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Oscillator-Like Unitary Representations of Non-Compact Groups with a Jordan Structure and the Non-Compact Groups of Supergravity

M. Günaydin[1] and C. Saçlioğlu[2*]

1 CERN, Geneva 23, Switzerland

2 Physikalisches Institut der Universität Bonn, Bonn, Federal Republic of Germany

"*Dedicated to Feza Gürsey on the occasion of his* 60*th birthday*"

Abstract. We give a general bosonic construction of oscillator-like unitary irreducible representations (UIR) of non-compact groups whose coset spaces with respect to their maximal compact subgroups are Hermitian symmetric. With the exception of $E_{7(7)}$, they include all the non-compact invariance groups of extended supergravity theories in four dimensions. These representations have the remarkable property that each UIR is uniquely determined by an irreducible representation of the maximal compact subgroup. We study the connection between our construction, the Hermitian symmetric spaces and the Tits–Koecher construction of the Lie algebras of corresponding groups. We then give the bosonic construction of the Lie algebra of $E_{7(7)}$ in SU(8), SO(8) and U(7) bases and study its properties. Application of our method to $E_{7(7)}$ leads to reducible unitary representations.

1. Introduction

Recently, Cremmer and Julia [1] have discovered a set of non-compact invariance groups in the bosonic sectors of $N=5,6,8$ extended supergravity theories in four dimensions, thereby generalizing the non-compact invariance group of the $N=4$ theory found by Cremmer, Ferrara and Scherk [2]. The vector field strengths in these theories and their duals get transformed into each other under the action of the non-compact group G and form a linear representation, whereas the scalar fields transform non-linearly as the coset space G/H where H is the maximal compact subgroup of G. The full invariance has the form $G_{\text{global}} \otimes H_{\text{local}}$ as in the two-dimensional generalized σ models [3].

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Julia and Cremmer conjectured that the composite gauge fields associated with H_{local} may become dynamical at the quantum level just as in the two-dimensional CP^N models [1]. Ellis, Gaillard Maiani and Zumino (EGMZ) have extended this idea and postulated that in $N = 8$ supergravity in addition to the vector bound states other bound states (fermionic as well as bosonic) form whose effective interactions at low energies correspond to a spontaneously broken grand unified theory based on SU(5) with three families of quarks and leptons [4, 5]. Again in analogy with CP^N models [6] it was suggested that the bound states in extended supergravity theories may fall into linear representations of G_{global} [7]. Since the global invariance for $N = 4$, 5, 6, 8 supergravity theories are all non-compact, their unitary representations are infinite dimensional. In fact, an infinite set of bound states seems to be needed for giving superheavy masses to the unwanted helicity states in the EGMZ program [5] or the extensions thereof [8].

In a previous publication we have given a construction of a class of oscillator-like unitary representations of some non-compact groups including those appearing in extended supergravity theories [9]. Our purpose in this paper is to present an extension of our method for constructing unitary irreducible representations (UIR) and point out its connection to other mathematical structures; in particular to Jordan triple systems [10] and Hermitian symmetric spaces [11]. The plan of the paper is as follows: in Sect. 2 we give the bosonic construction of the Lie algebras of Ref. [9] in a generalized form which allows one to construct larger classes of UIRs. Specifically this section contains a construction of the Lie algebras of $SP(2n, \mathbb{R})$, $SO(2n)^*$, $SU(m, n)$ and $SO(m, n)$ in terms of boson annihilation and creation operators, some of which are well known in the literature [12, 13]. We then present, in a generalized form, the extension of the standard construction which yields only the Lie algebras of the non-compact groups of supergravity. This extension uses boson operators transforming exactly like the vector fields in the corresponding supergravity theories. (The construction of $E_{7(7)}$ is deferred to Sect. 5.) In Sect. 3 we point out that with the exception of $SO(m, n) (m \neq 2$ and $n \neq 2)$ all the Lie algebras of Sect. 2 decompose as $L = L^+ \oplus L^0 \oplus L^-$, where L^0 is the Lie algebra of the maximal compact subgroup H that contains an Abelian U(1) factor, L^+ and L^- space are conjugate to each other and carry opposite U(1) charges. This decomposition shows that the coset space G/H is a Hermitian symmetric space and the Lie algebra L can be constructed from a so-called Hermitian Jordan triple system. This Jordan structure is discussed and the Tits–Koecher construction of Lie algebras from Jordan triple systems is given. In Sect. 4 we formulate our general method for constructing UIRs for non-compact groups with a Jordan structure in the Fock space of the corresponding boson operators. Section 5 contains the construction of the Lie algebra of $E_{7(7)}$ which does not have a Jordan structure with respect to its maximal compact subgroup SU(8). Rewriting the Lie algebra of $E_{7(7)}$ in the SO(8) basis we show its triality properties. We then give the U(7) basis of $E_{7(7)}$ and indicate its connection to the Kantor construction of the Lie algebras of the E series in terms of antisymmetric tensors of rank three [14, 15]. This suggests a possible link between their emergence in $N = 8$ extended supergravity theories in various dimensions and the Kantor construction.

In the last section we show how applying our methods to the case of $E_{7(7)}$ leads to infinitely reducible unitary representations, which may still be of relevance to supergravity [16]. We then mention a method due to Gell–Mann for constructing UIRs of $E_{7(7)}$ on certain coset spaces of its maximal compact subgroup SU(8) [17, 18]. We conclude with the suggestion that in addition to the unitary representations constructed by using boson operators transforming like the vector fields one can construct further classes of unitary representations using boson operators transforming like the scalar fields in supergravity theories via the operator methods developed by Gursey and his collaborators [19].

2. Bosonic Construction of the Lie Algebras of a Class of Non-Compact Groups

In this section we give a bosonic construction of the Lie algebras of the non-compact groups Sp($2n$, $\mathbb{R}$), SO($2n$)*, SU(n, m) and SO(n, m) in a more general form than the one considered in Ref. [9]. This generalization is trivial on the Lie algebra level in the sense that it corresponds to taking direct sums, but as we shall see later, it leads to the construction of a much larger class of UIRs by our methods. The Lie algebras SP($2n$,$\mathbb{R}$), SO($2n$)*, SU(n,m) and SO(n,2) have a jordan structure with respect to their maximal compact subalgebra as explained in the next section. The non-compact groups that come up in extended super-gravity theories in four dimensions all have a Jordan structure with respect to their maximal compact subgroups. The only exception is the non-compact group $E_{7(7)}$ of $N = 8$ supergravity [1] which does not have a Jordan structure with respect to its maximal compact subgroup SU(8). We treat the bosonic construction of $E_{7(7)}$ separately in a later section.

Consider N pairs of boson annihilation and creation operators $a_i(K)$, $b_i(K)$ and $a_i^\dagger(K)$, $b_i^\dagger(K)$, where $i = 1,\ldots,n$ denotes a U(n) index and $K = 1,\ldots,N$ labels the different pairs which can be infinitely many in certain cases of physical interest as is shown in the Appendix. We shall denote the creation operators by upper indices; thus $a_i(K)^\dagger = a^i(K)$, $b_i(K)^\dagger = b^i(K)$. They obey the canonical commutation relations

$$\begin{aligned}
[a_i(K), a^j(L)] &= \delta_i^j \delta^{KL}, \\
[b_i(K), b^j(L)] &= \delta_i^j \delta^{KL}, \\
[a_i(K), a_j(L)] &= 0 = [b_i(K), b_j(L)].
\end{aligned} \tag{2.1}$$

The U(n) generators are then

$$I_n^m = \mathbf{a}^m \cdot \mathbf{a}_n + \mathbf{b}_n \cdot \mathbf{b}^m, \tag{2.2}$$

where the dot product represents a sum over the generation index K, i.e.,

$$\mathbf{a}^m \cdot \mathbf{a}_n \equiv \sum_{K=1}^{N} a^m(K) a_n(K).$$

The U(n) algebra can be extended to the Lie algebra of a non-compact group with a maximal compact subgroup U(n):

a) U(n) → SP($2n$, $\mathbb{R}$):

The symmetric diboson operators

$$S_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j + \mathbf{a}_j \cdot \mathbf{b}_i; \; S^{ij} = \mathbf{a}^i \cdot \mathbf{b}^j + \mathbf{a}^j \cdot \mathbf{b}^i \tag{2.3}$$

together with the I^m_n obey the commutation relations

$$
\begin{aligned}
[S_{ij}, S^{kl}] &= \delta^l_j I^k_i + \delta^k_i I^l_j + \delta^k_j I^l_i + \delta^l_i I^k_j, \\
[I^m_n, S_{ij}] &= -\delta^m_i S_{nj} - \delta^m_j S_{in}, \\
[I^m_n, S^{ij}] &= \delta^n_i S^{mj} + \delta^j_n S^{im},
\end{aligned} \tag{2.4}
$$

which corresponds to the Lie algebra of Sp($2n$, ℝ) in a so-called split basis.

b) U(n) → SO($2n$)*:

The antisymmetric diboson operators

$$
\begin{aligned}
A_{ij} &= \mathbf{a}_i \cdot \mathbf{b}_j - \mathbf{a}_j \cdot \mathbf{b}_i, \\
A^{ij} &= \mathbf{a}^i \cdot \mathbf{b}^j - \mathbf{a}^j \cdot \mathbf{b}^i,
\end{aligned} \tag{2.5}
$$

and the I^m_n satisfy

$$
\begin{aligned}
[A_{ij}, A^{kl}] &= \delta^l_j I^k_i + \delta^k_i I^l_j - \delta^k_j I^l_i - \delta^l_i I^k_j, \\
[I^m_n, A_{ij}] &= -\delta^m_i A_{nj} - \delta^m_j A_{in}, \\
[I^m_n, A^{ij}] &= \delta^i_n A^{mj} + \delta^j_n A^{im}.
\end{aligned} \tag{2.6}
$$

This is the Lie algebra of SO($2n$)* with maximal compact subgroup U(n) in a split basis.

Instead of considering the particular combination I^m_n one can also take the operators

$$
\begin{aligned}
P^i_j &= \mathbf{a}^i \cdot \mathbf{a}_j - \frac{1}{n}\delta^i_j(\mathbf{a}^m \cdot \mathbf{a}_m), \\
R^i_j &= \mathbf{b}_j \cdot \mathbf{b}^i - \frac{1}{n}\delta^i_j(\mathbf{b}_m \cdot \mathbf{b}^m), \\
Q &= \mathbf{a}^m \cdot \mathbf{a}_m + \mathbf{b}_m \cdot \mathbf{b}^m,
\end{aligned} \tag{2.7}
$$

which generate the Lie algebra of S(U(n) × U(n)). These operators together with the non-symmetrized diboson operators

$$
\begin{aligned}
U_{ij} &= \mathbf{a}_i \cdot \mathbf{b}_j, \\
U^{ij} &= \mathbf{a}^i \cdot \mathbf{b}^j,
\end{aligned} \tag{2.8}
$$

give us the Lie algebra of SU(n,n).

If the indices of the boson operators a and b run differently, i.e., for a_i, $i = 1, \ldots, n$ and b_u, $\mu = 1, \ldots, m$, then the operators

$$
\begin{aligned}
P^i_j &= \mathbf{a}^i \cdot \mathbf{a}_j - \frac{1}{n}\delta^i_j(\mathbf{a}^m \cdot \mathbf{a}_m), \\
R^\mu_\nu &= \mathbf{b}_\nu \cdot \mathbf{b}^\mu - \frac{1}{m}\delta^\mu_\nu(\mathbf{b}_\lambda \cdot \mathbf{b}^\lambda), \\
N &= \frac{1}{n}\mathbf{a}^m \cdot \mathbf{a}_m + \frac{1}{m}\mathbf{b}_\lambda \cdot \mathbf{b}^\lambda, \\
& i, j, .. = 1, \ldots n; \quad \mu, \nu, \lambda \ldots = 1, \ldots, m,
\end{aligned} \tag{2.9}
$$

Together with the diboson operators

$$\begin{aligned} U_{i\mu} &= \mathbf{a}_i \cdot \mathbf{b}_\mu, \\ U^{i\mu} &= \mathbf{a}^i \cdot \mathbf{b}^\mu, \end{aligned} \tag{2.10}$$

generate the Lie algebra of SU(m, n)

$$\begin{aligned} [U_{i\mu}, U^{j\nu}] &= \delta_i^j R_\mu^\nu + \delta_\mu^\nu P_i^j + \delta_\mu^\nu \delta_i^j N, \\ [P_j^i, U_{k\mu}] &= -\delta_k^i U_{j\mu} + \frac{1}{n}\delta_j^i U_{k\mu}, \\ [P_j^i, U^{k\mu}] &= \delta_j^k U^{i\mu} - \frac{1}{n}\delta_j^i U^{k\mu}, \\ [P_j^i, P_l^k] &= \delta_j^k P_l^i - \delta_l^i P_j^k, \end{aligned} \tag{2.11}$$

$$\begin{aligned} [R_\nu^\mu, U_{i\lambda}] &= -\delta_\lambda^\mu U_{i\nu} + \frac{1}{m}\delta_\nu^\mu U_{i\lambda}, \\ [R_\nu^\mu, U^{i\lambda}] &= \delta_\nu^\lambda U^{i\mu} - \frac{1}{m}\delta_\nu^\mu U^{i\lambda}, \\ [N, U_{i\mu}] &= -\left(\frac{1}{m} + \frac{1}{n}\right) U_{i\mu}, \\ [N, U^{i\mu}] &= \left(\frac{1}{m} + \frac{1}{n}\right) U^{i\mu}, \\ [R_\nu^\mu, R_\delta^\lambda] &= \delta_\nu^\lambda R_\delta^\mu - \delta_\delta^\mu R_\nu^\lambda. \end{aligned} \tag{2.12}$$

The following subset of the above operators

$$i(P_j^i - P_i^j), \; i(R_\nu^\mu - R_\mu^\nu), \; (U_{i\mu} + U^{i\mu}),$$

generate the SO(n, m) subalgebra of SU(n, m). Of the non-compact groups SO(n, m) only those for which $n = 2$ or $m = 2$ have a Jordan structure as explained in the following section.

Now we repeat the above extension procedure by considering annihilation and creation operators transforming like the antisymmetric tensor representation of U(n). This is of interest as one obtains exactly the Lie algebras of the non-compact groups occurring as global symmetries in extended supergravity theories and nothing else [9]. We now have

$$\begin{aligned} [a_{ij}(K), a^{kl}(L)] &= \delta^{KL}(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k), \\ [b_{ij}(K), b^{kl}(L)] &= \delta^{KL}(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k), \\ [a_{ij}(K), a_{kl}(L)] &= 0 = [b_{ij}(K), b_{kl}(L)], \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} &a_{ij}(K) = -a_{ji}(K); \; b_{ij}(K) = -b_{ji}(K) \\ &i, j \ldots = 1, \ldots, n \quad K, L, \ldots = 1, \ldots, N. \end{aligned}$$

It is easy to see that the $n = 2$ and $n = 3$ cases revert to the standard construction in the form discussed under a) and b) above. New algebras are found only for $n = 4, 5, 6,$

7, 8 and the diboson operators that extend the U(n) generators $\mathbf{a}^{ik}\cdot\mathbf{a}_{jk}+\mathbf{b}_{jk}\cdot\mathbf{b}^{ik}$ to the Lie algebra of a non-compact group all have the form

$$A_{i_{n-4}\ldots i_n}\equiv\varepsilon_{i_1 i_2\ldots i_n}\mathbf{a}^{i_1 i_2}\cdot\mathbf{b}^{i_3 i_4},$$

and

$$A^{i_{n-4}\ldots i_n}\equiv\varepsilon^{i_1 i_2\ldots i_n}\mathbf{a}_{i_1 i_2}\cdot\mathbf{b}_{i_3 i_4}.$$

Thus for $n=4$ we have the dibosons

$$\begin{aligned} Q^+ &= \tfrac{1}{4}\varepsilon_{ijkl}\mathbf{a}^{ij}\cdot\mathbf{b}^{kl},\\ Q^- &= \tfrac{1}{4}\varepsilon^{ijkl}\mathbf{a}_{ij}\cdot\mathbf{b}_{kl}, \end{aligned}\tag{2.14}$$

which, together with $Q^0=\frac{1}{4}(\mathbf{a}^{ij}\cdot\mathbf{a}_{ij}+\mathbf{b}_{ij}\cdot\mathbf{b}^{ij})$ representing the trace of U(4) generate the SU(1, 1) algebra

$$\begin{aligned} [Q^+,Q^-] &= -2Q^0,\\ [Q^0,Q^+] &= Q^+,\\ [Q^0,Q^-] &= -Q^-. \end{aligned}\tag{2.15}$$

With the remaining SU(4) generators $T^i_j=\mathbf{a}^{in}\cdot\mathbf{a}_{jn}+\mathbf{b}_{jn}\cdot\mathbf{b}^{in}-\delta^i_jQ^0$ the resulting extension is U(4) → SU(4) × SU(1, 1).

For $n=5$ one has the first simple Lie algebra of a non-compact group. Again we split the U(5) generators into the trace

$$Q=\mathbf{a}^{ij}\cdot\mathbf{a}_{ij}+\mathbf{b}_{ij}\cdot\mathbf{b}^{ij},\tag{2.16}$$

and the SU(5) part:

$$T^i_j=\mathbf{a}^{ik}\cdot\mathbf{a}_{jk}+\mathbf{b}_{jk}\cdot\mathbf{b}^{ik}-\tfrac{1}{5}\delta^i_jQ,\tag{2.17}$$

and use the diboson operators

$$A_i=\frac{\sqrt{2}}{4}\varepsilon_{ijklm}\mathbf{a}^{jk}\cdot\mathbf{b}^{lm},\tag{2.18}$$

$$A^i=\frac{\sqrt{2}}{4}\epsilon^{ijklm}\mathbf{a}_{jk}\cdot\mathbf{b}_{lm},$$

to arrive at the algebra of SU(5, 1):

$$\begin{aligned} [A_i,A^j] &= T^j_i-\tfrac{3}{10}\delta^j_iQ,\\ [T^k_i,A^j] &= \delta^j_iA^k-\tfrac{1}{5}\delta^k_iA^j,\\ [T^k_i,A_j] &= -\delta^k_jA_i+\tfrac{1}{5}\delta^k_iA_j,\\ [Q,A_i] &= 4A_i,\\ [Q,A^i] &= -4A^i. \end{aligned}\tag{2.19}$$

Under the action of the SU(5, 1) group the boson operators $a_{ij}(K)$ and $b^{ij}(K)$ transform into each other and form a 20-dimensional representation of SU(5, 1).

Finally, we treat $n=6$: there are 15 diboson operators $\varepsilon_{ijklmn}\mathbf{a}^{kl}\cdot\mathbf{b}^{mn}$ and their conjugates. With the 36 generators of U(6) one thus has a total of 66 operators, suggesting SO(12)*. However, the algebra does not close, because unlike the U(5) case, SO(12)* does not have a 30 dimensional representation corresponding to $a_{ij}(K) \oplus b^{ij}(K)$. The simplest way to remedy the situation is to introduce two SU(6) singlet boson operators $v(K)$ and $w^\dagger(K)$ in order to build up the 32 dimensional semi-spinor of SO(12)*. Indeed this 32 dimensional representation of SO(12)* decomposes as $1 \oplus 15 \oplus \overline{15} \oplus 1$ with respect to SU(6). The correct diboson generators now are

$$A_{ij} = \tfrac{1}{4}\varepsilon_{ijklmn}\mathbf{a}^{kl}\cdot\mathbf{b}^{mn} + \frac{1}{\sqrt{2}}(\mathbf{a}_{ij}\cdot\mathbf{v} + \mathbf{b}_{ij}\cdot\mathbf{w}),$$

$$A^{ij} = \tfrac{1}{4}\varepsilon^{ijklmn}\mathbf{a}_{kl}\cdot\mathbf{b}_{mn} + \frac{1}{\sqrt{2}}(\mathbf{a}^{ij}\cdot\mathbf{v}^\dagger + \mathbf{b}^{ij}\cdot\mathbf{w}^\dagger), \tag{2.20}$$

whereas the SU(6) and U(1) generators are given by

$$\begin{aligned} T^i_j &= \mathbf{a}^{in}\cdot\mathbf{a}_{jn} + \mathbf{b}^{in}\cdot\mathbf{b}_{jn} - \tfrac{1}{6}\delta^i_j(\mathbf{a}^{mn}\cdot\mathbf{a}_{mn} + \mathbf{b}^{mn}\cdot\mathbf{b}_{mn}), \\ Q &= \mathbf{a}^{mn}\cdot\mathbf{a}_{mn} + \mathbf{b}_{mn}\cdot\mathbf{b}^{mn} - 6\mathbf{v}^\dagger\cdot\mathbf{v} - 6\mathbf{w}\cdot\mathbf{w}^\dagger. \end{aligned} \tag{2.21}$$

They satisfy the commutation relations of SO(12)*

$$\begin{aligned} [A_{ij}, A^{kl}] &= \tfrac{1}{2}(\delta^k_i T^l_j + \delta^l_j T^k_i - \delta^l_i T^k_j - \delta^k_j T^l_i) \\ &\quad - \tfrac{1}{12}(\delta^k_i \delta^l_j - \delta^k_j \delta^l_i)Q, \\ [T^i_j, A_{kl}] &= -\delta^i_k A_{jl} - \delta^i_l A_{kl} + \tfrac{1}{3}\delta^i_j A_{kl}, \\ [Q, A_{ij}] &= -4A_{ij}, \\ [T^i_j, A^{kl}] &= \delta^k_j A^{il} + \delta^l_j A^{ki} - \tfrac{1}{3}\delta^i_j A^{kl}, \\ [Q, A^{ij}] &= 4A^{ij}. \end{aligned} \tag{2.22}$$

From the basis we have chosen for the Lie algebras above, it may not be obvious what non-compact form of the respective groups we are dealing with. Since we are interested in constructing unitary representations we assume implicitly that we are working in a Hermitian basis, i.e., all the generators of our group are Hermitian operators. In the above bases this is not the case. Therefore, we must take suitable linear combinations of the operators above to go to the Hermitian basis in which all the generators H_i of the group are Hermitian and the structure constants f_{ijk} defined by $[H_i, H_j] = if_{ijk}H_k$ are all pure real. Then the operator $U(g) = \exp(iH_i w^i)$ representing a general group element is unitary (w^i are real group parameters). It is in this basis that we calculate the Killing metric so as to determine the form of non-compactness.

3. Lie Algebras with a Jordan Structure and Hermitian Symmetric Spaces

We define a simple Lie algebra with a Jordan structure as a Lie algebra L that has a three-dimensional graded form, i.e.,

$$L = L^- \oplus L^0 \oplus L^+, \tag{3.1}$$

where L^0 contains the generator Q of an Abelian U(1) factor such that $L^0 = H \oplus Q$ and $[Q, H] = 0$, $[L^+, L^+] = 0$, $[Q, L^\pm] = \pm L^\pm$. In addition we have a conjugation†

such that $(L^+)^\dagger \cong L^-, L^{0\dagger} \cong L^0$. Because of the grading and simplicity of the algebra we have $[L^0, L^+] \cong L^+, [L^0, L^-] \cong L^-, [L^+, L^-) \cong L^0$.

All simple Lie algebras L with a Jordan structure can be related to the so-called Jordan triple systems by the following simple method [20, 21]. Denote the elements of L that lie in the L^+ subspace by U_a and the elements of the L^- space by $U_a^\dagger$, where a belongs to some vector space V. Further denote the commutator of U_a and $U_b^\dagger$ as S_{ab}

$$S_{ab} \equiv [U_a, U_b^\dagger]. \tag{3.2}$$

Because of the grading $[U_a, U_b] = 0 = [U_a^\dagger, U_b^\dagger]$. Through the commutator of S_{ab} with U_c one defines a triple product (abc) in the vector space V:

$$[S_{ab}, U_c] = U_{(abc)}. \tag{3.3}$$

Then all the commutation relations can be expressed in terms of the triple product (abc) by using the Jacobi identities.

$$\begin{aligned} [S_{ab}, U_c^\dagger] &= -U_{(bac)}^\dagger, \\ [S_{ab}, S_{cd}] &= S_{a(dcb)} - S_{(cda)b}. \end{aligned} \tag{3.4}$$

Jacobi identities impose two conditions on the triple product:

$$(abc) = (cba), \tag{3.5a}$$

$$(ab(cdx)) - (cd(abx)) - (a(dcb)x) + ((cda)bx) = 0. \tag{3.5b}$$

These conditions define a Jordan triple system [10]. Therefore, given any Jordan triple system one can construct a Lie algebra with a Jordan structure as above. This construction of a Lie algebra with a Jordan structure is known as the Tits–Koecher construction [20, 21]. It has also been extended to Lie superalgebras with a Jordan structure [22, 23].

A Jordan algebra with a symmetric product $a \cdot b = \frac{1}{2}(ab + ba)$ defines a Jordan triple system with the triple product

$$(abc) \equiv a\cdot(b\cdot c) - b\cdot(a\cdot c) + (a\cdot b)\cdot c \tag{3.6}$$

that satisfies the conditions (3.5a) and (3.5b).

Below we list the Lie algebras L and their respective subalgebras L^0 that can be constructed from various Jordan algebras using the Jordan triple product (3.6).

Jordan Algebra	L^0	L
$J_n^{\mathbb{R}}$	$\mathrm{U}(n)$	$\mathrm{Sp}(2n)$
$J_n^{\mathbb{C}}$	$\mathrm{SU}(n) \times \mathrm{SU}(n) \times \mathrm{U}(1)$	$\mathrm{SU}(2n)$
$J_n^{\mathbb{H}}$	$\mathrm{U}(2n)$	$SO(4n)$
J_3^8	$E_6 \times \mathrm{U}(1)$	E_7
$\Gamma(d)$	$\mathrm{SO}(d+1) \times \mathrm{SO}(2)$	$\mathrm{SO}(d+3)$

where $J_n^{\mathbb{R}}, J_n^{\mathbb{C}}, J_n^{\mathbb{H}}$ denote the Jordan algebras of $n \times n$ real symmetric, complex Hermitian and quaternionic Hermitian matrices respectively. Here J_3^8 is the exceptional Jordan algebra of 3 × 3 Hermitian octonionic matrices. $\Gamma(d)$ denotes the Jordan algebra of γ matrices in d dimensions

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu} 1; \quad \mu, \nu = 1, \ldots d.$$

Rectangular $n \times m$ matrices over the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ define a Jordan triple system under the triple product [22]

$$(abc) \equiv a\bar{b}^T c + c\bar{b}^T a, \tag{3.7}$$

where the bar denotes conjugation over the underlying division algebra $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and T is transposition. The corresponding Lie algebras L and L^0 are

Jordan Triple System	L^0	L
$K^{\mathbb{R}}_{nm}$	$SU(n) \times SU(m) \times U(1)$	$SU(n+m)$
$K^{\mathbb{C}}_{nm}$	$SU(n) \times SU(n) \times SU(m) \times SU(m) \times U(1)$	$SU(n+m) \times SU(n+m)$
$K^{\mathbb{H}}_{nm}$	$SU(2n) \times SU(2m) \times U(1)$	$SU(2n+2m)$
$K^{\mathbb{O}}_{21}$	$SO(10) \times SO(2)$	E_6

where $K^{\mathbb{R},\mathbb{C},\mathbb{H}}_{mn}$ refers to the Jordan triple system of $n \times m$ matrices over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. In the case of (2×1) octonionic matrices $K^{\mathbb{O}}_{21}$ the triple product is modified to be $(abc) = \{(a\bar{b}^T)c + (b\bar{a}^T)c - b(\bar{a}^T c)\} + \{a \leftrightarrow c\}$ due to the non-associativity of octonions.

In the above we have denoted all the Lie algebras L^0 and L with the compact form of the corresponding groups. In general they will be Lie algebras of the non-compact form depending on the underlying Jordan triple system. All simple Lie algebras with a Jordan structure can be constructed from a suitable triple system [24].

Now if we denote the groups corresponding to the Lie algebras L and L^0 as G and H, then the coset space G/H is a Hermitian symmetric space. Hermitian symmetric spaces are all Kählerian and they can in general be represented in the form of a tensor product [25].

$$M_0 \times M_1 \times M_2 \cdots \times M_r,$$

where M_0 is the quotient of a complex Euclidean space by a discrete group of pure translations and each $M_i (i > 0)$ is one of the following Riemannian symmetric spaces:

$$SU(p+q)/S(U(p) \times U(q)), \qquad SO(2n)/U(n),$$

$$SO(n+2)/SO(n) \times SO(2), \qquad E_6/SO(10) \times SO(2),$$
$$Sp(2n)/U(n), \qquad E_7/E_6 \times U(1).$$

From this classification it follows that all simple Lie algebras have a Jordan structure with respect to some suitable subalgebra except for the Lie algebras of G_2, F_4 and E_8. For the detailed study of the connection between symmetric spaces and Jordan triple systems we refer the reader to Refs. [11] and [26].

All the Lie algebras of the non-compact groups [except for SO(m, n), where both m and n are different from 2] considered in the previous section have a Jordan structure with respect to the Lie algebra of their maximal compact subgroups. Of the remaining Lie algebras of the non-compact groups with a Jordan structure with respect to their maximal compact subgroup, the Lie algebras of $E_{6(-14)}$ and $E_{7(-25)}$ can be similarly constructed from boson operators. In the next section we give a general method for constructing certain classes of UIRs of non-compact groups with a Jordan structure with respect to their maximal compact subgroup.

4. Oscillator-Like Unitary Irreducible Representations of Non-Compact Groups with a Jordan Structure

The Lie algebras of the non-compact groups constructed above have a Jordan structure with respect to the Lie algebra of their maximal compact subgroup:

$$L = L^- \oplus L^0 \oplus L^+ ; \quad (L^+)^\dagger \cong L^-, \quad (L^0)^\dagger \cong L^0,$$

where the L^- and L^+ subspaces correspond to the non-compact generators constructed in terms of diboson annihilation and creation operators. In Ref. [9], we have given a construction of a certain class of UIRs of non-compact groups with a Jordan structure in the case when L^- (and L^+) generators were constructed in terms of diboson annihilation (and creation) operators only. Here we give the same construction in a more general form when we have an arbitrary number of pairs of boson operators $a(K)$, $b(K)$, $k = 1, 2, \ldots, N$ instead of a single pair as was done in Ref. [9]. On the Lie algebra level this extension is trivial in the sense that it gives us a direct sum of N copies of the same Lie algebra. However, this simple extension enables us to construct larger classes of UIRs of the respective groups [27].

Consider now the Fock space constructed from the tensor product of Fock spaces of individual boson operators. The vacuum $|0\rangle$ in our Fock space will be a tensor product of the individual vacua $|0)$

$$|0\rangle \equiv |0)|0)\ldots.|0). \tag{4.1}$$

It is annihilated by all the annihilation operators.

$$a_i(K)|0\rangle = 0 = b_i(K)|0\rangle, \quad K = 1, \ldots, N. \tag{4.2}$$

Choose a set of states $|\psi_A\rangle$ in our Fock space which is annihilated by all the diboson operators in the L^- space and transform as a certain representation of the maximal compact subgroup generated by L^0:

$$L^- |\psi_A\rangle = 0. \tag{4.3}$$

Then the infinite number of states generated by applying the operators L^+ on $|\psi_A\rangle$ form the basis of a unitary representation of the non-compact group G:

$$|\psi_A\rangle, \quad L^+|\psi_A\rangle, (L^+)^2|\psi_A\rangle, \ldots. \tag{4.4}$$

Now if $|\psi_A\rangle$ are chosen such that they transform like an irreducible representation of the maximal compact subgroup generated by L^0, then the corresponding representation of the non-compact group is also irreducible. The proof of this theorem, which was given in Ref. [9], is very simple and uses the Jordan structure of the non-compact group in a crucial manner.

With the exception of the second construction of SO(12)* [see Eq. (2.20)] the L^- spaces of all the Lie algebras constructed in Sect. 2 involve diboson annihilation operators only. The states in our Fock space that are annihilated by L^- involving diboson annihilation operators only, are a linear combination of the states of the form

$$[a^i(1)]^{m_1}[a^j(2)]^{m_2}\cdots[\cdots][a^k(N)]^{m_N}|0\rangle, \tag{4.5}$$

and of the form

$$[b^i(1)]^{n_1}[b^j(2)]^{n_2}\ldots[b^k(N)]^{n_N}|0\rangle. \tag{4.6}$$

These states transform in general like a reducible representation of the maximal compact subgroup H. However, using suitable projection operators one can project out the irreducible components. The possible irreducible representations of H that can be constructed this way depends on the number N pairs of boson operators a and b. For example in the case when $a(K)$ and $b(K)$, $(K=1,\ldots,N)$ transform like the fundamental representation of SU(n), then the irreducible representations of SU(n) that one can obtain by this method have Young tableaux with at most N rows. This is simply due to the fact that the largest totally antisymmetric representation of SU(n) that one can construct from N copies of boson operators is of rank N. Of course, if $N \geqq n$ any representation of SU(n) can be constructed by repeated application of the creation operators followed by a projection operator. In Young tableaux notation we have

$$[a^i(K)]^{m_K}|0\rangle \Rightarrow (m_K,0,0,\ldots), i,j=1,2,\ldots n \tag{4.7a}$$

$$[a^i(K)]^{m_K}[a^j(L)]^{m_L}\;|0\rangle \Rightarrow (m_K+m_L,0,0\,..)+(m_K,m_L,0,\ldots), \tag{4.7b}$$

$$\begin{aligned}&[a^i(1)]^{m_1}[a^j(2)]^{m_2}\ldots[a^l(N)]^{m_N}|0\rangle\\ &\Rightarrow (m_1,0,\ldots)\otimes(m_2,\ldots)\otimes\ldots\otimes(m_N,0,0,..),\end{aligned} \tag{4.7c}$$

where $(m_1,m_2,\ldots,m_n)$ denotes a representation with a Young tableau which has m_i number of boxes in the i^{th} row. The maximal compact subgroups in our case have a U(1) factor whose generator in most cases corresponds to the boson number operator. Each one of the states constructed above has a definite U(1) charge.

The remarkable feature of the unitary representations above is that they are uniquely determined by the initial state $|\psi_A\rangle$ that is annihilated by the L^- space and the irreducibility of the representation follows directly from the irreducibility of $|\psi_A\rangle$ under the maximal compact subgroup H. This is a general property of the representations belonging to the discrete series [28]. Furthermore the condition $L^-|\psi_A\rangle$ could be interpreted as an holomorphicity condition and thus we would expect the above representations to belong to the holomorphic discrete series [29].

We should note that the use of boson operators for constructing UIRs of non-compact groups is certainly not new in physics. They have been used from time to time to construct representations of certain non-compact groups of physical interest, using most often one set of boson operators which leads to only two UIRs [see Ref. [12] for a comprehensive list of references]. What our formulation does is to give a unified treatment of the oscillator-like representations of all non-compact groups with a Jordan structure in the most general form. The irreducibility of the resulting representations that was proven in individual cases by the brute force method of calculating all the Casimir operators follows simply from condition (4.3) and the Jordan structure [9]. Furthermore, the use of an arbitrary number of pairs of boson operators enables us to construct infinite classes of UIRs.

5. Bosonic Construction of the Lie Algebra of Non-Compact Group $E_{7(7)}$

The Lie algebra of $E_{7(7)}$ with a maximal compact subgroup SU(8) was constructed in Ref. [9] in terms of a pair of boson annihilation and creation operators transforming like the antisymmetric tensor representations of SU(8). Here we give the same construction using an arbitrary number N pairs of boson operators. They satisfy the commutation relations:

$$\begin{aligned} [a_{ij}(K), a^{kl}(L)] &= \delta^{KL}(\delta_i^k \delta_j^l - \delta_j^k \delta_i^l), \\ [b_{ij}(K), b^{kl}(L)] &= \delta^{KL}(\delta_i^k \delta_j^l - \delta_j^k \delta_i^l), \\ [a_{ij}(K), a_{kl}(L)] &= 0 = [b_{ij}(K), b_{kl}(L)], \\ i,j,k,l\ldots = 1,2,\ldots 8\,; &\quad K,L,\ldots = 1,\ldots,N, \end{aligned} \tag{5.1}$$

and

$$a_{ij}(K) = -a_{ji}(K); \quad b_{ij}(K) = -b_{ji}(K).$$

The SU(8) generators are taken as

$$T^i_j = \mathbf{a}^{im}\cdot\mathbf{a}_{jm} + \mathbf{b}_{jm}\cdot\mathbf{b}^{im} - \tfrac{1}{8}\delta^i_j(\mathbf{a}^{kl}\cdot\mathbf{a}_{kl} + \mathbf{b}_{kl}\cdot\mathbf{b}^{kl}), \tag{5.2}$$

where the dot product again denotes summation over the generation index $K = 1,\ldots,N$. They satisfy the commutation relations

$$[T^i_j, T^k_l] = \delta^k_j T^i_l - \delta^i_l T^k_j. \tag{5.3}$$

Now the 133 dimensional adjoint representation of E_7 decomposes under the SU(8) subgroup as

$$\underline{133} = \underline{63} \oplus \underline{70},$$

where $\underline{63}$ stands for the adjoint representation of SU(8) and $\underline{70}$ corresponds to the totally antisymmetric rank four tensor representation. This suggests that as the 70 non-compact generators of $E_{7(7)}$ we take

$$V_{ijkl} = \mathbf{a}_{[ij}\cdot\mathbf{b}_{kl]} + \tfrac{1}{4}\varepsilon_{ijklmnpq}\mathbf{a}^{mn}\cdot\mathbf{b}^{pq}, \tag{5.4}$$

where indices inside brackets [ ] are all antisymmetrized. The operator V_{ijkl} is totally antisymmetric in its indices and satisfies

$$V^\dagger_{ijkl} = \frac{1}{4!}\varepsilon^{ijklmnpq}V_{mnpq} \equiv V^{ijkl}, \tag{5.5}$$

which reflects the fact that the representation $\underline{70}$ of SU(8) is self-conjugate. The operators V_{ijkl} do indeed close into the generators of SU(8):

$$
\begin{aligned}
&[V_{ijkl}, V^{abcd}] = -\tfrac{1}{6}\varepsilon_{ijklmpqr} T^m_n \varepsilon^{abcdnpqr},\\
&[T^m_n, V_{ijkl}] = \tfrac{1}{6}\varepsilon_{ijklnpqr} V^{mpqr} - \tfrac{1}{2}\delta^m_n V_{ijkl},\\
&[T^m_n, V^{ijkl}] = -\tfrac{1}{6}\varepsilon^{ijklmpqr} V_{npqr} + \tfrac{1}{2}\delta^m_n V^{ijkl},\\
&i, j, k, \ldots = 1, \ldots 8.
\end{aligned}
\tag{5.6}
$$

That the resulting Lie algebra is that of E_7 follows from the fact that it is the only simple Lie algebra of dimension 133. To determine whether it is the Lie algebra of non-compact $E_{7(7)}$ with the maximal compact subgroup SU(8) one has to look at the Killing form. The Killing metric turns out to be

$$
\begin{aligned}
&g_{T^i_j, T^k_l} = 36\,\delta^i_l \delta^k_j + 1002\,\delta^i_j \delta^k_l,\\
&g_{V_{ijkl}, V_{mnpq}} = -36\,\varepsilon_{ijklmnpq},\\
&g_{T^i_j, V_{mnpq}} = 0,
\end{aligned}
\tag{5.7}
$$

showing that in a Hermitian basis it gives us the Lie algebra of $E_{7(7)}$.

The Killing metric determines the quadratic Casimir operator C_2 up to an overall constant. For $E_{7(7)}$ we choose this constant such that

$$
C_2 = \tfrac{1}{6}\left(\tfrac{1}{24} V_{ijkl} V^{ijkl} - T^k_l T^l_k\right).
\tag{5.8}
$$

This construction of $E_{7(7)}$ from boson operators corresponds to an operator formulation of a realization of E_7 in a 56 dimensional space by *H*. Freudenthal [30]. Remarkably enough the Casimir operator C_2 is exactly the quartic symplectic invariant on the 56 dimensional fundamental representation space given by Freudenthal. In our case the operators a_{ij} (b_{ij}) and b^{ij}(a^{ij}) get transformed into each other under the action of $E_{7(7)}$ and form the 56 dimensional fundamental representation.

The non-compact group $E_{7(7)}$ is the global invariance group of the bosonic sector of the largest possible supergravity theory ($N = 8$) in four dimensions. The natural SO(8) symmetry of $N = 8$ supergravity first gets extended to a global SU(8) symmetry via a chiral-dual transformation, i.e., it acts on the spinor fields of the theory as chiral transformations and on the spin one fields as duality rotations which transform the electric and magnetic field strengths into each other [1]. Then this compact SU(8) symmetry is enlarged to the non-compact $E_{7(7)}$ which is realized non-linearly over the 70 scalar fields of the theory corresponding to the coset space $E_{7(7)}$/SU(8) [1]. Thus, it would be interesting to rewrite $E_{7(7)}$ Lie algebra in an SO(8) basis. Doing this turns out to be the same as going to the Hermitan basis:

Consider the following Hermitian linear combinations of $E_{7(7)}$ generators:

$$
\begin{aligned}
&A_{mn} = i(T^m_n - T^n_m) = -A_{nm}; \quad && A^\dagger_{mn} = A_{mn},\\
&S_{mn} = T^m_n + T^n_m = S_{nm}; && S^\dagger_{mn} = S_{mn},\\
&S_{ijkl} = V_{ijkl} + V^{ijkl}; && S^\dagger_{ijkl} = S_{ijkl},\\
&A_{ijkl} = i(V_{ijkl} - V^{ijkl}); && A^\dagger_{ijkl} = A_{ijkl}.
\end{aligned}
\tag{5.9}
$$

Note that since we are working in an SO(8) basis we are not making any distinction between upper and lower indices. These operators satisfy the commutation relations:

$$\begin{aligned}
[A_{ijkl}, A_{\alpha\beta\gamma\delta}] &= \tfrac{1}{6}(\varepsilon_{ijklm\mu\nu\rho}\varepsilon^{n\mu\nu\rho}_{\alpha\beta\gamma\delta} - \varepsilon_{ijklm\mu\nu\rho}\varepsilon_{\alpha\beta\gamma\delta n\mu\nu\rho})A_{mn} \\
[S_{mn}, S_{ijkl}] &= \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}A_{n\mu\nu\rho}A_{n\mu\nu\rho} + \tfrac{1}{6}\varepsilon_{ijkln\mu\nu\rho}A_{m\mu\nu\rho} - \delta_{mn}A_{ijkl}, \\
[A_{mn}, S_{ijkl}] &= \tfrac{1}{6}\varepsilon_{ijkln\mu\nu\rho}S_{m\mu\nu\rho} - \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}S_{n\mu\nu\rho}, \\
[S_{mn}, A_{ijkl}] &= \tfrac{1}{6}\varepsilon_{ijkln\mu\nu\rho}S_{m\mu\nu\rho} + \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}S_{n\mu\nu\rho} - \delta_{mn}S_{ijkl}, \\
[A_{mn}, A_{ijkl}] &= -\tfrac{1}{6}\varepsilon_{ijkln\mu\nu\rho}A_{m\mu\nu\rho} + \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}A_{n\mu\nu\rho}, \\
[S_{ijkl}, S_{\alpha\beta\gamma\delta}] &= -(\tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}\varepsilon^{n\mu\nu\rho}_{\alpha\beta\gamma\delta} + \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}\varepsilon_{\alpha\beta\gamma\delta n\mu\nu\rho})A_{mn}, \\
[S_{ijkl}, A_{\alpha\beta\gamma\delta}] &= (-\tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}\varepsilon^{n\mu\nu\rho}_{\alpha\beta\gamma\delta} + \tfrac{1}{6}\varepsilon_{ijklm\mu\nu\rho}\varepsilon_{\alpha\beta\gamma\delta n\mu\nu\rho})S_{mn}, \\
i,j,k,\ldots,\alpha,\beta,\ldots &= 1,\ldots,8; \quad \varepsilon^{mnpq}_{ijkl} \equiv \tfrac{1}{24}\varepsilon_{ijkl\mu\nu\rho\lambda}\varepsilon^{\mu\nu\rho\lambda mnpq}.
\end{aligned} \tag{5.10}$$

The number of independent A_{ijkl} is 35 and so is the number of independent S_{ijkl}. Since $\sum_{n=1}^{8} S_{nn} = 0$, there are also 35 independent S_{mn}. Therefore, under the SO(8) subgroup generated by A_{mn} the adjoint representation of E_7 decomposes as

$$\underline{133} = \underline{28} \oplus \underline{35}^V \oplus \underline{35}^L \oplus \underline{35}^R, \tag{5.11}$$

which corresponds to the decomposition of the generators as

$$133 = A_{mn} \oplus S_{mn} \oplus A_{ijkl} \oplus S_{ijkl}. \tag{5.12}$$

These three 35 dimensional representations of SO(8) are all inequivalent and are related by the principle of triality [31] which generalizes the well-known triality among the three 8 dimensional representations 8^V, 8^L and 8^R of SO(8), namely the vectors 8^V, left-handed spinors 8^L and right-handed spinors 8^R, respectively [32]. In fact the exceptional group F_4 has the same structure with respect to its SO(8) subgroup, i.e., its adjoint representation 52 decomposes as

$$\underline{52} = \underline{28} \oplus \underline{8}^V \oplus \underline{8}^L \oplus \underline{8}^R. \tag{5.13}$$

From the following Kronecker products

$$\begin{aligned}
&\underline{8}^i \otimes \underline{8}^i = \underline{1} + \underline{28} + \underline{35}^i, \quad i = V, L, R, \\
&\underline{8}^i \otimes \underline{8}^i = \underline{1} + \underline{28} + \underline{35}^k, \\
&\qquad i,j,k = V, L, R \text{ taken in cyclic order.}
\end{aligned} \tag{5.14}$$

it follows that the representations 35^V, 35^L and 35^R correspond to symmetric traceless tensors in vector, left-handed spinor and right-handed spinor indices in eight dimensions, respectively. The triality principle also implies that

$$\begin{aligned}
&\underline{35}^i \otimes \underline{35}^i = \underline{28} + \underline{35}^i + \ldots, \\
&\underline{35}^i \otimes \underline{35}^j = \underline{35}^k + \ldots, \\
&\qquad i,j,k = V, L, R \text{ in cyclic order.}
\end{aligned} \tag{5.15}$$

A unitary operator representing an element of the group $E_{7(7)}$ can now be written as

$$U(g) = e^{i(w_{ij}A_{ij} + v_{ij}S_{ij} + w_{ijkl}A_{ijkl} + v_{ijkl}S_{ijkl})},$$
$$U^\dagger(g)U(g) = 1, \tag{5.16}$$

where w_{ij}, v_{ij}, w_{ijkl} and v_{ijkl} are real parameters with the same tensorial properties as the respective operators, i.e.,

$$\begin{aligned} w_{ij} &= -w_{ji}; \quad v_{ij} = v_{ji} \text{ and } v_{ii} = 0, \\ w_{ijkl} &= -\tilde{w}_{ijkl} \equiv -\tfrac{1}{24}\varepsilon_{ijklmnpq}w_{mnpq}, \\ v_{ijkl} &= \tilde{v}_{ijkl} \equiv \tfrac{1}{24}\varepsilon_{ijklmnpq}v_{mnpq}. \end{aligned} \tag{5.17}$$

Now if we define the boson number operator as $N = \mathbf{a}^{mn}\cdot\mathbf{a}_{mn} + \mathbf{b}^{mn}\cdot\mathbf{b}_{mn}$ we find that the SU(8) generators have zero boson number

$$[N, T^m_n] = 0, \tag{5.18}$$

and the non-compact generators V_{ijkl} do not have a well-defined boson number since they involve disoson creation as well as annihilation operators. This is a reflection of the exceptional feature of $E_{7(7)}$, whose adjoint representation decomposes as the adjoint plus a real irreducible representation with respect to a maximal compact unitary subgroup. [The only other group with this property is SO(6,1).] The operator N which lies outside of $E_{7(7)}$ does however still generate an automorphism of its Lie algebra.

A maximal rank compact subgroup with respect to which the additional generators in $E_{7(7)}$ split into complex representations is U(7). If we take as the SU(7) generators T^A_B, where $A, B = 1,\ldots,7$ and as the U(1) generator $T^8_8 = -T^A_A$, then under this U(7) the Lie algebra of $E_{7(7)}$ decomposes as

$$\begin{aligned} &T^i_j = (T^A_B \oplus T^8_8) \oplus T^A_8 \oplus T^8_A = (\underline{48} + \underline{1}) + \underline{7} + \underline{\bar{7}}, \\ &V_{ijkl} = V_{ABCD} \oplus V_{ABC8} = \underline{35} \oplus \underline{\overline{35}}, \\ &i,j,k,l = 1,..8; \quad A,B,C,D = 1,..7. \end{aligned} \tag{5.19}$$

Denoting the generators V_{ABC8} as A_{ABC} and V^{ABC8} as A^{ABC} we can write the $E_{7(7)}$ Lie algebra in the U(7) basis as a direct sum

$$\begin{aligned} L &= F_A \oplus A_{ABC} \oplus (T^A_B \oplus T^8_8) \oplus A^{ABC} \oplus F^A, \\ L &= L^{-2} \oplus L^{-1} \oplus L^0 \oplus L^{+1} \oplus L^{+2}, \end{aligned} \tag{5.20}$$

where F_A stands for T^8_A and F^A for T^A_8. We see that with respect to the U(1) generator T^8_8 the Lie algebra has a five dimensional graded structure. This is a more general structure than the Jordan structure and all simple Lie algebras have a five dimensional graded structure with respect to a suitable maximal subalgebra. We shall call this type of a five dimensional graded structure a Kantor structure [14]. The construction of Lie (super) algebras from Jordan (super) triple systems has been extended to this more general case [14, 22]. The ternary algebra that gives us the Lie algebra of E_7 in a U(7) basis in this more general construction corresponds to antisymmetric tensors of rank three in seven dimensions [14]. In fact, Kantor's construction yields the Lie algebra of the exceptional group E_d when the underlying ternary algebra is the antisymmetric tensors of rank three in d dimensions with a suitable triple product [15]. This construction gives finite dimensional Lie algebras E_d for $d \leqq 8$ and leads to infinite dimensional Lie algebras [14] for $d > 8$. Now the $N = 8$ extended supergravity theory in d dimensions has non-compact $E_{d(d)}$ as its global invariance group [33], and all these theories are obtained from the 11 dimensional simple supergravity theory by dimensional reduction. The fundamental boson field that enters this latter theory is an antisymmetric tensor of rank three. Thus the Kantor construction of E series suggests a possible link between the emergence of these groups and the presence of antisymmetric rank three tensor fields in $N = 8$ supergravity theories.

6. Unitary Representations of $E_{7(7)}$

If we apply the methods of Sect. 4 to $E_{7(7)}$ for the construction of unitary representations we find that the resulting representations are reducible. This can be seen easily as follows. Consider a set of states $|\psi_A\rangle$ transforming as an irreducible representation of the maximal compact subgroup SU(8) that is constructed by acting on the vacuum state with the creation operators $\mathbf{a}^{ij}$ and $\mathbf{b}^{ij}$. By repeated application of the non-compact generators V_{ijkl} on $|\psi_A\rangle$ we can generate an infinite set of states:

$$|\psi_A\rangle,\ V_{ijkl}|\psi_A\rangle,\ V_{ijkl}V_{mnpq}|\psi_A\rangle, \ldots \tag{6.1}$$

which form the basis of a unitary representation of $E_{7(7)}$. The V_{ijkl} transform as the self-conjugate representation $\underline{70}$ of SU(8) and the product

$$V_{ijkl}V_{mnpq} = \tfrac{1}{2}\{V_{ijkl}, V_{mnpq}\} + \tfrac{1}{2}[V_{ijkl}, V_{mnpq}]$$

transforms as the reducible $(1 + 720 + 1764)_{\mathrm{sym}} + 63_{\mathrm{antisym}}$ representation of SU(8). The fact that the product contains a singlet of SU(8) means that every irreducible representation of SU(8) that occurs in the infinite set of states (6.1) will reappear again after two applications of the V's. Thus the multiplicity of an irreducible representation of SU(8) that occurs in the unitary representation defined over the set of states (6.1) is infinite. This means that the resulting unitary representation is infinitely reducible as a consequence of the well-known fact that the multiplicity of an irreducible representation of the maximal compact subgroup inside an UIR of a non-compact group is less than or equal to its dimension [34]. Though reducible, these representations may still be of relevance for physical applications [5, 16].

Application of our method to the second construction of SO(12)* [see Eqs. (2.20)–(2.22)] gives reducible unitary representations as well. In this case even though we have a Jordan structure there are no states transforming like an irreducible representation of U(6) that is annihilated by the L^- space.

The fact that one gets infinitely reducible unitary representations in the case of SO(12)* and $E_{7(7)}$ suggests the use of coherent states to construct UIRs of these groups. In fact there is a method due to Gell–Mann for constructing a class of UIRs of some non-compact groups on certain coset spaces of their maximal compact subgroup [17, 18]. His method does apply to $E_{7(7)}$ and is particularly simple for determining the multiplicities of the irreducible representations of the maximal compact subgroup inside a UIR of the non-compact group. For example, one possible coset space on which to realize UIRs of $E_{7(7)}$ is SU(8)/Sp(8). In this case the multiplicity of an irreducible representation of SU(8) in an UIR of $E_{7(7)}$ constructed by his method is determined by the number of Sp(8) singlets that representation contains [17]. The reason why Gell–Mann's method cannot be applied to our construction of $E_{7(7)}$ is due to the fact that the boson operators we use transform linearly under the SU(8) subgroup rather than non-linearly as some coset space of SU(8) satisfying certain criteria [18]. On the other hand our construction of the non-compact groups of supergravity parallels very closely their emergence in supergravity and the boson operators we use correspond to the vector fields in these theories. The reducibility of some of the resulting representations seems to be necessary for the compatibility of supersymmetry with the non-compact invariance in $N=4-8$ supergravity theories [16, 35].

The only basic bosonic fields that transform non-linearly in supergravity theories are the scalar fields which sit on the coset space G/H of the non-compact group G with respect to its maximal compact subgroup. The action of G on the scalar fields can be represented as a generalized linear fractional transformation [1] indicating that the scalars can be considered as a Gelfand-Z-basis of G [36]. Gürsey and his collaborators have given an operator formulation of the construction of UIRs in a Z-basis á la Gelfand in the case of some smaller non-compact groups [19]. By these operator techniques one can construct new classes of unitary representations of these groups using operators corresponding to the scalar fields in supergravity theories, which may be of relevance as well for physical applications.

Note Added

We would like to thank the referee for bringing to our attention two related works on the unitary representations of non-compact groups, namely

1) M. Kashiwara and M. Vergne, "On the Segal–Shale–Weil Representations and Harmonic Polynomials," Inventiones Math. **44** (1978) 1–47.
2) R. Howe, "Classical Invariant Theory," and "Transcending Classical Invariant Theory," Yale University Preprints, unpublished.

Of particular relevance in the work of Kashiwara and Vergne is the construction of a series of new unitary irreducible representations with highest weight vectors of the group $U(p, q)$ and the metaplectic group $Mp(n)$, which is the two-sheeted covering group of the symplectic group, by decomposing tensor products of harmonic representations into irreducible components. For the non-compact groups of supergravity their method corresponds to the construction of unitary irreducible representations in terms of scalar fields as suggested in the last paragraph of Sect. 6 above. R. Howe's papers which deal mainly with invariant theory stress in particular the analogy, noticed by previous authors, between Clifford algebras and the spin representations on the one hand and the Weyl algebra and the oscillator representation on the other.

Appendix—Current Algebra of Diboson Fields

As an illustration of how our boson operators can be related to a field theory context, we construct a current algebra of free boson fields. We will also see in this example that the so-far unspecified vectorial character of our operators $\mathbf{a}_1$ acquires a specific meaning: the components of the vector represent various Fourier modes $a_i(\mathbf{k})$. Another point of interest is that the construction involves a bosonic version of the Pauli–Gürsey (PG) transformation [37].

We first start with a set of boson fields φ_i transforming (for simplicity) as an irreducible representation of a global symmetry group H. The bosons can be scalars or vectors, but in the latter case the study of the algebra is simpler if they are taken to be massive, so that gauge related quantization problems are avoided. The generators of H can be then immediately written down as integrated charges obtained from the standard currents, which are always of the form $\varphi_i^\dagger \partial_\mu \varphi_j$.

Next, we extend H by PG-like transformations which mix φ_i and $\varphi_k^\dagger$. This gives rise to new currents of the form $\varphi_i \partial_\mu \varphi_j$ and $\varphi_i^\dagger \partial_\mu \varphi_j^\dagger$. The original PG of course applies to fermions ψ and leaves the free massless Lagrangian and even certain gauge couplings invariant (up to a total divergence). The new difermions $\int dv \psi\psi$ and $\int dv \psi^\dagger \psi^\dagger$, together with the usual $\int dv \psi^\dagger \psi$ charges, can sometimes close into a new algebra of a compact group. In contrast our bosonic version of PG and the resulting diboson generators only leave the equaltime free boson field commutators invariant and can lead to both compact and non-compact groups.

The main features of the construction can be understood with the example of a single complex scalar field. The diboson charges are most conveniently written by defining the two-component field $\psi^T = (\psi_1, \psi_2)$ with [38]

$$\psi_1 = \sqrt{\frac{m}{2}}\left(\varphi + \frac{i}{m}\dot{\varphi}\right),$$
$$\psi_2 = \sqrt{\frac{m}{2}}\left(\varphi - \frac{i}{m}\dot{\varphi}\right), \tag{A.1}$$

so that the similarity to fermions becomes more evident. The equal-time commutation relations between $\dot{\varphi}$ and $\varphi^\dagger$ now translate into (other combinations vanish)

$$[\psi_\alpha(\bar{x}, 0), \psi_\beta^\dagger(\bar{y}, 0)] = (\tau_3)_{\alpha\beta} \delta^3(\bar{x} - \bar{y}), \tag{A.2}$$

indeed displaying a similarity to harmonic oscillator operators or spinor field anticommutation relations: there is no (i) on the right-hand side and the fields ψ and $\psi^\dagger$ appear on the left-hand side, instead of the less symmetrical φ, $\varphi^\dagger$ pair. The τ_3 signature on the right-hand side originates from the fact that $\psi_1(\psi_2)$ acts like an annihilation (creation) operator; for example, for the $E^2 = m^2$ modes $\psi_1 \sim a_k$ and $\psi_2 \sim a_k^{c\dagger}$. In this formalism the U(1) charge operator is

$$Q_3 = \tfrac{1}{2}\int d^3x (\psi^\dagger \tau_3 \psi + \psi^T \tau_3 \psi^*)$$
$$= \tfrac{1}{2}\sum_k (a_k^\dagger a_k + a_k a_k^\dagger - a_k^{c\dagger} a_k^c - a_k^c a_k^{c\dagger}). \tag{A.3}$$

As mentioned before, massive spin one is no problem: we just let $\varphi \to A_b, \dot{\varphi} \to E_b$, $\psi \to \psi_b (b = 1, 2, 3)$ and sum over the space index b in Q_3. The resulting Q_3 is, of course, nothing but the usual $\int dv(A_b^* E_b + E_b^* A_b)$. Thus it is not surprising that similar but more complicated non-compact symmetries hold for vector bosons in extended supergravity. Since the vector index can be trivially added, we go on with our simple scalar example.

In the ψ basis one can introduce a bosonic version of the Pauli–Gürsey transformation of the form

$$\psi'_\alpha = a\psi_\alpha + bM_{\alpha\beta}\psi^\dagger_\beta, \tag{A.4}$$

where a, b are in general complex and $M_{\alpha\beta}$ is a 2×2 matrix. This does not leave the Lagrangian invariant, but it is canonical in the sense of preserving the field commutators when M satisfies [39]

$$(\tau_3)_{\alpha\beta} = |a|^2(\tau_3)_{\alpha\beta} - |b|^2 M_{\alpha\mu}(\tau_3)_{\mu\nu} M^\dagger_{\nu\beta}. \tag{A.5}$$

This gives rise to two possibilities and conditions:

$$\begin{aligned} &\text{a) } M = I \text{ or } \tau_3; \quad |a|^2 - |b|^2 = 1, \\ &\text{b) } M = \tau_1 \text{ or } \tau_2; \quad |a|^2 + |b|^2 = 1. \end{aligned} \tag{A.6}$$

Conditions (a) and (b) suggest the groups SU(1, 1) and SU(2) respectively. Indeed, taking $M = I$, applying Eq. (5) on Eq. (3), and then picking from the results the diboson SU(1, 1) generators

$$\begin{aligned} Q_+ &= \int \frac{dv}{\sqrt{2}} \psi^\dagger \tau_3 \psi^* = \frac{1}{\sqrt{2}} \sum_k (a_k^\dagger a_{-k}^\dagger e^{ziwt} - a_k^c a_{-k}^c e^{-ziwt}), \\ Q_- &= \int \frac{dv}{\sqrt{2}} \psi^T \tau_3 \psi = \frac{1}{\sqrt{2}} \sum_k (a_k a_{-k} e^{-ziwt} - a_k^{c\dagger} a_{-k}^{c\dagger} e^{-ziwt}), \end{aligned} \tag{A.7}$$

this expectation for (a) is verified. Of course $a_k (a_k^c)$ represent particle (antiparticle) destruction operators. The exponential time dependence can be absorbed into the operators through $a_k \to a_k e^{-iwt}$, hence it will not be written down again. Note that $M = \tau_3$ duplicates the algebra and $M = \tau_1$ gives charges such as $\int dv\varphi\varphi$ or $\int dv\dot{\varphi}\dot{\varphi}$ with different Lorentz transformation properties; we consider them no further. Here $M = \tau_2$ gives vanishing charges on account of Bose statistic, but when fields ψ_i representing a $U(n)$ group are considered, $n(n-1)$ combinations of the type $\psi_j i\tau_2 \psi_i, \psi^k \tau_2 \psi^l$ become possible, extending the $U(n)$ to *compact* SO(2*n*). These have the form

$$\begin{aligned} Q_{ij} &= \int \frac{dv}{\sqrt{2}} \psi_i^\dagger i\tau_2 \psi_j = \frac{1}{\sqrt{2}} \sum_k a_i(k) a_j^{c\dagger}(k) - a_j(k) a_i^{c\dagger}(k), \\ Q^{ij} &\equiv (Q_{ij})^\dagger. \end{aligned}$$

The Lie algebra of compact bosonic E_7 can also be represented through such operators. Returning to the set $Q_\pm, Q_3$, we find

$$[Q_+, Q_-] = -2Q_3,$$

$$[Q_3, Q_+] = Q_+, \qquad (A.8)$$
$$[Q_3, Q_-] = -Q_-,$$

where the clear SU(1, 1) signature is seen in the first commutator. The extension of this current algebra representation to cases where the initial symmetry is bigger than U(1) can obviously be effected by re-interpreting our former expressions: for example, $\mathbf{a}_i \cdot \mathbf{b}_j$ could now be taken to mean $\sum_k a_i(\mathbf{k}) a_j(-\mathbf{k})$ in the non-compact cases and $\mathbf{a}_i \mathbf{b}_j^\dagger$ be identified with $\sum_k a_i^\dagger(\mathbf{k}) a_j^c(\mathbf{k})$.

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