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Some Results for SU(N) Gauge-Fields on the Hypertorus

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Abstract. We show how to prove and to understand the formula for the "Pontryagin" index P for SU(N) gauge fields on the Hypertorus T^4, seen as a four-dimensional euclidean box with twisted boundary conditions. These twists are defined as gauge invariant integers modulo N and labelled by $n_{\mu\nu}$ $(=-n_{\nu\mu})$. In terms of these we can write ($\nu \in \mathbb{Z}$)

$$P=\frac{1}{16\pi^2}\int \mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu})d_4x=\nu+\left(\frac{N-1}{N}\right)\cdot\frac{n_{\mu\nu}\tilde{n}_{\mu\nu}}{4}.$$

Furthermore we settle the last link in the proof of the existence of zero action solutions with all possible twists satisfying $\frac{n_{\mu\nu}\tilde{n}_{\mu\nu}}{4}=\kappa(n)=0(\mathrm{mod}\,N)$ for arbitrary N.

1. Introduction

A long standing problem is proving quark confinement in QCD [1]. To simplify the picture a first step in this direction would be to show confinement of static quarks. In this way the problem reduces to an understanding of the behaviour of electric flux strings in quarkless QCD, thus working in pure SU(N) gauge theories, where up to present energies $N=3$. Usually this boils down to studying the behaviour of the vacuum expectation value of the Wilson loop operator [1, 2].

But some time ago 't Hooft [3] introduced another elegant method for studying flux strings. By putting the gauge fields on a four dimensional euclidean box, one can imitate a quark source on one side and an antiquark source on the other side of the box by introducing socalled twisted boundary conditions. These boundary conditions force electric flux into the box, just as gauge invariance forces an electric flux string in-between a quark and antiquark source. Similarly one can introduce magnetic flux in the box, and the great strength of the method is its electric-magnetic duality properties.

Essential is that all fields transform trivially under the centre Z_N of SU(N). Effectively the gauge group is thus $SU(N)/Z_N$ and the twists are labelled by six

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integers $n_{\mu\nu}=-n_{\nu\mu}$ ($\mu=1$, 2, 3 or 4) defined modulo N. Here $n_{\mu\nu}$ describes a winding number in the (μ,ν)-plane, as an element of the first homotopy group $\pi_1(\mathrm{SU}(N)/Z_N)\cong Z_N$. The gauge fields are in fact connections for a $\mathrm{SU}(N)/Z_N$ – principal fiber bundle on the hypertorus T^4. This will be discussed in Sect. 2.

The topology of the gauge fields is not completely specified by the twist, but there is also the "2nd Chern class' related to instanton type configurations and given by (the coupling constant g is put equal to 1):

$$P=\frac{1}{16\pi^2}\int d_4x\,\mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu}), \tag{1.1}$$

$$\tilde{G}_{\mu\nu}=\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G_{\alpha\beta} \quad \text{(the dual of } G_{\mu\nu}). \tag{1.2}$$

In Sect. 3 we shall explain why P (or actually $2NP$) should really be called the 1st Pontryagin index. 't Hooft [4] used invariance arguments to show the following identity:

$$P=\nu+\left(\frac{N-1}{N}\right)\kappa(n), \tag{1.3}$$

$$\kappa(n)=\tfrac{1}{4}\tilde{n}_{\mu\nu}n_{\mu\nu}=\tfrac{1}{8}\varepsilon_{\mu\nu\alpha\beta}n_{\alpha\beta}n_{\mu\nu}\,. \tag{1.4}$$

This will be proved in Sect. 3, where we will resolve the interpretation of this formula, especially what happens if we add a multiple of N to $n_{\mu\nu}$.

For finding solutions to the euclidean equations of motion the importance of P comes from the Schwarz-inequality for the action $S(A)$:

$$S(A)=\tfrac{1}{2}\int \mathrm{Tr}(G_{\mu\nu}G_{\mu\nu})d_4x \geqq 8\pi^2|P|\,. \tag{1.5}$$

Recently 't Hooft [5] constructed a very special type of solutions with minimal nontrivial action.

In Sect. 4 we complete his proof for the existence of orthogonal twist ($\kappa(n)=0(\mathrm{mod}\,N)$) zero action configurations for *arbitrary* N.

In an appendix we prove a formula for the 1st Pontryagin (2nd Chern) index for a general four dimensional compact(ifiable) manifold in terms of the transition functions only.

Our primary aim in this paper is a precise understanding of the twist-dependence for the Pontryagin number. For this we need an explicit realization of the topological structure of the fiber bundles over T^4, yielding at the same time a classification of $\mathrm{SU}(N)/Z_N$ bundles over T^4. Classification of fiber bundles is well known to mathematicians, however even for this problem it is a non trivial exercise, using K-theory, as demonstrated by C. Nash in his preprint entitled: "Gauge potentials and bundles over the 4-torus" (St. Patrick's college, January 1982). We thank him for correspondence on this subject.

2. The Structure of the Gauge Fields on the Hypertorus

From the requirement of periodicity for gauge invariant quantities we have [5]: (A_λ is the gauge potential in the fundamental representation)

$$\begin{aligned} A_\lambda(x_\mu=a_\mu)&=[\Omega_\mu]A_\lambda(x_\mu=0)\\ &=\Omega_\mu A_\lambda(x_\mu=0)\Omega_\mu^{-1}-i\Omega_\mu\partial_\lambda\Omega_\mu^{-1}\,. \end{aligned} \tag{2.1}$$

The euclidean box is defined by $0 \leqq x_\mu \leqq a_\mu$, $\mu = 1,2,3,4$. When the argument of a function on the box is put equal to x_μ, we mean that x_μ is fixed and the other coordinates are arbitrary. $[\Omega_\mu]$ is the action of a SU(N) gauge transformation Ω_μ, independent of x_μ. All other fields ψ satisfy the same formal periodic boundary condition $\psi(x_\mu = a_\mu) = [\Omega_\mu]\psi(x_\mu = 0)$.

As long as all these fields transform trivially under the centre Z_N of SU(N) we have from the consistency of writing $\psi(x_\mu = a_\mu,\ x_\nu = a_\nu)$ in two ways in terms of $\psi(x_\mu = 0,\ x_\nu = 0)$ [through $\psi(x_\mu = a_\mu,\ x_\nu = 0)$ and $\psi(x_\mu = 0,\ x_\nu = a_\nu)$]:

$$Z_{\mu\nu} = \Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = 0)\Omega_\mu^{-1}(x_\nu = 0)\Omega_\nu^{-1}(x_\mu = a_\mu)\,, \tag{2.2}$$

where $Z_{\mu\nu}$ is an element of the centre Z_N of SU(N). We write $Z_{\mu\nu} \in Z_N$ in terms of the twist integers $n_{\mu\nu} \in \mathbb{Z}(\operatorname{mod} N)$:

$$Z_{\mu\nu} = \exp(2\pi i n_{\mu\nu}/N)\,. \tag{2.3}$$

Clearly $n_{\mu\nu} = -n_{\nu\mu}$, they are independent of the coordinates, and are gauge invariant, since under an arbitrary gauge transformation $\Omega(x)$ $(\psi' = [\Omega]\psi)$ we have:

$$\Omega'_\mu = \Omega(x_\mu = a_\mu)\Omega_\mu \Omega^{-1}(x_\mu = 0)\,. \tag{2.4}$$

The gauge functions are really defined modulo the centre of the gauge group, so the gauge group is actually SU$(N)/Z_N$. The multiple transition functions Ω_μ then take their values in SU$(N)/Z_N$. In this representation we have (2.2) with $Z_{\mu\nu} = 1$ [the identity in SU$(N)/Z_N$], which becomes a multiple cocycle condition. We borrowed the terminology from the theory of fiber bundles [6] and we will explain how the above structure defines a SU$(N)/Z_N$ principal fiber bundle on the hypertorus T^4 parametrized by: $0 \leqq x_\mu \leqq a_\mu$, $\mu = 1,2,3,4$; with $x_\mu = 0$ identified with $x_\mu = a_\mu$.

We will first discuss the case of the two dimensional torus T^2. We need a covering of T^2 with open sets U_i, however it is more advantageous to reduce the overlapping regions to a minimum area. This is done by taking the closure of U_i and reducing their size to a minimum, such that they still cover the manifold completely. We will denote such a covering by $\{U_i^c\}$. For T^n the minimal number of such sets is 2^n and in Fig. 1 we specify the situation for T^2. We can take $\delta = \varepsilon$ (δ and ε are defined in Fig. 1) but it should always be understood in the limit $\delta \downarrow \varepsilon$.

A fiber bundle is specified [6, 7] by the transition functions $\Omega_{ij} = \Omega_{ji}^{-1}$ on $U_i \cap U_j$ (in our case $U_i^c \cap U_j^c$), such that:

$$A_\mu^{(i)}(x) = [\Omega_{ij}(x)]A_\mu^{(j)}(x)\,, \qquad x \in U_i^c \cap U_j^c\,. \tag{2.5}$$

And consistency requires these transition functions to satisfy the cocycle condition:

$$\Omega_{ij}(x)\Omega_{jk}(x) = \Omega_{ik}(x)\,, \qquad x \in U_i^c \cap U_j^c \cap U_k^c\,. \tag{2.6}$$

In Fig. 1 the relevant transition functions for T^2 are indicated. Gauge transformations are specified by:

$$A_\mu^{(i)\prime} = [g_i]A_\mu^{(i)}\,. \tag{2.7}$$

So

$$\Omega'_{ij} = g_i \Omega_{ij} g_j^{-1}\,.$$

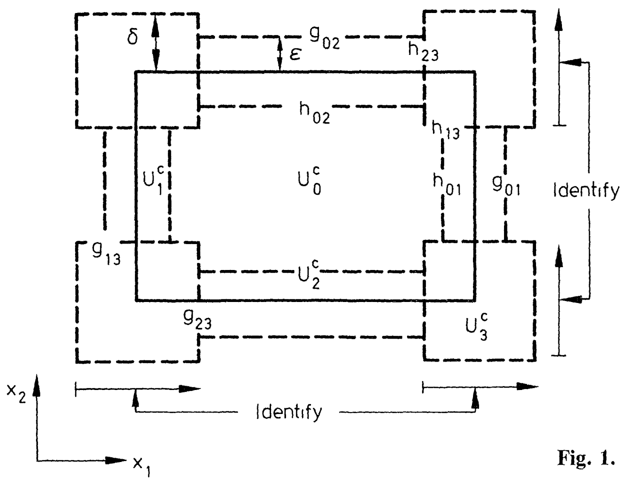

Fig. 1. The fiber bundle structure of T^2

It is now easy to see that when we take the limit $\varepsilon\to 0$, $\delta\to 0$:

$$\begin{aligned} A_\lambda^{(0)}(x_\mu=a_\mu) &= [h_{0\mu}]A_\lambda^{(\mu)} = [h_{0\mu}g_{0\mu}^{-1}][g_{0\mu}]A_\lambda^{(\mu)} \\ &= [h_{0\mu}g_{0\mu}^{-1}]A_\lambda^{(0)}(x_\mu=0). \end{aligned} \tag{2.8}$$

So we have the following expression for the multiple transition functions Ω_μ in terms of the ordinary transition functions:

$$\Omega_\mu = h_{0\mu}g_{0\mu}^{-1}. \tag{2.9}$$

Using the cocycle conditions (2.6) many times and the fact that h_{13}, h_{23}, g_{13}, g_{23} are independent of the coordinates in the limit $\varepsilon\to 0$, $\delta\to 0$, one derives the consistency condition (2.2) with $Z_{\mu\nu}=1$. On the other hand it is not difficult to show that given Ω_μ satisfying the consistency condition one can construct a fiber bundle structure in the above sense.

The gauge freedom in (2.7) now reduces to $g_0=\Omega$, g_μ independent of x_μ, such that:

$$\begin{aligned} g'_{0\mu} &= \Omega(x_\mu=0)g_{0\mu}g_\mu^{-1}, \\ h'_{0\mu} &= \Omega(x_\mu=a_\mu)h_{0\mu}g_\mu^{-1}. \end{aligned} \tag{2.10}$$

This gives with (2.9) the transformation property (2.4). Generalization of the above to T^4 is obvious (also the validity for general T^n and gauge group G with centre Z_G is obvious). We thus proved that 't Hooft's method of introducing $SU(N)$ gauge fields on the hypertorus transforming trivially under the centre Z_N defines a $SU(N)/Z_N$ principal fiber bundle structure on the hypertorus with connection A_μ (in local coordinates). This puts the theory in the right mathematical framework. We would however like to stress that this framework is discussed for the sake of completeness. The essential ingredient is the ansatz (2.9) and the accompanying gauge freedom (2.10), which can be made for $SU(N)/Z_N$ as well as $SU(N)$ multiple transition functions and without the assumption of an underlying bundle structure.

We can use the freedom in choosing either $g_{0\mu}$ or $h_{0\mu}$ to construct a closed loop in $SU(N)/Z_N$. With the aid of the consistency condition one easily checks that the following choice does the job:

$$g_{0\mu}(x_\nu=0)=1\,,\qquad g_{0\mu}(x_\nu=a_\nu)=\Omega_\nu(x_\mu=0)\,,\qquad (\mu\leftrightarrow\nu)\,. \tag{2.11}$$

The loop along the boundary of T^2 [the solid line in Fig. 1, where $(1,2)$ is identified with (μ,ν)] is mapped into a closed loop in $SU(N)/Z_N$. Its homotopy type as an element of $\pi_1(SU(N)/Z_N)$ is precisely $n_{\mu\nu}$, since we can pull up the loop to $SU(N)$ but then it jumps by an element of the centre Z_N. Having the same definition for $g_{0\mu}$ and $h_{0\mu}$ as $SU(N)$ functions this jump occurs at (a_μ,a_ν) and is exactly $Z_{\mu\nu}=\exp(2\pi i n_{\mu\nu}/N)$. (The correspondence of $n_{\mu\nu}$ with the homotopy type follows most trivially from the construction of $SU(N)$ as the universal covering group of $SU(N)/Z_N$ [8].)

3. The Pontryagin Number on T^4

The formula (1.1) is the usual form for (minus) the 2nd Chern number for a $SU(N)$ gauge theory[1]. However we saw that in general the gauge theory on T^4 does not define a $SU(N)$-principal fiber bundle structure and thus P need not be an integer. It is nevertheless integer, when there is no twist, since in Sect. 2 we showed that in that case we had a suitable fiber bundle structure. Note that this is consistent with (1.3).

By putting N cubes next to each other in the μ-direction each twist in the (μ,ν)-plane vanishes. From this one can derive that PN^3 is integer. Together with invariance arguments concerning the dependence of P on $n_{\mu\nu}$, the explicit calculation of P for a sample of configurations led 't Hooft [9] to a formula like (1.3).

Let us first put the terminology right. When $n_{\mu\nu}\neq 0$ we have transition functions in $SU(N)/Z_N$ but these are not elements of $GL(k,\mathbb{C})$ for any k, necessary [10] for using Chern classes. However the adjoint representation of $SU(N)$ is a faithful representation of $SU(N)/Z_N$. So provided we transform the gauge potentials to the adjoint representation, we can chose transition functions in $SO(N^2-1)\subset GL(N^2-1,\mathbb{R})$ and $P_{\rm ad}$ [equals (1.1) but with the gauge fields in the adjoint reprexentation] is just the integer 1st Pontryagin number.

If V_i $i=1,\ldots,N^2-1$ are the generators of $SU(N)$, $\mathrm{Tr}(V_iV_j)=2\delta_{ij}$, and L_i the generators of the adjoint representation, we can write $\mathrm{ad}(A_\mu)=-iA^a_\mu L_a$ if $A_\mu=A^a_\mu\frac{V_a}{2}$, from this one easily finds [11]:

$$P_{\rm ad}=\frac{\mathrm{Tr}(L_a^2)}{\mathrm{Tr}((iV_a/2)^2)}P=2NP\in\mathbb{Z}\,. \tag{3.1}$$

With (1.3) this would imply that $P_{\rm ad}$ misses the odd integers. This missing is obvious since the adjoint representation does not cover the whole of $GL(N^2-1,\mathbb{R})$.

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1 See the appendix for a proper definition

We will work alternatively in SU(N) or its adjoint representation, whichever is the most suitable. Translation from one to the other is straightforward.

The Pontryagin index is determined by the topology of the fiber bundle only, which is specified by the transition functions [6]. Therefore it should be a function of these transition functions only, which is generally proved by construction in the appendix. For T^4 we have:

Lemma (3.1).

$$P=\frac{1}{24\pi^2}\sum_{\mu}\int d_3\sigma_\mu\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}((\Omega_\mu\partial_\nu\Omega_\mu^{-1})(\Omega_\mu\partial_\alpha\Omega_\mu^{-1})(\Omega_\mu\partial_\beta\Omega_\mu^{-1}))$$

$$+\frac{1}{8\pi^2}\sum_{\mu,\nu}\int d_2S_{\mu\nu}\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}((\Omega_\nu^{-1}\partial_\alpha\Omega_\nu)_{x_\mu=a_\mu}(\Omega_\mu\partial_\beta\Omega_\mu^{-1})_{x_\nu=0})$$

with

$$\int d_3\sigma_1=\int_0^{a_2}dx_2\int_0^{a_3}dx_3\int_0^{a_4}dx_4\,,\qquad \int d_2S_{12}=\int_0^{a_3}dx_3\int_0^{a_4}dx_4$$

etc.

Proof. In the appendix it is shown how to extract this formula from the general case. Here we show it by a direct calculation.

$$P=\frac{1}{16\pi^2}\int d_4x\,\mathrm{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu})$$

$$=\frac{1}{16\pi^2}\sum_{\mu}\int d_3\sigma_\mu[K_\mu(x_\mu=a_\mu)-K_\mu(x_\mu=0)]$$

with

$$K_\mu=2\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(A_\nu\partial_\alpha A_\beta-\frac{2}{3i}A_\nu A_\alpha A_\beta\right). \tag{3.2}$$

Inserting the boundary condition (2.1) we find:

$$P=\frac{1}{24\pi^2}\sum_{\mu}\int d_3\sigma_\mu\varepsilon_{\mu\nu\alpha\beta}\left\{\frac{1}{3}\mathrm{Tr}((\Omega_\mu\partial_\nu\Omega_\mu^{-1})(\Omega_\mu\partial_\alpha\Omega_\mu^{-1})(\Omega_\mu\partial_\beta\Omega_\mu^{-1}))\right.$$

$$\left.+\partial_\nu\,\mathrm{Tr}\left(\frac{1}{i}(\Omega_\mu^{-1}\partial_\alpha\Omega_\mu)A_\beta(x_\mu=0)\right)\right\}. \tag{3.3}$$

We use the consistency condition (2.2) to calculate

$$\Omega_\nu^{-1}(x_\mu=0)(\Omega_\mu^{-1}\partial_\alpha\Omega_\mu)_{x_\nu=a_\nu}\Omega_\nu(x_\mu=0)\,,$$

which can be used together with the boundary condition (2.1) to show that:

$$\left[\mathrm{Tr}\left(\frac{1}{i}(\Omega_\mu^{-1}\partial_\alpha\Omega_\mu)_{x_\nu=a_\nu}A_\beta(x_\mu=0,x_\nu=a_\nu)\right)\right.$$

$$\left.-\mathrm{Tr}\left(\frac{1}{i}(\Omega_\mu^{-1}\partial_\alpha\Omega_\mu)_{x_\nu=0}A_\beta(x_\mu=x_\nu=0)\right)\right]$$

$$-[\mu\leftrightarrow\nu]=\mathrm{Tr}((\Omega_\nu^{-1}\partial_\alpha\Omega_\nu)_{x_\mu=a_\mu}(\Omega_\mu\partial_\beta\Omega_\mu^{-1})_{x_\nu=0})-(\mu\leftrightarrow\nu)\,.$$

Inserting this in (3.3) completes the proof. □

From this lemma one can easily compute P when the multiple transition functions are mutually commuting. By a suitable global gauge transformation they can be simultaneously diagonalized. Here P is gauge invariant so it is certainly invariant under a global transformation. The above configuration, which will be called abelian, can be chosen in $H=U(1)^{N-1}$ the maximal abelian (Cartan) subalgebra of SU(N) generated by:

$$T_a=\mathrm{diag}(1,1,\ldots,1,-N+a,0,\ldots,0),$$
$$\text{(The first } N-a \text{ entries are 1)}$$
$$\mathrm{Tr}(T_a)=0, \qquad \mathrm{Tr}(T_aT_b)=(N-a+1)(N-a)\delta_{ab}. \tag{3.4}$$

Write

$$\Omega_\mu=\exp\left(\frac{2\pi i}{N}f_\mu^a T_a\right), \tag{3.5}$$

then the topology is specified by $n_{\mu\nu}^{(a)}$ $a=1,2\ldots N-1$, $n_{\mu\nu}^{(1)}\in\mathbb{Z}$, $n_{\mu\nu}^{(a)}\in N\mathbb{Z}$, $a\neq 1$, and $n_{\mu\nu}=n_{\mu\nu}^{(1)}(\mathrm{mod}\,N)$. Note that now $n_{\mu\nu}^{(a)}$ are genuine integers and one cannot transform or deform *within H* configurations with different $n_{\mu\nu}^{(a)}$ but equal $n_{\mu\nu}$ into each other. The winding numbers are given by:

$$[f_\mu^a(x_\nu=a_\nu)-f_\mu^a(x_\nu=0)]-[f_\nu^a(x_\mu=a_\mu)-f_\nu^a(x_\mu=0)]=n_{\mu\nu}^{(a)}. \tag{3.6}$$

Lemma (3.2). *For an abelian configuration as above with winding numbers* $n^{(a)}$, $a=1,2\ldots N-1$ *we find:*

$$P=\sum_{a=1}^{N-1}\frac{(N-a+1)(N-a)}{N^2}\kappa(n^{(a)})=\left(\frac{N-1}{N}\right)\kappa(n)+\mathbb{Z}.$$

Proof. Insert (3.5) into the formula of Lemma (3.1) to find:

$$P=\sum_{\mu,\nu}\int d_2S_{\mu\nu}\varepsilon_{\mu\nu\alpha\beta}(\partial_\alpha f_\nu^a)_{x_\mu=a_\mu}(\partial_\beta f_\mu^b)_{x_\nu=0}\frac{\mathrm{Tr}(T^aT^b)}{2N^2}$$
$$=\varepsilon_{\mu\nu\alpha\beta}\frac{n_{\nu\alpha}^{(a)}}{2}\frac{n_{\mu\beta}^{(b)}}{2}\frac{(N-a+1)(N-a)}{2N^2}\delta_{ab}=\sum_a\frac{(N-a+1)(N-a)}{N^2}\kappa(n^{(a)}).$$

Together with $n_{\mu\nu}^{(a)}=\delta_{a1}n_{\mu\nu}+Nl_{\mu\nu}^{(a)}$, $l_{\mu\nu}^{(a)}\in\mathbb{Z}$ we find

$$P=\left(\frac{N-1}{N}\right)\kappa(n)+\sum_{a=1}^{N-1}(N-a+1)(N-a)\kappa(l^{(a)})$$
$$+\left(\frac{N-1}{4}\right)\varepsilon_{\mu\nu\alpha\beta}n_{\mu\nu}l_{\alpha\beta}^{(1)}=\left(\frac{N-1}{N}\right)\kappa(n)+\mathbb{Z}. \quad \square$$

We would like to stress that one cannot reach all possible values of P, with an abelian set of multiple transition functions. For example, for $n_{\mu\nu}=0(\mathrm{mod}\,N)$ we have $P\in 2\mathbb{Z}$.

We will now make use of the gauge invariance of P by making suitable choices of $g_{0\mu}$ and $h_{0\mu}$, as defined in Sect. 2. This enables us to comprehend the topological structure of the fiber bundle and in the general case to compute P in terms of topological invariants. (An application of this principle was the interpretation of $n_{\mu\nu}$ as winding numbers in Sect. 2.)

Theorem (3.1). *Given arbitrary multiple transition functions Ω_μ, we can choose $h_{0\mu}$ and $g_{0\mu}$ as follows:*

$$h_{0\mu}=U(x_\mu=a_\mu)\omega_\mu\,,\qquad g_{0\mu}=U(x_\mu=0)\,, \tag{3.7}$$

$$\omega_\mu=\exp\left(\frac{\pi i}{N}\sum_\nu\frac{n_{\mu\nu}x_\nu}{a_\nu}T_1\right), \tag{3.8}$$

where U is a gauge function on the boundary of the four-dimensional box.

Proof. First take any Ω_μ, $n_{\mu\nu}$ and try to specify $g_{0\mu}$ and $h_{0\mu}$ on the skeleton of the boundary of the four-dimensional box (the skeleton is defined to be the edges of the 8 cubes specified by $x_\mu=0$, and $x_\mu=a_\mu$, $\mu=1$, 2, 3, and 4). Where ever more than one gauge function is specified on this skeleton, we demand them to be equal.

Let us first use the gauge freedom (2.4) to bring Ω_μ in a suitable form. It is not difficult to see [using that SU(N) is (simply) connected and π_2 (SU(N))$=0$] that one can choose a gauge transformation $\Omega(x)$, such that:

$$\begin{gathered}\Omega(x=0)=1\,,\\ \Omega(x_\mu=a_\mu, x_\nu=0\,;\forall\nu\neq\mu)=\Omega_\mu(0)\,.\end{gathered} \tag{3.9}$$

This proves that we can restrict ourselves to:

$$\Omega_\mu(0)=1\,,\qquad \mu=1,2,3,4\,. \tag{3.10}$$

Note that ω_μ is already in this form.

With (2.9) the functions $g_{0\mu}$ and $h_{0\mu}$ on the skeleton are specified by giving $g_{0\mu}$ on the edges of the cube $x_\mu=0$. Here we work for a moment in the adjoint representation avoiding jumps with elements of Z_N, now hidden in the homotopy type of each square on the skeleton (compare Sect. 2).

If we want to specify U in terms of $g_{0\mu}$ and $h_{0\mu}$ we should have the conditions:

$$\begin{aligned}&h_{0\mu}(x_\nu=a_\nu)=h_{0\nu}(x_\mu=a_\mu)\,,\\ &h_{0\mu}(x_\nu=0)=g_{0\nu}(x_\mu=a_\mu)\,,\\ &g_{0\mu}(x_\nu=0)=g_{0\nu}(x_\mu=0)\,.\end{aligned} \tag{3.11}$$

If we demand the following condition to be met:

$$g_{0\mu}(x_\nu=a_\nu)=\Omega_\nu(x_\mu=0)g_{0\mu}(x_\nu=0)\,, \tag{3.12}$$

the first two conditions in (3.11) are superficial and can be derived from (2.9), (2.2), (3.12) and the last condition in (3.11). Choosing $g_{0\mu}(x_\nu=x_\lambda=0)=1$ now completely fixes $g_{0\mu}$ on the skeleton, which is indicated in Fig. 2. It is an easy but tedious exercise to show that at the corners of the cube all definitions of $g_{0\mu}$ represent the same value. For this one uses the consistency condition and the gauge choice (3.10).

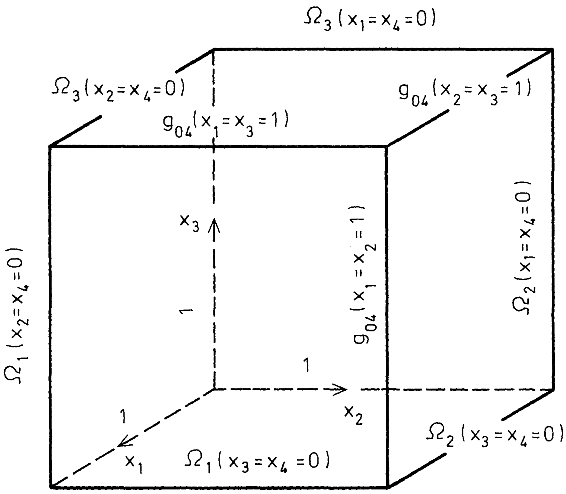

Fig. 2. The choice of $g_{0\mu}$ on the skeleto of the boundary of the four-dimensiona box. It is understood that $g_{0\mu}(x_\nu=a_\nu)=\Omega_\nu(x_\mu=0)g_{0\mu}(x_\nu=0)$. We only display g_{04} and take for convenience $a_\mu=1$

We now choose $\tilde{h}_{0\mu}$ and $\tilde{g}_{0\mu}$ in terms of ω_μ as above and introduce $\hat{h}_{0\mu}, \hat{g}_{0\mu}, \hat{\Omega}_\mu$ by "dividing out" the abelian twist carrying configuration

$$\begin{aligned}&\omega_\mu=\tilde{h}_{0\mu}\tilde{g}_{0\mu}^{-1},\\&\hat{h}_{0\mu}=h_{0\mu}\tilde{h}_{0\mu}^{-1},\\&\hat{g}_{0\mu}=g_{0\mu}\tilde{g}_{0\mu}^{-1},\qquad \hat{\Omega}_\mu=\hat{h}_{0\mu}\hat{g}_{0\mu}^{-1}.\end{aligned}\tag{3.13}$$

Here $\hat{h}_{0\mu}$ and $\hat{g}_{0\mu}$ are defined in terms of $\hat{\Omega}_\mu$ as above, but since the homotopy type of each square in the (μ,ν) plane for the Ω_μ and ω_μ configuration is equal to $n_{\mu\nu}$, the same homotopy type for the $\hat{\Omega}_\mu$ configuration is cancelled exactly.

So $\hat{h}_{0\mu}, \hat{g}_{0\mu}$ are genuine SU(N) functions specifying U:

$$\begin{aligned}&U(x_\mu=0)=\hat{g}_{0\mu},\\&U(x_\mu=a_\mu)=\hat{h}_{0\mu},\end{aligned}\tag{3.14}$$

as a continuous function on the skeleton.

We can extend $U(x_\mu=x_\nu=0)$ restricted to the edges of the cube $x_\mu=0$ continuously to the square $x_\mu=x_\nu=0$, for all six possible combinations fixing U on all sides of the cubes $x_\mu=0$ [and $x_\mu=a_\mu$ by using (3.12) and (2.9)]. Finally, since $\pi_2(\mathrm{SU}(N))=0$, we can continuously extend U inside the cubes $x_\mu=0$; (2.9) then fixes U inside the cubes $x_\mu=a_\mu$.

Putting things together we have, restricted to the skeleton:

$$g_{0\mu}=U(x_\mu=0)\tilde{g}_{0\mu},\qquad h_{0\mu}=U(x_\mu=a_\mu)\tilde{h}_{0\mu}.\tag{3.15}$$

These cannot be extended to the whole of the cubes $x_\mu=0$, respectively $x_\mu=a_\mu$, because of the nontrivial homotopies. However

$$\Omega_\mu=h_{0\mu}g_{0\mu}^{-1}=U(x_\mu=a_\mu)\omega_\mu U(x_\mu=0)^{-1}$$

makes the choice (3.7) possible.

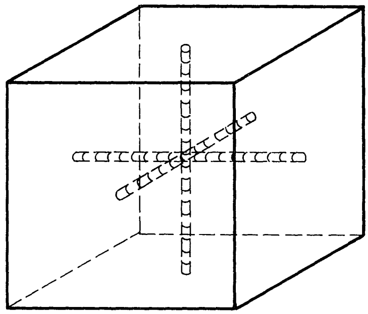

Fig. 3. The singularity structure of $A(x_\mu=0)$

We can however make a maximal continuous extension $\tilde{A}$ of $\tilde{g}_{0\mu}$ and $\tilde{h}_{0\mu}$ to the cubes such that

$$\tilde{A}(x_\mu=0)=\tilde{g}_{0\mu} \quad \text{and} \quad \tilde{A}(x_\mu=a_\mu)=\tilde{h}_{0\mu}. \tag{3.16}$$

We necessarily have line singularities restricted in their topology by (2.9), (3.11), and (3.12). A possible choice is depicted in Fig. 3. We can do the same for $g_{0\mu}$ and $h_{0\mu}$ defining A. If we choose the position of the line singularities the same as for $\tilde{A}$, (3.15) tells us that $A\tilde{A}^{-1}$ has a removable singular structure. □

U is a gauge function on the boundary of the four-dimensional hypercube (which is homotopic to S^3) so the homotopy type of U is an element of $\pi_3(\mathrm{SU}(N))=\mathbb{Z}$ specified by $\nu(U)$. Given Ω_μ we can have different choices of U belonging to different continuous extensions of U from the skeleton to the boundary of the four-dimensional box. But by construction they can be continuously deformed into each other. So ν is a unique function of Ω_μ, which topology is thus completely specified by $n_{\mu\nu}$ and ν. The gauge invariance of ν is obvious from the transformation property of U [using (2.4)]

$$U'=\Omega U. \tag{3.17}$$

And it is easy to see that any two configurations of Ω_μ with the same $n_{\mu\nu}$ and ν are gauge equivalent. Thus P should be a unique function of ν and $n_{\mu\nu}$.

Since P is invariant under continuous deformations of the transition functions we have:

$$\begin{aligned} P(\Omega_\mu) &= P(\omega_\mu)+P(\hat{\Omega}_\mu) \\ &= \frac{(N-1)}{N}\kappa(n)+\nu, \end{aligned} \tag{3.18}$$

where we used Lemma (3.2) and the fact that:

$$\nu(U)=\frac{1}{24\pi^2}\sum_\mu \int d_3\sigma_\mu \varepsilon_{\mu\nu\alpha\beta}[\mathrm{Tr}((U\partial_\nu U^{-1})(U\partial_\alpha U^{-1})(U\partial_\beta U^{-1}))]_{x_\mu=0}^{x_\mu=a_\mu}. \tag{3.19}$$

In the appendix we will prove that $P(\hat{\Omega}_\mu)=\nu(U)$ as a simple application of the general formula.

Now we also understand what happens if we add a multiple of N to $n_{\mu\nu}$ (yielding $n'_{\mu\nu}$). We will have different ω'_μ in Theorem (3.1) in constructing U'.

However by applying the theorem to ω'_μ itself we have $\tilde{g}'_{0\mu} = V(x_\mu = 0)\tilde{g}_{0\mu}$, $\tilde{h}'_{0\mu} = V(x_\mu = a_\mu)h_{0\mu}$ [compare (3.15)] and consequently we can continuously deform UV into U', so $\nu(U') = \nu(U) + \nu(V)$. On the other hand applying (3.18) to ω'_μ we have $\left(\frac{N-1}{N}\right)\kappa(n') = \left(\frac{N-1}{N}\right)\kappa(n) + \nu(V)$.

So both expressions for $P(\Omega_\mu)$ are equal. However the abelian contribution differs by an integer which is precisely the homotopy type of the gauge transformation (defined on the boundary of the box only) transforming ω_μ into ω'_μ. This is necessarily a nonabelian configuration.

4. Orthogonal Twist, Zero Action Configurations

Let us assume the existence of selfdual solutions for all possible topologies. This then implies that there are solutions with different electric and magnetic fluxes but equal action. The behaviour of $\kappa(n)$ is responsible for this. It boils down to having $n'_{\mu\nu} \neq n_{\mu\nu} (\mathrm{mod}\, N)$ and $\kappa(n) = \kappa(n') (\mathrm{mod}\, N)$.

A striking consequence is then the existence of zero action solutions with nontrivial topology, discovered on the lattice by Groeneveld [16] a.o., and called twist eating configurations by them.

Zero action implies $P = 0$ so these zero action configurations necessarily have $\kappa(n) = 0 (\mathrm{mod}\, N)$. Since zero action implies $G_{\mu\nu} = 0$ there is a gauge such that $A_\mu = 0$. In this gauge the multiple transition functions Ω_μ are constant. We call the twist $n_{\mu\nu}$ satisfying $\kappa(n) = 0 (\mathrm{mod}\, N)$ orthogonal, so we proved [5]:

Theorem (4.1). *There exist zero action, orthogonal twist solutions iff there are* $\Omega_\mu \in \mathrm{SU}(N)$ *such that* [2] $[\Omega_\mu, \Omega_\nu] = \exp(2\pi i n_{\mu\nu}/N)$. □

With the remarks following Theorem 3.1 it is not hard to show uniqueness up to constant gauge transformations! In the box we thus have leading perturbative contributions to the ground state energies in a certain $(\mathbf{e}, \mathbf{m})$ sector.

Ambjørn and Flyvbjerg [14] first showed the existence of zero classical energy solutions in the continuum with arbitrary magnetic flux $(\mathbf{m})$ by constructing $[\Omega_k, \Omega_j] = \exp(2\pi i \varepsilon_{kjl} m_l/N)$, which is in fact in the form of the above theorem for time independent configurations. From the mathematical point of view it is the interplay between the multiconnectedness of the hypertorus and the topology of $\mathrm{SU}(N)/Z_N$ which causes this behaviour [14]. From this formal point of view zero energy solutions on general three spaces and for general gauge groups are studied in Ref. 15. This boils down to the search for zero action solutions in three dimensional euclidean gauge theories.

Recently 't Hooft [5] proved the existence of Ω_μ as in Theorem (5.1) for N not divisible by a prime squared. We will combine his ideas and those of Ref. 14 to complete the proof for general N. The essential part is proving the result for $N = p^e$, $e \in \mathbb{N}$ (p will denote a prime). Let us first review the case where $e = 1$.

Let $P, Q \in \mathrm{SU}(N)$ be such that [5, 14]:

$$[P, Q] = e^{2\pi i/N}. \tag{4.1}$$

2 $[\Omega_\mu \Omega_\nu] = \Omega_\mu \Omega_\nu \Omega_\mu^{-1} \Omega_\nu^{-1}$, the group commutator

Then defining

$$\Omega_\mu = P^{s_\mu} Q^{t_\mu}, s_\mu, t_\mu \text{ integers},$$
$$[\Omega_\mu, \Omega_\nu] = \exp(2\pi i(s_\mu t_\nu - s_\nu t_\mu)/N), \tag{4.2}$$

the problem is reduced to finding s_μ and t_μ such that

$$\hat{n}_{\mu\nu} = s_\mu t_\nu - s_\nu t_\mu, \qquad n_{\mu\nu} = \hat{n}_{\mu\nu} (\mathrm{mod}\, N). \tag{4.3}$$

This automatically demands orthogonal twist, because $\kappa(\hat{n}) = 0$. Now one can transform by an SL(4, $\mathbb{Z}$) transformation [*here* we do not need to restrict to SL(4, Z_N)] $n_{\mu\nu}$ to the standard form:

$$n'_{12} = n'_{13} = n'_{34} = 0. \tag{4.4}$$

Let $\{X_{\mu\nu}\}$ be an element of SL(4, $\mathbb{Z}$) [or SL(4, Z_N)], then:

$$n_{\mu\nu} \to X_{\mu\mu'} X_{\nu\nu'} n_{\mu'\nu'},$$
$$s_\mu \to X_{\mu\mu'} s_{\mu'}, \qquad t_\mu \to X_{\mu\mu'} t_{\mu'}, \tag{4.5}$$
$$\kappa(n) \to (\det X) \kappa(n) = \kappa(n).$$

So $\kappa(n) = n'_{14} n'_{23} = 0 (\mathrm{mod}\, N)$, if N is prime this implies either $n'_{14} = 0 (\mathrm{mod}\, N)$ or $n'_{23} = 0 (\mathrm{mod}\, N)$ for which one can solve (4.3) easily [5].

It is not hard to see that in general a necessary condition for the solvability of (4.3) is the condition (for convenience we uniquely label $0 \leqq n_{\mu\nu} < N$ if $\mu < \nu$) that the greatest common divisor (g.c.d.) of $n_{\mu\nu}$ and N [notation: g.c.d. $(n_{\mu\nu}, N)$] equals 1. This is because (4.3) would imply the existence of $l_{\mu\nu} \in \mathbb{Z}$ such that $\kappa(n + Nl) = 0$ or solvability of $\frac{\kappa(n)}{N} + \frac{1}{2} n_{\mu\nu} \tilde{l}_{\mu\nu} + N\kappa(l) = 0$. It is also sufficient because taking $\hat{n} = n + lN$ we can solve for s_μ and t_μ by bringing $\hat{n}$ in the form (4.4) and using $\kappa(\hat{n}) = 0$ as above (now as a genuine equation over $\mathbb{Z}$, and not Z_N).

It remains to prove solvability of $\kappa(n + Nl) = 0$ for $\kappa(n) = 0 (\mathrm{mod}\, N)$ if g.c.d. $(n_{\mu\nu}, N) = 1$ or more general:

Lemma (4.1). *If g.c.d.* $(n_{\mu\nu}, N) = 1$ *we can find* $l_{\mu\nu} \in \mathbb{Z}$, *and* $q \in \mathbb{Z}$ *such that* $\kappa(n) = \kappa(n + Nl) + Nq$, $0 \leqq \kappa(n + Nl) < N$, *and* $0 \leqq n_{\mu\nu} < N$, $\forall \mu < \nu$.

Proof.

$$\kappa(n + Nl) + Nq - \kappa(n) = \tfrac{1}{2} N n_{\mu\nu} \tilde{l}_{\mu\nu} + \tfrac{1}{4} N^2 l_{\mu\nu} \tilde{l}_{\mu\nu} + Nq = 0.$$

Put $m_i = \varepsilon_{ijk} n_{jk}$, $a_i = \varepsilon_{ijk} l_{jk}$, $k_i = n_{i4}$, $b_i = l_{i4}$; so to solve:

$$(\mathbf{m} \cdot \mathbf{b}) + (\mathbf{k} \cdot \mathbf{a}) + N(\mathbf{a} \cdot \mathbf{b}) + q = 0.$$

Transforming to the standard form (4.4) we have g.c.d. $(n_{\mu\nu}, N) = 1$ iff g.c.d. $(n'_{13}, n'_{14}, n'_{24}, N) = 1$ [since we use SL(4, $\mathbb{Z}$) transformations only]. So to solve:

$$m'_2 b'_2 + k'_1 a'_1 + k'_2 a'_2 + N(a'_1 b'_1 + a'_2 b'_2 + a'_3 b'_3) + q = 0,$$

choosing $a'_3 = 1$, $b'_1 = 0$, $b'_3 = b - a'_2 b'_2$ this boils down to solving:

$$b'_2 m'_2 + a'_1 k'_1 + a'_2 k'_2 + bN + q = 0$$

for fixed m'_2, k'_1, k'_2, N. The solvability follows from the elementary algebraic relation (a consequence of Euler's remainder theorem):

$$\sum (q_i \mathbb{Z}) = \text{g.c.d.}(q_i)\mathbb{Z}. \quad \square \tag{4.6}$$

We thus have the following simple corollary:

Corollary. *When g.c.d.* $(n_{\mu\nu}, N) = 1 (0 \leqq n_{\mu\nu} < N, \mu < \nu)$ *and* $\kappa(n) = 0 (\text{mod}\, N)$ *we can find* Ω_μ *as in* (4.2), *satisfying the condition in Theorem* (4.1). $\square$

Before proving the existence of Ω_μ for the case $N = p^e$ we want to show that this is sufficient to deal with the case of general N. We follow the method of Ref. 14, splitting general N in its prime factors: $N = \prod_i p_i^{e_i}$ we have ($\oplus$ is the direct sum, $\otimes$ is the tensor product):

$$\begin{gathered} Z_N \cong \bigoplus_i Z_{N_i}, \qquad N_i = p_i^{e_i}, \\ \text{SU}(N) \supset \bigotimes_i \text{SU}(N_i). \end{gathered} \tag{4.7}$$

We thus decompose $n_{\mu\nu}$ and correspondingly Ω_μ as follows:

$$\begin{aligned} n_{\mu\nu} &= \sum (N p_i^{-e_i}) n_{\mu\nu}^{(i)}, \qquad n_{\mu\nu}^{(i)} \in Z_{N_i}, \\ \Omega_\mu &= \bigotimes_i \Omega_\mu^{(i)}, \qquad \Omega_\mu^{(i)} \in \text{SU}(N_i). \end{aligned} \tag{4.8}$$

From the decomposition of Ω_μ and the assumed existence of $\Omega_\mu^{(i)}$ belonging to $n_{\mu\nu}^{(i)}$ we have:

$$\begin{aligned} [\Omega_\mu, \Omega_\nu] &= \left[\bigotimes_i \Omega_\mu^{(i)}, \bigotimes_i \Omega_\nu^{(i)}\right] = \bigotimes_i [\Omega_\mu^{(i)}, \Omega_\nu^{(i)}] = \prod_i \exp(2\pi i n_{\mu\nu}^{(i)}/N_i) \\ &= \exp(2\pi i (\textstyle\sum N p_i^{-e_i} n_{\mu\nu}^{(i)})/N) = \exp(2\pi i n_{\mu\nu}/N). \end{aligned}$$

There is however one vital thing we forgot to prove. For the existence of $\Omega_\mu^{(i)} \in \text{SU}(N_i)$ we need $\kappa(n_{\mu\nu}^{(i)}) = 0 (\text{mod}\, N_i)$. Let us write $N = N_1 N_2$, g.c.d. $(N_1, N_2) = 1$ then we can uniquely decompose $n_{\mu\nu} \in Z_N$ into $n_{\mu\nu} = N_2 n_{\mu\nu}^{(1)} + N_1 n_{\mu\nu}^{(2)}$, $n_{\mu\nu}^{(i)} \in Z_{N_i}$, from this we have

$$\kappa(n) = N_2^2 \kappa(n^{(1)}) + N_1^2 \kappa(n^{(2)}) (\text{mod}\, N) = 0 (\text{mod}\, N).$$

This necessarily implies $\kappa(n^{(i)}) = 0 (\text{mod}\, N_i)$, as one can easily deduce from the general solution of $\kappa(n^{(i)})$. Splitting the N_i's up further we can deduce the general case.

We now finish the proof of the existence of zero action solutions for arbitrary orthogonal twist and N by the following theorem:

Theorem (4.2). *If N equals a prime to some positive power $e (N = p^e)$, we can find $\Omega_\mu \in \text{SU}(N)$ such that*

$$[\Omega_\mu, \Omega_\nu] = \exp(2\pi i n_{\mu\nu}/N)$$

for all $n_{\mu\nu} \in Z_N$ satisfying $\kappa(n) = 0 (\text{mod}\, N)$.

Proof. We can write g.c.d. $(n_{\mu\nu}) = r \cdot p^f$, g.c.d. $(r, p) = 1$, $f \in \mathbb{N}$. Let us first reduce the case $2f > e$ to $2f \leqq e$. We have both $\kappa(n) = k \cdot p^e$ ($\kappa(n) = 0 (\text{mod}\, N)$) and $\kappa(n) = l \cdot p^{2f}$ (g.c.d. $(n_{\mu\nu}) = r \cdot p^f$), so k contains p^{2f-e} as a factor. Furthermore $f < e$ so we can define $n'_{\mu\nu} = n_{\mu\nu} p^{-(2f-e)}$ and $N' = N p^{-(2f-e)} = p^{2(e-f)}$, and it is easy to deduce

$\kappa(n')=0(\mathrm{mod}\,N')$ and g.c.d. $(n'_{\mu\nu})=r\cdot p^{e-f}$. So $f'=e-f$ and $e'=2(e-f)$. When we can solve for $\Omega'_\mu\in SU(N')$ we choose $\Omega_\mu=\Omega'_\mu(\otimes 1_{N'})^{(N-N')/N'}$. Thus we can assume $2f\leqq e$ from now on.

There is a pair (μ_0,ν_0) $\mu_0<\nu_0$ such that $\tilde{n}_{\mu_0\nu_0}p^{-f}$ is relative prime to p. We split $n_{\mu\nu}\in Z_N$ in $n^{(i)}_{\mu\nu}\in Z_{N_i}$ (i=1,2) (with $N_1=p^{e-f}$, $N_2=p^f$) according to $n_{\mu\nu}=N_2 n^{(1)}_{\mu\nu}+N_1 n^{(2)}_{\mu\nu}$, with:

$$n^{(1)}_{\mu_0\nu_0}=n_{\mu_0\nu_0}p^{-f}-kp^{e-2f}=-n^{(1)}_{\nu_0\mu_0}, \qquad n^{(2)}_{\mu_0\nu_0}=-n^{(2)}_{\nu_0\mu_0}=k\in Z_{N_2}$$

and

$$n^{(1)}_{\mu\nu}=n_{\mu\nu}p^{-f}, \qquad n^{(2)}_{\mu\nu}=0$$

for the other pairs (μ,ν). Now

$$\kappa(n^{(1)})=\left[\frac{\kappa(n)}{N}-k(\tilde{n}_{\mu_0\nu_0}p^{-f})\right]p^{e-2f}$$

and we can choose k such that $\kappa(n^{(1)})=0(\mathrm{mod}\,N_1)$. We can apply the corollary to find $\Omega^{(1)}_\mu\in SU(N_1)$. Furthermore $s^{(2)}_\mu=k\delta_{\mu\mu_0}$, $t^{(2)}_\mu=\delta_{\mu\nu_0}$ gives $\Omega^{(2)}_\mu\in SU(N_2)$ as in (4.2). Putting things together $\Omega_\mu=\Omega^{(1)}_\mu\otimes\Omega^{(2)}_\mu\in SU(N)$ yields the desired commutation relations. □

The existence of Ω_μ as in Theorem (4.1) for each N are not only important for zero action solutions, 't Hooft [5] also used them extensively in constructing solutions with general twist, and in particular self dual solutions with minimal nontrivial action.

5. Conclusions

We showed that 't Hooft's method for introducing gauge fields on the four-dimensional box is equivalent to a principal fiber bundle on $T^4=S^1\times S^1\times S^1\times S^1$, with structure group $SU(N)/Z_N$. It is the invariance of the fields in the box under Z_N which gave us the rich topological structure we found. Equivalently we can take as structure group the adjoint of $SU(N)$. Then the topological structure associated with Z_N is hidden in winding numbers.

One has a Pontryagin number expressible in the topological invariants of the bundle. A new feature is that one can have configurations with equal Pontryagin number but different topology. Intimately related with this is the existence of twist eating configurations for arbitrary N and orthogonal twist as we proved in detail. The explicit dependence of the Pontryagin number on the magnetic flux **m** is such that one recovers the equivalent of Witten's result [4, 13]; switching on the θ-parameter [12] gives a θ-dependent electric flux. When we can do dynamics in the box it is possible to study things like oblique confinement, introduced by 't Hooft [13] in the continuum theory, in the box.

It is hoped that our complete understanding of the topology of gauge fields on the hypertorus will be helpful in constructing explicit solutions of the euclidean equations of motion. We would like to be able to restrict ourselves to (anti-)self dual solutions. The solutions 't Hooft [5] constructed are only self dual if the ratio of the a_μ satisfy certain conditions. It is thus conjectured that if this is not the case the solutions are unstable.

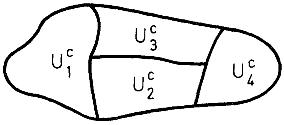

Fig. 4. A generic choice of sets U_i^c satisfying the conditions in the appendix. We suppressed two dimensions

Once we have the classical solutions one can do semiclassical computations of e.g. the relevant free energies, and hope that the weak coupling results can be pushed far enough to be relevant for QCD.

Appendix

Here we prove a general theorem for the 1st Pontryagin or 2nd Chern class of a 4-dimensional compact manifold without boundary (generalization to other manifolds is obvious). We apply the result to a proof of Lemma (3.1) and $P(\hat{\Omega}_\mu)=v(U)$ [compare (3.18) and (3.19)].

Notation. Let M be a compact manifold of real dimension 4 without a boundary and $\{U_i\}$ a covering of M with open sets.

We choose $\{U_i^c\}$ a set of closed sets with $U_i^c \subset U_i$, which still cover M, but satisfy the following properties: $U_i^c \cap U_j^c = \partial U_i^c \cap \partial U_j^c$ and

$$U_i^c \cap U_j^c \cap U_k^c = \partial(U_i^c \cap U_j^c) \cap \partial U_k^c ,$$

consequently these are 3 respectively 2-dimensional submanifolds. (See Fig. 4 for a generic situation.) Furthermore our convention is such that $[U_i^c \cap U_j^c]_i$ denotes $U_i^c \cap U_j^c$ with the orientation of ∂U_i^c and $[U_i^c \cap U_j^c \cap U_k^c]_i$ denotes $U_i^c \cap U_j^c \cap U_k^c$ with the orientation of $\partial [U_i^c \cap U_j^c]_i$. A local section of the connection 1-form on U_i is denoted by ω_i $(=iA_\mu^{(i)}dx_\mu^{(i)})$, and $\Omega_i = d\omega_i + \omega_i \wedge \omega_i$ is the curvature 2-form on U_i. Here $\wedge$ is the wedge product of p-forms. The wedge product for n equal p-forms is written as $(p\text{-form})^n$. Finally ω_i transforms under a change of coordinate patch, with transition functions g_{ij} defined on $U_i \cap U_j$, according to

$$\omega_j = g_{ij}^{-1}\omega_i g_{ij} + g_{ij}^{-1} dg_{ij} .$$

Lemma (A.1).

$$\begin{aligned} -\int_M \mathrm{Tr}(\Omega \wedge \Omega) &= \tfrac{1}{6}\sum_{i,j} \int_{[U_i^c \cap U_j^c]_i} \mathrm{Tr}((g_{ij} dg_{ij}^{-1})^3) \\ &\quad + \tfrac{1}{6}\sum_{i,j,k} \int_{[U_i^c \cap U_j^c \cap U_k^c]_i} \mathrm{Tr}((g_{ij}^{-1} dg_{ij}) \wedge ((dg_{jk}) g_{jk}^{-1})) \\ &= \tfrac{1}{3}\sum_{i<j} \int_{[U_i^c \cap U_j^c]_i} \mathrm{Tr}((g_{ij} dg_{ij}^{-1})^3) \\ &\quad + \sum_{i<j<k} \int_{[U_i^c \cap U_j^c \cap U_k^c]_i} \mathrm{Tr}((g_{ij}^{-1} dg_{ij}) \wedge ((dg_{jk}) g_{jk}^{-1})) \equiv 8\pi^2 P(g_{ij}) . \end{aligned}$$

Proof.

$$\begin{aligned} \int_M \mathrm{Tr}(\Omega \wedge \Omega) &= \sum_i \int_{U_i^c} \mathrm{Tr}(\Omega_i \wedge \Omega_i) = \sum_i \int_{U_i^c} dK_i \\ &= \tfrac{1}{2}\sum_{i,j} \int_{[U_i^c \cap U_j^c]_i} (K_i - K_j) , \end{aligned}$$

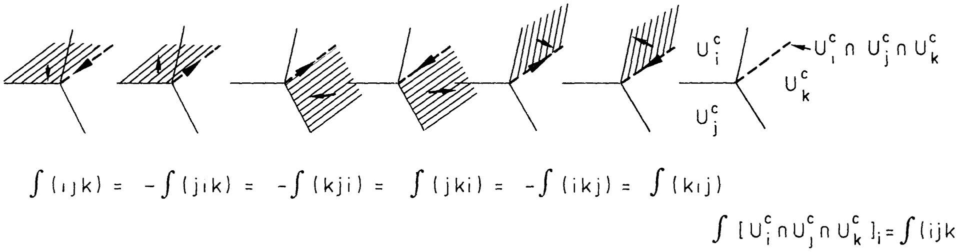

Fig. 5. Constructing the right sign in the expression for L_{ijk}. We suppressed one dimension

where

$$\mathrm{Tr}(\Omega_i \wedge \Omega_i) = dK_i \quad \text{and} \quad K_i = \mathrm{Tr}(\omega_i \wedge d\omega_i + \tfrac{2}{3}\omega_i \wedge \omega_i \wedge \omega_i).$$

A straightforward calculation gives, using $\omega_j = g_{ij}^{-1}\omega_i g_{ij} + g_{ij}^{-1} dg_{ij}$:

$$K_j - K_i = \tfrac{1}{3}\mathrm{Tr}((g_{ij} dg_{ij}^{-1})^3) + dL_{ij}, \qquad L_{ij} = \mathrm{Tr}(\omega_i \wedge (dg_{ij}) g_{ij}^{-1}).$$

Combining these results we find:

$$-\int_M \mathrm{Tr}(\Omega \wedge \Omega) = \tfrac{1}{6}\sum_{i,j} \int_{[U_i^c \cap U_j^c]_i} \mathrm{Tr}((g_{ij} dg_{ij}^{-1})^3) + \tfrac{1}{2}\sum_{i,j} \int_{\partial[U_i^c \cap U_j^c]_i} L_{ij}.$$

$$\sum_{i,j} \int_{\partial[U_i^c \cap U_j^c]_i} L_{ij} = \sum_{i,j,k} \int_{[U_i^c \cap U_j^c \cap U_k^c]_i} L_{ij}$$

$$= \sum_{i,j,k} \tfrac{1}{6}\left\{ \int_{(ijk)} L_{ij} + \int_{(jik)} L_{ji} + \int_{(ikj)} L_{ik} + \int_{(kij)} L_{ki} + \int_{(kji)} L_{kj} + \int_{(jki)} L_{jk} \right\}$$

$$= \sum_{i,j,k} \tfrac{1}{6} \int_{(ijk)} (L_{ij} - L_{ji} - L_{ik} + L_{ki} - L_{kj} + L_{jk}) \equiv \tfrac{1}{6} \sum_{i,j,k} \int_{(ijk)} L_{ijk}.$$

(See Fig. 5, for convenience we defined $(ijk) \equiv [U_i^c \cap U_j^c \cap U_k^c]_i$.)

Expressing ω_j and ω_k in ω_i and using the cocycle condition $g_{ij} g_{jk} = g_{ik}$ to show that

$$2(dg_{ij}) g_{ij}^{-1} - 2(dg_{ik}) g_{ik}^{-1} = g_{ik}((dg_{kj}) g_{kj}^{-1}) g_{ik}^{-1} - g_{ij}((dg_{jk}) g_{jk}^{-1}) g_{ij}^{-1}$$

one finds that the ω_i dependence drops out and we are left with:

$$L_{ijk} = \mathrm{Tr}(g_{ij}^{-1}(dg_{ij}) \wedge (dg_{jk}) g_{jk}^{-1}) + \mathrm{Tr}(g_{jk}^{-1}(dg_{jk}) \wedge (dg_{ji}) g_{ji}^{-1})$$
$$\equiv \hat{L}_{ijk} + \hat{L}_{jki}, \qquad \hat{L}_{ijk} \equiv \mathrm{Tr}(g_{ij}^{-1}(dg_{ij}) \wedge (dg_{jk}) g_{jk}^{-1}).$$

So

$$\tfrac{1}{6}\sum_{i,j,k} \int_{(ijk)} L_{ijk} = \tfrac{1}{6}\sum_{i,j,k} \left\{ \int_{(ijk)} \hat{L}_{ijk} + \int_{(jki)} \hat{L}_{jki} \right\}$$
$$= \tfrac{1}{3}\sum_{i,j,k} \int_{(ijk)} \mathrm{Tr}(g_{ij}^{-1}(dg_{ij}) \wedge (dg_{jk}) g_{jk}^{-1}). \quad \square$$

The 1st Pontryagin Number is defined for a real vector bundle with transition functions $g_{ij} \in \mathrm{GL}(N, \mathbb{R})$ by [10]:

$$P_1(\Omega) = \int_M \frac{\mathrm{Tr}(\Omega \wedge \Omega^\dagger)}{8\pi^2}. \tag{A.1}$$

We have furthermore $\Omega = \frac{i}{2} G_{\mu\nu} dx_\mu \wedge dx_\nu$ (in local coordinates) which puts (A.1) in the form (1.3). Since we will always have an O(N) vector bundle [10], $\Omega^\dagger = -\Omega$ and Lemma (A.1) gives $P_1(\Omega)$ in terms of transition functions only.

The 2nd Chern number is defined for a complex vector bundle with transition functions $g_{ij} \in \mathrm{GL}(N, \mathbb{C})$ by [10]:

$$C_2(\Omega) = \frac{1}{8\pi^2} \int_M (\mathrm{Tr}(\Omega \wedge \Omega) - \mathrm{Tr}(\Omega) \wedge \mathrm{Tr}(\Omega)). \tag{A.2}$$

For SU(N) vector bundles $\mathrm{Tr}(\Omega) = 0$ and we have $C_2(\Omega) = -P$, as defined in (1.3) in terms of $G_{\mu\nu}$. Since we claimed a general formula for $C_2(\Omega)$ in terms of the transition functions only we have to deal with the last term in (A.2):

Lemma (A.2).

$$-\int_M \mathrm{Tr}(\Omega) \wedge \mathrm{Tr}(\Omega) = \sum_{i<j<k} \int_{[U_j^c \cap U_j^c \cap U_k^c]_i} \mathrm{Tr}(g_{ij}^{-1} dg_{ij}) \wedge \mathrm{Tr}((dg_{jk}) g_{jk}^{-1}) \equiv 8\pi^2 Q(g_{ij}).$$

Proof. We can use the same technique as in Lemma (A.1), we now have $\mathrm{Tr}(\Omega_i) = d\,\mathrm{Tr}(\omega_i)$ so

$$\mathrm{Tr}(\Omega_i) \wedge \mathrm{Tr}(\Omega_i) = dK_i, \qquad K_i = \mathrm{Tr}(\omega_i) \wedge d\,\mathrm{Tr}(\omega_i),$$
$$K_j - K_i = dL_{ij}, \qquad L_{ij} = \mathrm{Tr}(\omega_i) \wedge \mathrm{Tr}(g_{ij}^{-1} dg_{ij}),$$

where we used that $d\,\mathrm{Tr}(g_{ij}^{-1} dg_{ij}) = 0$. Again define

$$L_{ijk} = L_{ij} - L_{ji} - L_{ik} + L_{ki} - L_{kj} + L_{jk}.$$

The cocycle condition this time implies

$$\mathrm{Tr}(g_{ij}^{-1} dg_{ij}) + \mathrm{Tr}(g_{jk}^{-1} dg_{jk}) - \mathrm{Tr}(g_{ik}^{-1} dg_{ik}) = 0$$

and also guarantees here absence of ω_i dependence. We find:

$$\begin{aligned} L_{ijk} &= \mathrm{Tr}(g_{ij}^{-1} dg_{ij}) \wedge \mathrm{Tr}(g_{jk}^{-1} dg_{jk}) - \mathrm{Tr}(g_{ik}^{-1} dg_{ik}) \wedge \mathrm{Tr}(g_{kj}^{-1} dg_{kj}) \\ &= \hat{L}_{ijk} - \hat{L}_{ikj}, \qquad \hat{L}_{ijk} \equiv \mathrm{Tr}(g_{ij}^{-1} dg_{ij}) \wedge \mathrm{Tr}(g_{jk}^{-1} dg_{jk}). \end{aligned}$$

So as in Lemma (A.1) we have:

$$\begin{aligned} -\int_M \mathrm{Tr}(\Omega) \wedge \mathrm{Tr}(\Omega) &= \tfrac{1}{2} \sum_{i,j} \int_{[U_i^c \cap U_j^c]_i} K_j - K_i \\ &= \tfrac{1}{12} \sum_{i,j,k} \left(\int_{(ijk)} \hat{L}_{ijk} + \int_{(ikj)} \hat{L}_{ikj} \right). \quad \square \end{aligned}$$

Theorem (A.1). (i) *The first Pontryagin number for an* O(N) *vector bundle with transition function* $g_{ij} \in \mathrm{O}(N)$ *on a four-dimensional manifold is given by:* $P_1(\Omega) = P(g_{ij})$, *defined in Lemma* (A.1).

(ii) *The second Chern number for a complex vector bundle with transition functions* $g_{ij} \in \mathrm{GL}(N, \mathbb{C})$ *on a four-dimensional manifold is given by:* $C_2(\Omega) = Q(g_{ij}) - P(g_{ij})$, *Q defined as in Lemma* (A.2). $\square$

Let us now use this theorem for proving Lemma (3.1). For T^4 we take:

$$U_0^c = \{\varepsilon \leqq x_\mu \leqq a_\mu - \varepsilon;\, \mu = 1, 2, 3, 4\},$$

and for $\mu=1,2,3,4$

$$U_\mu^c=\{|x_\mu|\leqq\varepsilon, \varepsilon\leqq x_\nu\leqq a_\nu-\varepsilon;\ \forall\nu\neq\mu\},$$

taking $\varepsilon\to 0$, with transition functions $h_{0\mu}$ and $g_{0\mu}$ as defined in Sect. 2. We do not need the explicit form of the other sets U_i^c since the relevant intersections can be described by the above alone.

It is obvious that we can read off from Fig. 1 the different contributions to $P(g_{ij})$ (in the limit $\varepsilon\to 0$, $\delta\to 0$. In this figure U_3^c should be labelled different, but there will be no confusion in the following). It is easy to see that we have for T^4:

$$\frac{1}{24\pi^2}\sum_{i<j}\int_{[U_i^c\cap U_j^c]_i}\mathrm{Tr}((g_{ij}dg_{ij}^{-1})^3)=\frac{1}{24\pi^2}\sum_\mu\int_{x_\mu=0}\mathrm{Tr}((h_{0\mu}dh_{0\mu}^{-1})^3-(g_{0\mu}dg_{0\mu}^{-1})^3). \tag{A.3}$$

The second term in $P(g_{ij})$ needs somewhat more discussion. We will treat the case where $x\in U_i^c\cap U_j^c$ has fixed x_1 and x_2 and we can read off the contributions from Fig. 1:

$$\begin{aligned}\mathrm{Tr}(h_{01}^{-1}dh_{01}\wedge(dh_{13})h_{13}^{-1})_{x_2=a_2}&+\mathrm{Tr}(h_{02}^{-1}dh_{02}\wedge(dg_{23})g_{23}^{-1})_{x_1=0}\\&+\mathrm{Tr}(g_{01}^{-1}dg_{01}\wedge(dg_{13})g_{13}^{-1})_{x_2=0}\\&+\mathrm{Tr}(g_{02}^{-1}dg_{02}\wedge(dh_{23})h_{23}^{-1})_{x_1=a_1}-(1\leftrightarrow 2).\end{aligned} \tag{A.4}$$

It is conjectured that in general one can eliminate h_{13}, g_{13}, h_{23}, g_{23} by using $(\mathrm{mod}\, Z_N)$:

$$\begin{aligned}h_{13}&=h_{01}^{-1}(x_2=a_2)h_{02}(x_1=a_1)h_{23},\\h_{23}&=g_{02}^{-1}(x_1=a_1)h_{01}(x_2=0)g_{13},\\g_{13}&=g_{01}^{-1}(x_2=0)g_{02}(x_1=0)g_{23},\\g_{23}&=h_{02}^{-1}(x_1=0)g_{01}(x_2=a_2)h_{13}.\end{aligned} \tag{A.5}$$

We will only specialize to the situations mentioned in the beginning. First we compute $P(\hat{\Omega}_\mu)$ as defined in Theorem 3.1, see (3.18). By a gauge transformation we can choose $g_{0\mu}=U(x_\mu=0)$, $h_{0\mu}=U(x_\mu=a_\mu)$. Equation (A.5) implies thus $h_{13}=h_{23}=g_{13}=g_{23}$ and (A.4) is identical zero. So $P(\hat{\Omega}_\mu)$ equals the contribution in (A.3). Comparing with (3.19) we find $P(\hat{\Omega}_\mu)=\nu(U)$.

In proving Lemma (3.1) we choose $g_{0\mu}=1$, $h_{0\mu}=\Omega_\mu$. Using (A.5) we can eliminate h_{13}, g_{13}, h_{23}, g_{23} in the following steps. Put into (A.4) $h_{13}=h_{02}(x_1=0)g_{23}$ and $h_{23}=h_{01}(x_2=0)g_{13}$ to find for the second term in $P(g_{ij})$:

$$\begin{aligned}&\mathrm{Tr}((\Omega_2^{-1}d\Omega_2)_{x_1=a_1}\wedge(\Omega_1 d\Omega_1^{-1})_{x_2=0})\\&\quad+\mathrm{Tr}((\Omega_2(x_1=a_1)\Omega_1(x_2=0))^{-1}d(\Omega_2(x_1=a_1)\Omega_1(x_2=0))\wedge g_{13}dg_{13}^{-1})-(1\leftrightarrow 2).\end{aligned}$$

Furthermore we have $g_{13}=g_{23}$ and using the consistency condition (2.2) the second term cancels the same term with 1 and 2 interchanged. Combining (A.3) with the above result generalized from the (1, 2) to the (μ,ν) contribution, correctly taking care of the orientations, we find the expression of Lemma (3.1).

While completing this work we found that M. Lüscher [17] derived a formula of the type in Lemma (3.1) in a completely different context.

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