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Renormalization Group for a Critical Lattice Model

Effective Interactions Beyond the Perturbation Expansion or Bounded Spins Approximation

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Abstract. New methods are developed for the study of the Kadanoff-Wilson renormalization group for critical lattice systems of unbounded spins. The methods are based on a combination of expansion and analyticity techniques and are applied to a nonlocal hierarchical model of the dipole gas. They remove the main obstacle against the use of the block spin strategy in more realistic models such as ϕ_d^4, the dipole gas and the anharmonic crystal.

1. Introduction

In the previous publications [4–6] the present authors have developed methods for a non-perturbative analysis of the Kadanoff-Wilson renormalization group (RG) in the context of massless lattice theories such as the dipole gas and ϕ_d^4, $d \geqq 4$, at the critical point. It was soon realized that the main problem was the simultaneous control of the clustering and positivity properties of the effective interactions. To study these questions separately, we introduced a hierarchical approximation to the systems (see [4] for more details and the motivation). We considered different cases depending on whether the fields describing fluctuations on a given distance scale were bounded or unbounded in magnitude and whether they had covariance totally local or one with an exponential falloff (as in the real models). The bounded nonlocal model was suitable for the study of clustering [4, 5] whereas the unbounded local model dealt with the positivity problem [6]. The first case was solved by use of a cluster expansion to compute the effective potential, the solution of the second one was based on the study of the analyticity improving properties of the RG transform.

In the present paper we combine these two ideas to carry out the analysis in the unbounded nonlocal case. We prove that for a wide class of potentials the RG

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drives the system to the line of massless Gaussian fixed points. We show this for the potentials but, as in [5], the analysis could be extended to the correlation functions giving their infrared behavior. Once combined with a careful study of the Gaussian piece of the effective interactions, the methods developed here lend themselves to the study of more realistic models mentioned above. Briefly, the idea here is that if the Gibbs factor is analytic in fields in a polystrip, then the RG transformation expands the analyticity region due to the scaling involved, producing a contractive effect. This is exhibited by analyzing the fluctuation field integral by means of a partially resummed cluster expansion. The resummation is done in the regions where the block spin field is large and the expansion would not converge. The (usually large) contributions from these regions are estimated by iteration of an *a priori* bound showing that they are not too large and do not spoil the positivity of the action.

Section 2 of the paper gives the description of the cluster expansion and its partial resummation. Section 3 states the inductive bounds for the effective interactions and shows how they carry over to the next step.

2. Set-Up of the Expansion

We work with the periodic boundary conditions. Our field $\psi=(\psi_x)$ (the block spin field of the model on some scale) lives on the periodic lattice $\Lambda_N:=\mathbb{Z}^d_{L^N}$, (viewed also as $]-\frac{1}{2}L^N, \frac{1}{2}L^N[^d\bigcap\mathbb{Z}^d$), see [4, Sect. 2]. For notational convenience L is taken odd. The (effective) interaction for ψ is an even functional $V(\psi)$, $V(0)=0$, invariant under periodic translations of ψ. The block spin field on the next scale $\phi\equiv(\phi_x)$ lives on Λ_{N-1} and is related to ψ by

$$\psi_x=L^{-d/2}\phi_{[L^{-1}x]}+\mathscr{A}(x-L[L^{-1}x])Z_{[L^{-1}x]}, \tag{1}$$

where Z is by definition the fluctuation field (on Λ_{N-1}). Here $\mathscr{A}$ is a fixed function bounded by 1, with mean zero, supported by the $L\times\ldots\times L$ block in $\mathbb{Z}^d$ centered at the origin. Also, $[L^{-1}x]$ denotes the integral point nearest to $L^{-1}x$. The effective interaction V' on the next scale is a functional of the block spin field ϕ obtained by averaging out Z in the Gibbs factor for ψ

$$\exp[-V'(\phi)]=\int\exp[-V(\psi)]\,dv_{\Gamma_{N-1}}(Z)/(\phi=0), \tag{2}$$

where $dv_\Gamma(Z)$ is a Gaussian measure with covariance Γ, and we have normalized the expression so that $V'(0)=0$.

The covariances Γ_N will be obtained from a single covariance $\Gamma=(\Gamma_{xy})_{x,y\in\mathbb{Z}^d}$ such that

$$\Gamma_{x+a\,y+a}=\Gamma_{xy}, \tag{3}$$

$$\Gamma_{xx}=1, \tag{4}$$

and

$$|\Gamma_{xy}|\leqq\exp[-\mathrm{Ad}(x,y)], \tag{5}$$

with A sufficiently big.

Throughout this paper we use the distance $d(x,y)=\sum_i|x^i-y^i|$ on $\mathbb{R}^d$; Γ_N will be the average of $\Gamma|_{\Lambda_N\times\Lambda_N}$ over periodic translations by vectors in $\mathbb{Z}^d_{L^N}$. It is straightforward that Γ_N satisfy the periodic versions of (3)–(5). Our estimates will be uniform in N.

We shall consider various subsets of periodic lattices D, X, Y and the like, which we always assume to be built of the blocks of L^d sites centered at points of $L\mathbb{Z}^d$, unless otherwise stated or obvious from the definition. For any general lattice subset B, $\bar{B}$ will denote the smallest subset containing B built of the blocks, $L^{-1}B := \{[L^{-1}x] : x \in B\}$, $LB := \overline{\{Lx : x \in B\}}$. For X built of blocks, by $|X|$ we shall denote the number of blocks contained in X and by $\mathscr{L}(X)$ the length divided by L of the shortest connected graph on the centers of the blocks contained in X. Then X is said to be connected if the graph on the centeres of the blocks building it, obtained by joining those corresponding to the nearest neighbor (n.n.) blocks, is connected (i.e. the centers of blocks of X cannot be divided into two non-empty subsets with no graph lines between them).

Suppose at the beginning that

$$V = \sum_{\emptyset \neq Y \subset \Lambda_N} V_Y + \sum_{x \in \Lambda_N} v_x, \tag{6}$$

where V_Y, v_x are even, vanish at zero, and depend only on $\psi|_Y$ and ψ_x respectively. Then for $D \subset \Lambda_N$, partly resumming the Mayer expansion for $\exp[-V]$,

$$\exp[-V] = \sum_{\{Y_\alpha\}} \prod_\alpha (\exp[-V_{Y_\alpha}] - 1) \prod_x \exp[-v_x],$$

we obtain

$$\exp[-V] = \sum_{\{X_j\}} \prod_j g_{X_j}^{D \cap X_j} \exp\left[- \sum_{Y : Y \cap \left(\bigcup_j X_j\right) = \emptyset} V_Y - \sum_{x \notin D} v_x \right], \tag{7}$$

where the sum over $\{X_j\}$ runs over the sets of disjoint subsets $X_j \subset \Lambda_N$ such that $\bigcup_j X_j \supset D$ and that $D \cap X_j$ are built from connected components (c.c.) of D. Here

$$g_{X_j}^{D \cap X_j} = \sum_{\{Y_\gamma\}} \prod_\alpha (\exp[-V_{Y_\alpha}] - 1) \prod_{x \in D \cap X_j} \exp[-v_x], \tag{8}$$

where $\sum_{\{Y_\alpha\}}$ is restricted as follows: $(D \cap X_j) \bigcup \left(\bigcup_\alpha Y_\alpha \right) = X_j$ and X_j cannot be divided into two subsets (built of blocks) such that each c.c. of $D \cap X_j$ and each Y_α is in either one. In the future whenever such restrictions appear we shall shortly state that X_j is to be connected with respect to c.c. of $D \cap X_j$ and Y_α.

In the sequel we shall use the representation (7) for the Gibbs factor after $n - n_0$ RG transformations (2) for ψ such that $|\psi_x| < n^2$ on $\Lambda \backslash D$. Hence g will contain the information about the behavior of V for large fields.

Notice the relation between g_X^D and $g_{X_1}^{D_1}$ with $D_1 \supset D$:

$$g_{X_1}^{D_1} = \sum_{\{X_j\}} \prod_j g_{X_j}^{D \cap X_j} \sum_{\{Y_\alpha\}} \prod_\alpha (\exp[-V_{Y_\alpha}] - 1) \prod_{x \in D_1 \backslash D} \exp[-v_x]. \tag{9}$$

Here $\sum_{\{X_j\}}$ runs over the sets of disjoint subsets of X_1 such that $D \cap X_j$ is built from c.c. of D and $\bigcup_j X_j \supset D$. Also $\sum_{\{Y_\alpha\}}$ is over sets of subsets of $X_1 \Big\backslash \bigcup_j X_j$ such that X_1 is connected with respect to c.c. of D_1, X_j, and Y_α.

To produce representation (6) and (7) for V' we shall cluster-expand (2) by the method of [9, 2]. Let us order the points of $\mathbb{Z}^d$. Let y be the first point of Λ_{N-1}. For $s\in[0,1]$ define the covariance Γ^s (we omit the subscript $N-1$)

$$\Gamma^s_{x_1x_2}=\begin{cases}\Gamma_{x_1x_2} & \text{if either } x_1=x_2=y \text{ or } x_1\neq y \text{ and } x_2\neq y,\\ s\Gamma_{x_1x_2} & \text{otherwise}.\end{cases} \tag{10}$$

Then

$$\begin{aligned}\int F(Z)d\nu_\Gamma(Z)&=\int F(Z)d\nu_{\Gamma^0}(Z)+\int_0^1 ds\frac{d}{ds}\int F(Z)d\nu_{\Gamma^s}(Z)\\ &=\int F(Z)d\nu_{\Gamma^0}(Z)+\sum_{y'\neq y}\int_0^1 ds\int\left(\frac{\delta}{\delta Z_y}\Gamma_{yy'}\frac{\delta}{\delta Z_{y'}}F(Z)\right)d\nu_{\Gamma^s}(Z).\end{aligned} \tag{11}$$

Iterating (10) and (11) we obtain

$$\int F(Z)d\nu_\Gamma(A)=\sum_{\{\bar y_\alpha\}}\int\prod_\alpha S(\bar y_\alpha)F(Z)d\nu_{\Gamma^s}(Z), \tag{12}$$

where

$$\bar y_\alpha=(y^1_\alpha,\dots,y^{n_\alpha}_\alpha)$$

is a sequence of different points in Λ_{N-1}, y^1_α being the first of them in the fixed order,

$$\underline{y}_\alpha=\{y^1_\alpha,\dots,y^{n_\alpha}_\alpha\},$$

$\sum_{\{\bar y_\gamma\}}$ runs over the sets $\{\bar y_\alpha\}$ such that $\{\underline{y}_\alpha\}$ forms a partition of Λ_{N-1},

$$S(y)=1, \tag{13}$$

$$S(\bar y_\alpha)=\int ds^1_\alpha\dots ds^{n_\alpha-1}_\alpha K(\bar y_\alpha,s_\alpha)\quad\text{for}\quad n_\alpha>1, \tag{14}$$

$$K(\bar y_\alpha,s_\alpha)=\prod_{i=2}^{n_\alpha}K(i,s_\alpha), \tag{15}$$

$$K(i,s_\alpha)=\sum_{1\leqq j\leqq i\leqq n_\gamma}s^j_\alpha\dots s^{i-2}_\alpha\frac{\partial}{\partial Z_{y^j_\alpha}}\Gamma_{y^j_\alpha y^i_\gamma}\frac{\partial}{\partial Z_{y^i_\alpha}}, \tag{16}$$

$$\Gamma^s_{y^i_\gamma y^i_\gamma}=\Gamma_{y^i_\gamma y^i_\gamma}, \tag{17}$$

$$\Gamma^s_{y^j_\gamma y^i_\gamma}=\Gamma^s_{y^i_\gamma y^j_\gamma}=s^j_\alpha\dots s^{i-1}_\alpha\Gamma_{y^i_\gamma y^j_\gamma}\quad\text{for}\quad j<i, \tag{18}$$

$$\Gamma^s_{y^i_\alpha y^j_\beta}=0\quad\text{for}\quad\alpha\neq\beta. \tag{19}$$

Besides (12) our cluster expansion will also contain the Mayer expansion

$$\prod_{|Y|>1}\exp[-V_Y]=\sum_{\{Y_\beta\}}(\exp[-V_Y]-1), \tag{20}$$

and the partition of unity specifying the region where the fluctuation field Z is large:

$$1=\sum_{R\subset\Lambda_{N-1}}\chi_R(Z), \tag{21}$$

where

$$\chi_R(Z) = \prod_{y \in R} 1(|Z_y| \geqq Bn^2) \prod_{y \notin R} 1(|Z_y| < Bn^2). \tag{22}$$

The superposition of (12), (20), and (21) for the right hand side of (2) gives

$$\exp[-V'(\phi)] = \sum_R \sum_{\{\bar{y}_\alpha\}} \sum_{\substack{\{Y_\beta\} \\ |Y_\beta| > 1}} \int \Big(\prod_\alpha S(\bar{y}_\alpha) \prod_\beta (\exp[-V_Y(\psi)] - 1) \\ \cdot \prod_{y \in \Lambda_{N-1}} \exp\Big[-V_{\Delta_y}(\psi) - \sum_{x \in \Delta_y} v_x(\psi) \Big] \Big) \chi_R(Z) d\nu_{\Gamma^s}(Z) / (\phi = 0), \tag{23}$$

where Δ_y is the block in Λ_N centered at Ly. Now we have to exhibit factorization properties of the expansion (23). Let $\tilde{R}$ contain $\bar{R}$ and the blocks being the n.n. of those in $\bar{R}$. Consider the subset

$$X = \tilde{R} \cup \Big(\bigcup_\beta L^{-1} Y_\beta \Big)^- \cup \Big(\bigcup_{\alpha : n_\alpha > 1} y_\alpha \Big)^- \tag{24}$$

of Λ_{N-1}. Draw a graph on the centers of the blocks building it the following way: there is a line for two different points if

one is the center of a block in $\bar{R}$ and the other of its n.n. block,

both blocks centered on them contain points in a single $L^{-1} Y_\beta$ or a single y_α.

Use the c.c. of this graph to select the clusters (polymers) X_ζ in X. Fixing $\{X_\zeta\}$ and performing the rest of the summation, we obtain for $\exp[-V']$ the expression involving the partition function of a system of polymers [11, 12]:

$$\exp[-V'(\phi)] = \Big(\sum_{\{X_\zeta\} \text{disjoint}} \prod_\zeta \varrho_{X_\zeta}(\phi) / (\phi = 0) \Big) \exp\Big[-\sum_x v'_x(\phi) \Big], \tag{27}$$

where for $X \subset \Lambda_{N-1}$ the polymer activity

$$\varrho_X(\phi) := \sum_R \sum_{\{\bar{y}_\alpha\}} \sum_{\{Y_\beta\}} \int \Big(\prod_\alpha S(\bar{y}_\alpha) \prod_\beta (\exp[-V_{Y_\beta}(\phi)] - 1) \\ \cdot \prod_{y \in X} \exp\Big[-V_{\Delta_y}(\psi) - \sum_{x \in \Delta_y} v_x(\psi) \Big] \Big) \chi_R(Z_X) d\nu_{\Gamma^s}(Z_X) \Big/ \Big(\prod_{x \in X} \exp[-w'_x(\phi)] \Big), \tag{28}$$

with $Z_X = Z|_X$, and

$$\exp[-w'_y(\phi)] := \int \exp\Big[-V_{\Delta_y}(\psi) - \sum_{x \in \Delta_y} v_x(\psi) \Big] \chi_\emptyset(Z_y) d\nu_{\Gamma_{yy}}(Z_y), \tag{29}$$

$$v'_y(\phi) := w'_y(\phi) - w'_y(0). \tag{30}$$

In (28) the sum over R, $\{\bar{y}_\alpha\}$ and $\{Y_\beta\}$ is restricted as follows: $\tilde{R} \subset X$, $\{y_\alpha\}$ is a partition of X, $L_\beta^{-1} Y \subset X$, $|Y_\beta| > 1$ and the graph on the centers of blocks in X drawn according to the rules described above is connected. Using (27) we would like to obtain an analogue of (6) for V'. This is possible if the activities $\varrho_{X_\zeta}(\phi)$ are small. For $Y \subset \Lambda_{N-1}$ put (see [1, Chap. II])

$$W'_Y(\phi) := -\sum_{\substack{(X_\zeta)_{\zeta=1}^{\Xi} \\ \cup_\zeta X_\zeta = Y}} \frac{1}{\Xi!} \sum_{\gamma_c} \prod_{l \in \gamma_c} A(l) \prod_\zeta \varrho_{X_\zeta}(\phi), \tag{31}$$

where $\sum_{\gamma_c}$ runs over connected graphs on vertices $\{1, \ldots, \Xi\}$ and

$$A(\zeta_1, \zeta_2) = \begin{cases} -1 & \text{if } X_{\zeta_1} \cap X_{\zeta_2} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases} \tag{32}$$

Now, with

$$V'_Y(\phi) := W'_Y(\phi) - W'_Y(0), \tag{33}$$

$$V' = \sum_{Y \subset \Lambda_{N-1}} V'_Y + \sum_{x \in \Lambda_{N-1}} v'_x \tag{34}$$

holds at least provided that the polymer activities $\varrho_{X_\zeta}(\phi)$ are small so that the series in (31) converges. Actually the activities are not small for large values of ϕ, and neither (6) nor (34) makes sense. What will be shown to make sense is (7) and its analogue for V' with D containing the region where the field is large. However we find it worthwhile from the pedagogical point of view to arrive at the final expansion by cluster-expanding everywhere and partially resumming the expansion, regardless of the fact that these two steps are only formal.

As a result of the partial resummation done exactly as in (7), we obtain for $D' \subset X'$

$$g'^{D'}_{X'} = \sum_{\{Y_\alpha\}} \prod_\alpha (\exp[-V'_{Y_\alpha}] - 1) \prod_{x \in D'} \exp[-v'_x], \tag{35}$$

with the same restrictions on $\{Y_\alpha\}$. Now using (33) and (31) we obtain

$$\begin{aligned} \left(\sum_{x \in D'} \exp v'_x(\phi)\right) g'^{D'}_{X'}(\phi) &= \sum_{\{Y_\alpha\}} \sum_{\substack{(n_\alpha, m_\alpha) \\ n_\alpha + m_\alpha > 1}} \prod_\alpha (n_\alpha! m_\alpha!)^{-1} (-W'_{Y_\alpha}(\phi))^{n_\alpha} (W'_{Y_\alpha}(0))^{m_\alpha} \\ &= \sum_{(X_1, \ldots, X_N; Y_1, \ldots, Y_M)} (N! M!)^{-1} \sum_\gamma \prod_{l \in \gamma} A(l) \prod_{i=1}^N \varrho_{X_i}(\phi) \prod_{j=1}^M W'_{Y_j}(0) \\ &= \sum_{\substack{\{X_1, \ldots, X_N\} \\ \text{disjoint}}} \sum_{(Y_1, \ldots, Y_M)} \prod_{i=1}^N \varrho_{X_i}(\phi) M!^{-1} \prod_{j=1}^M W'_{Y_j}(0), \end{aligned} \tag{36}$$

where the sums over $X_1, \ldots, Y_M$ are restricted by requiring that X' be connected with respect to them and the c.c. of D'.

Introduce now for $D' \subset \tilde{X}$

$$\varrho^{D'}_{\tilde{X}} := \sum_{\{X_i\}} \prod_i \varrho_{X_i} \prod_{x \in D'} \exp[-v'_x], \tag{37}$$

where the sum runs over the sets $\{X_i\}$ of disjoint X_i such that $\tilde{X}$ is connected with respect to X_i and c.c. of D'. Notice that for empty D'

$$\varrho^{\emptyset}_{\tilde{X}} = \varrho_{\tilde{X}}. \tag{38}$$

We may rewrite (36) as

$$g'^{D'}_{X'}(\phi) = \sum_{\{X_1, \ldots, \tilde{X}_N\}} \sum_{(Y_1, \ldots, Y_M)} \prod_{i=1}^N \varrho^{D' \cap \tilde{X}_i}_{\tilde{X}_i}(\phi) M!^{-1} \prod_{j=1}^M W'_{Y_j}(0), \tag{39}$$

with restrictions requiring that $\tilde{X}_i$ be disjoint, $D'\cap\tilde{X}_i$ be built of c.c. of D' or be empty, $\bigcup_i \tilde{X}_i \supset D'$ and that X' be connected with respect to $\tilde{X}_i$ and Y_j.

The polymer activities $\varrho_{\tilde{X}}^{D'}$ may be obtained by selecting differently the clusters of the cluster expansion: working with (25) we could have considered the set

$$\tilde{X} = D' \cup \tilde{R}\Big(\bigcup_\beta L^{-1} Y_\beta\Big)^- \cup \Big(\bigcup_{\alpha : n_\alpha > 1} y_\alpha\Big)^- \tag{40}$$

and have drawn additionally lines between the centers of the n.n. blocks in D' when building the graph used to select the c.c. of (40). This would have given

$$\exp[-V'(\phi)] = \Bigg(\sum_{\{\tilde{X}_\zeta\}} \prod_\zeta \varrho_{\tilde{X}_\zeta}^{D'\cap\tilde{X}_\zeta}(\phi) \Bigg/ \Bigg(\begin{matrix}\phi=0\\ D'=\emptyset\end{matrix}\Bigg)\Bigg) \prod_{x\notin D'} \exp[-v'_x(\phi)]\,, \tag{41}$$

where $\tilde{X}_\zeta$ are disjoint, $\tilde{X}_\zeta \cap D'$ are built of c.c. of D' or are empty and $\bigcup_\zeta \tilde{X}_\zeta \supset D'$. The polymer activities are

$$\begin{aligned}\varrho_{\tilde{X}}^{D'}(\phi) = \sum_R \sum_{\{y_\alpha\}} \sum_{\{Y_\beta\}} \int \Bigg(\prod_\alpha S(\bar{y}_\alpha) \prod_\beta (\exp[-V_{Y_\beta}(\psi)]-1) \prod_{y\in\tilde{X}} \exp\Bigg[-V_{\Delta_y}(\psi) - \sum_{x\in\Delta_y} v_x(\psi)\Bigg]\Bigg)\\ \cdot \chi_R(Z_{\tilde{X}}) d\nu_{\Gamma^s}(Z_{\tilde{X}}) \Bigg/ \Bigg(\prod_{x\in\tilde{X}\setminus D'} \exp[-v'_x(\phi)] \prod_{x\in\tilde{X}} \exp[-w'_y(0)]\Bigg).\end{aligned} \tag{42}$$

Here the sum over R, $\{\bar{y}_\alpha\}$ and $\{Y_\beta\}$ is restricted similarly as in (28), except that now the graph on the centers of the blocks in $\tilde{X}$ with lines for the centers of n.n. blocks in D' added is to be connected. It is straightforward that (37) and (42) give the same object.

The crucial property of the activities $\varrho_{\tilde{X}}^{D'\cap\tilde{X}}(\phi)$ is that they are sufficient to give $g_{X'}^{\prime D'\cap X'}(\phi)$ and $V'_Y(\phi)$ for $D'\cap Y=\emptyset$ [see (39) and (33), (31), (38)] and that in turn they may be expressed by a Z integral of expressions involving only $g_X^{D\cap X}(\psi)$ and $V_Y(\psi)$ for $D\cap Y=\emptyset$ with $D=L(D'\cup R)$. Partially resumming $\prod_\beta(\exp[-V_{Y_\beta}]-1)$ in (42) we obtain

$$\begin{aligned}\varrho_{\tilde{X}}^{D'}(\phi) = \sum_R \sum_{\{X_1,\dots,X_N\}} \sum_{\{\bar{y}_\alpha\}} \sum_{\{Y_\beta\}} \int \Bigg(\prod_\alpha S(\bar{y}_\alpha) \prod_{i=1}^N g_{X_i}^{D\cap X_i}(\psi) \prod_\beta (\exp[-V_{Y_\beta}(\psi)]-1)\\ \cdot \prod_{y\in\tilde{X}\setminus\bigcup_i L^{-1}X_i} \exp[-V_{\Delta_y}(\psi)] \prod_{x\in L\tilde{X}\setminus D} \exp[-v_x(\psi)]\Bigg)\\ \cdot \chi_R(Z_{\tilde{X}}) d\nu_{\Gamma^s}(Z_{\tilde{X}}) \Bigg/ \Bigg(\prod_{x\in\tilde{X}\setminus D'} \exp[-v'_x(\phi)] \prod_{x\in\tilde{X}} \exp[-w'_x(0)]\Bigg),\end{aligned} \tag{43}$$

with the following restrictions on the sums: $\tilde{R}\subset\tilde{X}$, $L^{-1}X_i\subset\tilde{X}$, X_i are disjoint, $D\cap X_i$ are built from c.c. of D, $\bigcup_i X_i \supset D$, $\underline{y}_\alpha$ are disjoint, $\bigcup_\alpha \underline{y}_\alpha = \tilde{X}$, $L^{-1}Y\subset\tilde{X}$, $Y_\beta\cap X_i=\emptyset$, $|Y_\beta|>1$ and the graph on the centers of blocks of $\tilde{X}$ obtained by drawing lines between the centers of blocks in $\bar{R}$ and their n.n. and between two different points such that both blocks centered at them contain points in a single $L^{-1}X_i$, $L^{-1}Y_\beta$ or $\underline{y}_\alpha$, is connected.

If ϕ in (43) is small outside $D':|\phi_x|<\frac{1}{2}L^{d/2}(n+1)^2$ for $x\notin D'$, then ψ under the integral is also small outside $D:|\psi_x|<n^2$ for $x\notin D$. This follows from (1), the definition $D=L(D'\cup R)$ and

$$\tfrac{1}{2}(n+1)^2+Bn^2<n^2\,, \tag{44}$$

which holds for $n\geqq n_0$ if B is chosen properly.

We shall still have to rewrite (43) in a slightly changed form. Let $v_x(\phi)=c_2\phi_x^2+\tilde{v}_x(\phi)$, where $\frac{d^2}{d\phi_x^2}\tilde{v}_x(0)=0$. Similarly $v'_x(\phi)=c'_x+\tilde{v}'_x(\phi)$. Now

$$\begin{aligned}\varrho_{\tilde{X}}^{D'}(\phi)=&\sum_R\sum_{\{X_1,\dots,X_N\}}\sum_{\{\bar{y}_\alpha\}}\sum_{\{Y_\beta\}}\prod_{x\in R\setminus D'}\exp c'_2\phi_x^2\prod_{x\in\tilde{X}\setminus(D'\cup R)}\exp[(c'_2-c_2)\phi_x^2]\\ &\cdot\int\left(\prod_\alpha S(\bar{y}_\alpha)\prod_{i=1}^N g_{X_i}^{D\cap X_i}(\psi)\prod_\beta(\exp[-V_{Y_\beta}(\psi)]-1)\prod_{y\in X\setminus\bigcup_i L^{-1}X_i}\exp[-V_{\Delta_y}(\psi)]\right.\\ &\cdot\left.\prod_{x\in L\tilde{X}\setminus D}\exp[-\tilde{v}_x(\psi)]\prod_{y\in\tilde{X}\setminus(D'\cup R)}\exp\left[-c_2\left(\sum_x\mathscr{A}(x)^2\right)Z_y^2\right]\right)\\ &\cdot\chi_R(Z_{\tilde{X}})dv_{\Gamma^s}(Z_{\tilde{X}})\Big/\left(\prod_{x\in\tilde{X}\setminus D'}\exp[-\tilde{v}'_x(\phi)]\prod_{x\in\tilde{X}}\exp[-w'_x(0)]\right),\end{aligned} \tag{45}$$

with the same restrictions on the sums as in (43).

Equations (29), (30), and (44) or (45) together with (39) and (33), (31), (38) define new v'_x, V'_Y, and $g'^{D'}_{X'}$ in terms of the old ones. For those expressions to make sense it is enough that the activities $\varrho_X(\phi)$ be small only when $D\cap X=\emptyset$ or when $\phi=0$, that is for small block spin fields. This will be shown to hold for all interations of the RG transformation. From the way the expressions for v'_x, V'_Y, $g'^{D'}_{X'}$ were obtained it is obvious on the level involving manipulations with ill-defined formal power series that the analogues of (7) and (9) hold again. To show that with full rigour, we have to proceed in another way, having chosen the formal one since it was more instructive. Let us sketch here the other argument leaving the details as an exercise to the reader.

i) In (2) we perform the expansion (12) and insert the partition of unity (21).

ii) We use (7) with $D=L(D'\cup R)$ to express the Gibbs factor.

iii) The Mayer expansion of $\exp\left[-\sum_{\substack{Y:Y\cap\left(\bigcup_j X_j\right)=\emptyset\\ |Y|>1}}V_Y(\psi)\right]$ is performed.

iv) We select the polymers decomposing

$$\tilde{R}\cup\left(\bigcup_j L^{-1}X_j\right)^-\cup\left(\bigcup_\beta L^{-1}Y_\beta\right)^-\cup\left(\bigcup_{\alpha:n_\alpha>1}\underline{y}_\alpha\right)^-,$$

with the help of the graph joining the centers of the n.n. blocks, one in $\bar{R}$, the other in $\tilde{R}$ or of the blocks, both containing points in a single $L^{-1}X_j$, $L^{-1}Y_\beta$ or $\underline{y}_\alpha$. The activities of these polymers are $\varrho_{\tilde{X}}^{D'\cap X}(\phi)$ as given by (43) or (45).

v) In the expression (41) which results this way, we Mayer expand $\prod_Y\exp[-W'_Y(0)]$, which is the $\begin{pmatrix}\phi=0\\ D'=\emptyset\end{pmatrix}$ term. Then $\tilde{X}_\zeta$ and Y_β forming clusters X'_j

connected with respect to $\tilde{X}_\zeta$ and Y_β around c.c. of D' give rise to the term $\prod_j g_{X'_j}^{\prime D'\cap X'_j}$ of $\exp[-V']$. The resummation of the other term produces

$$\exp\left[-\sum_{Y:Y\cap\left(\bigcup_j X'_j\right)=\emptyset} V'_Y - \sum_{x\notin D'} v'_x\right].$$

Thus we obtain (39) together with an analogue of (7).

vi) One shows using (43) and (9) that for $D'_1\supset D'$

$$\varrho_{\tilde{X}_1}^{D'_1}(\phi)=\sum_{\{\tilde{X}_j\}}\prod_j \varrho_{\tilde{X}_j}^{D'\cap\tilde{X}_j}(\phi)\prod_{x\in D'_1\setminus D'}\exp[-v'_x(\phi)], \tag{46}$$

where we sum over sets of disjoint $\tilde{X}_j$ such that $\tilde{X}_j\subset\tilde{X}_1$, $D'\cap\tilde{X}_j$ are built of c.c. of D' or are empty, $\bigcup_j\tilde{X}_j\supset D$ and $\tilde{X}_1$ is connected with respect to $\tilde{X}_j$ and c.c. of D'_1.

vii) The analogue of (9) follows from (39) and (46).

3. The Estimates

We start formulating inductive assumptions on v_x, V_Y, g_X^D. For $X_1, X_2\subset\Lambda_N$, $X_1\cap X_2=\emptyset$, define

$$B(X_1,X_2;r_1,r_2):=\{(\phi_x)_{x\in X_1\cup X_2}:|\mathrm{Im}\,\phi_x|<r_1 \text{ for } x\in X_1, |\phi_x|<r_2; \text{ for } x\in X_2\}. \tag{1}$$

Our inductive assumptions will involve the following parameters: $\delta, L, A, r, \kappa, E, n$. We shall always assume that $0<\delta<1$, $L>L_0(\delta)$, $A>A_0(\delta,L)$, $r>r_0(\delta,L,A)$, $\frac{1}{2}\kappa_0(\delta,L,A,r)<\kappa<\kappa_0(\delta,L,A,r)$, $E>E_0(L,A,r)$, $n>n_0(\delta,L,A,r,E)$ with appropriate choice of L_0, A_0, r_0, κ_0, E_0, and n_0.

Assumptions

1. $v_x(\phi)\equiv v(\phi_x)$ is an even analytic function on $B(\emptyset,\{x\};0,\frac{3}{2}n^2)$, real for real ϕ_x, $v(0)=0$, $v(\phi_x)=c_2\phi_x^2+\tilde{v}(\phi_x)$, where $\frac{d^2}{d\phi_X^2}\tilde{v}(0)=0$, $|c_2|<\frac{1}{2}\kappa$, $|\tilde{v}|<\delta^n$.

2. $V_Y(\phi)$ is an even analytic function on $B(\emptyset, Y;0,\frac{3}{2}n^2)$, real for real ϕ, $V_Y(0)=0$, $|V_Y|<\delta^n\exp[-A\mathscr{L}(Y)]$.

3. $g_X^D(\phi)$ is an even analytic function on $B(D, X''D;n^2,n^2)$,

$$\left|\frac{\partial^{|M|}}{\partial\phi^M}g_X^D(\phi)\right|\leqq M!\,r^{-|M|}\exp\left[\kappa\sum_{x\in D}|\phi_x|^2-A\mathscr{L}(X)+E|D|\right],\qquad M=(m_x)_{x\in X}.$$

We shall prove that v'_x, V'_Y, $g_{X'}^{D'}$ satisfy the same assumptions with $n\to n+1$, $\kappa\to\kappa'=\kappa(1+O(n^{-2}))$ and the other parameters not changed.

The Estimates of the Local Potential

By (2.29)

$$w'_y(\phi)=c_2\phi_y^2-\log\int\exp\left[-V_{\Delta_y}(\psi)-\sum_{x\in\Delta_y}\tilde{v}_x(\psi)\right]$$
$$\cdot\exp\left[-c_2\left(\sum_x\mathscr{A}(x)^2\right)Z_y^2\right]\chi_\emptyset(Z_y)d\nu_{\Gamma_{yy}}(Z_y). \tag{2}$$

It is straightforward that

$$|w_y'(0)| \leqq O(\kappa_0) \tag{3}$$

and

$$c_2' - c_2 = \frac{d^2}{d\phi_y^2} w_y'(0) - c_2 \leqq O(\delta^n). \tag{4}$$

Hence

$$|c_2'| < \tfrac{1}{2}|\kappa'|. \tag{5}$$

Now $w_y'(\phi)$ is analytic for $|\phi_y| < \frac{3}{2} \cdot \frac{1}{2} L^{d/2}(n+1)^2$, see (2.44). For $|\phi_y| < \frac{3}{2}(n+1)^2$

$$|\tilde{v}'(\phi)| \leqq \sum_{m=4}^{\infty} m!^{-1} \left| \frac{d^m}{d\phi_y^m} w_y'(0) \right| (\tfrac{3}{2}(n+1)^2)^m. \tag{6}$$

But

$$w_y'(\phi) = -\log\left(1 + \left\langle \exp\left[-V_{\Delta_y}(\psi) - \sum_{x \in \Delta_y} \tilde{v}_x(\psi)\right] - 1 \right\rangle\right) + \text{const}, \tag{7}$$

where $\langle - \rangle$ is the expectation in the probability measure

$$N^{-1} \exp\left[-c^2\left(\sum_x \mathscr{A}(x)^2\right) Z_y^2\right] \chi_\theta(Z_y) dv_{\Gamma_{yy}}(Z_y)$$

and

$$\left| \exp\left[-V_{\Delta_y}(\psi) - \sum_{x \in \Delta_y} \tilde{v}_x(\psi)\right] - 1 \right| \leqq (L^d + 1)\delta^n(1 + C\delta^n) \tag{8}$$

for $|\phi_y| < \frac{3}{4}\mathrm{L}^{d/2}(\mathrm{n}+1)^2$. Hence, for $m > 0$ by the Cauchy formula,

$$\left| \frac{d^m}{d\phi_y^m} w_y'(0) \right| \leqq m!(\tfrac{3}{4}L^{d/2}(n+1)^2)^{-m}(L^d + 1)\delta^n(1 + C\delta^n). \tag{9}$$

(6) and (9) give

$$|\tilde{v}'(\phi)| \leqq (\tfrac{1}{2}L^{-d/2})^{-4}(1 - 2L^{-d/2})^{-1}(L^d + 1)\delta^n(1 + C\delta^n) \leqq \delta^{n+1}. \tag{10}$$

Thus v' possesses the required properties. We also see the mechanism which drives the local potentials down (except for the quadratic terms). This is the expansion of the analyticity region by factor $O(L^{d/2})$ each time the RG transformation is applied. We shall use this mechanism still once more below when estimating V_Y' with small Y.

The Estimate of the Polymer Activities

In order to show that V_Y' and $g_X'^D$ satisfy Assumptions 2 and 3 we shall have to estimate $\varrho_X^{D'}(\phi)$ as given by (2.45). From (2.1), (2.44), and (2.43) or (2.45) it is obvious that $\varrho_X^{D'}(\phi)$ is analytic in

$$B(D', X \backslash D'; \tfrac{1}{2}L^{d/2}(n+1)^2, \tfrac{1}{2}L^{d/2}(n+1)^2).$$

Let us write

$$\prod_\alpha S(\bar{y}_\alpha) = \prod_\alpha \int ds_\alpha \sum_{\tau_\alpha} f_{\tau_\alpha}(s_\alpha) \prod_{l \in \tau_\alpha} \frac{\partial}{\partial Z_{l_-}} \Gamma_{l_- l_+} \frac{\partial}{\partial Z_{l_+}}, \tag{11}$$

where τ_α is a graph (tree) on $\bar{y}_\alpha$ with lines $l=(l_-, l_+)$, l_- earlier than l_+ in $\bar{y}_\alpha$, such that each y_α^i for $i>1$ appears once and only once as l_+. Then

$$f_{\tau_\alpha}(s_\alpha) = \prod_{l\in\tau_\alpha} s_l\,, \tag{12}$$

where, if $l=(y_\alpha^i, y_\alpha^j)$, then $s_l = s^i \dots s^{j-2}$ ($s_l=1$ if $j-i\leqq 1$). Let μ be the family of points

$$(l_-, l_+)_{l\in\bigcup_\alpha \tau_\alpha}\,, \quad \mu=\bigcup_y \mu(y)$$

[each point y appearing so many times ($|\mu(y)|$) as there is derivatives over Z_y in (11)]. Let

$$\{\mu_i\}_{i=1}^N, \{\mu_\beta\}, \{\mu_{y\gamma}\}_{y\in\tilde{X}\setminus\bigcup_\iota L^{-1}X_\iota}, \{\mu_{x\delta}\}_{x\in L\tilde{X}\setminus D}, \{\mu_y\}_{y\in\tilde{X}\setminus(D'\cup R)}$$

be the partition of μ determined according to which term in (2.45) is differentiated over Z_y's.

$$\begin{aligned}
\varrho_{\tilde{X}}^{D'}(\phi) = &\sum_R \sum_{\{X_\iota\}} \sum_{\{\bar{y}_\alpha\}} \sum_{\{Y_\beta\}} \sum_{\{\tau_\alpha\}} \sum_{\{\mu_I\}} \prod_{x\in R\setminus D'} \exp[c'\phi_x^2] \prod_{x\in\tilde{X}\setminus(D'\cup R)} \exp[(c_2'-c_2)\phi_x^2] \\
&\cdot \int \prod_\alpha ds_\alpha f_{\tau_\alpha}(s_\alpha) \prod_{l\in\tau_\alpha} \Gamma^s_{l_-l_+} \int \prod_{i=1}^N \frac{\partial^{|\mu_i|}}{\partial Z^{\mu_i}} g_{X_i}^{D\cap X_i}(\psi) \prod_\beta \frac{\partial^{|\mu_\beta|}}{\partial Z^{\mu_\beta}} (\exp[-V_{Y_\beta}(\psi)]-1) \\
&\cdot \prod_{y\in\tilde{X}\bigcup_\iota L^{-1}X_\iota} \left(\prod_\gamma\left(-\frac{\partial^{|\mu_{y\gamma}|}}{\partial Z^{\mu_{y\gamma}}} V_{\Delta_y}(\psi)\right)\right) \exp[-V_{\Delta_y}(\psi)] \prod_{x\in L\tilde{X}\setminus D} \left(\prod_\delta\left(-\frac{\partial^{|\mu_{x\delta}|}}{\partial Z^{\mu_{x\delta}}} \tilde{v}_x(\psi)\right)\right) \\
&\cdot \exp[-\tilde{v}_x(\psi)] \prod_{y\in\tilde{X}\setminus(D'\cup R)} \frac{\partial^{|\mu_y|}}{\partial Z^{\mu_y}} \exp\left[-c_2\left(\sum_x \mathscr{A}(x)^2\right) Z_y^2\right] \chi_R(Z_{\tilde{X}}) d\nu_{\Gamma^s}(Z_{\tilde{X}}) \\
&\cdot \left(\prod_{x\in\tilde{X}\setminus D'} \exp[-\tilde{v}_x'(\phi)] \prod_{x\in\tilde{X}} \exp[-w_x'(0)]\right)^{-1}.
\end{aligned} \tag{13}$$

The expressions with derivatives under the integrals in (13) are bounded as follows

$$\left(\mu! := \prod_y |\mu(y)|!\right)$$

$$\prod_{i=1}^N \left|\frac{\partial^{|\mu_i|}}{\partial Z^{\mu_i}} g_{X_i}^{D\cap X_i}(\psi)\right| \leqq \prod_i \mu_i!(Cr^{-1})^{|\mu_i|} \exp\left[\kappa \sum_{x\in D} |\phi_x|^2 - A\sum_i \mathscr{L}(X_i) + E|D|\right], \tag{14}$$

$$\left|\frac{\partial^{|\mu_\beta|}}{\partial Z^{\mu_\beta}} (\exp[-V_{Y_\beta}(\psi)]-1)\right| \leqq \mu_\beta!(Cr^{-1})^{|\mu_\beta|}\delta^n(1+C\delta^n)\exp[-A\mathscr{L}(Y_\beta)], \tag{15}$$

$$\left|\frac{\partial^{|\mu_{y\gamma}|}}{\partial Z^{\mu_{y\gamma}}} V_{\Delta_y}(\psi)\right| \leqq \mu_{y\gamma}!(Cr^{-1})^{|\mu_{y\gamma}|}\delta^n, \tag{16}$$

$$\left|\frac{\partial^{|\mu_{x\delta}|}}{\partial Z^{\mu_{x\delta}}} \tilde{v}_x(\psi)\right| \leqq \mu_{x\delta}!(Cr^{-1})^{|\mu_{x\delta}|}\delta^n, \tag{17}$$

$$\left|\frac{\partial^{|\mu_y|}}{\partial Z^{\mu_y}} \exp\left[-c_2\left(\sum_x \mathscr{A}(x)^2\right) Z_y^2\right]\right| \leqq \mu_y!(Cr^{-1})^{|\mu_y|} e^{C\kappa Z_y^2}, \tag{18}$$

where C depends only on L.

Using Assumptions (3), (4), and (14)–(18), we obtain

$$|\varrho_{\tilde{X}}^{D'}(\phi)| \leqq \sum_{\substack{R,\{X_i\},\{\bar{y}_\alpha\}\\ \{Y_\beta\},\{\tau_\alpha\},\{\mu_I\}}} \exp\left[\kappa \sum_{x\in D'} |\phi_x|^2 + EL^d|D'|\right] \prod_i \exp[-A\mathscr{L}(X_i)]$$
$$\cdot \prod_\beta \exp[-A\mathscr{L}(Y_\beta)] \prod_{l\in \bigcup_\alpha \tau_\alpha} \Gamma_{l_- l_+} \mu! C^\mu r^{-|\mu|}$$
$$\cdot \prod_\beta [\delta^n(1+C\delta^n)] \exp[C\kappa|\tilde{X}|] \exp[E|LR| + 2\kappa n^4 |L(R\backslash D')|]$$
$$\cdot \int \prod_\alpha ds_\alpha f_{\tau_\alpha}(s_\alpha) \int \exp\left[C\kappa \sum_{y\in\tilde{X}} Z_y^2\right] \chi_R(Z_{\tilde{X}}) d\nu_{\Gamma^s}(Z_{\tilde{X}}). \tag{19}$$

It is straightforward to show that the Z integral on the right hand side of (19) is bounded by

$$\exp[C\kappa|\tilde{X}|] \prod_{y\in R} \exp[-\tfrac{1}{4}B^2 n^4]. \tag{20}$$

The sums in (19) will be estimated by the method of combinatoric coefficients [7]:

$$\sum_\alpha |f_\alpha| \leqq \sup_\alpha c_\alpha |f_\alpha| \quad \text{if} \quad \sum_\alpha c_\alpha^{-1} \leqq 1, \qquad c_\alpha > 0. \tag{21}$$

Let us start with the sum over $\{\mu_I\}$.

i) We first sum over the choices of $\{\mu_i\}$, $\bigcup_\beta \mu_\beta$, $\left\{\bigcup_\gamma \mu_{y\gamma}\right\}$, $\left\{\bigcup_\delta \mu_{x\delta}\right\}$, and $\{\mu_y\}$. This is controlled by the combinatoric coefficient $(4+L^d)^{|\mu|}$.

ii) We sum for each Y_β over the choices of the subsets Q of $L_\beta^{-1} Y$ of points y with $|\mu_\beta(y)| > 0$. The coefficient $\prod_\beta 2^{|Y_\beta|}$ controls this sum.

iii) For each y we sum over the partitions of $\bigcup_\beta \mu_\beta(y)$ into $\{\mu_\beta(y)\}_{\beta: y\in Q_\beta}$. This is controlled by the coefficient

$$\prod_y 2^{\sum_\beta |\mu_\beta(y)| - 1} \left(\sum_\beta |\mu_\beta(y)|\right)!$$

iv) For each y we sum over the partitions of $\bigcup_\gamma \mu_{y\gamma}$ into $\{\mu_{y\gamma}\}$. The coefficient is

$$C^{\sum_{y,\gamma} |\mu_{y\gamma}|} \prod_y \left(\sum_\gamma |\mu_{y\gamma}|\right)!$$

v) For each x we sum over the partitions of $\bigcup_\delta \mu_{x\delta}$ into $\{\mu_{x\delta}\}$. The relevant coefficient is

$$C^{\sum_{y,\delta} |\mu_{x\delta}|} \prod_x \left(\sum_\delta |\mu_{x\delta}|\right)!.$$

Altogether we pick up a coefficient which may be bounded by

$$C^{|\mu|} \mu! \prod_\beta 2^{|Y_\beta|}. \tag{22}$$

Inserting (20) and estimating the sum over $\{\mu_I\}$ in (19) with the use of (22), we get

$$|\varrho_{\tilde{X}}^{D'}(\phi)| \leqq \sum_{\substack{R,\{X_i\},\{\bar{y}_\alpha\}\\ \{Y_\beta\},\{\tau_\alpha\}}} \max_{\{\mu_I\}} \left\{ \exp\left[\kappa \sum_{x\in D'} |\phi_x|^2 + EL^d|D'|\right] \prod_i exp[-A\mathscr{L}(X_i)] \right.$$
$$\cdot \prod_\beta (\delta^n C^{|Y_\beta|} \exp[-A\mathscr{L}(Y_\beta)]) \prod_{l\in \cup_\alpha \tau_\alpha} \Gamma_{l_- l_+} \mu!^2 C^{|\mu|} r^{-|\mu|}$$
$$\left. \cdot \exp[C\kappa|\tilde{X}|] \prod_{y\in R} \exp[-\tfrac{1}{6}B^2 n^4] \int \prod_\alpha ds_\alpha f_{\tau_\alpha}(s_\alpha) \right\}. \tag{23}$$

An easy estimate [4, Lemma 2] yields for $\varepsilon > 0$

$$(1-\varepsilon)\mathscr{L}(X_i) \geqq (1+\varepsilon)\mathscr{L}(\overline{L^{-1}X_i}) - C, \tag{24}$$
$$(1-\varepsilon)\mathscr{L}(Y_\beta) \geqq (1+\varepsilon)\mathscr{L}(\overline{L^{-1}Y_\beta}) - C, \tag{25}$$
$$(1-\varepsilon)d(l_-, l_+) \geqq (1+\varepsilon)d([L^{-1}l_-],[L^{-1}l_+]) - C. \tag{26}$$

Hence by virtue of the connectivity properties of $\tilde{X}$

$$\prod_{i=1}^N \exp[-(1-\varepsilon)A\mathscr{L}(X_i)] \prod_\beta \exp[-(1-\varepsilon)A\mathscr{L}(Y_\beta)] \prod_l \exp[-(1-\varepsilon)Ad(l_-, l_+)]$$
$$\cdot \exp[-\varepsilon B^2 n^4 |\tilde{R}|] \leqq C(A)^{N+|\mu|} \prod_\beta C(A) \exp[-(1+\varepsilon)A\mathscr{L}(\tilde{X})]. \tag{27}$$

Another easy estimate [8, Lemma 10.2] gives

$$\mu!^2 \leqq C^{|\mu|} \prod_{l\in \cup_\alpha \tau_\alpha} \exp[d(l_-, l_+)]. \tag{28}$$

Inserting (27) and (28) into (23) and using (2.5), we get

$$|\varrho_{\tilde{X}}^{D'}(\phi)| \leqq \sum_{\substack{R,\{X_i\},\{\bar{y}_\alpha\}\\ \{Y_\beta\},\{\tau_\alpha\}}} \max_{\mu_I} \left\{ \exp\left[\kappa \sum_{x\in D'} |\phi_x|^2 + EL^d|D'|\right] \exp[-(1+\varepsilon)A\mathscr{L}(\tilde{X})] \right.$$
$$\cdot \prod_i (C(A)\exp[-\varepsilon A\mathscr{L}(X_i)]) \prod_\beta (C(A)\delta^n C^{|Y_\beta|}$$
$$\cdot \exp[-\varepsilon A\mathscr{L}(Y_\beta)]) \prod_{l\in \cup_\alpha \tau_\alpha} exp\left[-\frac{\varepsilon}{2}\mathrm{Ad}(l_-, l_+)\right]$$
$$\left. \cdot (C(A)r^{-1})^{|\mu|} \exp[C\kappa|\tilde{X}|] \prod_{y\in R} \exp\left[-\frac{1}{10}B^2 n^4\right] \int ds_\alpha f_{\tau_\alpha}(s_\alpha) \right\}. \tag{29}$$

Now

$$\sum_{\{\bar{y}_\alpha\},\{\tau_\alpha\}} \prod_l \exp\left[-\frac{\varepsilon}{2}\mathrm{Ad}(l_-, l_+)\right] \prod_\alpha \int ds_\alpha f_{\tau_\alpha}(s_\alpha)$$
$$\leqq \sum_{\{\underline{y}_\alpha\},\{\tau_\alpha\}} \prod_l \exp\left[-\frac{\varepsilon}{4}\mathrm{Ad}(l_-, l_+)\right] \prod_\alpha \int ds_\alpha f_{\tau_\alpha}(s_\alpha)$$
$$\leqq \sum_{\{\underline{y}_\alpha\}} \prod_\alpha \left(\exp\left[-\frac{\varepsilon}{4}A\mathscr{L}(\underline{y}_\alpha)\right] \sum_{\{\tau_\alpha\}} \int ds_\alpha f_{\tau_\alpha}(s_\alpha) \right)$$
$$\leqq C^{|\tilde{X}|}, \tag{30}$$

where we have denoted by $\mathscr{L}(\underline{y}_\alpha)$ the length of the shortest tree on $\underline{y}_\alpha$ and have used [10, Lemma 5]. Hence the sum over $\{\bar{y}_\alpha\}$ and $\{\tau_\alpha\}$ contributes the overall factor $C^{|\tilde{X}|}$. A similar factor is contributed by the sum over R.

The sum over $\{Y_\beta\}$ is controlled in steps.

i) For any $Y\subset \Lambda_N$ we fix one of its blocks. First we sum over the sets Q of points $y\in\tilde{X}$ such that block Δ_y is the fixed block of some Y_β. This is also controlled by the combinatoric coefficient $C^{|\tilde{X}|}$.

ii) For each $y\in Q$ we sum over the possible numbers $b(y)$ of Y_β for which Δ_y is the fixed block. This contributes the factor $\prod\limits_y 2^{b(y)}=\prod\limits_\beta 2$.

iii) Next we sum over the choices of Y_β with one block fixed. This contributes $\exp[C\mathscr{L}(Y_\beta)]$ with C sufficiently big.

The sum over $\{X_i\}$ is controlled by $C^{|D'\cup R|}\prod\limits_i \exp[C\mathscr{L}(X_i)]$.

Altogether $\sum\limits_{\substack{R,\{X_i\},\{\underline{y}_\alpha\}\\ \{Y_\beta\},\{\tau_\alpha\}}}$ is replaced by the combinatoric coefficient

$$C^{|\tilde{X}|}\prod_i \exp[C\mathscr{L}(X_i)]\prod_\beta(2\exp[C\mathscr{L}(Y_\beta)])\,. \tag{31}$$

Since moreover

$$C^{|\tilde{X}|}\leqq C^{|\mu|}\prod_i C^{|X_i|}\prod_\beta C^{|Y_\beta|}\prod_{y\in R} C\,, \tag{32}$$

and

$$C(A)r^{-1}\leqq 1\,, \tag{33}$$

we get finally

$$|\varrho_{\tilde{X}}^{D'}(\phi)|\leqq\exp\left[\kappa\sum_{x\in D'}|\phi_x|^2+\mathrm{C(L,A,E)|D'|}\right]\exp[-(1+\varepsilon)A\mathscr{L}(\tilde{X})]G^{-|\tilde{X}|} \tag{34}$$

on $B(D',\tilde{X}\backslash D';\frac{1}{2}L^{d/2}(n+1)^2,\frac{1}{2}L^{d/2}(n+1)^2)$, where G may be taken big.

We shall need a more refined bound for $D'=\emptyset$ on $B(\emptyset,X;0,\frac{3}{2}(n+1)^2)$:

$$|\varrho_X(\phi)-\varrho_X(0)|\leqq\delta^n\exp[-(1+\varepsilon)A\mathscr{L}(X)]G^{-|X|}$$
$$\text{for}\quad |X|>2^d\quad\text{or}\quad X\leqq 2^d,\quad X\ \text{disconnected}\,, \tag{35}$$

$$|\varrho_X(\phi)-\varrho_X(0)|\leqq CL^{-1}\delta^n\exp[-A\mathscr{L}(X)]$$
$$\text{for}\quad |X|\leqq 2^d,\quad X\ \text{connected}\,, \tag{36}$$

where C is L independent. Notice that when estimating in (13) the terms with $R\neq\emptyset$, with more than one Y in $\{Y_\beta\}$, or with one Y_β but $\mu\neq\emptyset$, we could have extracted the additional $\frac{1}{2}\delta^n$ factor. Thus we are left only with the terms with $R=\emptyset$ and no Y_β or with $R=\emptyset$, one Y_β and $\mu=\emptyset$. From the terms with $R=\emptyset$ and no Y_β, only those with $\mu=\bigcup\limits_y \mu_y$ do not carry the additional $\frac{1}{2}\delta^n$ factor. If we subtract their value at $\phi=0$, we may extract the additional factor $Cr^{-1}\delta^n C^{|X|}$ which will do the job. In the terms with $R=\emptyset$, one Y_β, $\mu=\emptyset$ for which $\mathscr{L}(Y_\beta)>\mathscr{L}(\overline{L^{-1}Y_\beta})$ we may use $\exp[-\varepsilon A\mathscr{L}(Y_\beta)]$ to extract the additional small factor we need. The other ones with $\mathscr{L}(Y_\beta)=\mathscr{L}(\overline{L^{-1}Y_\beta})$ have $|Y_\beta|\leqq 2^d$. For $d=2$ they are drawn on Fig. 1.

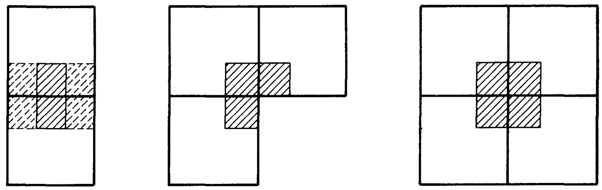

Fig. 1

In this case we shall use the fact that

$$|\exp[-V_{Y_\beta}(\psi)]-1|\leqq \delta^n(1+C\delta^n)\exp[-A\mathscr{L}(Y_\beta)]$$

and is analytic for $|\phi_x|<\frac{1}{2}L^{d/2}(n+1)^2$ and that odd derivatives of $\exp[-V_{Y_\beta}(\psi)]$ over ϕ at zero vanish when integrated over Z. The analyticity in a bigger region yields contraction, as in the case of the local potential, see (6), (9), and (10):

$$\left|\sum_{\substack{\{n(y)\}\\ y\in L^{-1}Y_\beta,\ \sum_y n(y)\geqq 2}} \frac{1}{\prod_y n(y)!}\,\frac{\partial^{\Sigma n(y)}}{\prod_y \partial\phi_y^{n(y)}}(\exp[-V_{Y_\beta}(\psi)]-1)\prod_y \phi_y^{n(y)}\right|$$

$$\leqq \sum_{\substack{\{n(y)\}\\ \sum_y n(y)\geqq 2}} \prod_y (|\phi_y|^{n(y)}(\tfrac{1}{2}L^{d/2}(n+1)^2)^{-n(y)})\delta^n(1+C\delta^n)\exp[-A\mathscr{L}(Y_\beta)]$$

$$\leqq \sum_{\substack{\{n(y)\}\\ \sum_y n(y)\geqq 2}} (\tfrac{1}{3}L^{d/2})^{-\sum_y n(y)}\delta^n(1+C\delta^n)\exp[-A\mathscr{L}(Y_\beta)]\leqq CL^{-d}\delta^n\exp[-A\mathscr{L}(Y_\beta)]. \tag{37}$$

The sum over Y_β gives at most the factor L^{d-1}. Hence the right hand side $CL^{-1}\delta^n\exp[-A\mathscr{L}(X)]$ of (36).

$g_{X'}^{\prime D'}$ and V_Y'. The Proof Completed

Having bounded $\tilde{\varrho}_X^D(\phi)$, we may continue for a while estimating $g_{X'}^{D'}(\phi)$ as given by (2.39). Notice that (2.31) yields

$$|W_Y'(0)|\leqq \sum_{\substack{(X_\zeta)_{\zeta=1}^{\Xi}\\ \bigcup_\zeta X_\zeta=Y}} \frac{1}{\Xi!}\left|\sum_{\gamma_c}\prod_{l\in\gamma_c}A(l)\right|\prod_\zeta|\varrho_{X_\zeta}(0)|$$

$$\leqq \exp[(1+\varepsilon)A\mathscr{L}(Y)]G^{-|Y|}, \tag{38}$$

where we have used the standard bound proven, e.g. by means of a Kirkwood-Salzburg equation [1, 3]

$$\sum_{\substack{(X_\zeta)_{\zeta=1}^{\Xi}\\ \bigcup X_\zeta=Y}} \frac{1}{\Xi!}\left|\sum_{\gamma_c}\prod_{l\in\gamma_c}A(l)\right|\prod_\zeta\exp[-C\mathscr{L}(X_\zeta)-C'|X_\zeta|]\leqq C', \tag{39}$$

with

$$C'=\sum_{\substack{X \text{ containing}\\ \text{a fixed block}}}\exp[-C\mathscr{L}(X)]. \tag{40}$$

Equation (2.39) together with (34) and (38) gives, after an easy argument involving combinatoric coefficients,

$$|g'^{D'}_{X'}(\phi)| \leqq \exp\left[\kappa \sum_{x\in D'} |\phi_x|^2 + C(L,A,E)|D'|\right] \exp[-(1+\varepsilon)A\mathscr{L}(X')]G^{-|X'|} \tag{41}$$

on $B(D', X'\backslash D'; \frac{1}{2}L^{d/2}(n+1)^2, \frac{1}{2}L^{d/2}(n+1)^2)$. This is not yet sufficient since $C(L,A,E)$ is much bigger than E.

To improve (41) with help of (2.9), first we have to estimate $V'_Y(\phi)$ on $B(\emptyset, Y; 0, \frac{3}{2}(n+1)^2)$. By (2.31) and (2.33)

$$V'_Y(\phi) = -\sum_{\substack{(X_\zeta)_{\zeta=1}^{\Xi} \\ \bigcup_\zeta X_\zeta = Y}} \frac{1}{\Xi!} \sum_{\gamma_c} \prod_{l\in\gamma_c} A(l) \sum_{\zeta_0=1}^{\Xi} \left(\prod_{\zeta<\zeta_0} \varrho_{X_\zeta}(0)\right) (\varrho_{X_{\zeta_0}}(\phi) - \varrho_{X_{\zeta_0}}(0)) \left(\prod_{\zeta>\zeta_0} \varrho_{X_\zeta}(\phi)\right). \tag{42}$$

Using (39) we obtain

$$|V'_Y(\phi)| \leqq C' \sup \sum_{\zeta_0=1}^{\Xi} \left(\prod_{\zeta<\zeta_0} |\varrho_{X_\zeta}(0)|\right) |\varrho_{X_{\zeta_0}}(\phi) - \varrho_{X_{\zeta_0}}(0)| \left(\prod_{\zeta>\zeta_0} |\varrho_{X_\zeta}(\phi)|\right) \cdot \prod_\zeta \exp[C\mathscr{L}(X_\zeta) + C'|X_\zeta|], \tag{43}$$

where the supremum is taken over $(X_\zeta)_{\zeta=1}^{\Xi}$ such that $\bigcup_\zeta X_\zeta = Y$ and $\sum_{\gamma_c} \prod_{l\in\gamma_c} A(l) \neq 0$. Now (34)–(36) show that

$$|V'_Y(\phi)| \leqq \delta^{n+1} \exp[-A\mathscr{L}(Y)]G^{-\mathscr{L}(Y)} \tag{44}$$

on $B(\emptyset, Y; 0, \frac{3}{2}(n+1)^2)$. This is the only step which forces us to choose big L.

We may finish now the estimation of $g'^{D'}_{X'}(\phi)$. Suppose that for given ϕ, $D\subset D'$ is such that

$$\sum_{x\in\Delta} |\phi_x|^2 \geqq \tfrac{1}{2}(n+1)^4 \quad \text{for the blocks} \quad \Delta\subset D, \tag{45}$$

$$|\mathrm{Im}\,\phi_x| < \tfrac{1}{2}L^{d/2}(n+1)^4 \quad \text{for} \quad x\in D, \tag{46}$$

$$|\phi_x| < \tfrac{3}{2}(n+1)^2 \quad \text{for} \quad x\in X'\backslash D. \tag{47}$$

Then using (2.9), (41), and (44), we obtain

$$|g'^{D'}_{X'}(\phi)| \leqq \exp\left[\kappa \sum_{x\in D} |\phi_x|^2 + C(L,A,E)|D|\right] \sum_{\{X_j\}} \prod_j (\exp[-(1+\varepsilon)A\mathscr{L}(X_j])$$
$$\cdot \sum_{\{Y_\alpha\}} \prod_\alpha (\delta^n \exp[-A\mathscr{L}(Y_\alpha)]G^{-\mathscr{L}(Y_\alpha)}) \prod_{x\in D'\backslash D} e^{2/3\kappa|\phi_x|^2 + C}, \tag{48}$$

with the restrictions on $\sum_{\{X_j\},\{Y_\alpha\}}$ as in (2.9). The sum over $\{X_j\}$ is controlled by the coefficient $2^{|D|} \prod_j \exp[C\mathscr{L}(X_j)]$. Since

$$\sum_{x\in D} |\phi_x|^2 \geqq \tfrac{1}{2}(n+1)^4 |D|, \tag{49}$$

we may write

$$\exp\left[\kappa \sum_{x\in D} |\phi_x|^2 + (C(L,A,E) + \log 2)|D|\right] \leqq \exp\left[\kappa'' \sum_{x\in D} |\phi_x|^2\right], \tag{50}$$

$$\kappa'' = \kappa(1 + O(n^{-4})). \tag{51}$$

The sum over $\{Y_\alpha\}$ is estimated by use of the coefficient

$$2^{\left|X'\setminus\bigcup_j X_j\right|}\prod_\alpha (2\exp[(\mathscr{L}(Y_\alpha)])\leqq 2^{|D'|}\prod_\alpha (2^{|Y_\alpha|+1}\exp[C\mathscr{L}(Y_\alpha)]).$$

Notice also that

$$\prod_j \exp[-A\mathscr{L}(X_j)]\prod_\alpha \exp[-A\mathscr{L}(Y_\alpha)]\prod_{\text{c.c. } D_i' \text{ of } D'} \exp[-A\mathscr{L}(D_i')]\leqq \exp[-A\mathscr{L}(X')],\tag{52}$$

and that

$$\prod_i \exp[A\mathscr{L}(D_i')]\leqq \exp[(E-C)|D'|]\tag{53}$$

(E does not depend on n!). Hence on $B(D',X'\setminus D';\frac{1}{2}L^{d/2}(n+1)^2,\frac{3}{2}(n+1)^2)$

$$|g_{X'}^{\prime D'}(\phi)|\leqq \exp\left[\kappa''\sum_{x\in D}|\phi_x|^2+\tfrac{2}{3}\kappa\sum_{x\in D'\setminus D}|\phi_x|^2-A\mathscr{L}(X')+(E-C)|D'|\right].\tag{54}$$

Now we are ready to apply the Cauchy integral formula to estimate $\frac{\partial^{|\mu|}}{\partial\phi^\mu}g_{X'}^{\prime D'}(\phi)$. Let D be the smallest subset of D' (built of blocks) such that $|\phi_x|<(n+1)^2$ on $X\setminus D$. Then if $|z_x|=r$

$$\sum_{x\in\Delta}|\phi_x+z_x|^2\geqq\tfrac{1}{2}(n+1)^4 \quad\text{for}\quad \Delta\subset D,$$

$$|\mathrm{Im}(\phi_x+z_x)|<\tfrac{1}{2}L^{d/2}(n+1)^4 \quad\text{for}\quad x\in D,$$

$$|\phi_x+z_x|<\tfrac{3}{2}(n+1)^2 \quad\text{for}\quad x\in X'\setminus D,$$

and (54) holds for $\phi+z$. Hence for $\phi\in B(D',X'\setminus D';(n+1)^2,(n+1)^2)$,

$$\left|\frac{\partial^{|\mu|}}{\partial\phi^\mu}g_{X'}^{\prime D'}(\phi)\right|\leqq\mu!\,r^{-\mu}$$
$$\cdot\sup_{(z_x):|z_x|=r}\exp\left[\kappa''\sum_{x\in D}|\phi_x+z_x|^2+\tfrac{2}{3}\kappa\sum_{x\in D'\setminus D}|\phi_x+z_x|^2-A\mathscr{L}(X')+(E-C)|D'|\right].\tag{55}$$

Now, if $|\phi_x|\geqq(n+1)^2$, then

$$|\phi_x+z_x|^2=|\phi_x|^2\left|1+\frac{z_x}{\phi_x}\right|^2\leqq|\phi_x|^2\left(1+\frac{r}{(n+1)^2}\right)^2\leqq|\phi_x|^2(1+O(n^{-2})),$$

and if $|\phi_x|<(n+1)^2$,

$$|\phi_x+z_x|^2\leqq|\phi_x|^2+2(n+1)^2r+r^2.$$

Thus

$$\kappa''\sum_{x\in D}|\phi_x+z_x|^2\leqq\kappa'\sum_{x\in D}|\phi_x|^2,\tag{56}$$

with

$$\kappa'=\kappa''(1+O(n^{-2}))=\kappa(1+O(n^{-2})).\tag{57}$$

For $x\in D'\backslash D$

$$|\phi_x+z_x|^2\leqq\tfrac{3}{2}|\phi_x|^2+3r^2\,.$$

Hence

$$\tfrac{2}{3}\kappa\sum_{x\in D'\backslash D}|\phi_x+z_x|^2\leqq\kappa\sum_{x\in D'\backslash D}|\phi_x|^2+C|D'|\,. \tag{58}$$

Inequalities (55), (56), and (58) give finally

$$\left|\frac{\partial^{|\mu|}}{\partial\phi^\mu}g'^{D'}_{X'}(\phi)\right|\leqq\mu!\,r^{-\mu}\exp\left[\kappa'\sum_{x\in D'}|\phi_x|^2-A\mathscr{L}(X')+E|D'|\right], \tag{59}$$

ending the proof of the fact that Assumptions 1–3 hold for v', V'_Y, g'^D_X with $n\to n+1$ and $\kappa\to\kappa'=\kappa(1+O(n^{-2}))$.

If we start with an interaction satisfying the assumptions, e.g. $V(\phi)=\lambda\sum_x\phi_x^4$ with small λ or $V(\phi)=z\sum_x(1-\cos\beta^{1/2}\phi_x)$ with z small, mimicking the anharmonic crystal and dipole gas interactions, respectively, we are driven under successive RG transformations to one of the line of the fixed points corresponding to the interactions $\sum_x c\phi_x^2$. In the next publication we shall extend these results to the $\lambda(\nabla\phi)^4$ and $z(1-\cos\beta^{1/2}\nabla\phi)$ lattice models, i.e. to the true anharmonic crystal and the dipole gas. The ideas developed here will have to be supplemented with the detailed study of the Gaussian part of the effective interactions.

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