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Symplectic Structure, Lagrangian, and Involutiveness of First Integrals of the Principal Chiral Field Equation

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Abstract. We deal with a form of the chiral equation, for which first integrals can be written explicitly. For these equations, we find a symplectic structure, the Lagrangian and first integrals in involution.

Although chiral fields have been thoroughly studied (for a bibliography, see [1, 2]), when we became interested in the question whether local first integrals are in involution, we found that the situation here is obscure. This situation is complicated by the fact that there are two forms of the chiral field equations; the transition from one of them to the other can be performed by a transformation of the Bäcklund type (a "gauge transformation"), the symplectic and the Lagrangian structures being written for one of these forms and local first integrals for the other. It is very difficult to establish the connection between the symplectic structure and the first integrals.

In the present paper we concentrate solely on one of these forms of the chiral equations (which is less popular), namely on the one in which the first integrals can be written explicitly. For these equations we suggest a symplectic structure (apparently a new one), find the Lagrangian and first integrals, and prove their involutiveness.

This paper is very close to [3] and requires only a slight extension of the apparatus developed in this work. Although all the definitions and assertions are formulated independently, some proofs are replaced by references to [3].

The present paper is a natural continuation of the series of papers by Gelfand and the author (e.g. [4, 5]).

1. Equations for Principal Chiral Fields

We consider the matrix equation

$$\bar{U}_t - \bar{V}_x = [\bar{U}, \bar{V}], \tag{1.1}$$

where $\bar{U} = \zeta^{-1} U_0 + U_1 + \zeta A$, $\bar{V} = -\zeta^{-1} U_0 + U_1 + \zeta A$; U_0, U_1, A are $N \times N$ matrices, and A is a constant diagonal matrix with distinct diagonal elements. The

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equations must be satisfied identically with respect to the parameter ζ. Equation (1.1) is an example of the Zakharov-Shabat equations, see [6] (also called zero curvature equations). Equation (1.1) is equivalent to the system of equations

$$\dot{U}_0 = -U_0' + 2[U_0, U_1], \qquad \dot{U}_1 = U_1' - 2[A, U_0]. \tag{1.2}$$

Usually the name "chiral equations" is related to another system of equations:

$$M_\eta + N_\xi = 0, \qquad M_\eta - N_\xi = [N, M]. \tag{1.3}$$

Let us elucidate the connection between them. First we pass to the cone variables $\xi = x + t$, $\eta = x - t$ in (1.2):

$$U_{0\xi} = -[U_1, U_0], \qquad U_{1\eta} = [A, U_0]. \tag{1.4}$$

If U_0 and U_1 are solutions of (1.4), then h can be found from

$$U_1 + A = h^{-1} h_\xi, \qquad U_0 = h^{-1} h_\eta.$$

[Equations (1.4) guarantee compatibility of these equations.] Let $M = 2hAh^{-1}$, $N = 2hV, h^{-1}$. Then M and N satisfy (1.3). Thus each solution of (1.2) generates a solution of (1.3) for which M has a constant spectrum and vice versa. The transition from one of the equations to the other is called a "gauge transformation."

Remark. An additional requirement can be imposed on the matrices U_0, U_1, and A, namely, they can be considered as belonging to a Lie subalgebra of the whole algebra $gL(N)$, but this is of no importance to us.

2. Symplectic Structure

The symbol $\mathscr{A}$ will denote the differential algebra of the polynomials in matrix elements of some matrices $U_0, \ldots, U_n$ and their derivatives with respect to x. Let $\tilde{\mathscr{A}} = \mathscr{A}/\partial\mathscr{A}$ ($\partial = d/dx$) be the space of the formal integrals (also called functionals) $\tilde{f} = \int f\,dx$, where $\int f\,dx$ means f modulo exact derivatives. As in [3], let R_- consist of matrices $X = \sum_0^n X_i \zeta^{-i-1}$ with the commutator

$$[X, Y]_z = \overline{[X, Y](\zeta^{n+1} + z)}, \tag{2.1}$$

where z is a fixed parameter on which the Lie algebra R_- depends, and the bar symbolizes cutting out a segment of a series in ζ^{-1} from ζ^{-1} to ζ^{-n-1}. Let R_+ consist of the matrices $m = \sum_0^n m_i \zeta^i$; this is the dual to R_- with respect to the coupling

$$(m, X) = \operatorname{tr} \operatorname{res} \int mX\,dx = \operatorname{tr} \int \sum_0^n m_i X_i\,dx,$$

where res denotes the coefficient in ζ^{-1}. One of the elements of R_+ is $U = \sum_0^n U_k \zeta^k$.

Two extreme values of z will play the main role: $z=\infty$ and $z=0$. We denote

$$[X,Y]^{(1)}=\overline{[X,Y]}, \qquad [X,Y]^{(2)}=\overline{[X,Y]\zeta^{n+1}}. \tag{2.2}$$

A differentiation in $\mathscr{A}$ can be assigned to every $m\in R_+$:

$$\xi_m=\sum_{i=0}^{\infty}\sum_{k=0}^{n}\sum_{j,l=1}^{N} m_{k,jl}^{(i)}\frac{\partial}{\partial U_{k,jl}^{(i)}}=\sum_{i=0}^{\infty}\sum_{k=0}^{n}\operatorname{tr} m_k^{(i)}\partial/\partial U_k^{*(i)},$$

where the asterisk denotes the transpose of a matrix, the superscript (i) denotes the i^{th} derivative with respect to x, and jl are matrix indices. Let

$$\partial/\partial U^{*(i)}=\sum_{k=0}^{n}(\partial/\partial U_k^{*(i)})\zeta^{-k-1}\in R.$$

Then

$$\xi_m=\sum_{i=0}^{n}\operatorname{tr}\operatorname{res} m^{(i)}\frac{\partial}{\partial U^{*(i)}}.$$

The differentiations ξ_m commute with $\partial=d/dx$ and can be transferred to $\tilde{\mathscr{A}}$. They will also be called "vector fields." For $\tilde{f}=\int f dx\in\tilde{\mathscr{A}}$ we have

$$\xi_m\tilde{f}=\int\xi_m f dx=\operatorname{tr}\operatorname{res}\int m\frac{\delta f}{\delta U^*}dx, \qquad \frac{\delta f}{\delta U^*}=\sum\frac{\delta f}{\delta U_k^*}\zeta^{-k-1}\in R_-, \tag{2.3}$$

where $\delta f/\delta U_k^*$ is a matrix consisting of the variational derivatives with respect to the elements of U_k.

Let us define the mapping $X\in R_-\mapsto M_X\in R_+$:

$$M_X=\widetilde{[L,X](\zeta^{n+1}+z)},$$

where $L=U+A\zeta^{n+1}+\partial\cdot\zeta$, A is the diagonal matrix defined earlier, and the wavy line symbolizes cutting out a segment of a series in ζ from ζ° to ζ^n. The two extreme cases of this definition are

$$M_X^{(1)}=\widetilde{[L,X]}, \qquad M_X^{(R)}=\widetilde{[L,X]\zeta^{n+1}}. \tag{2.4}$$

In the case $m=M_X$ the differentiation ξ_m will be denoted as ξ_X.

Lemma 1.

$$[\xi_X,\xi_Y]=\xi_{[X,Y]_z+\xi_X Y-\xi_Y X}. \tag{2.5}$$

The proof is the same as in [3]. It should be noted that the only distinction between the above definitions and those in [3] is the coefficient ζ in the term $\partial\cdot\zeta$ in L. This does not affect the proof of this and the following assertions in this section.

Due to this lemma the vector fields ξ_X form a Lie algebra. Let us define a differential 2-form ω:

$$\omega(\xi_X,\xi_Y)=\operatorname{tr}\operatorname{res}\int M_X Y dx. \tag{2.6}$$

Proposition. *The form ω is closed.*

The proof is given in [3]. Now to each functional $\tilde{f}=\int f dx\in\tilde{\mathscr{A}}$ a Hamiltonian vector field can be assigned. Namely, let

$$\tilde{f}\mapsto\xi_{\tilde{f}}=\xi_{\delta f/\delta U^*}. \tag{2.7}$$

Lemma 2.

$$\xi_Y \tilde{f} = -\omega(\xi_{\tilde{f}}, \xi_Y).$$

The proof follows immediately from (2.6) and (2.3).

The Poisson bracket is defined thus:

$$\{\tilde{f}, \tilde{g}\} = \xi_{\tilde{f}} \tilde{g} = \omega(\xi_{\tilde{f}}, \xi_{\tilde{g}}) = \operatorname{tr} \operatorname{res} \int M_{\delta f/\delta U^*} \cdot \frac{\delta g}{\delta U^*} dx. \tag{2.8}$$

The differential equation corresponding to a Hamiltonian $\tilde{f}$ is

$$\dot{U} = M_{\delta f/\delta U^*}. \tag{2.9}$$

We have constructed a more general theory (for an arbitrary n) than is necessary here. The theory admits further generalisation by replacing $\partial \cdot \zeta$ by a more general term $\partial \cdot \zeta^p$, $p \leqq n$. This is useful for the study of more general Zakharov-Shabat equations than (1.1).

The case of chiral fields corresponds to $n=1$. Then we have $M_X = M_{X,0} + M_{X,1} \cdot \zeta$; for the first symplectic structure ($z=\infty$) we have

$$M^{(1)}_{X,0} = [U_1, X_0] + X'_0 + [A, X_1], \qquad M^{(1)}_{X,1} = [A, X_0], \tag{2.10}$$

and

$$\{\tilde{f}, \tilde{g}\}^{(1)} = \operatorname{tr} \int \left\{ \left(\left[U_1, \frac{\delta f}{\delta U_0^*} \right] + \left(\frac{\delta f}{\delta U_0^*} \right)' + \left[A, \frac{\delta f}{\delta U_1^*} \right] \right) \frac{\delta g}{\delta U_0^*} + \left[A, \frac{\delta f}{\delta U_0^*} \right] \frac{\delta g}{\delta U_1^*} \right\} dx. \tag{2.11}$$

For the second one ($z=0$) we have

$$M^{(2)}_{X,0} = [U_0, X_1], \qquad M^{(2)}_{X,1} = [U_1, X_1] + X'_1 + [U_0, X_0], \tag{2.12}$$

and

$$\{\tilde{f}, \tilde{g}\}^{(2)} = \operatorname{tr} \int \left\{ \left[U_0, \frac{\delta f}{\delta U_1^*} \right] \frac{\delta g}{\delta U_0^*} + \left(\left[U_1, \frac{\delta f}{\delta U_1^*} \right] + \left(\frac{\delta f}{\delta U_1^*} \right)' + \left[U_0, \frac{\delta f}{\delta U_0^*} \right] \right) \frac{\delta g}{\delta U_1^*} \right\} dx. \tag{2.13}$$

Remark. For the first symplectic structure, the last of Eq. (2.9) (corresponding to ζ^n) is $\dot{U}_n = [A, \delta f/\delta U_0^*]$. Thus $\operatorname{diag} \dot{U}_n = 0$. This makes it possible to reduce Eq. (2.9) to the submanifold $\operatorname{diag} U_n = 0$ and to put

$$U_n = [A, \varphi_0]. \tag{2.14}$$

3. The Hamiltonian for the Principal Chiral Field

It will be shown that Eq. (1.2) can be written as (2.9) with a Hamiltonian $\tilde{f}$. We shall use the first symplectic structure. As to the second structure, we shall make a relevant remark on it at the end of the section.

Equation (2.14) for $n=1$ has the form $U_1 = [A, \varphi_0]$. Let

$$H = \operatorname{tr} \{ \tfrac{1}{3} [[A, \varphi_0], \varphi'_0] \varphi_0 + \varphi'_0 U_0 - \tfrac{1}{2} \varphi_0'^2 - U_0^2 \}. \tag{3.1}$$

Theorem. *Equation* (2.9) *for* $f=H$ *goes into* (1.2).

Proof. From

$$\delta\tilde{f}=\operatorname{tr}\int\left(\frac{\delta f}{\delta\varphi_0^*}\delta\varphi_0+\frac{\delta f}{\delta U_0^*}\delta U_0\right)dx=\operatorname{tr}\int\left(\frac{\delta f}{\delta U_1^*}\delta U_1+\frac{\delta f}{\delta U_0^*}\delta U_0\right)dx$$
$$=\operatorname{tr}\int\left(\frac{\delta f}{\delta U_1^*}[A,\delta\varphi_0]+\frac{\delta f}{\delta U_0^*}\delta U_0\right)dx=\operatorname{tr}\int\left(-\left[A,\frac{\delta f}{\delta U_1^*}\right]\delta\varphi_0+\frac{\delta f}{\delta U_0^*}\delta U_0\right)dx,$$

we have

$$\frac{\delta f}{\delta\varphi_0^*}=-\left[A,\frac{\delta f}{\delta U_1^*}\right].$$

Now

$$\begin{aligned}\delta H=\operatorname{tr}\{&\tfrac{1}{3}[[A,\delta\varphi_0],\varphi_0']\varphi_0+\tfrac{1}{3}[[A,\varphi_0],\delta\varphi_0']\varphi_0\\&+\tfrac{1}{3}[[A,\varphi_0],\varphi_0']\delta\varphi_0+\delta\varphi_0'U_0+\varphi_0'\delta U_0-\varphi_0'\delta\varphi_0'-2U_0\delta U_0\}\\=\operatorname{tr}\{&[[A,\varphi_0],\varphi_0']-U_0'+\varphi_0''\}\delta\varphi_0+\operatorname{tr}(\varphi_0'-2U_0)\delta U_0+\partial(\ ),\end{aligned}$$

where $\partial(\ )$ denotes an exact differential. Hence

$$\frac{\delta H}{\delta\varphi_0^*}=[[A,\varphi_0],\varphi_0']-U_0'+\varphi_0'',\ \frac{\delta H}{\delta U_0^*}=\varphi_0'-2U_0. \tag{3.2}$$

Equation (2.9) for $n=1$ takes the form

$$\dot{U}_0=\left[U_1,\frac{\delta f}{\delta U_0^*}\right]+\left(\frac{\delta f}{\delta U_0^*}\right)'+\left[A,\frac{\delta f}{\delta U_1^*}\right],\quad \dot{U}_1=\left[A,\frac{\delta f}{\delta U_0^*}\right]. \tag{3.3}$$

Substituting $-\delta f/\delta\varphi_0^*$ for $[A,\delta f/\delta U_1^*]$ and using (3.2), we obtain the required system of Eqs. (1.2).

Changing slightly the Hamiltonian, namely putting

$$\hat{H}=\operatorname{tr}\{\tfrac{1}{3}[[A,\varphi_0],\varphi_0']\varphi_0+\varphi_0'U_0-\tfrac{1}{2}\varphi_0'^2\}, \tag{3.4}$$

we obtain

$$\frac{\delta\hat{H}}{\delta\varphi_0^*}=[[A,\varphi_0],\varphi_0']-U_0'+\varphi_0'',\qquad \frac{\delta\hat{H}}{\delta U_0^*}=\varphi_0',$$

and Eq. (2.9) goes into

$$\dot{U}_0=U_0',\qquad \dot{U}_1=U_1'. \tag{3.5}$$

This simplest equation corresponds to the vector field $\xi=\partial$. For our further aims the Hamiltonian $H+\hat{H}$ will be of particular interest. The corresponding equations are

$$\dot{U}_0=2[U_0,U_1],\qquad \dot{U}_1=2U_1'-2[A,U_0]. \tag{3.6}$$

Remark. With the second symplectic form the situation is more complicated. Instead of (3.3) we have

$$\dot{U}_0=\left[U_0,\frac{\delta f}{\delta U_1^*}\right],\quad \dot{U}_1=\left[U_1,\frac{\delta f}{\delta U_1^*}\right]+\left(\frac{\delta f}{\delta U_1^*}\right)'+\left[U_0,\frac{\delta f}{\delta U_0^*}\right]. \tag{3.7}$$

Equation (3.6) can be obtained from (3.7) without difficulty: it is only necessary to take $f=\mathrm{tr}(U_1^2+2AU_0)$. However we cannot find a Hamiltonian for the vector field $\xi=\partial$ since the equation $U_0'=[U_0,X]$ has no solution in $\mathscr{A}$. In terms of the Kirillov-Kostant's orbit theory the field $\xi=\partial$ is not tangent to the orbit. This means that in the second Hamiltonian structure equation (1.2) is not Hamiltonian, at least within the algebraic framework outlined here.

4. Lagrangian

Equation (2.9) for the first symplectic structure will be obtained from a variational principle.

The matrix U_n is already replaced by φ_0, according to (2.13). Now all the matrices U_i will be expressed in terms of φ_i. Let $\varphi=\sum_0^n \varphi_i\zeta^{-i-1}\in R_-$ and let

$$U+A\zeta^{n+1}+\partial\cdot\zeta=\widetilde{(1+\varphi)^{-1}(A\zeta^{n+1}+\partial\cdot\zeta)(1+\varphi)}. \tag{4.1}$$

(In [3] ζ was not involved in the term $\partial\cdot\zeta$.) The last equation (corresponding to the term with ζ^n) coincides with (2.13). Let

$$\mathscr{L}=-f+\mathrm{tr}\,\mathrm{res}(1+\varphi)^{-1}A\zeta^{n+1}\dot\varphi+\tfrac{1}{2}\mathrm{tr}\,\dot\varphi_0\varphi_0'. \tag{4.2}$$

Theorem. *The equation* $\delta\mathscr{L}/\delta\varphi_0^*=0$ *is equivalent to* (2.9).

Proof. We have $\delta f=\mathrm{tr}\,\mathrm{res}\,(\delta f/\delta U^*)\delta U+\partial(\ )$, where $\partial(\ )$ is the derivative of a 1-form which is unimportant to us now. Thus

$$\delta\mathscr{L}=\mathrm{tr}\,\mathrm{res}\left\{-\frac{\delta f}{\delta U^*}\delta U+(1+\varphi)^{-1}A\zeta^{n+1}\delta\dot\varphi\right.$$
$$\left.-(1+\varphi)^{-1}\delta\varphi(1+\varphi)^{-1}A\zeta^{n+1}\dot\varphi+\tfrac{1}{2}\mathrm{tr}\,\delta\dot\varphi_0\cdot\varphi_0'+\tfrac{1}{2}\mathrm{tr}\,\dot\varphi_0\delta\varphi_0'\right\}.$$

Substituting

$$\delta U=\widetilde{(1+\varphi)^{-1}\delta\varphi(1+\varphi)^{-1}(A\zeta^{n+1}+\partial\cdot\zeta)(1+\varphi)}+\widetilde{(1+\varphi)^{-1}(A\zeta^{n+1}+\partial\cdot\zeta)\delta\varphi},$$

we get

$$\delta\mathscr{L}=\mathrm{tr}\,\mathrm{res}\left(\left[U+A\zeta^{n+1}+\partial\cdot\zeta,\frac{\delta f}{\delta U^*}\right]-\dot U\right)(1+\varphi)^{-1}\delta\varphi+\partial(\ )+\partial_t(\ )$$

($\partial_t=\partial/\partial t$). Hence

$$\frac{\delta\mathscr{L}}{\delta\varphi^*}=\widetilde{\left(\left[U+A\zeta^{n+1}+\partial\cdot\zeta,\frac{\delta f}{\delta U^*}\right]-\dot U\right)(1+\varphi)^{-1}}.$$

This means that $\delta\mathscr{L}/\delta\varphi_i^*$ are connected with $\dot U_i-(M_{\delta f/\delta U^*})_i$ by a triangular transformation with unities on the diagonal. Therefore $\delta\mathscr{L}/\delta\varphi^*=0$ is equivalent to $\dot U-M_{\delta f/\delta U^*}=0$. □

In the particular case of the chiral fields $n=1$, $f=H$ we have

$$\mathscr{L}=\mathrm{tr}(-\tfrac{1}{3}[[A,\varphi_0],\varphi_0']\varphi_0-\varphi_0'U_0+\tfrac{1}{2}\varphi_0'^2+U_0^2$$
$$-\varphi_0A\dot\varphi_1-\varphi_1A\dot\varphi_0+\varphi_0^2A\dot\varphi_0+\tfrac{1}{2}\dot\varphi_0\varphi_0'), \tag{4.3}$$

and

$$U_0=[A,\varphi_1]-\varphi_0[A,\varphi_0]+\varphi_0', \qquad U_1=[A,\varphi_0].$$

5. Resolvent of an Operator Bundle

The following definitions will be used for the construction of first integrals of the above equations. A resolvent of the operator $L=U+A\zeta^{n+1}+\partial\cdot\zeta$ is a series $\mathfrak{R}(\zeta)=\sum_0^\infty R_k\zeta^{-k-1}$ (the elements of the matrices R_k belong to $\mathscr{A}$) which satisfies the equation

$$[L,\mathfrak{R}]=0. \tag{5.1}$$

Although the definition of the operator L here is slightly different from [3], the theory of the resolvent remains exactly the same. Therefore we confine ourselves to stating theorems and referring the reader to [3] for the proofs.

Let

$$\mathfrak{R}_p=\sum_{k=0}^{n} R_{p+k}\zeta^{-k-1}\in R_-, \qquad \tilde{\mathfrak{R}}(z,\zeta)=\sum_{l=0}^{\infty} z^{-l}\mathfrak{R}_{-r_0+(n+1)l}\in R_-((z^{-1})),$$

where r_0 is a fixed integer $0\leqq r_0\leqq n$ and z is a new parameter. It is clear that $\mathfrak{R}(\zeta)=\tilde{\mathfrak{R}}(\zeta^{n+1},\zeta)\cdot\zeta^{r_0}$. We shall call $\tilde{\mathfrak{R}}(z,\zeta)$ the polarization of the resolvent. The mapping M defined in Sect. 2 can be extended to $R_-((z^{-1}))$.

Proposition. *The polarizations of the resolvents are all the solutions of the equation*

$$M_{\mathfrak{R}}=[L,\tilde{\mathfrak{R}}](\zeta^{n+1}+z)=0. \tag{5.2}$$

Lemma. *Equation* (5.2) *is equivalent to the recurrence relations*

$$M^{(1)}_{\mathfrak{R}_{-r_0+(n+1)l}}+M^{(2)}_{\mathfrak{R}_{-r_0+(n+1)(l+1)}}=0$$

for any fixed r_0.

This is an obvious corollary of the previous proposition (to this end one has to expand (5.2) into series with respect to ζ).

Corollary. *For any natural r the resolvents satisfy the equation*

$$M^{(1)}_{\mathfrak{R}_r}+M^{(2)}_{\mathfrak{R}_{r+n+1}}=0. \tag{5.3}$$

Theorem (on the existence of resolvents). *For any constant diagonal matrix B, there exists a resolvent $\mathfrak{R}$ for which $R_0=B$, and there are no constants in other R_k.*

We denote such a resolvent by $\mathfrak{R}^{(B)}$.

Theorem (on the variational derivatives)

$$\frac{\delta}{\delta U^*}\operatorname{tr}U_\zeta\mathfrak{R}\Big|_r=(-r+1)\mathfrak{R}_{r-1}. \tag{5.4}$$

The subscript r denotes the coefficient in ζ^{-r}, and the subscript ζ denotes the derivative with respect to ζ.

Now let

$$H_r^{(B)} = \frac{1}{-r+1} \operatorname{tr} U_\zeta \mathfrak{R}^{(B)}\Big|_r . \tag{5.5}$$

Theorem (on the involutiveness of H_r). *For any constant diagonal matrices B and C, we have*

$$\int \operatorname{tr} \operatorname{res} M^{(1)}_{\mathfrak{R}_r^{(B)}} \cdot \mathfrak{R}_s^{(C)} dx = -\int \operatorname{tr} \operatorname{res} M^{(2)}_{\mathfrak{R}_r^{(B)}} \cdot \mathfrak{R}_s^{(C)} dx = 0 .$$

In particular, due to (5.4) this means that $\{H_r^{(B)}, H_s^{(C)}\} = 0$ for both symplectic structures.

6. First Integrals of Chiral Field Equations

Theorem. *The Hamiltonian $H + \hat{H}$ (see Sect. 3) is one of the coefficients H_r defined above.*

Proof. Let us find several first terms of the resolvent $\mathfrak{R}^{(-2A)}$. From the recurrence formula

$$[A, R_{i+1}] + [U_1, R_i] + R_i' + [U_0, R_{i-1}] = 0 \tag{6.1}$$

we find, in succession, $R_0 = -2A$, $R_1 = -2U_1 = -2[A, \varphi_0]$, $R_2 = 2\varphi_0' - 2U_0$ (outside the diagonal). Further, we have

$$[A, R_3] + [U_1, R_2] + R_2' + [U_0, R_1] = 0$$

and, according to Eq. (5.4), $\delta H_4/\delta U_0^* = R_2$, and $\delta H_4/\delta U_1^* = R_3$. If H_4 is taken as a Hamiltonian, Eq. (3.3) take the form

$$\begin{aligned} \dot{U}_0 &= [U_1, R_2] + R_2' + [A, R_3] = -[U_0, R_1] = 2[U_0, U_1] , \\ \dot{U}_1 &= [A, R_2] = 2[A, \varphi_0'] - 2[A, U_0] = 2U_1' - 2[A, U_0] , \end{aligned}$$

i.e. the system of Eq. (3.6) is obtained. This means that $H + \hat{H} = H_4$ with $B = -2A$. □

Lemma. *The Hamiltonian $\hat{H}$ commutes (is in involution) with all H_r.*

Proof. The vector field corresponding to $\hat{H}$ is ∂ and therefore commutes with all ξ_{H_r}. □

Theorem. *H_r are first integrals (in involution) of the chiral field Eq.* (1.6).

Proof. $H + \hat{H} = H_4$ commutes with all H_r. $\hat{H}$ also commutes with them, and hence H commutes with all H_r. □

Remark. If we act from the very beginning in the cone variables ξ, η, Eq. (1.1) will be replaced by

$$\bar{\bar{U}}_\eta - \bar{\bar{V}}_\xi = [\bar{\bar{V}}, \bar{\bar{U}}] , \tag{6.2}$$

where $\bar{\bar{U}} = -(U_1 + \zeta A)$, $\bar{\bar{V}} = -U_0 \zeta^{-1}$. A resolvent is now a solution of the equation $R_\xi = [\bar{\bar{U}}, R]$. This coincides with a resolvent of the simplest matrix differential operator $L = \partial + U + \zeta A$. The quantities $\int \operatorname{tr} AR d\xi|_r$ are first integrals of (6.2) which is easy to prove (see also [1]). They are in involution with respect to the Poisson bracket $\int \left[A, \frac{\delta f}{\delta U_1^*}\right] \frac{\delta g}{\delta U_1^*} d\xi$, but it is unclear whether this Poisson bracket has any connection to Eq. (6.2). The cone variables are characteristic to this equation, i.e. the equation is not one of the Cauchy-Kovalevsky type and therefore cannot be Hamiltonian.

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