r_{45}=r_{52}+r_{23}+r_{35}=r_{12}+r_{26}+r_{61}=\pi, \tag{5.18}$$

$$r_{13}+r_{41}+r_{34}=\pi. \tag{5.19}$$

We shall also need following quantities associated with a triangle $(\theta_i, \theta_j, \theta_k)$:

$$2\alpha_{ijk}=\theta_i+\theta_j+\theta_k-\pi, \tag{5.20}$$

$$2s_{ijk}=r_{ij}+r_{jk}+r_{ki}, \tag{5.21}$$

$$e_i=\exp\tfrac{1}{2}\theta_i, \qquad x_i=\cos\tfrac{1}{2}\theta_i, \qquad y_i=\sin\tfrac{1}{2}\theta_i, \tag{5.22}$$

$$\begin{aligned} &F_{ijk}=\sin\tfrac{1}{2}(s_{ijk}+\alpha_{ijk}), \qquad G_{ijk}=\sin\tfrac{1}{2}(s_{ijk}-\alpha_{ijk}) \\ &H_{ijk}=\tfrac{1}{2}\sin\alpha_{ijk}, \qquad L_{ijk}=\sin\tfrac{1}{2}r_{ij}, \\ &M_{ijk}=\sin\tfrac{1}{2}(\alpha_{ijk}+s_{ijk}-r_{ij}), \\ &N_{ijk}=\sin\tfrac{1}{2}(\alpha_{ijk}-s_{ijk}+r_{ij}). \end{aligned} \tag{5.23}$$

From (5.18). (5.19), and (5.21) it is apparent that

$$s_{132}+s_{416}+s_{345}+s_{265}=2\pi. \tag{5.24}$$

First Identity

Substituting the forms given in (5.7)–(5.17) for the functions $\sigma, \ldots, V$ into the first identity (5.2), cancelling common factors, and using the definitions (5.22) and (5.23), the identity becomes

$$\begin{aligned} &\mu G_{132}G_{416}G_{345}H_{265}+H_{132}H_{416}H_{345}F_{265} \\ &\quad=\lambda\mu L_{132}L_{416}L_{345}G_{265}+\lambda N_{132}N_{416}N_{345}H_{265}, \end{aligned} \tag{5.25}$$

where

$$\lambda=x_1x_4x_3, \qquad \mu=y_1y_4y_3. \tag{5.26}$$

Each of H_{265}, F_{265}, G_{265} is defined by (5.23) in terms of a sine function. For these quantities, it is convenient to write $\sin u$ as $\mathrm{Im}[\exp iu]$ (or, for G_{265}, as $-\mathrm{Im}[\exp(-iu)]$. The resulting expressions can be factored, using (5.20) and (5.24), into a product of 3 terms associated with the other triangles, e.g.

$$F_{265}=\mathrm{Im}(A_{132}A_{416}A_{345}), \tag{5.27}$$

where

$$A_{ijk}=\exp[i(\pi+\theta_k-2s_{ijk})/4]. \tag{5.28}$$

(I use the convention that where i occurs as an index, it is an integer between 1 and 6; elsewhere it is the square root of -1.)

Subtracting the right hand side from the left hand side, it follows that (5.25) can be written as

$$\mathrm{Im}(J)=0\,, \tag{5.29}$$

where

$$\begin{aligned} J=&-\tfrac{1}{2}i\mu\hat{G}_{132}\hat{G}_{416}\hat{G}_{345}-e_1e_3e_4\hat{H}_{132}H_{416}\hat{H}_{345}\\ &-i\lambda\mu(e_1e_3e_4)^{-1}\hat{L}_{132}\hat{L}_{416}\hat{L}_{345}-\tfrac{1}{2}\lambda\hat{N}_{132}\hat{N}_{416}\hat{N}_{345}\,, \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} &\hat{G}_{ijk}=G_{ijk}\exp[\tfrac{1}{2}i(\pi+\theta_k+r_{ij})]\,,\\ &\hat{H}_{ijk}=H_{ijk}\exp[i(\theta_k-\theta_i-\theta_j-\pi+2r_{ij}-2s_{ijk})/4]\\ &\hat{L}_{ijk}=L_{ijk}\exp[\tfrac{1}{2}i(\alpha_{ijk}+s_{ijk})]\,,\\ &\hat{N}_{ijk}=N_{ijk}\exp[\tfrac{1}{2}i(\pi+\theta_k)] \end{aligned} \tag{5.31}$$

Each of $\hat{G}_{ijk},\ldots,\hat{N}_{ijk}$ is a property of a single spherical triangle, with angles θ_i, θ_j, θ_k. We can express them in terms of two angles and an included side; in particular of θ_i, θ_j, and r_{ij}. This will give J as a function only of θ_1, θ_4, θ_3 and r_{13}, r_{41}, r_{34}. These variables are independent except for the simple relation (5.19), so we should be able to verify explicitly that (5.29) is satisfied.

In fact we can simplify this procedure. Using only (5.20), (5.23), and (5.31), it is readily seen that each of $\hat{G}_{ijk}$, $\hat{H}_{ijk}$, $\hat{L}_{ijk}$, $\hat{N}_{ijk}$ is a linear combination of the expressions

$$\begin{gathered} \exp[i(2s_{ijk}+\theta_k)/4]\,,\qquad \exp[-i(2s_{ijk}+\theta_k)/4]\,,\\ \exp[i(-2s_{ijk}+3\theta_k)/4]\,, \end{gathered} \tag{5.32}$$

with coefficients that are simple explicit functions of θ_i, θ_j, and r_{ij}.

We can relate the three expressions (5.32) by using spherical trigonometry. The duals of Eqs. (25) and (26) of Sect. 138 of TL are

$$\begin{gathered} \sin s=\sin c\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\,\mathrm{cosec}\tfrac{1}{2}C\,,\\ \cos s=[-\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B+\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos c]\,\mathrm{cosec}\tfrac{1}{2}C\,. \end{gathered} \tag{5.33}$$

From these it follows that

$$e^{-is}\sin\tfrac{1}{2}C=[\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\,e^{-ic}-\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B]\,. \tag{5.34}$$

Expanding $\sin\frac{1}{2}C$ on the left hand side in terms of $\exp(\pm\frac{1}{2}iC)$, then multiplying by $\exp[i(2s+C)/4]$ and replacing A, B, C, s, c by θ_i, θ_j, θ_k, s_{ijk}, r_{ij}, we obtain a linear relation between the expressions (5.32). Thus we can write $\hat{G}_{ijk},\ldots,\hat{N}_{ijk}$ as linear combinations of any two of the expressions (5.32), with coefficients that are simple explicit functions of θ_i, θ_j, and r_{ij}.

More conveniently, we can write them as linear combinations of

$$\begin{aligned} p_{ijk}&=\sin[\tfrac{1}{2}(s_{ijk}+\alpha_{ijk}-r_{ijk})]\,,\\ q_{ijk}&=\exp(\tfrac{1}{2}ir_{ij})\sin[\tfrac{1}{2}(s_{ijk}+\alpha_{ijk})] \end{aligned} \tag{5.35}$$

[which are themselves linear combinations of the expressions (5.32)]. We find that

$$\begin{aligned}\hat{G}_{ijk}&=2x_ix_jp_{ijk}+[-\cos\tfrac{1}{2}(\theta_i-\theta_j)+i\sin\tfrac{1}{2}(\theta_i+\theta_j)]q_{ijk},\\ \hat{H}_{ijk}&=x_ix_jp_{ijk}-y_iy_jq_{ijk},\\ \hat{L}_{ijk}&=-p_{ijk}+q_{ijk},\\ \hat{N}_{ijk}&=-[\cos\tfrac{1}{2}(\theta_i-\theta_j)+i\sin\tfrac{1}{2}(\theta_i+\theta_j)]p_{ijk}+2y_iy_jq_{ijk}.\end{aligned}\tag{5.36}$$

Substituting these expressions into (5.30) and re-arranging, we obtain (after many cancellations)

$$2J=-x_1x_3x_4p_{132}p_{416}p_{345}-iy_1y_3y_4q_{132}q_{416}q_{345}.\tag{5.37}$$

Using (5.19), we find from (5.35) that $q_{132}q_{416}q_{345}$ is pure imaginary. Since x_j, y_j, p_{ijk} are real, it follows that J is real. We have therefore verified (5.29), and hence the identities (5.2) and (3.9).

It is interesting to note that after (5.36) we have only used the definitions (5.35) to note that $p_{132}p_{416}p_{345}$ and $iq_{132}q_{416}q_{345}$ are real. Thus (5.29) is true for any expression J given by (5.30) and (5.36), the only restrictions on p_{ijk} and q_{ijk} being these two reality conditions.

Second Identity

The same general techniques can be used to verify the second identity (5.3), but there is no longer such a simple symmetry between triangles (132), (416), and (345).

Substituting into (5.3) the expressions for σ, $S,\ldots,V$ given in (5.7)–(5.17), cancelling common factors and using the definitions (5.23), we obtain

$$\begin{aligned}&y_iG_{132}G_{146}H_{345}H_{265}+y_5H_{132}H_{146}F_{345}F_{265}\\ &\quad=\varrho y_1M_{231}M_{641}L_{345}L_{265}+\varrho y_5L_{231}L_{641}N_{345}N_{265},\end{aligned}\tag{5.38}$$

where

$$\varrho=x_2x_3x_4x_6.\tag{5.39}$$

Writing the sine functions associated with triangle (256) as imaginary parts of exponentials, and using (5.18)–(5.20) and (5.24) to share out these exponentials between the other three triangles, (5.38) can be written as

$$\mathrm{Im}(K)=0,\tag{5.40}$$

where

$$\begin{aligned}K=&\tfrac{1}{2}iy_1\hat{G}_{132}\hat{G}_{146}\hat{\hat{H}}_{345}-e_1\hat{H}_{132}\hat{H}_{146}\hat{\hat{F}}_{345}\\ &+iy_1x_3x_4\hat{M}_{132}\hat{M}_{146}\hat{\hat{L}}_{345}+e_1^{-1}x_3x_4\hat{Z}_{132}\hat{Z}_{146}\hat{\hat{N}}_{345},\end{aligned}\tag{5.41}$$

$\hat{G}$, $\hat{H}$ being defined by (5.36), and $\hat{M},\ldots,\hat{\hat{N}}$ by

$$\begin{aligned}\hat{M}_{ijk}&=x_kM_{kji}\exp[-\tfrac{1}{2}i(\pi+r_{ik})],\\ \hat{Z}_{ijk}&=x_kL_{kji}\exp[i(\pi+\theta_i+\theta_k-\theta_j+2s_{ijk}-2r_{ik})/4],\\ \hat{\hat{H}}_{345}&=H_{345}\exp[\tfrac{1}{2}i(\theta_5+r_{34}-\pi)],\\ \hat{\hat{F}}_{345}&=y_5F_{345}\exp[\tfrac{1}{2}i(\alpha_{345}-s_{345}+r_{34})],\\ \hat{\hat{N}}_{345}&=y_5N_{345}\exp[\tfrac{1}{2}i(\alpha_{345}+s_{345})].\end{aligned}\tag{5.42}$$

Again we look for linear relations between the functions associated with each triangle, the coefficients being explicit functions of $\theta_1, \theta_3, \theta_4, r_{13}, r_{14}, r_{34}$. Using (5.34), we find that

$$\begin{aligned} \hat{M}_{ijk} &= x_i e_j p_{ijk} - y_i y_j q_{ijk} - i x_i y_j q^*_{ijk}, \\ \hat{Z}_{ijk} &= x_i p_{ijk} + i y_i q_{ijk}, \\ \hat{\hat{F}}_{345} &= \hat{\hat{H}}_{345} + x_3 x_4 L_{345}, \\ \hat{\hat{N}}_{345} &= \hat{\hat{H}}_{345} - y_3 y_4 L_{345}, \end{aligned} \tag{5.43}$$

q^*_{ijk} being the complex conjugate of q_{ijk}.

Substituting these expressions, and the expressions (5.36) for $\hat{G}$ and $\hat{H}$, into (5.41), one finds that K can be written as

$$K = K_1 + K_2 + K_2^*, \tag{5.44}$$

where

$$K_1 = i y_1 q_{132} q_{146} [\tfrac{1}{2} e_3^{-1} e_4^{-1} \hat{\hat{H}}_{345} - x_3 x_4 y_3 y_4 L_{345}], \tag{5.45}$$

$$\begin{aligned} K_2 = \{ & \tfrac{1}{2} x_1 (y_1 x_3 y_4 + y_1 y_3 x_4 - x_1 x_3 x_4 - x_1 y_3 y_4) p_{132} p_{146} \\ & + i x_1 y_1 (y_3 q_{132} - i e_3^{-1} p_{132})(y_4 q_{146} - i e_4^{-1} p_{146}) - y_1^2 y_3 y_4 q_{132} q^*_{146} \} x_1 x_3 x_4 L_{345} \end{aligned} \tag{5.46}$$

Plainly $K_2 + K_2^*$ is real, so to verify (5.40) we have only to show that K_1 is real. Using (5.42), (5.23), (5.35), (5.19) and the spherical trigonometric formula

$$\sin\theta_3 \sin\theta_4 \cos r_{34} = \cos\theta_5 + \cos\theta_3 \cos\theta_4$$

(Sect. 54 of TL), we can establish that

$$K_1 = \tfrac{1}{8} y_1 F_{132} F_{146} [\sin\theta_5 - \sin(\theta_3 + \theta_4) + \sin\theta_3 \sin\theta_4 \sin r_{34}]. \tag{5.47}$$

Thus K_1 is real; we have verified (5.40) and hence the identities (5.3) and (3.10).

I have assumed that $\theta_1, \ldots, \theta_6$ and $r_{12}, \ldots, r_{56}$ are real numbers: this is really just a notational device to avoid writing (5.30) and (5.41) twice, once as given and once with i replaced by $-i$. The identities (5.2) and (5.3) are basically algebraic, so must of course be true for complex values of the parameters as well as real ones (so long as consistent choices are made of the branches of multi-valued functions).

6. The Static Solution

In the first of his two papers, i.e. in ZI, Zamolodchikov considers the "static limit" of the tetrahedron equations. This can be thought of as the limit when the $(\theta_1, \theta_2, \theta_3)$ and $(\theta_2, \theta_5, \theta_6)$ triangles in Fig. 5 are infinitesimally small, in which case

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= -\theta_1 + \theta_4 + \theta_6 = \theta_3 + \theta_4 - \theta_5 \\ &= \theta_2 + \theta_5 + \theta_6 = \pi. \end{aligned} \tag{6.1}$$

The spherical quadrilateral becomes planar, as in Fig. 7 of ZII.

In this case the parameters Q_0, Q_1, Q_2, Q_3, R_0 in Table 1 vanish, for all four functions W, W', W'', W'''. Thus each W is determined by just seven parameters: $P_0, P_1, P_2, P_3, R_1, R_2, R_3$.

It is interesting to see if the tetrahedron equations (2.2) admit some more general solution than that found by Zamolodchikov, containing Zamolodchikov's as a special case. In general this is a very difficult problem, but one possible start is to attempt to generalize the static limit solution, i.e. to look for solutions of (2.2) such that the weight functions W, W', W'', W''' all have the form given in Table 1 (different functions having different values of the constants), with Q_0, Q_1, Q_2, Q_3, R_0 all equal to zero.

Many simplifications arise in this limit. For arbitrary $P_0, \ldots, R_3'''$ we can still use the spin symmetries of Sect. 3 to reduce the number of equations to 224, occurring in 112 pairs, each equation of a pair being obtained from the other by negating every Q_j. However, since we are taking every Q_j to be zero, this means that the two equations of a pair are identical, so there are only 112 distinct equations.

Each of these equations is of the form

$$\pm A \pm B = \pm C \pm D, \tag{6.2}$$

where each of A, B, C, D is a product of four of the parameters $P_0, \ldots, R_3'''$ (one for each of the four weight functions W, W', W'', W'''). Since R_0 in Table 1 is zero, some of A, B, C, D may vanish. Indeed, 18 of the 112 equations are simply

$$0+0=0+0. \tag{6.3}$$

This leaves us with 94 non-trivial distinct equations, which break up into the following four main sets:

(i) 28 equations of the form

$$A+0=C+0 \tag{6.4}$$

(i.e. one non-zero product on each side),

(ii) 12 equations of the form

$$A-B=0+0, \tag{6.5}$$

(iii) 36 equations of the form

$$A \pm B = C+0, \tag{6.6}$$

(iv) 18 equations of the form

$$A \pm B = C \pm D. \tag{6.7}$$

Obviously (2.2) is unchanged by multplying any of the four weight functions by a constant. Assuming that P_0 is non-zero, we can without further loss of generality choose it to be unity, for each of the functions W, W', W'', W'''. Thus

$$\mathrm{P}_0 = \mathrm{P}_0' = \mathrm{P}_0'' = \mathrm{P}_0''' = 1. \tag{6.8}$$

This leaves us with six available parameters for each function, giving us 24 in all.

We now seek to systematically solve the 94 equations for these 24 unknowns. The 40 equations of types (i) and (ii) involve only two products, so are quite easy to examine. Assuming that our remaining 24 parameters are all non-zero, we could linearize these equations by taking logarithms. It turns out that they are satisfied if and only if there exist 12 parameters $x_1, \ldots, x_6, y_1, \ldots, y_6$ such that

$$\begin{array}{lll}
P_1=(z_1z_3)^{1/2}, & P_2=(z_2z_3)^{1/2}, & P_3=(z_1z_2)^{1/2}, \\
R_1=(y_2/x_1x_3)^{1/2}, & R_2=(y_1/x_2x_3)^{1/2}, & R_3=(y_3/x_1x_2)^{1/2}, \\
P'_1=(z_1/z_4)^{1/2}, & P'_2=1/(z_4z_6)^{1/2}, & P'_3=(z_1/z_6)^{1/2}, \\
R'_1=(x_6/x_1y_4)^{1/2}, & R'_2=(y_1/y_4y_6)^{1/2}, & R'_3=(x_4/x_1y_6)^{1/2}, \\
P''_1=1/(z_3z_4)^{1/2}, & P''_2=(z_5/z_4)^{1/2}, & P''_3=(z_5/z_3)^{1/2}, \\
R''_1=(y_5/y_3y_4)^{1/2}, & R''_2=(x_3/y_4x_5)^{1/2}, & R''_3=(x_4/y_3x_5)^{1/2}, \\
P'''=(z_2z_6)^{1/2}, & P'''_2=(z_5z_6)^{1/2}, & P'''_3=(z_2z_5)^{1/2}, \\
R'''_1=(y_5/x_2x_6)^{1/2}, & R'''_2=(y_2/x_5x_6)^{1/2}, & R'''=(y_6/x_2x_5)^{1/2},
\end{array} \tag{6.9}$$

where for brevity I have introduced $z_1, \ldots, z_6$ such that

$$z_j=y_j/x_j, \tag{6.10}$$

and the $\frac{1}{2}$ powers are introduced for later convenience.

At this stage it may be noted that if the equations of type (ii) are changed from $A-B=0$ to $A+B=0$, then the combined set of 40 equations has no solutions with $P_1, \ldots, R'''_3$ all non-zero. This means that the sign factors *ab*, *ac*, *ad* in Table 1 are essential and that W cannot be chosen to have all its values non-negative. This is unfortunate from the viewpoint of statistical mechanics.

Now we substitute these expressions for $P_1, \ldots, R'''_3$ into the 36 equations of type (iii), and obtain

$$\begin{aligned}
y_1+y_2y_3&=x_2x_3\\
y_2+y_3y_1&=x_3x_1\\
y_3+y_1y_2&=x_1x_2\\
-y_1+y_4y_6&=x_4x_6\\
y_4-y_6y_1&=x_6x_1\\
y_6-y_1y_4&=x_1x_4\\
y_3-y_4y_5&=x_4x_5\\
y_4-y_5y_3&=x_5x_3\\
-y_5+y_3y_4&=x_3x_4\\
y_2+y_5y_6&=x_5x_6\\
y_5+y_6y_2&=x_6x_2\\
y_6+y_2y_5&=x_2x_5\\
y_2y_4+y_3y_6&=x_1x_5\\
y_3y_6-y_1y_5&=x_2x_4\\
y_1y_5+y_2y_4&=x_3x_6.
\end{aligned} \tag{6.11}$$

(The first 12 equations occur twice, the last 3 occur four times.)

Eliminating x_3 and y_3 between the first three equations gives

$$\Delta_1 = \Delta_2, \tag{6.12}$$

where

$$\Delta_j = (x_j^2 + y_j^2 - 1)/(x_j y_j). \tag{6.13}$$

By symmetry, it follows from the first three equations that $\Delta_1 = \Delta_2 = \Delta_3$. Similarly, the next three sets of three give $-\Delta_1 = \Delta_4 = \Delta_6$, $\Delta_3 = \Delta_4 = -\Delta_5$, $\Delta_2 = \Delta_5 = \Delta_6$.

The only solution of these equations is

$$\Delta_j = 0, \quad j = 1, \ldots, 6, \tag{6.14}$$

so that

$$x_j^2 + y_j^2 = 1. \tag{6.15}$$

We can therefore choose six unknowns $\theta_1, \ldots, \theta_6$ such that

$$x_j = \cos\tfrac{1}{2}\theta_j, \quad y_j = \sin\tfrac{1}{2}\theta_j \tag{6.16}$$

for $j = 1, \ldots, 6$. The first twelve equations in (6.11) then reduce to

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= \pi, \\ -\theta_1 + \theta_4 + \theta_6 &= \pi, \\ \theta_3 + \theta_4 - \theta_5 &= \pi, \\ \theta_2 + \theta_5 + \theta_6 &= \pi, \end{aligned} \tag{6.17}$$

(apart from additive multiples of 2π which can be absorbed into $\theta_1, \ldots, \theta_6$). These are precisely Eqs. (6.1), so we can regard $\theta_1, \ldots, \theta_6$ as the angles of a plane quadrilateral. Comparing these results with (3.11)–(3.15), we find that we have regained Zamolodchikov's solution in the static limit. Thus this is the only solution of (2.2) in which the weights W, W', W'', W''' have the form given in Table 1, with Q_0, Q_1, Q_2, Q_3, R_0 all zero.

The last three equations in (6.11), as well as all the type (iv) equations, are now satisfied automatically: indeed they have to be, as these are just special cases of the general equations which have been verified in Sects. 3–5.

7. Summary

The tetrahedron equations are given in (2.2) and Zamolodchikov's solution in Table 1 and Eqs. (3.11)–(3.15). For this solution, the 2^{14} tetrahedron equations reduce, first to 224 non-trivial distinct equations, and then to just two identities. In Sect. 5 I have proved these identities, thereby verifying Zamolodchikov's solution.

This solution is very special: it does contain three adjustable parameters for any particular Boltzmann weight function W, namely the three angles θ_1, θ_2, θ_3 of a spherical triangle. However, these parameters are probably "irrelevant" (in the language of renormalization group theory), just as the corresponding elliptic function argument u (or v) for the two-dimensional eight-vertex model is irrelevant [1]. If so, then no critical behaviour can be observed by varying these parameters. The solution also has the property that some weights occur in anti-symmetric pairs of opposite sign, which is unfortunate from the point of view of statistical mechanics.

Even so, it is remarkable that the tetrahedron equations have any non-trivial solutions at all, and this leads one to hope that other three-dimensional solutions may be found, perhaps by generalizing Zamolodchikov's result. In Sect. 6 I have attempted to do this in a modest way by restricting the weight functions to have the same symmetries, anti-symmetries and zero elements as Zamolodchikov's "static limit" solution. Unfortunately it turns out that no such generalization is possible: Zamolodchikov's is the only solution of this form. This is disappointing, but one can still hope that other, less restricted, generalizations or alternative solutions remain to be found.

Ultimately, of course, one is interested in statistical mechanics in calculating the partition-function per site $Z^{1/N}$. Zamolodchikov calls this the "unitarizing factor" (dropping the superfix $1/N$), and writes down the inversion equation for it in (5.2) of ZII. Unfortunately this determines $Z^{1/N}$ only if appropriate analyticity assumptions are made [16, 17], and it is not obvious what these are, or precisely how to use them. Again, one would like to generalize W to include a temperature-like variable. The analyticity assumptions could then be checked against low- or high-temperature series expansions, as can be done for two-dimensional exactly solved models [10].

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