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# The Existence of a Non-Minimal Solution to the SU(2) Yang-Mills-Higgs Equations on $\mathbb{R}^3$: Part II

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**Abstract.** This paper proves that there exists a finite action solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$ in the Bogomol'nyi-Prasad-Sommerfield limit which is not a solution to the first order Bogonol'nyi equations. The existence is established using Ljusternik-Šnirelman theory on non-contractible loops in the configuration space.

## I. Introduction

In the first paper in this series ([1], to be referred to as Part I), the author stated the following theorem:

**Theorem 1.1.** *There exists a smooth, finite action solution to the* SU(2) *Yang-Mills-Higgs equations in the Bogomol'nyi-Prasad-Sommerfield limit which does not satisfy the first order Bogomol'nyi equations.*

This sequel to Part I contains the proof of Theorem 1.1. The reader is referred to Sects. I.2, 3 for an introduction to Yang-Mills-Higgs theory. These sections also define the author's terminology and notation.

The proof of Theorem 1.1 is an application of Ljusternik-Šnirelman theory on the space of finite action field configurations with monopole number zero (denoted $\mathscr{C}_0$). Part I established that a solution to the Yang-Mills-Higgs equations (I.2.2, 3) with non-zero action exists in $\mathscr{C}_0$ if there exists $k>0$, and a non-trivial generator $e\in \Pi_k(\mathrm{Maps}((S^2,n);(S^2,n),e_*)$ such that

$$\inf_{c(y)\in \Lambda(e)} \left\{\sup_{y\in S^k} \mathscr{a}(c(y))\right\} < 8\pi . \tag{1.1}$$

Such a solution cannot satisfy the Bogomol'nyi equations (I.2.6). It is the purpose of this paper to establish that the above criteria is satisfied and Sects. 2–5 prove that (1.1) is satisfied for the generator of $\Pi_1(\mathrm{Maps}((S^2,n);(S^2,n)),e_*)$. It is also

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proved in Sects. 4–5, Theorem 4.4, that the only solutions to Eqs. (I.2.2, 3) in $\mathscr{C}$ which are local minima of (cf. Def. 4.3) are the solutions to the Bogomol'nyi equations (I.2.6). The full proof of Theorem 1.1 is exhibited in Sect. 6.

## II. The Trial Loop

This section and Sects. 3–5 investigate in detail the behavior of the action functional on loops in $\Lambda(e)$, where $e$ is the generator of $\Pi_1(\text{Maps}((S^2, n); (S^2, n)), e_*)$. The result is the following theorem:

**Theorem 2.1.** *Let e be the generator of* $\Pi_1(\text{Maps}((S^2; n), (S^2, n)), e_*)$. *There exists a loop* $c(y) \in \Lambda(e)$ *with* $\sup_{y \in S^1} \mathscr{a}(c(y)) < 8\pi$.

The loop $c(y)$ in Theorem 2.1 represents the following physical procedure: Create a monopole, anti-monopole pair from the vacuum (the configuration $(0, -\frac{1}{2}\sigma^3)$). Separate them a distance $d$, and then rotate the monopole with respect to the anti-monopole by $2\pi$ about their common axis. Finally, bring them together again. The action remains less than $8\pi$ due to the fact that monopoles and anti-monopoles attract.

To begin, consider $\Pi_1(\text{Maps}((S^2, n); (S^2, n)), e_*)$. Let $\{\sigma^i\}_{i=1}^3$ be a basis for $\mathfrak{su}(2)$ such that $\sigma^i\sigma^j = -\delta^{ij} - \varepsilon^{ijk}\sigma^k$, $(\sigma^i, \sigma^j) = 4\delta^{ij}$. Let $S$ be the interval $[0, 2\pi]$ with endpoints identified. Define $e(t; \hat{x}) \in C^0((S, \{0\}); (\text{Maps}((S^2; n); (S^2, n)), e_*))$ by

$$e(t, \hat{x}) = -\tfrac{1}{2}(\cos^2\theta + \sin^2\theta \cos t)\sigma^3 + \tfrac{1}{2}\sin\theta\cos\theta(1-\cos t)(\cos\phi\sigma^1 + \sin\phi\sigma^2) + \tfrac{1}{2}\sin\theta \sin t(\cos\phi\sigma^2 - \sin\phi\sigma^1). \tag{2.1}$$

Here $(\theta, \phi)$ are spherical coordinates.

**Lemma 2.2.** *The map* $e(t; \hat{x})$ *is a generator of* $\Pi_1(\text{Maps}((S^2, n); (S^2, n)), e_*)$.

*Proof of Lemma 2.2.* The groups $\Pi_3(S^2, n)$ and $\Pi_1(\text{Maps}(S^2, n); (S^2, n)), e_*)$ are isomorphic. The Hopf map $H$: $SU(2) \to S^2$ generates $\Pi_3(S^2, n)$. Represent a point $g \in SU(2)$ by the unitary matrix

$$g(\chi, \theta, \phi) = \cos\chi + \sin\chi(\cos\theta\sigma^3 - \sin\theta(\cos\phi\sigma^1 + \sin\phi\sigma^2)),$$

for $\chi \in [0, \pi]$. Then $H(g) = -\frac{1}{2}g\sigma^3g^{-1} \in S^2$ and $e(t; \theta, \phi) = H(g(t/2, \theta, \phi))$.

As described in Sect. I.4, the map $e(t)$ defines a noncontractible loop

$$c(e)(t) \in C^0((S^1, n); (\mathscr{C}_0, c_*)).$$

For convenience, the notation of Sect. I.4 will be changed. Let $S^1$ denote the interval $[-\pi, 3\pi]$ with endpoints identified. With this change, the construction of Sect. I.4 yields

$$c(e)(t) = \begin{cases} -\left(1 - \dfrac{(t+\pi)}{\pi}\beta(x)\right)\left(0, \dfrac{1}{2}\sigma^3\right), & t \in [-\pi, 0]; \\ (1-\beta(x))(-[e(t; \hat{x}), de(t; \hat{x})], e(t, \hat{x})), & t \in [0, 2\pi]; \\ -\left(1 - \dfrac{(3\pi - t)}{\pi}\beta(x)\right)\left(0, \dfrac{1}{2}\sigma^3\right), & t \in [3\pi, 2\pi]. \end{cases} \tag{2.2}$$

The loop $c(e)(t) \equiv c_0(t) = (A_0(t), \Phi_0(t))$ is a generator of $\Pi_1(\mathscr{C}_0, c_*)$.

*Proof of Theorem 2.1.* The multi-step procedure, below, establishes the theorem.

(1) A loop $a(t) = (A(t), \Phi(t)) \in C^0([0, 2\pi]; \mathscr{C}_0)$ is constructed in this section, which has $a(0) = a(2\pi)\,(\neq c_*)$ and satisfies $\lim_{|x| \to \infty} \Phi(t; x) \to e(t; \hat{x})$, uniformly in $t$. The loop $a(t)$ and its properties are summarized in Definition 2.3 and Proposition 2.4.

(2) In Sect. 3, a loop $b(t) \in C^0([0, 2\pi]; \mathscr{C}_0)$ is constructed from $a(t)$ which satisfies Statements (1)–(3) of Definition I.4.1 for $t \in [0, 2\pi]$. In addition, $b(0) = b(2\pi)$ and $\sup_{t \in [0, 2\pi]} \mathscr{A}(b(t)) < 8\pi$. This is Proposition 3.1.

(3) It is established in Sects. 4 and 5 that there exists a path $d(t) \in C^0([0, \pi], \mathscr{C}_0)$ such that: (a) $d(0) = b(0)$. (b) $d(\pi) = g(0, -\frac{1}{2}\sigma^3)$, where $g \in \mathscr{G}$ and $|g - 1|$ has compact support. (c) $\sup_{t \in [0, \pi]} \mathscr{A}(d(t)) < 8\pi$. (d) $g^{-1}d(-t)$, $t \in [-\pi, 0]$, and $g^{-1}d(t - 2\pi)$, $t \in [2\pi, 3\pi]$ satisfy Statements (1)–(3) of Definition I.4.1 in their respective domains of definition. This is Proposition 4.2.

(4) Then the loop

$$c(t) \equiv \begin{cases} g^{-1}d(-t), & t \in [-\pi, 0], \\ g^{-1}b(t), & t \in [0, 2\pi], \\ g^{-1}d(3\pi - t), & t \in [2\pi, 3\pi], \end{cases} \tag{2.3}$$

*is in $\Lambda$ and $\sup_{t \in S} \mathscr{A}(c(t)) < 8\pi$.*

*Construction of the Loop $a(t)$.* To simplify the construction, some coordinate systems are needed. Let $(x_1, x_2, x_3)$ be the cartesian coordinates on $\mathbb{R}^3$ centered at 0. Let $(r, \theta, \phi)$ be the spherical coordinates centered at 0, so $r = |x|$ and $\theta = \operatorname{Arc}\cos(x_3/|x|)$. Define $x_d = (0, 0, d)$ and let $(s, \omega, \phi)$ be the spherical coordinates, centered at $x_d$; so $s = |x - x_d|$ and

$$\omega = \operatorname{Arc}\cos((x - x_d)_3 / |x - x_d|).$$

Also needed are the cut-off functions $\beta_R(x) = \beta(x/R)$, and $\bar{\beta}_R(x) = \beta((x - x_d)/R)$, where $\beta(x)$ is given in Eq. (I.3.4).

The loop $a(t)$ is presented by giving the following data: (1) An open cover $\mathbb{R}^3 = \bigcup_{\alpha=1}^4 V_\alpha$, (2) Transition functions $g_{\alpha\beta} \in C^\infty(S \times (V_\alpha \cap V_\beta); \mathrm{SU}(2))$, which satisfy the appropriate cocycle conditions, (3) Configurations $a_\alpha \in C^\infty(S; \Gamma(A) \oplus \Gamma(g))_{V_\alpha}|$ which satisfy $a_\alpha = g_{\alpha\beta} \cdot a_\beta$ in $S \times (V_\alpha \cap V_\beta)$.

*Definition 2.3.* The loop $a(t) = (A(t), \Phi(t))$: For $t \in S$, $R > 2$, and $d \geqq 8R$, define $a(t) = a(t; R, d)$ as follows:

I) The open cover $\{V_\alpha = V_\alpha(R)\}$:

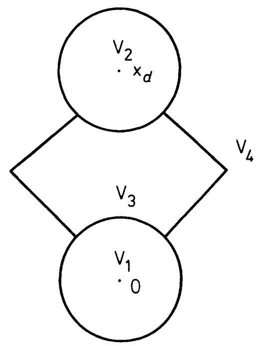
$$V_1 = \{x \in \mathbb{R}^3 : |x| < 2R\},$$

$$V_2 = \{x \in \mathbb{R}^3 : |x - x_d| < 2R\},$$

$$V_3 = \left\{x \in \mathbb{R}^3 : (|x| > R) \text{ and } (|x - x_d| > R) \text{ and } \left(\theta < \frac{\pi}{4}\right) \text{ and } (\omega > 3\pi/4)\right\},$$

$$V_4 = \left\{x \in \mathbb{R}^3 : (|x| > R) \text{ and } (|x - x_d| > R) \text{ and either } (\theta > \pi/8) \text{ or } \left(\omega < \frac{7\pi}{8}\right)\right\}. \tag{2.4a}$$

The cover is drawn schematically in the plane $x_2=0$ below:




(2.4b)

II) The transition functions: In

$$S\times(V_4\cap V_1):\ g_{41}(t)=\cos\theta/2(\cos\phi+\sin\phi\sigma^3)+\sin\theta/2(\cos t\sigma^2-\sin t\sigma^1),$$

$$S\times(V_4\cap V_2):\ g_{42}(t)=\sin\omega/2(\cos\phi+\sin\phi\sigma^3)+\cos\omega/2\sigma^2,$$

$$S\times(V_4\cap V_3):\ g_{43}(t)=\cos\phi+\sin\phi\sigma^3,$$

$$S\times(V_1\cap V_3):\ g_{13}(t)=\cos\theta/2-\sin\theta/2(\cos t\hat{\sigma}^2-\sin t\hat{\sigma}^1),$$

$$S\times(V_2\cap V_3):\ g_{23}(t)=\sin\omega/2-\cos\omega/2\hat{\sigma}^2,$$

where

$$\hat{\sigma}^1=\cos\phi\sigma^1+\sin\phi\sigma^2 \quad \text{and} \quad \hat{\sigma}^2=\cos\phi\sigma^2-\sin\phi\sigma^1. \tag{2.5}$$

III) The configurations: Define first

$$h=(2-\coth d+1/d)^{-1}[(1-\bar{\beta}_1)(\coth r-1/r)+(1-\beta_1)(\coth s-1/s)$$
$$-(1-\beta_1)(1-\bar{\beta}_1)(\coth d-1/d). \tag{2.6}$$

Let $a_\alpha(t)=(A_\alpha(t),\Phi_\alpha(t))$. Then in

$$S\times V_4:\ (A_4(t),\Phi_4(t))=\left(\frac{1}{2}(\cos\omega-\cos\theta)\sigma^3 d\chi,\frac{1}{2}h\sigma^3\right). \tag{2.7a}$$

$$S\times V_3:\ (A_3(t),\Phi_3(t))=\left(\frac{1}{2}(2+\cos\omega-\cos\theta)\sigma^3 d\chi,\frac{1}{2}h\sigma^3\right). \tag{2.7b}$$

$$S\times V_2:\ \Phi_2(t)=\frac{1}{2}h(-\cos\omega\sigma^3+\sin\omega\hat{\sigma}^1),$$

$$A_2(t)=\frac{1}{2}(1-\bar{\beta}_1)(1-\cos\theta)[-\cos\omega\sigma^3+\sin\omega\hat{\sigma}^2]d\phi$$
$$+\frac{1}{2}\left(1-\bar{\beta}_R\frac{s}{\sinh s}\right)[\sin^2\omega\sigma^3 d\phi-\hat{\sigma}^2 d\omega+\sin\omega\cos\omega\hat{\sigma}^1 d\phi]. \tag{2.7c}$$

$$S\times V_1:\ \Phi_1(t)=\frac{1}{2}h[\cos\theta\sigma^3+\sin\theta(\cos t\hat{\sigma}^1+\sin t\hat{\sigma}^2)],$$

$$A_1(t)=\frac{1}{2}(1-\beta_1)(1+\cos\omega)[\cos\theta\sigma^3+\sin\theta(\cos t\hat{\sigma}^1+\sin t\hat{\sigma}^2)]d\phi$$

$$+\frac{1}{2}\left(1-\beta_R\frac{r}{\sinh r}\right)[\sin^2\theta\sigma^3 d\phi+(\cos t\hat{\sigma}^2-\sin t\hat{\sigma}^1)d\theta$$

$$-\sin\theta\cos\theta(\cos t\hat{\sigma}^1+\sin t\hat{\sigma}^2)d\phi]. \tag{2.7d}$$

Loosely speaking $a(t)$ represents the Prasad-Sommerfield [2] $k=-1$ solution centered at $x_d$, and with a $t$-dependent rotation, the $k=1$ solution centered at $x_0$ [see Eq. (3.13)].

**Proposition 2.4.** *Let $\{a_\alpha(t), g_{\alpha\beta}(t), V_\alpha\}_{\alpha,\beta=1}^4$ be given by Definition 2.3. This data is smooth in the domain of definition. There exist gauge transformations*

$$\{f_\alpha(t)\in C^\infty(S\times V_\alpha;\mathrm{SU}(2))\}_{\alpha=1}^4$$

*such that:* (1) *$f_\alpha(t)a_\alpha(t)$ is smooth in $S\times V_\alpha$.* (2) *In $S\times(V_\alpha\cap V_\beta)$, $f_\alpha a_\alpha=f_\beta a_\beta$.* (3) *The loop $a(t)$, defined to be $f_\alpha(t)a_\alpha(t)$ on $S\times V_\alpha$ is in $C^0(S,\mathscr{C}_0)$.* (4) *With $a(t)=(A(t),\Phi(t))$ and $e(t)$ given by Eq.* (9.1), *$\lim_{|x|\to\infty}\Phi(t;x)\to e(t;\hat{x})$ uniformly in $t$ for $t\in[0,2\pi]$.*

*Proof of Proposition 2.4.* It is left to the reader to verify the cocycle conditions for $(a_\alpha, g_{\alpha\beta})$. As for the smoothness, consider first $a_4$. The only possible trouble is on the set $(\{\omega=\pi\}\cap\{\omega=0\})\cup\{x_d,0\}$, and this set does not intersect $V_4$. For the same reason the transition functions $g_{41}$, $g_{42}$, and $g_{43}$ are smooth. Next examine $a_3$. The only question arises near the $x_3$-axis, where $\cos\omega=-1+0(|\omega-\pi|^2)$ and $\cos\theta=1+0(|\theta|^2)$. Thus $A_3(t)=0$ on the $x_3$-axis, and is smooth there. Also, in $V_3$, $\sin\theta/2=0+0(|\theta|)$ and $\cos\omega/2=0+0(|\pi-\omega|)$ so both $g_{13}$ and $g_{23}$ are smooth as well. Consider $a_2$ in $S\times V_2$. The function $h$ is $0(|x-x_d|^2)$ as $x\to x_d$, so $\Phi_2(t)$ is smooth. As for $A_2(t)$, the first bracket is smooth as $(1-\cos\theta)$ is $0(|\theta|^2)$ near the ray $\theta=0$ and near $x_d$, $(1-\bar{\beta}_1)\equiv 0$. The second term, aside from the factor $\bar{\beta}_R$ which plays no role here, is the smooth Prasad-Sommerfield solution ([3], IV.1.15 and Eq. (3.13)) in spherical coordinates. The smoothness is guaranteed by the fact that $(1-s/\sinh s)$ is $0(s^2)$ as $s\to 0$. The analysis of $a_1(t)$ in $S\times V_1$ is similar.

The data $\{S\times V_\alpha; g_{\alpha\beta}\}$ defines a $C^\infty$ principal SU(2) bundle over $S\times\mathbb{R}^3$. Every such bundle is $C^\infty$ isomorphic to $S\times\mathbb{R}^3\times\mathrm{SU}(2)$. This implies the existence of $f_\alpha$'s satisfying statements (1) and (2) of Proposition 2.4.

Note that $|\Phi(t)|=h$ is gauge invariant, and the $\lim_{|x|\to\infty} h=1$. Therefore, Eq. (I.2.3) is satisfied uniformly with $t\in S$. As for the action, $\mathscr{A}(a(t))$ is finite for each $t\in S$. This calculation is done in Proposition 3.2. As $\mathscr{A}(\cdot)$ is $\mathscr{G}$-invariant, one can conclude from (2.7a) that the $t$-dependence of $\mathscr{A}(a(t))$ is due to the variations of the fields over a bounded set. Therefore $\mathscr{A}(a(t))$ is a continuous function of $t$.

The conclusion is that $a(t)\in C^0(S;\mathscr{C})$ given the topology of Definition I.2.1, and so Statement (3) of Proposition 2.4 is satisfied.

*Proof of Statement 4 of Proposition 2.4.* The straightforward proof of this statement is to explicitly construct the $\{f_\alpha\}_{\alpha=1}^4$ of Statements (1)–(3) of Proposition 2.4.

Let $l(x)$, $\bar{l}(x)$ be smooth cut-off functions, $0 \leqq l, \bar{l} \leqq 1$ and (1) $l=1$ ($\bar{l}=1$) if $x_3 < d/4$ $\left(x_3 > \frac{3d}{4}\right)$, (2) $l=0$ ($\bar{l}=0$) if $x_3 > d/2$ ($x_3 < d/2$). Schematically

$$\begin{array}{ccc} \bar{l}=1 & \cdot x_d & l=0 \\ \hline \\ \\ \hline \bar{l}=0 & \cdot 0 & l=1 \end{array} \tag{2.8}$$

In $S \times V_3$, define

$$\begin{aligned} q_3 = & [\sin(\pi/2 + \bar{l}(\omega/2 - \pi/2)) - \cos(\pi/2 + \bar{l}(\omega/2 - \pi/2))\hat{\sigma}^2] \\ & \cdot [\cos(l\theta/2) - \sin(l\theta/2)(\cos t\hat{\sigma}^2 - \sin t\hat{\sigma}^1)] . \end{aligned} \tag{2.9a}$$

For $\alpha \neq 3$, define on $S \times V_\alpha$ the matrix

$$q_\alpha \equiv 1 . \tag{2.9b}$$

The gauge transformation $q_3 \in C^\infty(S \times V_3; \mathrm{SU}(2))$. Indeed, the only questionable set is the $x_3$-axis. As

$$\cos(\pi/2 + \bar{l}(\omega/2 - \pi/2)) \sim \mathcal{O}(|\omega - \pi|)$$

near $\omega = \pi$ and $\sin(l\theta/2) \sim \mathcal{O}(|\theta|)$ near $\theta = 0$, there are no singularities.

Gauge transforming $a_\alpha$ by $q_\alpha$ on each $S \times V_\alpha$, $\alpha = 1, \ldots, 4$, one obtains

$$\bar{a}_\alpha(t) = q_\alpha(t) a_\alpha(t) , \tag{2.10}$$

and $\bar{a}_\alpha(t) \in C^\infty(S; \Gamma(A) \oplus \Gamma(\mathscr{g})_{V_\alpha}|)$. By construction,

$$q_3 = g_{13} \quad \text{on} \quad S \times (V_1 \cap V_3) \quad \text{and} \quad q_3 = g_{23} \quad \text{on} \quad S \times (V_2 \cap V_3) . \tag{2.11}$$

Due to (9.10, 11) and the cocycle conditions,

$$\bar{a}_1(t) = \bar{a}_3(t) \quad \text{on} \quad S \times (V_1 \cap V_3) \quad \text{and} \quad \bar{a}_2(t) = \bar{a}_3(t) \quad \text{on} \quad S \times (V_1 \cap V_2) . \tag{2.12}$$

Let $V = V_1 \cap V_2 \cap V_3$ and define

$$\bar{a}_5(t)(x) = \begin{cases} \bar{a}_1(t; x) & \text{for} \quad x \in V_1 , \\ \bar{a}_2(t; x) & \text{for} \quad x \in V_2 , \\ \bar{a}_3(t; x) & \text{for} \quad x \in V_3 . \end{cases} \tag{2.13}$$

It follows from (2.12) that $\bar{a}_5(t; x) \in C^0(S; \Gamma(A) \oplus \Gamma(\mathscr{g})|_{V_5})$.

The base manifold $\mathbb{R}^3 = V_4 \cup V_5$. The configuration $a(t)$ of Proposition 2.4 is represented by the data: $\{(S \times V_4, S \times V_5), g_{45}, (\bar{a}_4, \bar{a}_5)\}$, where on $S \times (V_4 \cap V_5)$,

$$g_{45} = (\cos\phi + \sin\phi\sigma^3) q_3^{-1} , \quad \text{and} \quad \bar{a}_4 = g_{45} \bar{a}_5 . \tag{2.14}$$

The fact that $g_{45} \in C^\infty(S \times (V_4 \cap V_5); \mathrm{SU}(2))$ is apparent if one knows that

$$\begin{aligned} & (1) \ (\cos\chi + \sin\chi\sigma^3)\hat{\sigma}^i = \sigma^i , \quad i = 1, 2 . \\ & (2) \ \cos(\pi/2 + \bar{l}(\omega/2 - \pi/2)) \text{ is } \mathcal{O}(|\omega - \pi|) \text{ near } \omega = \pi . \\ & (3) \ \sin(l\theta/2) \text{ is } \mathcal{O}(|\theta|) \text{ near } \theta = 0 . \end{aligned} \tag{2.15}$$

Consider the following gauge transformation in $C^\infty(S\times S^2;\mathrm{SU}(2))$:

$$\hat{f}_4(t;\hat{x}) = -(\cos^2\theta/2+\cos t\sin^2\theta/2)\sigma^2+\sin^2\theta/2\sin t\sigma^1 + 1/2\sin\theta[(1-\cos t)(\cos\phi-\sin\phi\sigma^3)-\sin t(\cos\phi\sigma^3+\sin\phi)]\,. \quad (2.16)$$

A short calculation reveals that

$$\tfrac{1}{2}\hat{f}_4\sigma^3\hat{f}_4^{-1} = e(t;\hat{x})\,, \quad (2.17)$$

where $e(t;\hat{x})$ is given in (9.1). The significance of $\hat{f}_4$ is given by

**Lemma 2.5.** *Statement* (4) *of Proposition 2.4 is true if there exists a smooth gauge transformation* $f_4\in C^\infty(S\times V;\mathrm{SU}(2))$ *which satisfies*

$$(1)\ \textit{For}\ x\in V_4\cap\{x\in\mathbb{R}^3:|x|<4d\}\,,\quad f_4(t;x)=g_{45}^{-1}(t;x)\,, \quad (2.18a)$$

$$(2)\ \textit{For}\ x\in\{x\in\mathbb{R}^3:|x|>16d\}\,,\quad f_4(t;x)=\hat{f}_4(t;\hat{x})\,. \quad (2.18b)$$

Proof of Lemma 2.5. Given such an $f_4$, set $a_4=f_4\bar{a}_4$ for $x\in V_4$, and $a_5=\bar{a}_5$ for $x\in V_5$. Because of (9.18a), one can define the loop $a(t)$ by

$$a(t)(x)=\begin{cases} a_4(t;x) & \text{for}\quad x\in V_4\,,\\ a_5(t;x) & \text{for}\quad x\in V_5\,,\end{cases}$$

and $a(t)\in C^0(S;\mathscr{C}_0)$. Using (2.7a) and (2.18b), one obtains that the

$$\lim_{|x|\to\infty}\Phi(t;x)=\lim_{|x|\to\infty}h(x)f_4(\tfrac{1}{2}\sigma^3)f_4^{-1}=e(t;\hat{x})\,.$$

Because $f_4(t;x)=\hat{f}_4(t;\hat{x})$ for $|x|>16d$, this limit is uniform in $t$. Therefore, Eq. (9.18) implies Statement (4) of Proposition 2.4, as claimed.

The construction of $f_4$ satisfying (9.18) is straightforward. Let $\beta(x)$ be the cut-off function of (I.3.4). The polar angle $\omega(x)$ is uniquely defined, for $|x|\geqq 4d$ by

$$\omega(x)=\operatorname{Arc}\cos\frac{x_3-d}{|x-x_d|}\,.$$

Define for $|x|\geqq 4d$

$$\tilde{\omega}(x)=\operatorname{Arc}\cos\frac{x_3-\beta(x/8d)d}{|x-\beta(x/8d)x_d|}\,. \quad (2.19)$$

Then $\tilde{\omega}$ is smooth in $V_4$ and if $|x|<4d$, $\tilde{\omega}=\omega$ and if $|x|\geqq 8d$, $\tilde{\omega}=\theta$.

Define smooth functions $\psi,\bar{\psi}$ on $V_4$ by

$$\psi(x)=\beta(x/16d)l\theta+[1-\beta(x/16d)]\theta\,,$$

$$\bar{\psi}(x)=\beta(x/16d)(\pi+\bar{l}(\tilde{\omega}(x)-\pi))+(1-\beta(x/16d))\theta\,. \quad (2.20)$$

The following facts are useful: In $V_4$,

(1) If $|x|>16d$, $\psi(x)=\bar{\psi}(x)=\theta$.

(2) If $|x|<8d$, $\psi(x)=l(x)\theta$, and $\bar{\psi}(x)=\pi+\bar{l}(\tilde{\omega}(x)-\pi)$.

(3) If $\theta=\omega=0$, $\psi(x)=\bar{\psi}(x)=0$ and both vanish as $0(|\theta|^2)$ as $\theta\to 0$.

(4) If $\theta=\omega=\pi$, $\psi(x)=\bar{\psi}(x)=\pi$ and both approach $\pi$ as $0(|\pi-\theta|^2)$ as $\theta\to\pi$. (2.21)

Consider the following map from $S \times V_4$ into SU(2):

$$f_4(t;x) = [\sin\tfrac{1}{2}\bar{\psi} - \cos\tfrac{1}{2}\bar{\psi}\hat{\sigma}^2] \cdot [\cos\tfrac{1}{2}\psi - \sin\tfrac{1}{2}\psi(\cos t\hat{\sigma}^2 - \sin t\hat{\sigma}^1)]\,[\cos\phi - \sin\phi\sigma^3]\,. \tag{2.22a}$$

By doing the matrix multiplication in (2.22a), one has

$$\begin{aligned} f_4(t;x) &= \sin\tfrac{1}{2}\bar{\psi}\cos\tfrac{1}{2}\psi(\cos\phi - \sin\phi\sigma^3) - \cos\tfrac{1}{2}\bar{\psi}\cos\tfrac{1}{2}\psi\sigma^2 \\ &\quad - \sin\tfrac{1}{2}\bar{\psi}\sin\tfrac{1}{2}\psi(\cos t\sigma^2 - \sin t\sigma^1) \\ &\quad + \cos\tfrac{1}{2}\bar{\psi}\sin\tfrac{1}{2}\psi(-\cos t(\cos\phi - \sin\phi\sigma^3) - \sin t(\cos\phi\sigma^3 + \sin\phi))\,. \end{aligned} \tag{2.22b}$$

The only points where $f_4$ is not clearly $C^\infty$ are those along the $x_3$-axis in $V_4$. But it follows from (3) and (4) of (2.21) that near the $x_3$-axis, $f_4$ behaves as

$$\begin{gathered} \tfrac{1}{2}\sin\theta(\cos\phi - \sin\phi\sigma^3) + \sigma^2 + \tfrac{1}{2}\sin\theta(-\cos t(\cos\phi - \sin\phi\sigma^3) \\ - \sin t(\cos\phi\sigma^3 + \sin\phi)) + \text{smooth terms}\,; \end{gathered}$$

and $f_4(t;x) \in C^\infty(S \times V_4; \mathrm{SU}(2))$. Because $\bar{\psi}$, $\psi = \theta$ for $|x| > 16d$, $f_4(t;x)$ as given in (2.22b) is equal to $\hat{f}_4$ of (2.17). Because $\psi = l\theta$ and $\bar{\psi} = \pi + \bar{l}(\omega - \pi)$ for $|x| < 4d$, $f_4(t,x)$ as given in (2.22a) is equal to $g_{45}^{-1}$ of (2.14). Appealing to Lemma 2.5 establishes Proposition 2.3.

## III. Action Estimates

The proof of Theorem 2.1 requires that the loop $a(t)$ of Definition 2.3 be modified. The changes make the behavior at large $|x|$ on $\mathbb{R}^3$ better without affecting the asymptotic limit of the Higgs field. The resulting loop, $b(t) \in C^0(S;\mathscr{C})$, has the following properties:

**Proposition 3.1.** *Let $c(e)(t) = (A_0(t), \Phi_0(t))$ be the configuration in Eq. (2.2). There exists a loop $b(t) = (A_0(t) + \omega(t), \Phi_0(t) + \eta(t)) \in C^0(S;\mathscr{C}_0)$ which satisfies (1) $\sup_{t \in S} \mathscr{a}(b(t)) < 8\pi$, (2) $(\omega(t), \eta(t)) \in \Gamma((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$ satisfy Statements (1)–(3) of Definition II.4.1 for $t \in [0, 2\pi]$.*

The proof of Statement (1) of Proposition 3.1 requires that $\mathscr{a}(a(t))$ be bounded by $8\pi$ also. The precise bound is given in the next proposition:

**Proposition 3.2.** *Let $a(t;R,d)$ be the loop of Definition 2.3. One can choose $R > 2$ and $d > 4R$ so that*

$$\sup_{t \in S} \mathscr{a}(a(t;R,d)) < 8\pi(1 - d^{-1} + d^{-3/2}) < 8\pi\,.$$

*Proof of Proposition 3.1, assuming Proposition 3.2.* Choose $R$ and $d$ so that $a(t;R,d)$ satisfies $\mathscr{a}(a(t)) < 8\pi - \delta$ for some $\delta > 0$ and all $t \in S$. Write $a(t) = (A(t), \Phi(t))$. Let $U = \{x \in \mathbb{R}^3 : |x| > 16d\}$. By construction [cf. Lemma 2.5 and Eq. (2.6)], when $x \in U$,

$$\Phi(t;x) - e(t;\hat{x}) = \tfrac{1}{2}(h-1)e(t;\hat{x})\,. \tag{3.1}$$

Define

$$\eta(t;x) = \Phi(t;x) - \Phi_0(t;x)\,, \tag{3.2}$$

with $\Phi_0$ given by (2.2). Then in $U$

$$|\eta(t;x)| \leqq \text{constant} \cdot |x|^{-1}, \tag{3.3}$$

and $\eta(t;x)$ satisfies Statement (2) of Definition I.4.2.

It follows from (3.1) that in $U$,

$$\Phi(t;x)/|\Phi(t;x)| = e(t;\hat{x}). \tag{3.4}$$

With $A_0(t)$ given by (2.2),

$$(\nabla_{A_0(t)} e(t))(x) = 0 \text{ in } U. \tag{3.5}$$

Thus, using (3.1), (3.2), and (3.5) one finds that

$$\nabla_{A_0(t)} \eta(t) = dh \cdot \Phi_0(t;x) \text{ in } U. \tag{3.6}$$

Computing $|dh|$, one obtains from (3.6) that

$$|\nabla_{A_0(t)} \eta(t) - \nabla_{A_0(t')} \eta(t')| \leqq z \cdot |t-t'|(1+|x|^2)^{-1},$$

and therefore Statement (3) of Definition I.4.1 is satisfied by $(A(t), \Phi(t))$.

The difference, $A(t) - A_0(t)$, does not satisfy Statement (1) of Definition I.4.1. Fortunately, $A(t)$ can be altered in such a way that the result satisfies both Statement (1) of Definition I.4.1, and the action estimate for $a(t)$, with $\delta/2$ replacing $\delta$. Using (2.7a) and Lemma 2.5, one finds that in $S \times U$, $\nabla_{A(t)} \Phi_0(t) = 0$. By construction, $\nabla_{A_0(t)} \Phi_0(t) = 0$ in $S \times U$ too, so

$$[A(t), \Phi_0(t)] = [A_0(t), \Phi_0(t)] \text{ in } U. \tag{3.7}$$

The conclusion is that $A^L(t) = A(t) - A_0(t)$ commutes with $\Phi_0(t)$.

The decay of $A(t)$ and $A^L(t)$ is estimated by the following device: Observe first that $a_4(t) = (A_4(t), \Phi_4(t))$ as given by (2.7a) satisfies

$$\begin{aligned} &|A_4(t)| \leqq \text{constant} \cdot |x|^{-1}, \\ &|\nabla A_4(t)| \leqq \text{constant} \cdot |x|^{-2} \quad \text{for} \quad t \in S \quad \text{and} \quad x \in U. \end{aligned} \tag{3.8}$$

The gauge transformation $f_4(t)$ of Lemma 2.5 is a function only of $t$ and the spherical angles $(\theta, \phi)$ in $S \times U$. As a consequence of this and (3.8),

$$\begin{aligned} &|A| = |f_4 A_4 f_4^{-1} + f_4 df_4^{-1}| \leqq \text{const} \cdot |x|^{-1}, \\ &|\nabla A| \leqq \text{const} \cdot |x|^{-2} \quad \text{for} \quad t \in S \quad \text{and} \quad x \in U. \end{aligned} \tag{3.9}$$

By construction, $A_0(t)$ satisfies the uniform bounds of Eq. (3.9) too. Therefore $A^L(t)$ also satisfies the bounds of Eq. (3.9).

Let $\varrho > 32d$. Define

$$B(t) = A(t) - (1 - \beta(x/\varrho))A^L(t), \tag{3.10}$$

where $\beta(x)$ is the cut-off function of (I.3.4). The connection $B(t)$ satisfies

$$\begin{aligned}
&(1)\ \omega(t)=B(t)-A_0(t)\in\Gamma^c(\{x\in\mathbb{R}^3:|x|<\varrho\};\mathscr{g}\otimes T^*),\\
&(2)\ \nabla_{B(t)}\Phi(t)=\nabla_{A(t)}\Phi(t),\\
&(3)\ \text{for } |x|<\varrho/2,\ F_{B(t)}=F_{A(t)},\\
&(4)\ \text{for } |x|>\varrho/2,\ |F_{B(t)}-F_{A(t)}|\leqq z\cdot|x|^{-2},\\
&\quad\ \text{where } z \text{ is independent of } t\in S \text{ and } \varrho.\\
&(5)\ \text{Let } b(t)=(B(t),\Phi(t)). \text{ Then } |\mathscr{a}(b(t))-\mathscr{a}(a(t))|\leqq z_1\cdot\varrho^{-1},\\
&\quad\ \text{where } z_1 \text{ is independent of } t\in S \text{ and } \varrho.
\end{aligned}\tag{3.11}$$

It follows from (3.11) that for $\varrho$ sufficiently large, Statements (1)–(3) of Definition I.4.1 hold for $b(t)=(B(t),\Phi(t))$ as does Proposition 3.1.

*Proof of Proposition 3.2.* The gauge invariance of $\mathscr{a}(\cdot)$ allows one to compute in any convenient gauge. Consider the following cover of $\mathbb{R}^3=U^3_{\alpha=1}B_\alpha$, where $B_1=\{|x|\leqq R\}$, $B_2=\{|x-x_d|\leqq R\}$, and $B_3=\mathbb{R}^3\backslash(B_1\cup B_2)$.

**Lemma 3.3.** *Let $a(t;R,d)=(A(t),\Phi(t))$ be as in Proposition 3.2. Then there exists $d_0<\infty$ such that if $d>d_0$ one can choose $R$ to make*

$$\tfrac{1}{2}\|\nabla_{A(t)}\Phi(t)\|_2^2<4\pi+\tfrac{1}{2}d^{-3/2}.\tag{3.12}$$

*Proof.* First recall that the Prasad-Sommerfield solution [2], $c_\pm\in\mathscr{C}_{\pm1}$, is given in spherical coordinates by

$$\begin{aligned}
\Phi_\pm&=\frac{1}{2}\left(\coth r-\frac{1}{r}\right)(\pm\cos\theta\sigma^3+\sin\theta(\cos t\hat\sigma^1+\sin t\hat\sigma)),\\
A_\pm&=\frac{1}{2}\left(1-\frac{r}{\sinh r}\right)[\sin^2\theta d\phi\sigma^3\pm(\cos t\hat\sigma^2-\sin t\hat\sigma^1)d\theta\\
&\quad\mp\sin\theta\cos\theta(\cos t\hat\sigma^1+\sin t\hat\sigma^2)d\phi].
\end{aligned}\tag{3.13}$$

Here $\pm$ refers to $k=\pm1$. Any $t\in[0,2\pi]$ is allowed as $c_\pm$ with two different values of $t$ are gauge equivalent. It is a fact that

$$\tfrac{1}{2}\|\nabla_{\Phi_\pm}A_\pm\|_2^2=2\pi.\tag{3.14}$$

Let $c_+(t)$ now denote (3.13) with the $+$ sign, centered at $x=0$, and $c_-$ denote (3.13) with the minus sign, at $t=0$, centered at $x=x_d$. So $c_-$ is (3.13) with the replacements $s\leftrightarrow r$ and $\omega\leftrightarrow\theta$. In what follows, assume $d\gg1$ *and for convenience, all constants independent of* $d$, $R>1$ are denoted by $\kappa$.

By (2.7a, b), the fields are abelian in $B_3$, and

$$\begin{aligned}
|\nabla_{A(t)}\Phi(t)|^2=|dh|^2&=(2-\coth d+1/d)^{-2}[|\nabla(\coth r-1/r)|^2+|\nabla(\coth s-1/s)|^2\\
&\quad+2(\nabla(\coth r-1/r),\nabla(\coth s-1/s))].
\end{aligned}\tag{3.15}$$

On the other hand, $\nabla_{A_+}\Phi_+$ is given explicitly in [3, IV.1.16], whence

$$|\nabla_{A_+}\Phi_+|^2\geqq|\nabla(\coth r-1/r)|^2-\frac{\kappa}{r^3}e^{-r}\ \text{in } B_3.\tag{3.16}$$

Together, (3.15) and (3.16) yield the estimate

$$|\nabla_{A(t)}\Phi(t)|^2 \leqq (2-\coth d+1/d)^{-2}(|\nabla_{A_+}\Phi_+|^2+|\nabla_{A_-}\Phi_-|^2 + 2(\nabla(\coth r-1/r), \nabla(\coth s-1/s))+\kappa r^{-3}e^{-r}). \tag{3.18}$$

In $B_1$, it is convenient to compute in the gauge specified in (2.7d). From (2.7d),

$$|A-A_+|\leqq \kappa(1/d^2+e^{-R}), \tag{3.19}$$

as $|(1+\cos\omega)d\psi|\leqq \kappa d^{-2}$ in $B_1$ and $(1-\beta_R)\frac{1}{\sinh r}(|rd\theta|+r\sin\theta|d\phi|)\leqq \kappa e^{-R}$ in $B_1$. Meanwhile in $B_1$,

$$\Phi=(2-\coth d+1/d)^{-1}\left[1+(1-\beta_1)\frac{(\coth s-\coth d-1/s+1/d)}{(\coth r-1/r)}\right]\Phi_+ . \tag{3.20}$$

The second term in the brackets above is bounded by $\kappa d^{-2}$ as is its derivative. Together (3.19, 20) imply that in $B_1$,

$$|\nabla_{A(t)}\Phi(t)|^2 \leqq (2-\coth d+1/d)^{-2}|\nabla_{A_+}\Phi_+|^2+\kappa\cdot(d^{-2}+e^{-R}). \tag{3.21}$$

A similar estimate with $c_+$ replaced by $c_-$ holds in $B_2$. Utilizing (3.21) and (3.18) one obtains by integrating that

$$\tfrac{1}{2}\|\nabla_{A(t)}\Phi(t)\|_2^2 \leqq (2-\coth d+1/d)^{-2}\Big[4\pi+\kappa R^3(e^{-R}+d^{-2}) + 2\int_{B_3}(\nabla(\coth r-1/r), \nabla(\coth s-1/s))\Big]. \tag{3.22}$$

Note that in $B_1$, $\coth r-1/r$ is smooth, while $|\nabla(\coth s-1/s)|\leqq \kappa\cdot d^{-2}$. The reciprocal is true in $B_2$. Therefore, at the expense of a new constant $\kappa$ in the second term in the brackets in (3.22),

$$\tfrac{1}{2}\|\nabla_{A(t)}\Phi(t)\|_2^2 \leqq (2-\coth d+1/d)^{-2}\Big[4\pi+\kappa R^3(e^{-R}+d^{-2}) + 2\int_{\mathbb{R}^3}(\nabla(\coth r-1-1/r), \nabla(\coth s-1/s)). \tag{3.23}$$

The extra $(-1)$ in the last term is to allow an integration by parts. Thus

$$\int(\nabla(\coth r-1-1/r), \nabla(\coth s-1/s)) \leqq \int|(\nabla(\coth r-1-1/r), \nabla\coth s)| - \int(\nabla(\coth r-1-1/r), \nabla(1/s)), \tag{3.24a}$$

$$\leqq \kappa\cdot d^{-2}+4\pi(1-\coth d+1/d). \tag{3.24b}$$

The contribution of the first term above is the $\kappa d^{-2}$. The second term in (3.24a) contributes $4\pi(1-\coth d+1/d)$ as $\frac{1}{4\pi}s^{-1}$ is the Green's function for the Laplacian on $\mathbb{R}^3$. Together (3.23, 4) imply that

$$\tfrac{1}{2}\|\nabla_{A(t)}\Phi(t)\|_2^2 \leqq (2-\coth d+1/d)^{-2}\left(4\pi\left(1+\frac{2}{d}\right)+\kappa R^3(e^{-R}+d^{-2})\right)$$
$$\leqq 4\pi+\kappa R^3(e^{-R}+d^{-2}). \tag{3.25}$$

Here the last line uses the fact that $(2-\coth d+1/d)^{-2}\leqq 1-2/d+\kappa d^{-2}$ for $d\gg 1$. By setting $R=\frac{1}{2}(1+\kappa)^{-1}d^{1/6}$ and then taking $d$ very large, one obtains Lemma 3.3.

**Lemma 3.4.** *Let* $a(t;R,d)=(A(t),\Phi(t))$ *be as in Proposition 3.2. Then there exists* $d_0<\infty$ *such that if* $d>d_0$, *one can choose* $R$ *to make* (3.12) *true, and in addition so that*

$$\tfrac{1}{2}\|F_{A(t)}\|_2^2<4\pi(1-2/d)+\tfrac{1}{2}d^{-3/2}.$$

*Proof.* As in the proof of Lemma 3.3, the sets $B_i$, $i=1,2,3$ are considered separately. Let $c_\pm$ be as before. Keep in mind that $F_{A_+}=*D_{A_+}\Phi_+$ and $F_{A_-}=-*D_{A_-}\Phi_-$.

Using (2.7a, b) one computes $F_{A(t)}$ in $B_3$ to be

$$|F_{A(t)}|^2=|(\sin\theta d\theta\wedge d\phi-\sin\omega d\omega\wedge d\phi)|^2,$$
$$|F_{A(t)}|^2=\left|\nabla\left(\frac{1}{r}\right)\right|^2+\left|\nabla\left(\frac{1}{s}\right)\right|^2-2\left(\nabla\left(\frac{1}{r}\right),\nabla\left(\frac{1}{s}\right)\right). \tag{3.26}$$

Comparing (3.26) with (3.15) and (3.18) one observes that in $B_3$,

$$|F_{A(t)}|^2\leqq|\nabla_{A_+}\Phi_+|^2+|\nabla_{A_-}\Phi_-|^2-2\left(\nabla\left(\frac{1}{r}\right),\nabla\left(\frac{1}{s}\right)\right)+\kappa\cdot e^{-r}. \tag{3.27}$$

In deriving (3.27), use has been made of the fact that in $B_3$, $|\nabla\coth r|\leqq\kappa\cdot e^{-r}$. Notice that in (3.26, 7) there is a minus sign in the cross term between $r$ and $s$, while in (3.15, 18) there is a plus sign.

In $B_1$, it is again convenient to compute in the gauge specified in (2.7d). One finds that $|A-A_+|$ satisfies in addition to (3.19), the bound

$$|\nabla(A-A_+)|\leqq\kappa(d^{-2}+e^{-R}). \tag{3.28}$$

The proof of (3.28) is straightforward. The following example indicates the manipulations that are involved:

$$(1-\beta_1)|\nabla(1+\cos\omega)d\phi|\leqq(1-\beta_1)(|\sin\omega d\phi\nabla\omega|+|(1+\cos\omega)\nabla d\phi|)\leqq\kappa d^{-2},$$

as $\sin\omega=0\left(\frac{(x_1^2+x_2^2)^{1/2}}{d}\right)$, $1+\cos\omega$ is $0\left(\frac{x_1^2+x_2^2}{d^2}\right)$ and $\nabla\omega$ is $0(d^{-1})$ in $B_1$. Using (3.28) one obtains in $B_1$ that

$$F_{A(t)}=F_{A_+}+D_{A_+}(A(t)-A_+)+(A(t)-A_+)\wedge(A(t)-A_+),$$

so

$$|F_{A(t)}|^2\leqq|F_{A_+}|^2+\kappa(d^{-2}+e^{-R})\text{ in }B_1. \tag{3.29}$$

A similar estimate holds in $B_2$ with $|F_{A_+}|^2$ replaced by $|F_{A_-}|^2$. Integrating (3.27, 9) over their respective domains, one finds that

$$\frac{1}{2}\|F_{A(t)}\|\leqq4\pi+\kappa R^3(d^{-2}+e^{-R})-2\int_{B_3}\left(\nabla\frac{1}{r},\nabla\frac{1}{s}\right). \tag{3.30}$$

The last term in (3.30) is crucial. Note that the integration can be extended to all of $\mathbb{R}^3$ at the expense of a new constant $\kappa$ in the second term. Using the fact that

$$\int_{B^3}\left(\nabla\frac{1}{s},\nabla\frac{1}{r}\right)=4\pi d^{-1}, \tag{3.31}$$

one obtains the estimate

$$\tfrac{1}{2}\|F_{A(t)}\|_2^2 \leqq 4\pi(1-2/d)+\kappa R^3(d^{-2}+e^{-R}). \tag{3.32}$$

Now choose $R=\frac{1}{2}(\kappa+1)^{-1}d^{1/6}$ and take $d$ very large to obtain Lemma 10.4. Together, Lemma 3.3 and 3.4 establish Proposition 3.2.

## IV. The Subset of $\mathscr{C}$ with $\mathscr{a}<8\pi$

The loop $b(t)$ of Proposition 3.1 is homotopically non-trivial with respect to the fixed basepoint $b(0)=b(2\pi)$. In order to complete the proof of Theorem 2.1, it must be established that $b(0)$ and $c_*$ are connected by a curve $d(t)$. The curve $d(t)$ must be sufficiently well-behaved asymptotically so that up to a $t$-independent gauge transformation, $(d^{-1}\circ b\circ d)(t)\in A$. In addition, $d(t)$ must obey the bound $\mathscr{a}(d(t))<8\pi$ for all $t$. The curve $d(t)$ will be an element of the following set:

*Definition 4.1.* The set $\mathscr{E}$ is defined to be $\mathscr{E}=\{c(t)=(A(t),\Phi(t))\in C^0([0,\pi];\mathscr{C}_0)$:
(1) There exists a compact set $K\subset\mathbb{R}^3$ such that $\operatorname{supp}|A(t)|\subset K$, for $t\in[0,\pi]$.
(2) $\Phi(t)=-\frac{1}{2}\sigma^3+\eta(t)$ and $\lim_{|x|\to\infty}|\eta(t)|(x)\to 0$, uniformly with $t\in[0,\pi]$.
(3) $d\eta(t)\in C^0([0,\pi],L_2(\mathscr{g}\otimes T^*))$.
(4) $c(\pi)=g(0,-1/2\sigma^3)$ where $g\in\mathscr{G}$ and $|g-1|$ has compact support in $\mathbb{R}^3$.$\}$

The existence of the curve $d(t)$, alluded to above follows from the first proposition.

**Proposition 4.2.** *Let* $c=(A,\Phi)\in\mathscr{C}_0$. *Suppose that* $c$ *satisfies* (1) $\mathscr{a}(c)<8\pi$, (2) $A\in\Gamma^c(\mathscr{g}\otimes T^*)$, (3) $\Phi=-1/2\sigma^3+\eta$ *and* $\lim_{|x|\to\infty}|\eta|(x)\to 0$, (4) $d\eta\in L_2(\mathscr{g}\otimes T^*)$. *Then there exists a curve* $d(t)\in\mathscr{E}$ *with* (a) $d(0)=c$, (b) $\sup_{t\in[0,\pi]}\mathscr{a}(d(t))=\mathscr{a}(c)$.

The immediate corollary of Theorem 4.2 and Proposition 3.2 is Theorem 2.1. This is straightforward, as $b(t)$ from Proposition (3.2) is such that $b(0)=b(2\pi)$ satisfies the conditions of Proposition 4.2. From $b(t)$ and the curve $d(t)$ of Proposition 4.2 and Definition 4.1, one constructs the loop

$$c(t)=\begin{cases} g^{-1}d(\pi+t), & \text{for } t\in[-\pi,0];\\ g^{-1}b(t), & \text{for } t\in[0,2\pi];\\ g^{-1}d(3\pi-t), & \text{for } t\in[2\pi,3\pi],\end{cases}$$

which satisfies all of the requirements of Theorem 2.1.

The proof of Proposition 4.2 requires knowledge of the fact that local minima of $\mathscr{a}(\cdot)$ on $\mathscr{C}$ satisfy the Bogomol'nyi equations. An analogous theorem for the

Yang-Mills equations on $S^4$ was proved by Bourguignon, Lawson, and Simons [4]. For the purposes here, the term local minimum is defined as follows:

*Definition 4.3.* A solution $c \in \mathscr{C}$ to Eqs. (I.2.2, 3) is a local minimum of $\alpha$ if the hessian $\mathscr{H}_c$ on $\Gamma((g \otimes T^*) \oplus g)$ satisfies $\mathscr{H}_c(\cdot) \geqq 0$.

The existence of local minima of $\alpha$ is summarized by

**Theorem 4.4.** *Let $c \in \mathscr{C}_k$ be a solution to Eqs.* (I.2.2, 3). *Then c is a local minimum of $\alpha$ iff Eq.* (I.2.6) *is satisfied, whence $\alpha(c) = 4\pi|k|$.*

The proof of Theorem 4.4 is the subject of Sect. 5. Its validity will be assumed in this section.

*Proof of Proposition 4.2, assuming Theorem 4.4.* The intuition behind the proof is as follows: A minimizing sequence beginning with $c$ can be translated as in Theorem I.7.1 so that it converges to a solution (Theorem I.5.6). The solution must be a local minima; and since $\alpha < 8\pi$, the solution must be trivial (Theorem 4.4).

The proof of Proposition 4.2 begins with

*Definition 4.5.* The set $\mathscr{E}_c \subset \mathscr{E}$ is $\mathscr{E}_c = \{d(t) \in \mathscr{E} : d(0) = c$ and there exists $t_1 > 0$ such that $\alpha(d(t)) < \alpha(c)$ for $t \in (0, t_1)$.$\}$ For $d(t) \in \mathscr{E}_c$, define $t_d$ to be the smallest $t > 0$ such that $\alpha(d(t_d)) = \alpha(c)$. If no such $t$ exists, set $t_d = \infty$. Define the configuration $\bar{d} = d(t_0)$, where $\alpha(d(t_0)) < \alpha(d(t))$ for $t < t_0$, $\alpha(d(t_0)) \leqq \alpha(d(t))$ for $t_0 \leqq t < t_d$. Finally, define

$$\alpha_\infty = \inf_{d(t) \in \mathscr{E}_c} (\alpha(\bar{d})). \tag{4.1}$$

The next series of results establish the relevant properties of $\mathscr{E}_c$.

**Lemma 4.6.** *Let c be given by Proposition 4.2, and suppose that $\alpha(c) > 0$. Then $\mathscr{E}_c \neq \emptyset$.*

*Proof of Lemma 4.6.* Note first that the curve $d(t) = \frac{\pi - t}{\pi}(A, \eta) + (0, -1/2\sigma^3)$ connects $c$ to $c_*$ by a continuous path in $\mathscr{E}$. Now suppose that $\mathscr{E}_c$ were empty. Then using Proposition I.5.2, one concludes that for all $\psi \in \Gamma((g \otimes T^*) \oplus g)$,

$$\frac{d}{dt}\alpha(c + t\psi)|_{t=0} = 0, \quad \text{and} \quad \frac{d^2}{dt^2}\alpha(c + t\psi)|_{t=0} \geqq 0. \tag{4.2}$$

By Theorem 4.4, this last equation is true iff $\alpha(c) = 0$ which is a contradiction. Hence, the lemma is true.

**Lemma 4.7.** *Let c be given by Proposition 4.2. Then either $c = (A, \Phi)$ satisfies*

$$\nabla_A^2 \Phi = 0, \tag{4.3}$$

*or there exists $d(t) \in \mathscr{E}_c$ and $t_0 < t_d$ such that $d(t_0) = (A, \Phi_0)$ satisfies* (4.3).

*Proof of Lemma 4.7.* The lemma is a direct consequence of Propositions I.4.8 and I.4.14 and the fact that the Higgs part of the action is strictly convex.

Crucial to the proof of Proposition 4.2 is the following *apriori* knowledge of $\alpha_\infty$:

**Proposition 4.8.** *Let c be given by Proposition 4.2. and* $\mathscr{a}_\infty$ *by Definition 4.5. Then* $\mathscr{a}_\infty = 0$.

The proof of Proposition 4.8 is deferred until the end of this section. Continuing with the proof of Proposition 4.2, it follows from Lemma 4.7 that there is no loss of generality to assume that the configuration $c$ satisfies (4.3) *apriori*. It is henceforth also assumed that $\mathscr{a}(c) > 0$.

It follows from Theorem I.4.5 and Lemma 4.6 that under these conditions, the following set is nonempty:

$$\bar{\mathscr{E}}_c = \{d(t) \in \mathscr{E}_c : d(t) \text{ satisfies (4.3) for all } t \in [0, \pi]\}. \tag{4.4}$$

It is also a consequence of Theorem I.4.5 that

$$\mathscr{a}_\infty = \inf_{d(t) \in \bar{\mathscr{E}}_c} \mathscr{a}(\bar{d}). \tag{4.5}$$

The curve $d(t) \in \mathscr{E}_c$ required by Proposition 4.2 is constructed in a two-step procedure. Assuming that $\mathscr{a}_\infty = 0$, it is established in Lemma 4.9 that given $\varepsilon > 0$, $c$ is connected by a path $d(t)$ (with $\mathscr{a}(d(t)) \leqq \mathscr{a}(c)$) to a configuration $d(t_2)$ with $\mathscr{a}(d(t_2)) < \varepsilon$. It is then proved that $d(t_2)$ is connected to a gauge transform of $c_*$ by a path with action less than $7\varepsilon$.

**Lemma 4.9.** *Let c be given by Proposition 11.2. Assume that c satisfies* (4.3), $\mathscr{a}(c) > 0$ *and that* $\mathscr{a}_\infty = 0$. *Given* $\varepsilon > 0$, *there exists* $d(t) = (A(t), \Phi(t)) \in \mathscr{E}_c$ *and* $t_2 \in (0, \pi)$ *such that*

(1) $\mathscr{a}(d(t)) < \mathscr{a}(c)$ *for* $t \in (0, t_2]$.
(2) $\mathscr{a}(d(t_2)) < \varepsilon$.
(3) $\eta(t_2) = \Phi(t_2) + \frac{1}{2}\sigma^3 \in \Gamma^c(\mathscr{g})$.
(4) $\| 1 - |\Phi(t_2)| \|_\infty < \varepsilon$.
(5) $\| \nabla_{A(t_2)} \Phi(t_2) \|_6 < \varepsilon$.

*Proof of Lemma 4.9.* Choose a sequence $\{d_i(t)\} \in \bar{\mathscr{E}}_c$ such that

(1) $\mathscr{a}(\bar{d}_i) \geqq \mathscr{a}(\bar{d}_{i+1})$,
(2) $\lim_{i \to \infty} \mathscr{a}(\bar{d}_i) = \mathscr{a}_\infty$. (4.6)

For $i$ sufficiently large, $d_i(t)$ with $t_2$ such that $\bar{d}_i = d_i(t_2)$ satisfies all but possibly Statement (3) of Lemma 4.9. In fact, Statement (4) above follows from Lemmas I.4.7 and I.7.4. Statement (5) is from Lemma I.4.7 also.

In order to exhibit a curve satisfying all of the requirements of Lemma 4.9, choose $d'(t) \in \bar{\mathscr{E}}_c$ to satisfy Statements (1), (2) and (4), (5) with $\varepsilon/2$ replacing $\varepsilon$. Let $t_1 \in (0, \pi)$ be such that $\bar{d}' = d'(t_1)$. Write $\bar{d}' = (A', \eta' - 1/2\sigma^3)$. As $\nabla_{A'}\sigma^3$ has compact support, it is a consequence of Propositions I.4.8 and I.4.14 that $\eta' \in K_{A'}$ and $\nabla_{A'}\eta' \in L_6$. The set

$$\bar{K} = \{\phi \in K_{A'} : \nabla_{A'}\phi \in L_6\} \tag{4.7}$$

is a Banach space with the norm

$$\|\phi\|_{\bar{K}}^2 = \|\nabla_{A'}\phi\|_2^2 + \|\nabla_{A'}\phi\|_6^2. \tag{4.8}$$

By construction, $\Gamma^c(\mathscr{g})$ is dense in $\bar{K}$. Using Lemma I.4.10 and the fact that $L_6^1(\mathbb{R}^3) \to C^0(\bar{\mathbb{R}}^3)$ [25], one infers that $\bar{K} \to C^0(\bar{\mathbb{R}}^3; \mathscr{g})$.

As $\Gamma^c(\mathscr{g})$ is dense in $\bar{K}$, one can choose $\eta\in\Gamma^c(\mathscr{g})$ sufficiently close to $\eta'$ in the norm (4.8) so that the following curve satisfies all of the requirements of Lemma 4.9:

$$d(t)=\begin{cases} d'(t), & \text{for} \quad t\in[0,t_1]. \\ \left(A',\eta'+2\dfrac{(t-t_1)}{(\pi+t_1)}(\eta-\eta')-1/2\sigma^3\right), & \text{for} \quad t\in\left[t_1,\dfrac{\pi+t_1}{2}\right]; \\ \dfrac{2(\pi-t)}{(\pi-t_1)}(A',\eta)+(0,-1/2\sigma^3) & \text{for} \quad t\in\left[\dfrac{\pi}{2}+t_1,\pi\right]. \end{cases} \tag{4.9}$$

For fixed $\varepsilon\ll 1/2$, let $d(t)\in\mathscr{E}_c$ satisfy Statements (1)–(5) of Lemma 4.9. Let $t_2\in(0,\pi)$ be fixed by Lemma 4.9 and denote $d(t_2)$ by $\bar{d}=(\bar{A},\bar{\Phi})$. By assumption, $|\bar{\Phi}|$ never vanishes, and there exists $R<\infty$ such that

$$\bar{\Phi}(x)=-\tfrac{1}{2}\sigma^3 \quad \text{for} \quad |x|>R. \tag{4.10}$$

As a function of $s\in[0,1]$, $f(s;\hat{x})=\bar{\Phi}(sR\hat{x})/|\bar{\Phi}|(sR\hat{x})$ defines a homotopy between the map $e_*\in\mathrm{Maps}(S^2;S^2)$ and the constant map $f(0,\hat{x}):S^2\to\bar{\Phi}(0)/|\bar{\Phi}|(0)\in S^2$. The homotopy lifting property of the fibration [5, Ch. 2] $0\to S^1\to \mathrm{SU}(2)\to S^2\to 0$ implies that there is a $C^\infty$ lifting of $f(s;\hat{x})$ to SU(2). As $\Pi_2(S^1)=(0)$, there is no loss of generality to assume that there exists $g(x)\in C^\infty(\mathbb{R}^3;\mathrm{SU}(2))$ satisfying

(1) $g(x)=1$ if $|x|>R+1$,

(2) $g(x)\bar{\Phi}(x)g^{-1}(x)=-\tfrac{1}{2}\sigma^3|\bar{\Phi}|$. (4.11)

Let $c'=g(\bar{A},\bar{\Phi})$. As the next Lemma states, $c'$ is path connected to $c_*$ by a curve with small action. For convenience, write $c'=(A',\Phi')$.

**Lemma 4.10.** *Let $d(t)\in\mathscr{E}_c$, $t_2\in(0,\pi)$ and $d(t_2)=(\bar{A},\bar{\Phi})$ satisfy Lemma 4.9 with $\varepsilon$ sufficiently small. Let $g$ be the gauge transformation of* (4.11). *There is a curve $d'(t)\in\mathscr{E}$ satisfying* (1) $d'(0)=c'=g(\bar{A},\bar{\Phi})$, (2) $d'(\pi)=(0,-1/2\sigma^3)$, (3) $\mathscr{a}(d'(t))<7\varepsilon$ *for* $t\in[0,\pi]$.

*Proof of Lemma 4.10.* Let $u=1-|\bar{\Phi}|$, $A^L=\tfrac{1}{4}\sigma^3(\sigma^3,A')$ and $A^T=A'-A^L$. The proof of the Lemma requires the fact that $|\nabla_{A'}\Phi'|^2=|\nabla u|^2+2|\bar{\Phi}|^2|A^T|^2$, from which one concludes using Lemma 4.9 that

(1) $\|\nabla u\|_p<\varepsilon$,

(2) $\||A^T|\|_p<3\varepsilon$ for $p\in[2,6]$. (4.12)

The path $d'(t)$ required by the lemma has three segments. For $t\in[0,\pi/3]$, let

$$d'(t)=\left(A',-\frac{1}{2}\left(1-\frac{3}{\pi}(\pi/3-t)u\right)\sigma^3\right). \tag{4.13}$$

Then when $t\in[0,\pi/3]$,

$$\mathscr{a}(d'(t))<\varepsilon+\|A^T\|_2^2<4\varepsilon. \tag{4.14}$$

In addition, $d'\left(\frac{\pi}{3}\right)=(A',-1/2\sigma^3)$.

Next for $t\in[\pi/3, 2\pi/3]$, let

$$d'(t)=\left(A^L+\left(2-\frac{3t}{\pi}\right)A^T, -1/2\sigma^3\right). \tag{4.15}$$

Then when $t\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$,

$$\begin{aligned}\mathscr{a}(d'(t))=&\frac{1}{2}\left\|F_A^L+\left[\left(2-\frac{3t}{\pi}\right)^2-1\right]A^T\wedge A^T\right\|_2^2+\frac{1}{2}\left(2-\frac{3t}{\pi}\right)\|F_A^T\|_2^2\\&+\left(2-\frac{3t}{\pi}\right)^2\|A^T\|_2^2,\end{aligned} \tag{4.16}$$

where $F_A^L=\frac{1}{4}\sigma^3(\sigma^3, F_A)$ and $F_A^T=F_A-F_A^L$. Using (4.12), one obtains from (4.16) that $\mathscr{a}(d'(t))\leqq 6\varepsilon+9\varepsilon^2<7\varepsilon$ for $t\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$ and $\varepsilon$ sufficiently small. In addition, $d'\left(\frac{2\pi}{3}\right)=\left(A^L, -\frac{1}{2}\sigma^3\right)$. Finally, define $d'(t)$ for $t\in\left[\frac{2\pi}{3},\pi\right]$ by $d'(t)=\left(\left(3-\frac{3t}{\pi}\right)A^L, -\frac{1}{2}\sigma^3\right)$. Since the field are now abelian, $\mathscr{a}(d'(t))<7\varepsilon$, $t\in\left[\frac{2\pi}{3},\pi\right]$ as well. This completes the proof of Lemma 4.10.

Assuming that $\mathscr{a}_\infty=0$, the requirements of Proposition 4.2 are satisfied by

$$\begin{cases} d(t), & t\in[0,t_2]\\ g^{-1}d'\left(\frac{(t-t_2)\pi}{(\pi-t_2)}\right), & t\in[t_2,\pi],\end{cases}$$

where $d(t)$ is given by Lemma 4.9 and $d'(t)$ is given by Lemma 4.10.

The proof of Proposition 4.2 is completed by establishing that $\mathscr{a}_\infty=0$. This is the last topic in Sect. 4.

*Proof of Proposition 4.8.* The proof is by contradiction, so suppose that $\mathscr{a}_\infty>0$. Let $\{d_i(t)\}\in\bar{\mathscr{E}}_c$ be a sequence satisfying (4.6). Let $\nabla\mathscr{a}_i(\cdot)=\nabla\mathscr{a}_{\bar{d}_i}(\cdot)$ etc.

**Lemma 4.11.** *Let c be as in Proposition 4.2 and satisfy* (4.3). *Under the assumption that $\mathscr{a}_\infty>0$, a sequence $\{d_i(t)\}\in\bar{\mathscr{E}}_c$ which satisfies* (4.6) *also satisfies* $\lim_{i\to\infty}\|\nabla\mathscr{a}_i\|_*\to 0$, *where $\|\cdot\|_*$ is the norm on $H^*_{\bar{d}_i}$.*

*Proof of Lemma 4.11.* The proof of the lemma is by contradiction. Assume that Lemma 4.11 is false. Then there exist a sequence $\{d_i(t)\}$ satisfying (4.6) and $\lim_{i\to\infty}\|\nabla\mathscr{a}_i\|_*>\delta>0$. One can conclude that for each $i$, there exists $\psi_i\in\Gamma^c((\mathscr{g}\otimes T^*)\oplus\mathscr{g})$ with $\nabla\mathscr{a}_i(\psi_i)<-\delta$ and $\|\psi_i\|_i=1$.

Hence, for $t<1$, one observe, using Proposition I.5.2, that

$$\begin{aligned}\mathscr{a}(\bar{d}_i+t\psi_i)&\leqq\mathscr{a}(\bar{d}_i)-t\delta+\tfrac{1}{2}t^2\kappa,\\&\leqq\mathscr{a}(\bar{d}_i)-t\delta/2\quad\text{for}\quad t<s=\delta\kappa^{-1}.\end{aligned} \tag{4.17}$$

The constant $\kappa$ is independent of $i$ by Proposition I.5.2. For $j$ sufficiently large, Eq. (4.17) implies that $a\left(\bar{d}_j+\frac{s}{2}\psi_j\right)<a_\infty$. Let $t_0$ be such that $\bar{d}_j=d_j(t_0)$, and consider

$$d(t)=\begin{cases} d_j(t), & \text{for } t\in[0,t_0], \\ \bar{d}_j+\dfrac{(t-t_0)}{(\pi-t_0)}s\psi_j, & \text{for } t\in[t_0,\tfrac{1}{2}(\pi+t_0)], \\ \dfrac{2(\pi-t)}{(\pi-t_0)}\left(\bar{d}_j+\dfrac{s}{2}\psi_j+(0,\tfrac{1}{2}\sigma^3)\right)-(0,\tfrac{1}{2}\sigma^3), & \text{for } t\in[\tfrac{1}{2}(\pi+t_0),\pi]. \end{cases} \tag{4.18}$$

The curve $d(t)\in\mathscr{E}_c$ and $a(\bar{d})<a_\infty$. This is a contradiction; hence the lemma is true.

A consequence of Lemma 4.11 is that the sequence $\{\bar{d}_i\}$ converges to a solution to Eq. (I.2.2, 3):

**Lemma 4.12.** *Let $c$ be given by Proposition 4.2 and satisfy* (4.3). *Under the assumption that $a_\infty>0$, there exists a sequence $\{d_i(t)\}\in\bar{\mathscr{E}}_c$ such that the sequence $\{\bar{d}_i\}$ converges strongly in $L^1_{2,\mathrm{loc}}$ to $d\in\mathscr{C}_0$. The configuration $d$ is a solution to Eqs.* (I.2.2, 3) *and $a(d)>0$.*

*Proof of Lemma 4.12.* Let $\{d_i(t)\}\in\bar{\mathscr{E}}_c$ be a sequence which satisfies (4.6). It follows from Lemma 4.11, and Theorems I.5.6, I.7.1, and I.8.1 that there exists a sequence of points $\{x_i\}\in\mathbb{R}^3$ such that the translated sequence $\{T_{x_i}\bar{d}_i\}$ converges strongly in $L^1_{2,\mathrm{loc}}$ to $d\in\mathscr{C}_0$. In addition, $d$ is a solution to Eqs. (I.2.2, 3) with $a(d)>0$. As $\mathbb{R}^3$ is path connected, there is no loss in generality to assume that each $x_i=0$.

It is a consequence of Theorem 4.4 that the hessian, $\mathscr{H}_d(\cdot)$ cannot be non-negative definite on $\Gamma^c((g\otimes T^*)\oplus g)$. The implications of this fact will yield a contradiction to the assumption that $a_\infty\neq 0$. In order to establish the contradiction, the following proposition is required.

**Proposition 4.13.** *Let $\{c_i\}\in\mathscr{C}$ be a sequence which converges strongly in $L^1_{2,\mathrm{loc}}$ to $c=(A,\Phi)\in\mathscr{C}$. Let $\psi\in\Gamma^c((g\otimes T^*)\oplus g)$ and suppose that*

$$(1)\ \|\psi\|_c=1.$$
$$(2)\ \mathscr{H}_c(\psi)=E. \tag{4.19}$$

*Then given $\varepsilon>0$, there exists $i(\varepsilon)$ such that for each $i>i(\varepsilon)$, there exists $\psi_i\in\Gamma^c((g\otimes T^*)\oplus g)$ satisfying*

$$(1)\ \|\psi_i\|_{c_i}=1,$$
$$(2)\ |\mathscr{H}_{c_i}(\psi_i)-E|<\varepsilon. \tag{4.20}$$

The proof of Proposition 4.13 is deferred momentarily in order to complete the proof of Proposition 4.8.

*Proof of Proposition 4.8 assuming Proposition 4.13: Completion.* One is required to demonstrate a contradiction resulting from the assumption that $a_\infty\neq 0$. Let $\{d_i(t)\}$ and $\{\bar{d}_i\}$ be the sequences of Lemma 4.12. According to Proposition 4.13 and Theorem 4.4, there exists $E<0$ and $\psi_i\in\Gamma^c((g\otimes T^*)\oplus g)$ such that for all $i$ sufficiently large, $\{\bar{d}_i,\psi_i\}$ and $E$ satisfy (4.20) with $\varepsilon=|E|/2$. A consequence of

Proposition I.5.2 is that for $t<1$,

$$\mathscr{a}(\bar{d}_i+t\psi_i)\leqq\mathscr{a}(\bar{d}_i)+t\|\nabla\mathscr{a}_i\|_*-\frac{t^2}{4}E+2\kappa t^3. \tag{4.21}$$

Using $(d_i(t),\psi_i)$ for $i$ sufficiently large, a curve $d'(t)\in\mathscr{E}_c$ can be constructed, which, as a consequence of Lemma 4.11 and (4.21), satisfies

$$\mathscr{a}(\bar{d}')<\mathscr{a}_\infty. \tag{4.22}$$

The curve $d'(t)$ is analogous to the curve $d(t)$ of (4.18), and the details are left to the reader. Equation (4.22) exhibits the required contradiction. Therefore $\mathscr{a}_\infty=0$.

*Proof of Proposition 4.13.* The notation of Sect. I.5 will be used here. Let $\{V_\alpha\}_{\alpha=1}^N$ be the part of the open cover of $\mathbb{R}^3$ (given by Definition I.5.5) that covers $U=\mathrm{supp}|\psi|$. Let $\{\beta_\alpha\}_{\alpha=1}^N$ be a partition of unity, subordinate to $\{V_\alpha\}_{\alpha=1}^N$. Let $\{g_\alpha(i),h_\alpha:V_\alpha\to\mathrm{SU}(2)\}$ be the gauge transformation of Definition I.5.5. In $V_\alpha$, define $\psi_\alpha\equiv h_\alpha^{-1}\psi h_\alpha$. So in $V_\alpha\cap V_\beta$,

$$\psi_\alpha=g_{\alpha\beta}\psi_\beta g_{\alpha\beta}^{-1}, \tag{4.23}$$

where $g_{\alpha\beta}$ is given in Definition I.5.5. For, each $i$, define

$$\psi_i'=\sum_\alpha g_\alpha^{-1}(i)\beta_\alpha\psi_\alpha g_\alpha(i)\in L_2^2(U;(\mathscr{g}\otimes T^*)\oplus\mathscr{g}) \tag{4.24}$$

(compare with Lemma I.5.9).

Using the gauge invariance, one can estimate the contribution to $\mathscr{H}_i(\psi_i')$ from $V_\alpha$. Let $g_{\alpha\beta}(i)=g_\alpha(i)g_\beta^{-1}(i)$. In $V_\alpha\cap V_\beta$ define $(\beta_\beta\psi_\beta)_\alpha(i)\equiv g_{\alpha\beta}(i)\beta_\beta\psi_\beta g_{\alpha\beta}^{-1}(i)$, $(\beta_\alpha\psi_\alpha)_\alpha(i)=\beta_\alpha\psi_\alpha$, and in $V_\alpha$ define:

$$\Psi_\alpha(i)=(\omega_\alpha(i),\eta_\alpha(i))\equiv\sum_\beta(\beta_\beta\psi_\beta)_\alpha(i). \tag{4.25}$$

Then the contribution to $\mathscr{H}_i(\psi_i')$ from $V_\alpha$ is

$$\begin{aligned}\int_{V_\alpha}&\{|D_{A_\alpha(i)}\omega_\alpha(i)|^2+|[\Phi_\alpha(i),\omega_\alpha(i)]|^2+|\nabla_{A_\alpha(i)}\eta_\alpha(i)|^2\\&+2(\omega_\alpha(i)\wedge\omega_\alpha(i),F_{A_\alpha(i)})+2([\omega_\alpha(i),\eta_\alpha(i)],\nabla_{A_\alpha(i)}\Phi_\alpha(i))\\&+2([\omega_\alpha(i),\Phi_\alpha(i)],\nabla_{A_\alpha(i)}\eta_\alpha(i))\}.\end{aligned} \tag{4.26}$$

The point of this exercise is that by assumption, $(A_\alpha(i),\Phi_\alpha(i))$ converges strongly in $L_2^1(V_\alpha)$ to $(A_\alpha,\Phi_\alpha)$, while $\Psi_\alpha(i)$ converges strongly in $L_2^2(V_\alpha)$ to $\psi_\alpha$. This implies convergence of (4.26) over each $V_\alpha$, and since there are a finite number of them,

$$\lim_{i\to\infty}|\mathscr{H}_i(\psi_i')-\mathscr{H}_c(\psi)|\to0. \tag{4.27}$$

The contribution to the norm $\|\psi_i'\|_i$ over each $V_\alpha$ can be shown to converge by a similar argument, so

$$\lim_{i\to\infty}|\,\|\psi_i'\|_i-1|\to0. \tag{4.28}$$

Therefore, given $\varepsilon>0$, for all $i$ sufficiently large, $\psi_i=\psi_i'/\|\psi_i'\|_i$ satisfies (4.20), proving Proposition 4.13.

## V. Local Minima

The proof of Theorem 4.4 is presented in this section. The proof is adapted from the proof by Bourguignon, Lawson, and Simons that all solutions to the SU(2) Yang-Mills equations on a bundle $P \to S^4$ which are local minima are self or anti-self dual.

The fact that solutions to Eq. (I.2.6) in $\mathscr{C}_k$ are local minima, and satisfy $\mathscr{a}(c) = 4\pi|k|$ is proved in [3, Chap. IV; see also 6.] This is the "if" part of Theorem 4.4. The "only if" part of the Theorem follows Bourguignon, Lawson, and Simons. The noncompactness of $\mathbb{R}^3$ is a problem when trying to adapt their proof. This problem is circumvented with the aid of the following *apriori* estimates.

**Proposition 5.1.** (Taubes [3, Chap. IV]). *Let $(A, \Phi) \in \mathscr{C}$ be a smooth, finite action solution to* (I.2.2, 3). *Then $\nabla_A F_A$ and $\nabla_A D_A \Phi$ are square integrable and* (1) $(1+|x|^2)(|F_A|(x)+|\nabla_A \Phi|(x)) \leqq$ *constant*, (2) $(1+|x|)(1-|\Phi|(x)) \leqq$ *constant*.

Assume that $c \in \mathscr{C}_k$ is a local minimum. Given a vector $\zeta = (\zeta_0, \zeta_i)_{i=1}^3$ of unit length in $\mathbb{R}^4$, set $\bar{\zeta} = \zeta_i dx^i \in \Gamma(T^*)$, define

$$\omega_\zeta = *(\bar{\zeta} \wedge (*F_A + D_A \Phi)) + \zeta_0 (*F_A + D_A \Phi),$$

$$\eta_\zeta = -*(\bar{\zeta} \wedge (F_A + *D_A \Phi)), \quad \text{and} \quad \psi_\zeta = (\omega_\zeta, \eta_\zeta). \tag{5.1}$$

It follows from Proposition 5.1 that $\mathscr{H}_c(\psi_\zeta)$ is well defined. One finds after an explicit calculation that

$$\mathscr{H}_c(\psi_\zeta) = 2\langle \omega_\zeta \wedge \omega_\zeta, F_A - *D_A \Phi \rangle_2 + 2\langle [\eta_\zeta, \omega_\zeta], *F_A - D_A \Phi \rangle_2 . \tag{5.2}$$

By varying $\zeta$, $\mathscr{H}_c(\psi_\zeta)$ defines a quadratic functional on the unit sphere in $\mathbb{R}^4$. Let $F_A + *D_A \Phi = u_i dx^i$ and $F_A - *D_A \Phi = v_i dx^i$. Then

$$\int_{\mathbb{R}^4} \delta(|\zeta|^2 - 1) d^4\zeta \mathscr{H}_c(\psi_\zeta) = \tfrac{1}{3} \int_{\mathbb{R}^3} d^3x \{ \varepsilon^{ijk}([u_i, u_j], v_k) + \varepsilon^{ijk}\varepsilon^{iln}\varepsilon^{jlm}([u_n, u_m], v_k) - 2\varepsilon^{ijk}([u_i, u_j] v_k)\} = 0 . \tag{5.3}$$

Now $\psi_\zeta$ is not in $\Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$, however, let $\beta_R$ be a smooth cut-off function with $\beta_R = 1$ when $|x| < R$, $\beta_R = 0$ if $|x| > 2R$ and $|\nabla \beta_R|(x) \leqq 2 \cdot R^{-1}$; then $\beta_R \psi_\zeta \in \Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g})$ for all $R < \infty$. In addition, the asymptotic decay given by Proposition 5.1 implies that there exists $R_0$ such that

$$|\mathscr{H}_c(\beta_R \psi_\zeta) - \mathscr{H}_c(\psi_\zeta)| < R^{-1/2}, \quad \text{for} \quad R > R_0 \quad \text{and all} \quad \zeta \in S^3 . \tag{5.4}$$

Therefore, since $\mathscr{H}_c(\beta_R \psi_\zeta) \geqq 0$ for all $R < \infty$, Eq. (5.4) implies that $\mathscr{H}_c(\psi_\zeta) \geqq 0$ as well. But this fact and (5.3) require $\mathscr{H}_c(\psi_\zeta) \equiv 0$ for all $\zeta \in S^3$.

Recall that $\mathscr{H}_c(\cdot)$ is, by assumption, a real, positive semi-definite quadratic form. Hence, the polarization identity implies that

$$\mathscr{H}_c(\psi_\zeta + \psi') - \mathscr{H}_c(\psi') = 0, \quad \text{for all} \quad \psi' \in \Gamma^c((\mathscr{g} \otimes T^*) \oplus \mathscr{g}) . \tag{5.5}$$

When written out in long hand, Eq. (5.5) means that for all $\zeta \in S^3$, and $x \in \mathbb{R}^3$, both

$$\varepsilon^{ijk}[(\varepsilon^{jmn}\zeta_m u_n + \zeta_0 u_j), v_k] + [\zeta_l u_l, v_i] = 0 ,$$

$$[\varepsilon^{kln}\zeta_l u_n + \zeta_0 u_k, v_k] = 0 . \tag{5.6}$$

Thus, since $(\zeta_0, \zeta_i) \in S^3$ is arbitrary, Eq. (5.6) implies that

$$[u_i, v_k] = 0, \qquad i, k = 1, 2, 3. \tag{5.7}$$

Let $\mathscr{g}_x^{+(-)}$ denote the subalgebra's of $\mathscr{su}(2)$ generated by $l^i u_i(x)(l^i v_i(x))$ for $l = l^i \frac{\partial}{\partial x^i} \in T_x \mathbb{R}^3$. By Lemma (7.22) of [7], either $\mathscr{g}_x^+$ or $\mathscr{g}_x^-$ is abelian. Without loss of generality, suppose that $\mathscr{g}_x^+$ is non-abelian at $x = \mathbb{R}^3$. This is an open condition, so there is an open neighborhood $U$ of $x$ such that $[\mathscr{g}_x^+, \mathscr{g}_x^+] \neq 0$ for all $x \in U$ and hence $[\mathscr{g}_x^-, \mathscr{g}_x^-] = 0$ in $U$. The solution $(A, \Phi)$ is necessarily real analytic on $\mathbb{R}^3$ ([3], Theorem IV.1.3) so $[\mathscr{g}_x^-, \mathscr{g}_x^-] = 0$ for all $x \in \mathbb{R}^3$. Now $v$ satisfies a second order partial differential equation, derivable from (I.2.2) ([3], Proposition IV.9.2), which in this case is:

$$-\nabla_A^2 v^i + [\Phi, [v^i, \Phi]] = 0, \tag{5.8}$$

where $-\nabla_A^2 = -\sum_{i=1}^{2} \nabla_{A_i} \nabla_{A_i}$ is the trace Laplacian. Upon contracting (5.8) with $v^i$ in the $L_2$ inner-product and integrating by parts one obtains

$$\langle \nabla_A v, \nabla_A v \rangle_2 + \langle [v, \Phi][v, \Phi] \rangle_2 = 0. \tag{5.9}$$

The integration by parts is justified as $v \in L_2$. Using Lemma I.4.10, one sees that $\|v\|_6 = 0$, so $v = 0$. Therefore $c$ satisfies Eq. (I.2.6). The fact that $\mathscr{a}(c) = 4\pi |k|$ follows from Theorem IV.1.5 of [3].

## VI. Conclusion

The proof of Theorem 1.1 can now be completed. Indeed, the results of Sects. 1–5 are summarized by the following theorem of which Theorem 1.1 is a corollary.

**Theorem 6.1.** *There exists a sequence of loops* $\{c_i(t)\} \in C^0((S^1, n); (\mathscr{C}_0, c_*))$ *which are homotopically non-trivial and such that:* (1) *The induced sequence* $\{\bar{c}_i\}$, *defined so that* $\mathscr{a}(\bar{c}_i) = \sup_{t \in S^1} \mathscr{a}(c_i(t))$, *converges strongly in* $L^1_{2,\mathrm{loc}}$ *to* $c \in \mathscr{C}_0$. (2) *The configuration* $c$ *satisfies the Yang-Mills-Higgs equations,* (I.2.2, 3), *and* $\mathscr{a}(c) > 0$. (3) *The configuration* $c$ *does not satisfy the Bogomol'nyi equations,* (I.2.6), *and* $c$ *is not a local minimum of* $\mathscr{a}$ *on* $\mathscr{C}_0$.

*Proof of Theorem 6.1.* By Proposition I.5.3, one can choose a good sequence of loops, $\{b_i(t)\} \in \Lambda(e)$, where $e$ is given in Eq. (2.1). Such loops are not null-homotopic in $C^0((S^1, n); (\mathscr{C}_0, c_*))$ (Lemma 2.1 and Theorems I.3.4 and I.3.5). Theorem I.4.4 states that $\mathscr{a}_\infty > 0$. A consequence of Theorem I.7.1 is that there is a sequence $\{x_i\} \in \mathbb{R}^3$ such that the translated sequence, $\{T_{x_i} \bar{b}_i\} \in \mathscr{C}_0$, converges strongly in $L^1_{2,\mathrm{loc}}$ to $c \in \mathscr{C}$. In addition, $\mathscr{a}(c) > 0$ and $c$ is a solution to Eqs. (I.2.2, 3). As translation by a vector $x \in \mathbb{R}^3$, $T_x : \mathscr{C} \to \mathscr{C}$ is continuous, the translated sequence of loops

$$\{c_i(t)\} = \{T_{x_i} b_i(t)\} \in C^0((S^1, n); (\mathscr{C}_0, c_*))$$

as well. By Theorem 2.1, $\mathscr{a}_\infty < 8\pi$ and Theorem I.8.1 is applicable. The conclusion is that $c \in \mathscr{C}_0$. Therefore Statements (1, 2) of Theorem 6.1 have been established.

By Theorem 4.4, a solution in $\mathscr{C}_0$ to the Bogomol'nyi equations must have zero action, and since $\mathscr{a}(c) > 0$, $c$ cannot satisfy the Bogomol'nyi equations. It is also a consequence of Theorem 4.4 that $c$ can not be a local minimum of $\mathscr{a}(\cdot)$ on $\mathscr{C}_0$ (Definition 4.3).

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