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A Criterion of Integrability for Perturbed Nonresonant Harmonic Oscillators. "Wick Ordering" of the Perturbations in Classical Mechanics and Invariance of the Frequency Spectrum

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Abstract. We introduce an analogue to the renormalization theory (of quantum fields) into classical mechanics. We also find an integrability criterion guaranteeing the convergence of Birkhoff's series and an algorithm for modifying the hamiltonian to fix the frequency spectrum of the quasi-periodic motions. We point out its possible relevance to the transition to chaos.

1. Introduction, Notations and Results

In a famous paper [1] Poincaré proved the generic nonexistence of analytic prime integrals for systems which are obtained by perturbing an integrable system. This system is described in its action-angle variables $(\mathbf{A}, \boldsymbol{\varphi})$ by an analytic hamiltonian $h_0(\mathbf{A})$ such that

$$\operatorname{rank} \frac{\partial^2 h_0}{\partial \mathbf{A} \partial \mathbf{A}}(\mathbf{A}) \geqq 2, \tag{1.1}$$

where $\mathbf{A} = (A_1, \ldots, A_\ell)$ denote the ℓ action variables, $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\ell)$ denote the ℓ conjugate angles, and we suppose that the system's phase space is

$$V \times \mathrm{T}^\ell, \tag{1.2}$$

where V is a closed sphere in R of radius r, fixed once and for all, and T^ℓ is the ℓ-dimensional torus. The number ℓ is the number of degrees of freedom.

The theorem of Poincaré deals with perturbations of the form

$$f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) = \sum_{\gamma \in Z^\ell} f_{0\gamma}(\mathbf{A}, \varepsilon) e^{i\gamma \cdot \boldsymbol{\varphi}}, \tag{1.3}$$

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where f_0 is an analytic function of $(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) \in V \times T^\ell \times [-a, a]$, for some $a > 0$, and is supposed *divisible* by ε: the right hand side of (1.3) is used to introduce our notation for the Fourier transform $f_{0\boldsymbol{\gamma}}$ of f_0 with respect to the $\boldsymbol{\varphi}$-variables: $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in Z^\ell$ are ℓ integers.

The usual way of phrasing Poincaré's theorem is by saying that "the perturbation series for the solutions of the Hamilton–Jacobi equation for

$$H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) = h_0(\mathbf{A}) + f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) \tag{1.4}$$

diverges, in general."

Apparently the cases ("rank 1" or "rank 0" respectively)

$$\begin{aligned} &\text{i) } h_0(\mathbf{A}) = h(\boldsymbol{\omega}_0 \cdot \mathbf{A}), \quad h' \neq 0, \ h'' \neq 0, \\ &\text{ii) } h_0(\mathbf{A}) = \boldsymbol{\omega}_0 \cdot \mathbf{A}, \end{aligned} \tag{1.5}$$

with $\boldsymbol{\omega}_0 \in R^\ell$ fixed, were not considered interesting enough by Poincaré to be worth studying.

Nevertheless his method can be adapted to show that in this case too the perturbation (1.3) will, in general, produce a new hamiltonian system which does not admit any nontrivial prime integrals which are analytic (in various senses) [3].

To be more precise, and even a bit more general, consider the hamiltonian on $V \times T^\ell$:

$$H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) = h_0(\mathbf{A}, \varepsilon) + f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) \tag{1.6}$$

with h_0 analytic on $V \times [-a, a]$ and f_0 analytic in $V \times T^\ell \times [-a, a]$.

The "integrability problem" is the following: does it exist a map $\mathscr{C}_\varepsilon : V \times T^\ell \times [-a', a'] \to R^\ell \times T^\ell$ analytic in $(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) \in V \times T^\ell \times [-a', a'], 0 < a' \leqq a$, canonical for every ε, and $h_\varepsilon : V \times [-a', a'] \to R$ such that

$$\text{i) } H_\varepsilon(\mathscr{C}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}')) = h_\varepsilon(\mathbf{A}') \quad \forall (\mathbf{A}', \boldsymbol{\varphi}', \varepsilon) \in V \times T^\ell \times [-a', a'], \tag{1.7}$$

$$\text{ii) } (\mathscr{C}_\varepsilon - \text{identity}) \text{ and } (h_\varepsilon(\mathbf{A}') - h_0(\mathbf{A}', 0)) \text{ are } \varepsilon\text{-divisible?} \tag{1.8}$$

Writing $\mathscr{C}_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}') = (\mathbf{A}, \boldsymbol{\varphi})$ in terms of its generating function Φ_ε, the relation between $(\mathbf{A}, \boldsymbol{\varphi})$ and $(\mathbf{A}', \boldsymbol{\varphi}')$ must be such that:

$$\begin{aligned} \mathbf{A} &= \mathbf{A}' + \frac{\partial \Phi_\varepsilon}{\partial \boldsymbol{\varphi}}(A', \varphi), \\ \boldsymbol{\varphi}' &= \boldsymbol{\varphi} + \frac{\partial \Phi_\varepsilon}{\partial \mathbf{A}'}(\mathbf{A}', \boldsymbol{\varphi}), \end{aligned} \tag{1.9}$$

with Φ_ε analytic on $V \times T^\ell \times [-a', a']$, and (without loss of generality):

$$\int_{T^\ell} \Phi_\varepsilon(\mathbf{A}', \boldsymbol{\varphi}) d\boldsymbol{\varphi} \equiv 0.$$

1 Note that $\mathscr{C}_\varepsilon$ maps $V \times T^\ell$ into $R^\ell \times T^\ell$ in general: nevertheless (1.7) makes sense for ε small, provided (1.8) holds, because of the analyticity assumption (and the fact that V is closed), which allows to extend h_0, f_0 outside $V \times T^\ell$

Then (1.8) says that:

$$\Phi_\varepsilon(\mathbf{A}',\varphi) = \varepsilon\Phi^{(1)}(\mathbf{A}',\varphi) + \varepsilon^2\Phi^{(2)}(\mathbf{A}',\varphi) + \ldots,$$
$$h_\varepsilon(\mathbf{A}') = h_0(\mathbf{A}',0) + \varepsilon h^{(1)}(\mathbf{A}') + \ldots, \tag{1.10}$$

and then (1.7) becomes in terms of Φ_ε, by the first relation in (1.9):

$$h_0\left(\mathbf{A}' + \frac{\partial\Phi_\varepsilon}{\partial\varphi}(\mathbf{A}',\varphi), \varepsilon\right) + f_0\left(\mathbf{A}' + \frac{\partial\Phi_\varepsilon}{\partial\varphi}(\mathbf{A}',\varphi), \varepsilon\right) = h_\varepsilon(\mathbf{A}'), \tag{1.11}$$

which after a universal development in powers of ε implies an infinite hierarchy of equations for the unknown functions $\Phi^{(1)},\ldots,h^{(1)},\ldots$.

It is immediately seen that the n^{th} equation has the form:

$$\omega_0(\mathbf{A}')\cdot\frac{\partial\Phi^{(n)}}{\partial\varphi}(\mathbf{A}',\varphi) + F^{(n)}(f_0^{(n)},\ldots,f_0^{(1)},h_0^{(n)},\ldots,h_0^{(0)},h^{(n+1)},\ldots,h^{(1)},\Phi^{(n-1)},\ldots,\Phi^{(1)})$$
$$= h^{(n)}(\mathbf{A}'), \tag{1.12}$$

where the upper indices denote derivatives with respect to ε evaluated at $\varepsilon = 0$, times $n!$, and we have set

$$\omega_0(A') = \frac{\partial h_0^{(0)}}{\partial \mathbf{A}'}(\mathbf{A}'), \tag{1.13}$$

and, finally, $F^{(n)}$ is a differential polynomial with coefficients involving powers of the derivatives $\partial/\mathrm{d}\varphi$ and $\partial/\partial\mathbf{A}'$.

For instance for $n = 1$ the $F^{(1)}$ is $F^{(1)}(f_0^{(1)}) \equiv f_0^{(1)}$ and (1.12) becomes:

$$\omega_0(\mathbf{A}')\cdot\frac{\partial\Phi^{(1)}}{\partial\varphi}(\mathbf{A}',\varphi) + f_0^{(1)}(\mathbf{A}',\varphi) + h_0^{(1)}(\mathbf{A}') = h^{(1)}(\mathbf{A}'), \tag{1.14}$$

which should determine $\Phi^{(1)}$ and $h^{(1)}$.

To discuss (1.12) introduce the notation

$$\bar{F}^{(n)}(\mathbf{A}') = \int_{T^\ell} F^{(n)}(\ldots)\frac{d\varphi}{(2\pi)^l}, \tag{1.15}$$

or, more generally:

$$F_\gamma^{(n)}(\mathbf{A}') = \int_{T^\ell} e^{-i\gamma\cdot\varphi}F^{(n)}(\ldots)\frac{d\varphi}{(2\pi)^\ell}. \tag{1.16}$$

Then we see that (1.12) implies

$$h^{(n)}(\mathbf{A}') = \bar{F}^{(n)}(\mathbf{A}') \tag{1.17}$$

(by integrating both sides with respect to φ) and

$$\Phi^{(n)}(\mathbf{A}',\varphi) = \sum_{\gamma\neq\mathbf{0}} \frac{F^{(n)}(\mathbf{A}')e^{i\gamma\cdot\varphi}}{-i\omega_0(\mathbf{A}')\cdot\gamma}. \tag{1.18}$$

A necessary condition for the existence of $\mathscr{C}_\varepsilon$ is, of course, that $F_\gamma^{(n)}(\mathbf{A}')$ can be written as

$$F_\gamma^{(n)}(\mathbf{A}') = (\boldsymbol{\omega}_0(\mathbf{A}')\cdot\boldsymbol{\gamma})\hat{F}_\gamma^{(n)}(\mathbf{A}'), \quad \forall 0 \neq \boldsymbol{\gamma}\in \mathbf{Z}^\ell, \tag{1.19}$$

i.e. $F_\gamma^{(n)}(\mathbf{A}')$ vanishes everytime $\boldsymbol{\omega}_0(\mathbf{A}')\cdot\boldsymbol{\gamma}$ does, i.e. everytime "a resonance takes place.'

We shall call (1.19) the "condition for the existence of perturbation theory" of f_0 with respect to h_0.

Definition. We say that the Hamilton–Jacobi equation (1.11) "admits a finite perturbation theory," if (1.19) is verified and if there are constants $\hat{F}_n, \xi > 0$ such that

$$|\hat{F}_\gamma^{(n)}(\mathbf{A}')| \leqq \hat{F}_n e^{-\xi|\gamma|} \quad \forall n \geqq 1, \forall \boldsymbol{\gamma}\in \mathbf{Z}^\ell, \forall \mathbf{A}\in V. \tag{1.20}$$

It is now clear that if (1.1) holds, the perturbation theory will, generically, not exist: e.g. if $f_{0\gamma}^{(1)}(\mathbf{A}) \neq 0$, $\forall\boldsymbol{\gamma}\in \mathbf{Z}^\ell$, $\forall\mathbf{A}\in V$, the condition (1.19) will not be satisfied already for $n=1$ (the right hand side vanishes, in fact, for some $\boldsymbol{\gamma}$ on a dense set in V). This remark is, essentially, the above-mentioned famous theorem of Poincaré[2].

A well-known theorem of Birkhoff provides an example of a hamiltonian h_0 which always admits a finite perturbation series, for *all* perturbations: it is the case (1.5) with $\boldsymbol{\omega}_0$ "nonresonant," i.e. such that

$$|\boldsymbol{\omega}_0\cdot\boldsymbol{\gamma}|^{-1} \leqq C|\boldsymbol{\gamma}|^\alpha \quad \forall \boldsymbol{\gamma}\in Z^\ell, \boldsymbol{\gamma} \neq \mathbf{0}, \tag{1.21}$$

for some $C > 0$, $\alpha > 0$. It is well known that almost all points in R^ℓ verify (1.21) with $C < +\infty$, $\alpha = \ell$, [2].

This theorem is easily proved by induction using (1.12), (1.18) and noting that (1.20) follows from the analyticity of $f_0^{(1)}$ in $\boldsymbol{\varphi}$ and from (1.21), in the starting case $n=1$, [2].

In his book on Mécanique Celeste, Poincaré raises the question of the integrability of systems admitting a perturbation series: in fact he examines in this light a few well-known integrable systems depending on a parameter. However he leaves the question open, contenting himself to say that "whenever (1.19) holds, nothing is against the integrability," see [1], p. 258.

No examples seem to be known of perturbations admitting a finite perturbation theory with respect to an h_0 verifying (1.1) with the rank equal to ℓ ("non-isochronous systems") but being, nevertheless, nonintegrable. In any case a simple example of a system with finite perturbation theory but not integrable, in the sense that (1.11) does not admit an analytic solution, can be easily constructed from a Birkhoff hamiltonian, like (1.5).

In fact, let $\ell = 2$, $\mathbf{A} = (A, B)$, $\boldsymbol{\varphi} = (\varphi, \psi)$ and $\boldsymbol{\omega}_0 = (\omega, 1)$ such that (1.21) holds, let f be a function on T^1 with nonzero Fourier coefficients, e.g.:

$$f(\varphi) = \sum_{n=1}^{\infty} e^{-\xi n}\cos n\varphi, \tag{1.22}$$

2 in [1] a stronger statement is proven concerning the nonexistence, in general, of an analytic prime integral, but in essence the main argument is the same as here

and consider $h_0(\mathbf{A}) = (\omega A + B)$ and

$$H_\varepsilon(A, B, \varphi, \psi) = (\omega A + B) + \varepsilon(B + f(\varphi)f(\psi)). \tag{1.23}$$

It is easy to check that if $|\gamma| = |\gamma_1| + |\gamma_2|$:

$$\Phi^{(k)}(A', B', \varphi, \psi) = \sum_{0 \neq \gamma \in Z^2} \frac{e^{-\xi|\gamma|}e^{i\gamma\cdot\varphi}}{-i(\omega\gamma_1 + \gamma_2)}\left(\frac{-\gamma_2}{\omega\gamma_2 + \gamma_1}\right)^{k-1}, \tag{1.24}$$

for $k = 1, 2, \ldots$.

However the series $\sum_{k=1}^{\infty} \varepsilon^k \Phi^{(k)}$ cannot converge, because (1.23) can be elementarily studied and it does not have, for all ε small, motions which are all quasi-periodic.

From the above example one can easily build [11] examples of systems with divergent, though finite, perturbation series: in fact one just forms the system on $V \times V' \times T^2 \times T'$ with hamiltonian defined by

$$H_\varepsilon(A, B, \varphi, \psi) + \tfrac{1}{2}(C_1^2 + \ldots + C_{\ell'}^2),$$

if $(A, B, \mathbf{C}, \varphi, \psi, \boldsymbol{\varphi}') \in V \times V' \times T^2 \times T^{\ell'}$, whose hessian matrix will have rank $\ell' < \ell = \ell' + 2$. An easy modification yields an example with rank $\ell - 1$.

Therefore in general we cannot expect that the perturbation series converges, even when it is finite in the sense of the above definition. In fact "in general" the series diverges: see introductory discussion in [3].

In this work we study the question of which could be the extra conditions necessary for the convergence of the perturbation series. We are able to exhibit a sufficient condition for the convergence of the perturbation series in the Birkhoff case:

$$h_0(\mathbf{A}, \varepsilon) = h(\boldsymbol{\omega}_0 \cdot \mathbf{A}) + \varepsilon h_0^{(1)}(\mathbf{A}) + \ldots, \tag{1.25}$$

with $h' \neq 0$ and $\boldsymbol{\omega}_0$ verifying (1.21). We recall that (1.25) is the only known case in which a condition for the existence of a finite perturbation theory is simple (namely there is no condition *at all*).

We prove

1. Proposition. *Let h_0 be given by (1.25) and let $\boldsymbol{\omega}_0$ verify (1.21): "the unperturbed system is a nonresonant isochronous oscillator." A sufficient condition for the convergence of the series (1.10) for ε small is that f_0 is such that* (see (*1.10*)):

$$h^{(k)}(\mathbf{A}') = \sigma^{(k)}(\boldsymbol{\omega}_0 \cdot \mathbf{A}'), \tag{1.26}$$

where $\sigma^{(k)}$ are functions of one variable. The condition is not necessary, obviously.

The method of proof almost yields a solution to the following problem:

2. Problem. In the general case, suppose that $h_0(\mathbf{A}, \varepsilon)$, and $f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon)$ are such that the perturbation theory of f_0 with respect to h_0 is finite and suppose that there are functions $\sigma^{(k)}$ of one variable such that:

$$h^{(k)}(\mathbf{A}) = \sigma^{(k)}(h_0^{(0)}(\mathbf{A})). \tag{1.27}$$

Is this sufficient for the convergence of the perturbation series?

The connection of Proposition 1 and of Problem 2 with the work of [10] is clear. Another question which we study in this paper is the following.

Given a hamiltonian of the form

$$H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) = h(\boldsymbol{\omega}_0 \cdot \mathbf{A}) + f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon), \tag{1.28}$$

with $h' \neq 0$, $\boldsymbol{\omega}_0$ verifying (1.21), is it possible to "renormalize" it by adding to it a function $N_{f_0}(\mathbf{A}, \varepsilon)$ analytic in $V \times [-a', a']$, $0 < a' \leqq a$, so that

$$H_\varepsilon^{\text{ren}}(\mathbf{A}, \boldsymbol{\varphi}) = H_\varepsilon(\mathbf{A}, \boldsymbol{\varphi}) - N_{f_0}(\mathbf{A}, \varepsilon) \tag{1.29}$$

is integrable and canonically conjugate to the unperturbed hamiltonian? The analogy of this problem with the renormalization problem of quantum fields is so striking that I take the liberty of setting:

$$:f_0: = f_0 - N_{f_0}, \tag{1.30}$$

whenever N_{f_0} exists, calling the operator : : "the Wick ordering of with respect to $h(\boldsymbol{\omega}_0 \cdot \mathbf{A})$."

It is possible to prove

3. Proposition. *Given the hamiltonian $h_0(A) = h(\boldsymbol{\omega}_0 \cdot A)$ with $\boldsymbol{\omega}_0$ verifying (1.21) and h analytic on $\mathscr{E} = \{E | E \in R, E = \omega_0 \cdot \mathbf{A}$ for some $\mathbf{A} \in V\}$, and given an analytic perturbation f_0 defined on $V \times T^\ell \times [-a, a]$, there is at most one function N_{f_0} such that $N_{f_0}(\mathbf{A}, \varepsilon)$ is analytic on $V \times [-a', a']$, $0 < a' \leqq a$, and*

$$h(\boldsymbol{\omega}_0 \cdot \mathbf{A}) + :f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon): \tag{1.31}$$

is integrable for small ε and analytically canonically conjugate with the unperturbed system $h(\boldsymbol{\omega}_0 \cdot \mathbf{A})$ with a transformation $\mathscr{C}_\varepsilon$ verifying (1.8). *There is an algorithm which allows us to construct for every f_0 a sequence of analytic functions on $V \times T^\ell$, $\{N_{f_0}^{(k)}\}_{k=1,2,\dots}$ such that N_{f_0} exists if and only if the series*

$$N_{f_0}(\mathbf{A}, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k N_{f_0}^{(k)}(\mathbf{A}) \tag{1.32}$$

converges for small ε.

Therefore we introduce the following definition:

Definition. Let $\boldsymbol{\omega}_0$ verify (1.21) and let $h_0(\mathbf{A}) = h(\boldsymbol{\omega}_0 \cdot \mathbf{A})$ as in Proposition 3.

Given f_0, as in Proposition 3, we say that f_0 is "h_0-renormalizable," if the series (1.32) converges.

We say that f_0 is "h_0-super-renormalizable," if $N_{f_0}^{(k)}(\mathbf{A}) \equiv 0$ for k large enough.

We think that, for aesthetic reasons, it should be true that

4. Conjecture. Let $h_0(\mathbf{A}) = \boldsymbol{\omega}_0 \cdot \mathbf{A}$ and $\boldsymbol{\omega}_0$ verify (1.21). Then every perturbation f_0 which is a polynomial in ε is h_0-renormalizable and (1.32) is an entire function in ε.

However we have only been able to prove that:

5. Proposition. *In the conditions of Proposition 3, given a perturbation f_0 analytic in $V \times T^\ell \times [-a, a]$ there exist for all $N > 0$, a positive constant a_N and an analytic function $\delta_N f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon)$ divisible by ε^{N+1} such that, for $|\varepsilon| < a_N$, the*

$$h_0(\boldsymbol{\omega}_0 \cdot \mathbf{A}) + f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) - \sum_{k=1}^{N} \varepsilon^k N_{f_0}^{(k)}(\mathbf{A}) - \delta_N f_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) \tag{1.33}$$

is integrable by a canonical map $\mathscr{C}_\varepsilon$ verifying (1.8) and conjugating (1.33) with $h(\boldsymbol{\omega}_0 \cdot \mathbf{A})$.

So any perturbation (renormalizable or not) "behaves as if it were renormalizable" to arbitrary accuracy in ε, with the drawback that a_N may tend to zero as $N \to \infty$.

We can also interpret Proposition 5 as saying that close to any perturbation there is another perturbation which is super-renormalizable.

Our construction of the :: operator can be relevant in the theory of the development of chaotic motions within a system with constant discrete spectrum, as well as in the design of the related numerical experiments. Such a problem has been recently introduced and discussed in the context of the theory of dissipative motions [4].

It is clear that the Wick ordering of the perturbations destroys the resonances and, therefore, the mechanism for the development of chaos must be, for Wick ordered systems, different from the usual one. We recall that the "usual mechanism" is based on the existence of resonances on an open dense set in phase space and on their invasion of large regions of phase space as ε increases.

In renormalized systems chaos occurs only because of the divergence of the perturbation series as ε grows: i.e., it probably occurs suddenly in the whole phase space as ε passes through a singularity of the perturbative series. At least if $N_{f_0}(\mathbf{A}, \varepsilon)$, (1.32), has a larger radius of convergence than that of the perturbation series, (see also Conjecture 4).

The reason why we say that the above Propositions 3, 5 may be relevant even for the design of numerical experiments is that the coefficients of N_{f_0} can be explicitly constructed by a recursive algorithm which is provided by the proof itself. For instance, to first order

$$N_{f_0}(\mathbf{A}, \varepsilon) = \varepsilon \bar{f}_0^{(1)}(\mathbf{A}) + O(\varepsilon^2), \tag{1.34}$$

i.e. to first order the "Wick ordering" coincides with the subtraction of the average of the perturbation's first order.

In this paper we give a detailed proof of Propositions 1, 3 and Proposition 5 is only sketched, since its proof is essentially a repetition, in most of the technical details, of the proof of Proposition 1.

The techniques used in this work are borrowed mainly from the classical theory of Kolmogorov–Arnold–Moser [5], see also [2] for an elementary exposition of it, and from [6], [7], [8], [9].

2. Proof of Proposition 1

For simplicity we shall only consider the case $h(\omega_0 \cdot \mathbf{A}) \equiv \omega_0 \cdot \mathbf{A}$. The general case can be treated in a similar fashion or it can be reduced to the above case by suitable transformations.[3]

We must make some quantitative statements about the analyticity of f_0.

3 To reduce $h_0(\mathbf{A}, \varepsilon)$ to the case when h_0 is ε-independent, just put the ε-dependent part of h_0 into f_0. To reduce the case $h(\boldsymbol{\omega}_0 \cdot \mathbf{A})$ to $\boldsymbol{\omega}_0 \cdot \mathbf{A}$, consider $h^{-1}(h(\boldsymbol{\omega}_0 \cdot \mathbf{A}) + \tilde{f}_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) = \boldsymbol{\omega}_0 \cdot \mathbf{A} + \tilde{\tilde{f}}_0(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon)$ and notice that the condition (1.26) is verified for the perturbation theory of this hamiltonian if and only if it is verified for the original hamiltonian

Therefore we choose to identify the torus T^ℓ with a subset of C^ℓ via the map

$$\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_\ell) \leftrightarrow \mathbf{z} = (e^{i\varphi_1}, \ldots, e^{i\varphi_\ell}). \tag{2.1}$$

This allows us to identify $V \times T^\ell$ as a subset of $C^{2\ell}$, and we regard an analytic function on $V \times T^\ell$ as a function which is holomorphic in the variables $(\mathbf{A}, \mathbf{z}) \in C^{2\ell}$ as $(\mathbf{A}, \mathbf{z})$ varies in a neighborhood of $V \times T^\ell$ thought of as a subset of $C^{2\ell}$, via (2.1).

If f is analytic on $V \times T^\ell$ we shall denote by $f(\mathbf{A}, \mathbf{z})$ the value of its holomorphic extension in the point $(\mathbf{A}, \mathbf{z})$ in the holomorphy domain.

We introduce the following complex domains, given $\rho_0, \xi_0, \theta_0 \in (0, 1)$:

$$\begin{aligned}
&B(\rho_0) = \{\mathbf{A} \mid \mathbf{A} \in C^\ell, \exists \mathbf{A}_0 \in V : |A_i - A_{0i}| < \rho_0, \forall i\}, \\
&C(\xi_0) = \{\mathbf{z} \mid \mathbf{z} \in C^\ell, e^{-\xi_0} < |z_i| < e^{\xi_0}, \forall i\}, \\
&D(\varepsilon_0) = \{\varepsilon \mid \varepsilon \in C, |\varepsilon| < \theta_0\}, \\
&\mathscr{E}(\rho_0) = \{E \mid E \in C, E = \boldsymbol{\omega}_0 \cdot \mathbf{A} \text{ for some } \mathbf{A} \in B(\rho_0)\}, \\
&W(\rho_0) \equiv B(\rho_0), W(\rho_0, \xi_0) = B(\rho_0) \times C(\xi_0), W(\rho_0, \xi_0, \theta_0) = B(\rho_0) \times C(\xi_0) \times D(\theta_0).
\end{aligned} \tag{2.2}$$

We denote for $\gamma \in Z^\ell, \mathbf{w} \in C^m, \mathbf{z} \in C^\ell$.

$$\begin{aligned}
&|\gamma| = \sum_{i=1}^{\ell} |\gamma_i|, \quad |\mathbf{w}| = \sum_{i=1}^{\ell} |w_i|, \\
&\mathbf{z} = \prod_{i=1}^{\ell} z_i^{\gamma_i}, \qquad \mathbf{z}e^{\mathbf{w}} = (z_1 e^{w_1}, \ldots, z_\ell e^{w_\ell}) \quad \text{if} \quad \ell = m, \\
&\frac{\partial}{\partial \boldsymbol{\varphi}} = \left(\frac{\partial}{\partial \varphi_1}, \ldots, \frac{\partial}{\partial \varphi_\ell}\right), \quad \frac{\partial}{\partial \varphi_j} = -iz_j \frac{\partial}{\partial z_j}.
\end{aligned} \tag{2.3}$$

Let K_0 be holomorphic on $\mathscr{E}(\rho_0) \times D(\theta_0)$ and suppose:

$$K_0(E, \varepsilon) = E + \sum_{k=1}^{\infty} \varepsilon^k K^{(k)}(E). \tag{2.4}$$

Let f_0 be holomorphic on $W(\rho_0, \xi_0, \theta_0)$, and let

$$f_0(\mathbf{A}, \mathbf{z}, \varepsilon) = \sum_{\gamma \in \mathbf{Z}'^\ell} f_{0\gamma}(\mathbf{A}) \mathbf{z}^\gamma \tag{2.5}$$

be the Laurent expansion of f_0 with respect to the $\mathbf{z}$.

We suppose K_0, f_0 real for $E, \mathbf{A}, \boldsymbol{\varphi}, \varepsilon$ real: eventually we are interested only in the case $K_0(E, \varepsilon) \equiv E$.

We shall study the following Hamilton–Jacobi equation:

$$K_0\left(\boldsymbol{\omega}_0, \left(\mathbf{A}' + \frac{\partial \Phi_\infty}{\partial \boldsymbol{\varphi}}(\mathbf{A}', \boldsymbol{\varphi})\right), \varepsilon\right) + f_0\left(\mathbf{A}' + \frac{\partial \Phi_\infty}{\partial \boldsymbol{\varphi}}(\mathbf{A}', \boldsymbol{\varphi}), \boldsymbol{\varphi}, \varepsilon\right) = K_\infty(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon) \tag{2.6}$$

where Φ_∞, K_∞ are unknown functions.

We ask whether (2.6) admits a solution Φ_∞ analytic in $W(\rho, \xi, \theta)$ and K_∞ analytic in $\mathscr{E}(\rho) \times D(\theta)$ for some $\rho, \xi, \theta > 0$.

We apply a recursive algorithm: we imagine to label the steps of the algorithm ("renormalization group") by an index $n = 1, 2, \ldots$ so that at the n-step we shall have built a canonical transformation $\mathscr{C}_{n-1}$ whose generating function Φ will be the n^{th} approximation to the solution of (2.6). The canonical transformation $\mathscr{C}^{(n-1)} = \mathscr{C}_{n-1}\,\mathscr{C}_{n-2}^{-1}$ (i.e. the "variation" of $\mathscr{C}_{n-1}$ with respect to $\mathscr{C}_{n-2}$) will change a hamiltonian of the form

$$H_{n-1}(\mathbf{A}, \boldsymbol{\varphi}) = K_{n-1}(\boldsymbol{\omega}_0 \cdot \mathbf{A}, \varepsilon) + f_{n-1}(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon), \tag{2.7}$$

defined on $W(\rho_{n-1}, \xi_{n-1}, \theta_{n-1})$ into

$$H_n(\mathbf{A}', \boldsymbol{\varphi}') = H_{n-1}(\mathscr{C}^{(n-1)}(\mathbf{A}', \boldsymbol{\varphi}')) = K_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}') + f_n(\mathbf{A}', \boldsymbol{\varphi}', \varepsilon), \tag{2.8}$$

defined on $W(\rho_n, \xi_n, \theta_n)$ such that

$$\mathscr{C}^{(n-1)} W(\rho_n, \xi_n, \theta_n) \subset W(\rho_{n-1}, \xi_{n-1}, \theta_{n-1}), \tag{2.9}$$

with ρ_n, ξ_n, θ_n being a suitable decreasing sequence of positive numbers.

Furthermore, denoting $[\quad]^{[\leqq p]}$ the truncation to order p of a (convergent) power series in ε:

$$K_n(E, \varepsilon) - K_{n-1}(E, \varepsilon) = \{\text{polynomial in } \varepsilon \text{ of degree } 2^n - 1, \text{ divisible by } \varepsilon^{2^{n-1}}\}, \tag{2.10}$$

$$f_n(\mathbf{A}, \boldsymbol{\varphi}, \varepsilon) = \{\text{function divisible by } \varepsilon^{2^n}\}, \tag{2.11}$$

$$\mathscr{C}^{(n-1)} - \text{identity} = \{\text{function divisible by } \varepsilon^{2^{n-1}}\}, \tag{2.12}$$

$$[\bar{f}_n(\mathbf{A}', \varepsilon)]^{[\leqq 2^{n+1}-1]} = \{\text{function of } \boldsymbol{\omega}_0 \cdot \mathbf{A}' \text{ and } \varepsilon\}, \tag{2.13}$$

where the bar denotes the average over $\boldsymbol{\varphi}$, see (1.15).

Since the algorithm is recursive and the $(K_0, f_0, \rho_0, \xi_0, \theta_0)$ are given at the beginning, we have only to explain how $K_{n+1}, f_{n+1}, \rho_{n+1}, \xi_{n+1}, \theta_{n+1}$ are constructed from $K_n, f_n, \rho_n, \xi_n, \theta_n$.

The construction of the n^{th} step depends on a small parameter which is rather arbitrary; we shall choose it once and for all as

$$\delta_n = \xi_0 / 2^5 (1 + n)^2. \tag{2.14}$$

The reader will realize that the only conditions dictating the choice of δ_n are that $6 \sum_{n=0}^{\infty} \delta_n < \xi_0$ and that δ_n does not go to zero too fast with n.

The construction of the n^{th} step proceeds as follows. Let $K' \equiv \partial K / \partial E$, and consider the equation:

$$K_n'(\boldsymbol{\omega}_0 \cdot \mathbf{A}, \varepsilon)\, \boldsymbol{\omega}_0 \cdot \frac{\partial \Phi}{\partial \boldsymbol{\varphi}}(\mathbf{A}', \boldsymbol{\varphi}, \varepsilon) + f_n(\mathbf{A}', \boldsymbol{\varphi}, \varepsilon) = F_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon), \tag{2.15}$$

("linearized Hamilton–Jacobi equation") which we wish to solve exactly to $O(\varepsilon^{2^{n+1}-1})$ with F_n being a polynomial in ε of order $\varepsilon^{2^{n+1}-1}$ divisible by ε^{2^n}. This means that we want to define Φ, F_n as solutions of

$$\boldsymbol{\omega}_0 \cdot \frac{\partial \Phi}{\partial \boldsymbol{\varphi}}(\mathbf{A}', \boldsymbol{\varphi}) = -\left[\frac{f_n(\mathbf{A}', \boldsymbol{\varphi}, \varepsilon)}{K_n'(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon)} - \frac{F_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon)}{K_n'(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon)}\right]^{[\leqq 2^{n+1}-1]} \tag{2.16}$$

with F_n as above.

Using (2.13), we see that if

$$f_n(\mathbf{A}', \varphi, \varepsilon) = -\left[\frac{f_n(\mathbf{A}', \varphi, \varepsilon) - \bar{f}_n(\mathbf{A}', \varepsilon)}{K_n'(\omega_0 \cdot \mathbf{A}', \varepsilon)}\right]^{[\leqq 2^{n+1}-1]}, \tag{2.17}$$

then we can take:

$$\Phi(\mathbf{A}', \mathbf{z}, \varepsilon) = \sum_{\gamma \neq \mathbf{0}} \frac{f_{n\gamma}(\mathbf{A}', \varepsilon)\mathbf{z}^{\gamma}}{i\omega_0 \cdot \gamma},$$
$$F_n(\omega_0 \cdot \mathbf{A}', \varepsilon) = [\bar{f}_n(\mathbf{A}', \varepsilon)]^{[\leqq 2^{n+1}-1]}. \tag{2.18}$$

The algorithm continues by using Φ to define implicitly via

$$\begin{array}{ccc} \mathbf{A} = \mathbf{A}' + \dfrac{\partial \Phi}{\partial \varphi}(\mathbf{A}', \varphi, \varepsilon) & & \mathbf{A} = \mathbf{A}' + \dfrac{\partial \Phi}{\partial \varphi}(\mathbf{A}', \varphi, \varepsilon) \\ & \text{or} & \\ \varphi' = \varphi + \dfrac{\partial \Phi}{\partial A'}(\mathbf{A}', \varphi, \varepsilon) & & \mathbf{z}' = \mathbf{z} \exp i \dfrac{\partial \Phi}{\partial \mathbf{A}'}(\mathbf{A}', \mathbf{z}, \varepsilon) \end{array} \tag{2.19}$$

a canonical map $\mathscr{C}^{(n)}$ on a set of the form $W(\rho_{n+1}, \xi_{n+1}, \theta_{n+1})$ such that

$$\mathscr{C}^{(n)} W(\rho_{n+1}, \xi_{n+1}, \theta_{n+1}) \subset W(\rho_n, \xi_n, \theta_n). \tag{2.20}$$

The actual construction of the map $\mathscr{C}^{(n)}$ involves the inversion of the first equation of (2.19) with respect to $\mathbf{A}'$ and the substitution of the solution into the second, to define $\mathscr{C}^{(n)-1}$ or, to define $\mathscr{C}^{(n)}$, we have to find the solution of the second equation of (2.19) with respect to $\mathbf{z}$ and substitute it into the first.

Since Φ is holomorphic and small with ε, it is clear that by taking $\rho_{n+1}, \xi_{n+1}, \theta_{n+1}$ much smaller than ρ_n, ξ_n, θ_n, the above inversions become totally trivial and $\mathscr{C}^{(n)}$ can be defined (see Appendix A for a precise statement of the implicit function theorems that we have in mind here).

Then setting $(\mathbf{A}, \mathbf{z}) = \mathscr{C}^{(n)}(\mathbf{A}', \mathbf{z}')$, we shall define

$$K_{n+1}(E, \varepsilon) = K_n(E, \varepsilon) + F_n(E, \varepsilon),$$
$$f_{n+1}(\mathbf{A}', \mathbf{z}', \varepsilon) = f_n(\mathscr{C}^{(n)}(\mathbf{A}', \mathbf{z}'), \varepsilon) + K_n(\omega_0 \cdot \mathbf{A}, \varepsilon) - K_n(\omega_0 \cdot \mathbf{A}', \varepsilon) - F_n(\omega_0 \cdot \mathbf{A}', \varepsilon) \tag{2.21}$$

At this point the algorithm will be continued provided K_{n+1}, f_{n+1}, $\mathscr{C}^{(n)}$, $[\bar{f}_{n+1}]^{[\leqq 2^{n+1}-1]}$ verify the properties (2.10) $\div$ (2.13) with $n+1$ replacing n.

From the above construction, see (2.18), it is clear that the only property which we must check is (2.13) with $n+1$ replacing n.

We claim that (2.13) holds too, as a consequence of our main hypothesis (1.26).

This can be proved inductively. Consider the composition $\mathscr{C}_n = \mathscr{C}^{(0)} \ldots \mathscr{C}^{(n)}$, which is defined on $W(\rho_{n+1}, \xi_{n+1}, \theta_{n+1})$, and let

$$(\mathbf{A}, \mathbf{z}) = \mathscr{C}_n(\mathbf{A}', \mathbf{z}'). \tag{2.22}$$

By the construction, we know that if $H_0 = K_0 + f_0$:

$$H_0(\mathscr{C}_n(\mathbf{A}', \mathbf{z}'), \varepsilon) = K_{n+1}(\omega_0 \cdot \mathbf{A}', \varepsilon) + f_{n+1}(\mathbf{A}', \mathbf{z}', \varepsilon) + O(\varepsilon^{2^{n+1}}), \tag{2.23}$$

with K_{n+1} being a polynomial in ε of order $\varepsilon^{2^{n+1}-1}$.

Furthermore the canonical transformation $\mathscr{C}^{(n+1)}$ generated by the first equation of (2.18) with n replaced by $n+1$ is such that if $(\mathbf{A}', \mathbf{z}') = \mathscr{C}^{(n+1)}(\mathbf{A}'', \mathbf{z}'')$, then

$$H_0(\mathscr{C}_n \mathscr{C}^{(n+1)}(\mathbf{A}'', \mathbf{z}''), \varepsilon) = K_{n+1}(\boldsymbol{\omega}_0 \cdot \mathbf{A}'', \varepsilon) + [\bar{f}_{n+1}(\mathbf{A}'', \varepsilon)]^{[\leqq 2^{n+2}-1]} + O(\varepsilon^{2^{n+2}}) \tag{2.24}$$

on $W(\bar{\rho}, \bar{\xi}, \bar{\theta})$ with $\bar{\rho}, \bar{\xi}, \bar{\theta}$ small enough. But by the uniqueness of perturbation theory it must be, on $W(\bar{\rho}, \bar{\xi}, \bar{\theta})$:

$$[\bar{f}_{n+1}(\mathbf{A}'', \varepsilon)]^{[\leqq 2^{n+2}-1]} = \sum_{k=2^{n+1}}^{2^{n+2}-1} \varepsilon^k \sigma^{(k)}(\boldsymbol{\omega}_0 \cdot \mathbf{A}), \tag{2.25}$$

which by the analyticity of f_{n+1} in $(\mathbf{A}', \mathbf{z}', \varepsilon)$ implies (2.13), with $n+1$ replacing n, on the whole domain $W(\rho_{n+1}, \xi_{n+1}, \theta_{n+1})$.

The above discussion shows that there is no obstacle to the iteration of the algorithm.

So the whole problem reduces to explicit estimates of ρ_n, ξ_n, θ_n as well as of the coefficients determining the actual sizes of the various $O(\ldots)$ found in the above discussion.

We will measure the size of the functions K_n, f_n by

$$\begin{aligned} \varepsilon_n &= \left\| \frac{\partial f_n}{\partial \mathbf{A}} \right\|_n + \rho_n^{-1} \left\| \frac{\partial f_n}{\partial \boldsymbol{\varphi}} \right\|_n, \\ E_n &= \left\| \frac{\partial K_n}{\partial E} - 1 \right\|_n, \end{aligned} \tag{2.26}$$

where $\| g \|_n$ denotes the supremum of g in $W(\rho_n, \xi_n, \theta_n)$ or in $\mathscr{E}(\rho_n) \times D(\theta_n)$, to shorten our notations, or more generally $\| g \|_\alpha$ will denote the supremum of g on the set on which it is regarded as defined, if the set depends on α.

To simplify the discussion we choose right away the sequences ρ_n, ξ_n, θ_n as

$$\rho_n = \rho_{n-1} e^{-6\delta_{n-1}}, \xi_n = \xi_{n-1} - 6\delta_{n-1}, \theta_n = \theta_{n-1} e^{-6\delta_{n-1}}, \tag{2.27}$$

so that $(\rho_\infty, \xi_\infty, \theta_\infty) = \lim_{n\to\infty} (\rho_n, \xi_n, \theta_n)$ exists and $\rho_\infty, \xi_\infty, \theta_\infty > 0$ (see 2.14)).

To estimate analytic functions and their derivatives we use "dimensional estimates" (i.e. essentially Cauchy's theorem).

The first equation of (2.18) allows us to estimate

$$\begin{aligned} \| \Phi \|_{\rho_n, \xi_n - \delta_n, \theta_n e^{-\delta_n}} &\leqq B_1' \delta_n^{-\alpha-\ell} C \| \hat{f}_n \|_{\rho_n, \xi_n, \theta_n e^{-\delta_n}} \\ &\leqq B_1'' \delta_n^{-\alpha-\ell} C \delta_n^{-1} \| (f_n - \bar{f}_n)/K_n' \|_{\rho_n, \xi_n, \theta_n} \\ &\leqq B_1'' \delta_n^{-\alpha-\ell-1} C (1 - E_n)^{-1} \left\| \frac{\partial f_n}{\partial \varphi} \right\|_{\rho_n, \xi_n, \theta_n} \leqq B_1 \delta_n^{-1-\ell-\alpha} C \varepsilon_n \rho_n, \end{aligned} \tag{2.28}$$

provided

$$E_n < 1/2, \tag{2.29}$$

and simple dimensional estimates have been used to estimate the truncation in $\hat{f}_n$. The constants, B_1, B_1', B_1'' depend only on ℓ, α.

So the map $(\mathbf{A}',\mathbf{z}')\to(\mathbf{A},\mathbf{z}')$ given by (2.19) is well defined on $W(\rho_n,\xi_n,-\delta_n,\theta_n e^{-\delta_n})$, where we control Φ by (2.28).

By applying two simple implicit function theorems (see Appendix A), choosing $\rho=\rho_n e^{-\delta_n},\xi=\xi_n-2\delta_n,\theta=\theta_n e^{-\delta_n},\delta=\delta_n$, we see that if

$$B\left\|\frac{\partial\Phi}{\partial\varphi}\right\|_{\rho,\xi,\theta}<\rho_n e^{-\delta_n}\delta_n^b,\qquad B\left\|\frac{\partial\Phi}{\partial\mathbf{A}}\right\|_{\rho,\xi,\theta}<\delta_n^b, \tag{2.30}$$

where $B,b>2$ are defined in Appendix B and depend only on ℓ, then we can define on $W(\rho_n e^{-2\delta_n},\xi_n-3\delta_n,\theta_n e^{-\delta_n})$ the canonical maps

$$\mathscr{C}^{(n)}:\begin{cases}\mathbf{A}=\mathbf{A}'+\Xi(\mathbf{A}',\mathbf{z}',\varepsilon)\\ \mathbf{z}=\mathbf{z}'\exp i\Delta(\mathbf{A}',\mathbf{z}',\varepsilon)\end{cases},\qquad \tilde{\mathscr{C}}^{(n)}:\begin{cases}\mathbf{A}'=\mathbf{A}+\Xi'(\mathbf{A},\mathbf{z})\\ \mathbf{z}'=\mathbf{z}\exp i\Delta'(\mathbf{A},z)\end{cases}, \tag{2.31}$$

such that on $W(\rho_n e^{-3\delta_n},\xi_n-4\delta_n,\theta_n e^{-\delta_n})$:

$$\Xi(\mathbf{A}',\mathbf{z}')\equiv\frac{\partial\Phi}{\partial\varphi}(\mathbf{A}',\mathbf{z})\equiv-\Xi'(\mathbf{A},\mathbf{z}),\qquad \Delta(\mathbf{A}',\mathbf{z}')\equiv-\frac{\partial\Phi}{\partial\mathbf{A}'}(\mathbf{A}',\mathbf{z})\equiv-\Delta'(\mathbf{A},\mathbf{z}), \tag{2.32}$$

and (by (2.30), (2.32) and $B,b>2$):

$$\mathscr{C}^{(n)},\tilde{\mathscr{C}}^{(n)}:W(\rho_n e^{-3\delta_n},\xi_n-4\delta_n,\theta_n e^{-\delta_n})\to W(\rho_n e^{-2\delta_n},\xi_n-3\delta_n,\theta_n e^{-\delta_n})$$
$$\mathscr{C}^{(n)}\tilde{\mathscr{C}}^{(n)}=\tilde{\mathscr{C}}^{(n)}\mathscr{C}^{(n)}=\text{identity, on } W(\rho_n e^{-3\delta_n},\xi_n-4\delta_n,\theta_n e^{-\delta_n}). \tag{2.33}$$

The condition (2.30) can be implied, using (2.28), some dimensional estimates and (2.29), by

$$B_2C\varepsilon_n\delta_n^{-b_2}<1,\qquad E_n<\tfrac{1}{2}, \tag{2.34}$$

with $B_2,b_2>2$ and suitably chosen (depending only on (ℓ,α)), $b_2>b_1+\ell+1$.

Then (2.32) and (2.30) imply on $W(\rho_n e^{-3\delta_n},\xi_n-4\delta_n,\theta_n e^{-\delta_n})$.

$$|\Xi|,|\Xi'|<B_3C\varepsilon_n\delta_n^{-b_3}\rho_n<\tfrac{1}{2}\delta_n\rho_n,\qquad |\Delta|,|\Delta'|<B_3C\varepsilon_n\delta_n^{-b_3}<\tfrac{1}{2}\delta_n, \tag{2.35}$$

with $B_3,b_3>1$ depending only on (α,ℓ).

There is an important relation which is verified by Δ,Δ' because Φ verifies (2.16):

$$K_n'(\omega_0\cdot\mathbf{A}',\varepsilon)\omega_0\cdot\Delta(\mathbf{A}',\mathbf{z}',\varepsilon)+K_n'(\omega_0\cdot\mathbf{A}',\varepsilon)\left[\frac{f_n(\mathbf{A}',\mathbf{z}',\varepsilon)-f_n(\mathbf{A}',\varepsilon)}{K_n'(\omega_0\cdot\mathbf{A}',\varepsilon)}\right]^{[\leq 2^{n+1}-1]}=0. \tag{2.36}$$

Next we define on $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$:

$$f_{n+1}(\mathbf{A}', \mathbf{z}', \varepsilon) = f_n(\mathbf{A}, \mathbf{z}, \varepsilon) + K_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}, \varepsilon) - K_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon) - F_n(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon), \quad (2.37)$$

and on $\mathscr{E}(\rho_n e^{-3\delta_n}) \times D(\theta_n e^{-3\delta_n})$:

$$K_{n+1}(E, \varepsilon) = K_n(E, \varepsilon) + F_n(E, \varepsilon). \quad (2.38)$$

In order to be able to set up the last two definitions we must check that as $(\mathbf{A}', \mathbf{z}')$ varies in $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$ the $(\mathbf{A}, \mathbf{z})$ varies in $W(\rho_n, \xi_n)$: this is insured by the right hand side of (2.35).

Finally we must estimate the sizes of K^{n+1}, f_{n+1}. Such estimates have a purely dimensional nature and are straightforward: we describe them in detail in Appendix B. There we prove the existence of constants B_4, B_5, b_4, b_5 depending only on α, C, ℓ and $\Omega = |\boldsymbol{\omega}_0|$ such that:

$$\begin{aligned} E_{n+1} &\leqq E_n + B_4 \varepsilon_n \delta_n^{-b_4}, \\ \varepsilon_{n+1} &\leqq B_5 (C\varepsilon_n^2 + \varepsilon_n e^{-\delta_n 2^{n+1}}) \delta_n^{-b_5}, \\ \| f_{n+1} \|_{\rho_{n+1}, \xi_{n+1}, \theta_{n+1}} &\leqq B_5 (\varepsilon_n^2 C \rho_n + \| f_n \|_{\rho_n, \xi_n, \theta_n} e^{-\delta_n 2^{n+1}}) \delta_n^{-b_5}, \end{aligned} \quad (2.39)$$

where $E_{n+1}, \varepsilon_{n+1}$ are defined by the (2.26) with $n+1$ replacing n, if ρ_{n+1}, ξ_{n+1}, θ_{n+1} are given by (2.27).

It is easy to show (from the first two equations of (2.39)) that if ε_0 is small enough, i.e. if for a suitable g:

$$\varepsilon_0 < g(\ell, \alpha, C, \Omega, \xi_0), \quad (2.40)$$

then ε_n tends to zero faster than $\exp - (\frac{3}{2})^n$ as $n \to \infty$, i.e. very fast, and (2.34) hold $\forall n$.

It is then clear that the limits

$$\begin{aligned} \mathscr{C}_\varepsilon &= \lim_{n \to \infty} \mathscr{C}^{(0)} \ldots \mathscr{C}^{(n-1)}, \\ K_\infty &= \lim_{n \to \infty} K_n, \end{aligned} \quad (2.41)$$

exist on $W(\rho_\infty, \xi_\infty, \theta_\infty)$ or on $\mathscr{E}(\rho_\infty) \times D(\theta_\infty)$ respectively, and

$$K_\infty(\boldsymbol{\omega}_0 \cdot \mathbf{A}', \varepsilon) = H_0(\mathscr{C}_\varepsilon(\mathbf{A}', \mathbf{z}'), \varepsilon) \quad (2.42)$$

for all $(\mathbf{A}', \mathbf{z}', \varepsilon) \in W(\rho_\infty, \xi_\infty, \theta_\infty)$.

The analyticity of $\mathscr{C}_\varepsilon$ and K_∞ comes from Vitali's theorem.

3. Proposition 3 and a Sketch of the Proof for Proposition 4

For simplicity we only consider the case

$$h_0(A) = \boldsymbol{\omega}_0 \cdot \mathbf{A}, \quad (3.1)$$

with $\boldsymbol{\omega}_0$ verifying (1.21).

If $N_{f_0}(\mathbf{A}, \varepsilon)$ does exist, then there is a $\mathscr{C}_\varepsilon$ transforming $H_\varepsilon^{\text{ren}} = h_0 + :f_0:$ into h_0 via (1.7), (1.8): this means that the Hamilton–Jacobi equation (1.11) for $H_\varepsilon^{\text{ren}}$ has a solution like (1.10) with $h^{(1)}, h^{(2)}, \ldots$ identically zero. Then a simple substitution of

(1.10) into (1.11) and a generalized power series expansion aimed at identifying the coefficients of ε^k in both sides yields:

$$\omega_0\cdot\frac{\partial\boldsymbol{\Phi}^{(n)}}{\partial\boldsymbol{\varphi}}(\mathbf{A}',\boldsymbol{\varphi})+\sum_{\substack{s;\mathbf{a};\boldsymbol{\kappa}^{(1)},\ldots,\boldsymbol{\kappa}^{(\ell)}\\ |s|+|\mathbf{a}|+\Sigma|\boldsymbol{\kappa}^{(i)}|=n\\ 1\leqq s\leqq n}}\left\{\frac{\partial^{\mathbf{a}}f_0^{(s)}}{\partial\mathbf{A}'^{\mathbf{a}}}(\mathbf{A}',\boldsymbol{\varphi})-\frac{\partial^{\mathbf{a}}N_{f_0}^{(s)}}{\partial\mathbf{A}'^{\mathbf{a}}}(\mathbf{A}')\right\}\cdot$$
$$\cdot\left\{\prod_{i=1}^{\ell}\frac{1}{a_i!}\left(\prod_{j=1}^{a_i}\frac{\partial\Phi^{(\kappa_j^{(i)})}}{\partial\varphi_i}(\mathbf{A}',\boldsymbol{\varphi})\right)\right\}+f_0^{(n)}(\mathbf{A}',\boldsymbol{\varphi})-N_{f_0}^{(n)}(\mathbf{A}')=0, \tag{3.2}$$

which for $n=1$ has to be interpreted as

$$\omega_0\cdot\frac{\partial\Phi^{(1)}}{\partial\mathbf{A}'}(\mathbf{A}',\boldsymbol{\varphi})+f_0^{(1)}(\mathbf{A}',\boldsymbol{\varphi})-N_{f_0}^{(1)}(\mathbf{A}')=0. \tag{3.3}$$

Then (3.2) determines uniquely $N_{f_0}^{(1)}(\mathbf{A}')$ as

$$N_{f_0}^{(1)}(\mathbf{A}')=\bar{f}_0^{(1)}(\mathbf{A}'), \tag{3.4}$$

and also $\partial\Phi^{(1)}/\partial\boldsymbol{\varphi}$ is uniquely determined by (3.3).

Then we proceed inductively, remarking that the sum in (3.2) only involves $\partial\boldsymbol{\Phi}^{(r)}/\partial\varphi$, $N_{f_0}^{(r)}$ with $1\leqq j\leqq n-1$: hence (3.2) uniquely determines $N_{f_0}^{(n)}$ as well as $\partial\boldsymbol{\Phi}^{(n)}/\partial\varphi$.

The above discussion is just a repetition of the proof of Birkhoff's theorem, of course, and it shows that the "counterterms" $N_{f_0}^{(u)}$ are uniquely determined (by the above algorithm) provided N_{f_0} exists. On the other hand the above algorithm (3.2), (3.3) permits us to define a sequence $N_{f_0}^{(k)}$ such that, if the series (1.32) converges for ε small enough, the hamiltonian (1.31) fulfills the criterion of convergence of Proposition 1. This completes the proof of Proposition 3.

Also the proof of Proposition 5 relies on the ideas appearing in the proof of Proposition 2.

Write the original hamiltonian in the form

$$H_0(\mathbf{A},\boldsymbol{\varphi},\varepsilon)=\omega_0\cdot\mathbf{A}+f_0(\mathbf{A},\boldsymbol{\varphi},\varepsilon), \tag{3.5}$$

considering, for simplicity, the case $h(E)\equiv E$.

Fix $N>0$ and define for $n\leqq N$ the functions $\Phi^{(1)},\ldots,\Phi^{(N)}$, $N_{f_0}^{(1)},\ldots,N_{f_0}^{(N)}$ by recursively solving (3.2).

Then we define

$$\Phi(\mathbf{A}',\boldsymbol{\varphi},\varepsilon)=\sum_{k=1}^{N}\varepsilon^k\Phi^{(k)}(\mathbf{A}',\boldsymbol{\varphi}), \tag{3.6}$$

and consider the map

$$\mathbf{A}=\mathbf{A}'+\frac{\partial\Phi}{\partial\boldsymbol{\varphi}}(\mathbf{A}',\boldsymbol{\varphi}),$$
$$\boldsymbol{\varphi}'=\boldsymbol{\varphi}+\frac{\partial\Phi}{\partial\mathbf{A}'}(\mathbf{A}',\boldsymbol{\varphi}). \tag{3.7}$$

For ε small enough (3.7) can be used to define (by the analytic implicit function theorems used in Sect. 2) two analytic canonical change of coordinates $\mathscr{C}$ and $\tilde{\mathscr{C}}$ defined on a domain containing $V \times T^{\ell}$ and inverse of each other on $V \times T^{\ell}$. We shall write $(\mathbf{A}, \boldsymbol{\varphi}) = \mathscr{C}(\mathbf{A}', \boldsymbol{\varphi}')$, and then we note that the hamiltonian

$$\bar{H}_0(\mathbf{A}, \boldsymbol{\varphi}) = H_0(\mathbf{A}, \boldsymbol{\varphi}) - \sum_{k=1}^{N} \varepsilon^k N_{f_0}^{(k)}(\mathbf{A}) \tag{3.8}$$

will appear, in the variables $(\mathbf{A}', \boldsymbol{\varphi}')$, as

$$H_1(\mathbf{A}', \boldsymbol{\varphi}') = \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f_1(\mathbf{A}', \boldsymbol{\varphi}', \varepsilon), \tag{3.9}$$

with f_1 analytic in $(\mathbf{A}', \boldsymbol{\varphi}') \in V \times T^{\ell}$, and in ε if ε is small enough (so that $\mathscr{C}, \tilde{\mathscr{C}}$ can be defined), say if

$$|\varepsilon| < a_N. \tag{3.10}$$

Furthermore f_1 is divisible by ε^{N+1}. We may and shall suppose $N = 2^m - 1$ for some $m > 0$.

So the proof of Proposition 5 is reduced to show that in (3.9) we can alter f_1 by a function δf_1 of order ε^{N+1} in ε, making $H_1 - \delta f_1$ integrable and conjugate to $\boldsymbol{\omega}_0 \cdot \mathbf{A}$.

The function δf_1 is constructed through a recursive algorithm very similar to the one used in Sect. 2.

We shall define $\rho_1, \xi_1, \theta_1 \in (0,1)$ so that H_1 can be regarded as holomorphic in $W(\rho_1, \xi_1, \theta_1)$. Then we modify (3.9) on $W(\rho_1, \xi_1, \theta_1)$ into:

$$\bar{H}_1 = \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f_1(\mathbf{A}', \boldsymbol{\varphi}', \varepsilon) - [\bar{f}_1(\mathbf{A}', \varepsilon)]^{[\leq 2^{m+1} - 1]} \tag{3.11}$$

using the notations of Sect. 2.

Then we define a canonical map $\mathscr{C}^{(1)}$ by the m^{th} step of the algorithm of Sect. 2: this allows us to change variables from $(\mathbf{A}', \boldsymbol{\varphi}')$ to $(\mathbf{A}'', \boldsymbol{\varphi}'')$ and to define on $W(\rho_2, \xi_2, \theta_2)$ a hamiltonian:

$$H_2(\mathbf{A}'', \boldsymbol{\varphi}'', \varepsilon) = \bar{H}_1(\mathscr{C}^{(1)}(\mathbf{A}'', \boldsymbol{\varphi}'', \varepsilon), \varepsilon) = \boldsymbol{\omega}_0 \cdot \mathbf{A}' + f_2(\mathbf{A}', \boldsymbol{\varphi}', \varepsilon), \tag{3.12}$$

with f_2 divisible by $\varepsilon^{2^{m+1}}$.

Then we make a new subtraction and change $\bar{H}_2$ into H_3 defined on $W(\rho_3, \xi_3, \theta_3)$, etc.

We claim that the relations between $(\rho_{n+1}, \xi_{n+1}, \theta_{n+1})$ and $(\rho_n, \xi_n, \theta_n)$ can be taken to be exactly the same as those in Sect. 2 and that the measure of f_n, which we call ε_n, and that of f_{n+1} will be related by the second equation of (2.39), possibly with new constants B, b. The proof of this statement is essentially identical to the one in Sect. 2.

Therefore if $|\varepsilon|$ is small enough we can iterate the above algorithm indefinitely and the composition

$$\mathscr{C}_\infty = \lim_{n \to \infty} \mathscr{C}\mathscr{C}^{(1)} \dots \mathscr{C}^{(n)} \tag{3.13}$$

will exist and define a canonical holomorphic map on $W(\rho_\infty, \xi_\infty, \theta_\infty)$, close to the identity within ε^{2^m}.

Also

$$\lim_{n\to\infty} H_n(\mathbf{A},\boldsymbol{\varphi},\varepsilon) = H_\infty(\mathbf{A},\boldsymbol{\varphi},\varepsilon) \equiv \boldsymbol{\omega}_0\cdot\mathbf{A} \tag{3.14}$$

will exist and, if $(\mathbf{A},\boldsymbol{\varphi})_1 \equiv \mathbf{A}$:

$$\tilde{H}(\mathbf{A},\boldsymbol{\varphi},\varepsilon) \equiv H_\infty(\mathscr{C}_\infty^{-1}(\mathbf{A},\boldsymbol{\varphi}),\varepsilon) \equiv \boldsymbol{\omega}_0\cdot(\mathscr{C}_\infty^{-1}(\mathbf{A},\boldsymbol{\varphi}))_1 = \boldsymbol{\omega}_0\cdot\mathbf{A} + O(\varepsilon^{2m}), \tag{3.15}$$

because $\mathscr{C}_\infty$ is close to the identity within ε^{2m}.

This completes the discussion.

Appendix A. Implicit Function Theorems

Let $\rho,\xi,\theta<1,\delta<\xi$. Use the notations of Sect. 2.

Lemma 1. *Let* $\mathbf{F}$ *be holomorphic in* $W(\rho,\xi,\theta)$, *and consider the map*

$$\mathbf{A} = \mathbf{A}' + \mathbf{F}(\mathbf{A}',\mathbf{z},\varepsilon). \tag{A.1}$$

There are constants $B, b > 2$ *depending only on* ℓ *such that if*

$$B\|F\|_{\rho,\psi,\theta} < \rho\delta^b, \tag{A.2}$$

then there is a holomorphic function Ξ' *on* $W(\rho e^{-\delta},\xi,\theta)$ *such that if* $\mathbf{A}'$ *is defined by*:

$$\mathbf{A}' = \mathbf{A} + \Xi'(\mathbf{A},\mathbf{z},\varepsilon) \qquad \text{on } W(\rho e^{-\delta},\xi,\theta), \tag{A.3}$$

then

$$\Xi'(\mathbf{A},\mathbf{z},\varepsilon) = -F(\mathbf{A}',\mathbf{z},\varepsilon). \tag{A.4}$$

Furthermore, if $\mathbf{F}$ *is real on* $V\times T^\ell\times[-\theta,\theta]$, *then* Ξ' *is also real there.*

In other words (A.1) can be inverted with respect to $\mathbf{A}'$ if (A.2) holds: the inverse form (A.3) as expressed by (A.4). Inequality (A.2) also provides a bound for Ξ' on $W(\rho e^{-\delta},\xi,\theta)$.

Lemma 2. *Let* $\mathbf{G}$ *be holomorphic in* $W(\rho,\xi,\theta)$, *and consider the map*

$$\mathbf{z}' = \mathbf{z}\exp i\mathbf{G}(\mathbf{A}',\mathbf{z},\varepsilon). \tag{A.5}$$

There are constants $B, b > 2$ *such that* B, b *depend only on* ℓ *and if*:

$$B\|\mathbf{G}\|_{\rho,\xi,\theta} < \delta^b, \tag{A.6}$$

there exists a function Δ *holomorphic in* $W(\rho,\xi-\delta,\theta)$ *such that if we define* $\mathbf{z}$ *as*

$$\mathbf{z} = \mathbf{z}'\exp i\Delta(\mathbf{A}',\mathbf{z}',\varepsilon) \qquad \text{on } W(\rho,\xi-\delta,\varepsilon), \tag{A.7}$$

then

$$\Delta(\mathbf{A}',\mathbf{z}',\varepsilon) = -G(\mathbf{A}',\mathbf{z},\varepsilon). \tag{A.8}$$

Furthermore if $\mathbf{G}$ *is real on* $V\times T^\ell\times[-\theta,\theta]$, *then such is* Δ.

In other words (A.5) is inverted in the form (A.7), as expressed by (A.8), provided (A.6) holds. The constants B, b in 1) or 2) can be taken equal.

The above two theorems are very easy (because of the strength of the hypotheses) and are left to the reader; alternatively see, for instance, appendices to [8].

Appendix B. Estimates of E_n, ε_n

Clearly, by dimensional estimates, if $\Omega \equiv |\omega_0|$:

$$E_{n+1} \leqq E_n + \left\| \frac{\partial F_n}{\partial E} \right\|_{\rho_{n+1}, \xi_{n+1}, \theta_{n+1}}$$
$$\leqq E^n + \left\| \frac{\omega_0}{|\omega_0|^2} \cdot \frac{\partial}{\partial \mathbf{A}} [\bar{f}_n(\mathbf{A}, \varepsilon)]^{[\leqq 2^{n+1}-1]} \right\|_{\rho_{n+1}, \xi_{n+1}, \theta_{n+1}} \leqq E_n + B'_\alpha \Omega^{-1} \varepsilon_n \delta_n^{-1}. \tag{B.1}$$

The estimate for ε_{n+1} is more delicate. In fact:

$$\begin{aligned} f_{n+1}(\mathbf{A}', \mathbf{z}', \varepsilon) &= K_n(\omega_0 \cdot (\mathbf{A}' + \Delta), \varepsilon) - K_n(\omega_0 \cdot \mathbf{A}', \varepsilon) \\ &\quad + f_n(\mathbf{A}' + \Xi, z' e^{i\Delta}, \varepsilon) - [\bar{f}_n(\mathbf{A}', \varepsilon)]^{[\leqq 2^{n+1}-1]} \\ &= \{K_n(\omega_0 \cdot (\mathbf{A}' + \Delta)) - K_n(\omega_0 \cdot \mathbf{A}', \varepsilon) - K'_n(\omega_0 \cdot \mathbf{A}', \varepsilon)\, \omega_0 \cdot \Delta\} \\ &\quad + \{K'_n(\omega_0 \cdot \mathbf{A}', \varepsilon) \omega_0 \cdot \Delta + [f_n(\mathbf{A}', \mathbf{z}' e^{i\Delta}, \varepsilon) - \bar{f}_n(\mathbf{A}', \varepsilon)]^{[\leqq 2^{n+1}-1]*}\} \\ &\quad + \{[f_n(\mathbf{A}' + \Xi, z' e^{i\Delta}, \varepsilon) - f_n(\mathbf{A}', \mathbf{z}' e^{i\Delta}, \varepsilon)]^{[\leqq 2^{n+1}-1]}\} \\ &\quad + \{[f_n(\mathbf{A}' + \Xi, \mathbf{z}' e^{i\Delta}, \varepsilon)]^{[\geqq 2^{n+1}-1]}\} \\ &\equiv f^{\mathrm{I}} + f^{\mathrm{II}} + f^{\mathrm{III}} + f^{\mathrm{IV}}, \end{aligned} \tag{B.2}$$

where the $*$ means that in the truncation operation one disregards the ε-dependence of the arguments other than the last (i.e. one considers Δ and Ξ' as ε-independent). The $f^{\mathrm{I}}, \dots, f^{\mathrm{IV}}$ are respectively defined by the last identity. We notice that the functions in (B.2) are well defined on $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$, see (2.37).

The basic relation (2.36) says that

$$f^{\mathrm{II}}(\mathbf{A}', \mathbf{z}', \varepsilon) = \left[K'_n(\omega_0 \cdot \mathbf{A}', \varepsilon) \left[\frac{f_n(\mathbf{A}', \mathbf{z}' e^{i\ \ }, \varepsilon) - \bar{f}_n(\mathbf{A}', \varepsilon)}{K'_n(\omega_0 \cdot \mathbf{A}', \varepsilon)} \right]^{[\leqq 2^{n+1}-1]*} \right]^{[\geqq 2^{n+1}-1]*} \tag{B.3}$$

with the same meaning for [ ]* as above.

By dimensional estimates and using $E_n < \frac{1}{2}$:

$$|f^{\mathrm{II}}| \leqq B_5^{(1)} \delta_n^{-1} e^{-\delta_n 2^{n+1}} \delta_n^{-1} \left\| \frac{\partial f_n}{\partial \varphi} \right\|_{\rho_n, \xi_n, \theta_n} \leqq B_5^{(1)} \delta_n^{-2} \varepsilon_n \rho_n e^{-\delta_n 2^{n+1}} \tag{B.4}$$

on $W(\rho_n e^{-2\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$, so that

$$\left\| \frac{\partial f^{\mathrm{II}}}{\partial \mathbf{A}'} \right\|_{n+1} + \frac{1}{\rho_{n+1}} \left\| \frac{\partial f^{\mathrm{II}}}{\partial \varphi} \right\|_{n+1} \leqq B_5^{(2)} \delta_n^{-2} \varepsilon_n e^{-\delta_n 2^{n+1}}. \tag{B.5}$$

Similarly on $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$, using (2.35):

$$\begin{aligned} |f^{\mathrm{III}}| &\leqq B_5^{(3)} \delta_n^{-1} \sup |f_n(\mathbf{A}' + \Xi, \mathbf{z}' e^{i\Delta}, \varepsilon) - f_n(\mathbf{A}', \mathbf{z}' e^{i\Delta}, \varepsilon)| \\ &\leqq B_5^{(4)} \delta_n^{-1} \varepsilon_n \sup |\Xi| < B_5^{(5)} C \varepsilon_n^2 \delta_n^{-b_5^{(1)}} \rho_n, \end{aligned} \tag{B.6}$$

where the sup is considered in $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$; so that

$$\left\|\frac{\partial f^{\mathrm{III}}}{\partial \mathbf{A}'}\right\|_{n+1} + \frac{1}{\rho_{n+1}}\left\|\frac{\partial f^{\mathrm{III}}}{\partial \varphi'}\right\|_{n+1} \leqq B_5^{(6)} C \varepsilon_n^2 \delta_n^{-b_5^{(2)}}. \tag{B.7}$$

Similarly on $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-\delta_n})$, using $E_n < \frac{1}{2}$:

$$|f^{\mathrm{I}}| \leqq B_5^{(7)} \frac{1}{\rho_n \delta_n \Omega} \Omega^2 \sup |\Delta|^2 \leqq B_5^{(8)} \Omega C^2 \varepsilon_n^2 \rho_n \delta_n^{-b_5^{(3)}}, \tag{B.8}$$

so that

$$\left\|\frac{\partial f^{\mathrm{I}}}{\partial \mathbf{A}'}\right\|_{n+1} + \frac{1}{\rho_{n+1}}\left\|\frac{\partial f^{\mathrm{I}}}{\partial \varphi'}\right\|_{n+1} \leqq B_5^{(9)} \Omega C^2 \varepsilon_n^2 \delta_n^{-b_5^{(4)}}. \tag{B.9}$$

It remains to estimate f^{IV}: since its φ-average is nonzero, we estimate directly its derivatives.

$$\left[\frac{\partial}{\partial \mathbf{A}'} f_n(\mathbf{A}', \Xi, \mathbf{z}' e^{i\Delta}, \varepsilon)\right]^{[>2^{n+1}-1]} \equiv \frac{1}{2\pi i}\oint \left(\frac{\varepsilon}{\lambda}\right)^{2^{n+1}} \frac{\partial}{\partial \mathbf{A}'} f_n(\mathbf{A}' + \Xi, \mathbf{z}' e^{i\Delta}, \lambda) \frac{d\lambda}{\lambda - \varepsilon}, \tag{B.10}$$

the integral being on the contour $|\lambda| = \theta_n e^{-\delta_n}$ and $|\varepsilon| < \theta_n e^{-2\delta_n}$: the Ξ and Δ are left depending on ε and not on λ to enforce the truncation that we consider.

It follows from (B.10): on $W(\rho_n e^{-3\delta_n}, \xi_n - 4\delta_n, \theta_n e^{-2\delta_n})$:

$$\begin{aligned}\left|\frac{\partial}{\partial \mathbf{A}'} f^{\mathrm{IV}}\right| &\leqq B_5^{(10)} e^{-\delta_n 2^{n+1}} \delta_n^{-1} \varepsilon_n \left[\sup\left(1 + \left|\frac{\partial \Xi}{\partial \mathbf{A}'}\right|\right) + \rho_n \left|\frac{\partial \Delta}{\partial \varphi'}\right|\right] \\ &\leqq B_5^{(11)} e^{-\delta_n 2^{n+1}} \varepsilon_n \delta_n^{-b_5^{(5)}},\end{aligned} \tag{B.11}$$

having used (2.35), (2.34) and dimensional bounds.

Analogously one bounds $\partial f^{\mathrm{IV}}/\partial \varphi'$. Collecting all the above estimates, (2.39) follows.

The above inequalities also show that

$$\| f_{n+1} \|_{n+1} \leqq B_5^{(12)} (C \varepsilon_n \rho_n + \| f_n \|_n e^{-\delta_n 2^{n+1}}) \delta_n^{-b_5^{(6)}}, \tag{B.12}$$

where the second term arises because (B.11) was not obtained by first estimating $|f^{\mathrm{IV}}|$ and then its derivative, but by directly estimating the derivative. A dimensional bound for f^{IV} is clearly, see (B.2):

$$|f^{\mathrm{IV}}| \leqq B_5^{(13)} \| f_n \|_n \delta_n^{-1} e^{-2^{n+1}\delta_n}. \tag{B.13}$$

All the $B_j^{(i)}, b_j^{(i)}$ above depend on ℓ only.

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