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Symmetric Random Walks in Random Environments

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Abstract. We consider a random walk on the d-dimensional lattice $\mathbb{Z}^d$ where the transition probabilities $p(x,y)$ are symmetric, $p(x,y)=p(y,x)$, different from zero only if $y-x$ belongs to a finite symmetric set including the origin and are random. We prove the convergence of the finite-dimensional probability distributions of normalized random paths to the finite-dimensional probability distributions of a Wiener process and find our an explicit expression for the diffusion matrix.

1. Formulation of the Problem and Results

We shall consider Markov chains whose phase space is the cubic d-dimensional lattice $\mathbb{Z}^d$. In the case of discrete time such chains are defined by their transition probabilities $p(x,y)$, $x\in\mathbb{Z}^d$, $y\in\mathbb{Z}^d$ which are replaced by differential transition probabilities $w(x,y)$, $x\in\mathbb{Z}^d$, $y\in\mathbb{Z}^d$ in the case of continuous time. We shall discuss the situation when $p(x,y)$ or $w(x,y)$ are random variables not depending on time. One says in these cases that one has a random walk in a random environment (see [1–2]).

There are many physical problems where one encounters similar random walks. We can mention some problems in crystallography (see [3]), and biophysics [4]. In this spirit one can discuss kinetic properties of Lorentz gas with random configurations of scatterers.

The one-dimensional case with possible transitions $x\to x\pm 1$ is mostly investigated from the mathematical point of view. The first results are due to Kesten, M. Kozlov, and Spitzer (see [1]). One can also mention the papers [5–6]. In [6] the case when $p(x,x+1)$ and $p(x,x-1)=1-p(x,x+1)$ are identically distributed was considered. An unexpected result of [6] is that the random walk can be highly nonuniform and a moving point spends an unusually large part of time in some regions of $\mathbb{Z}^1$. The positions of these regions and the distribution of time depend on a realization of probabilities $p(x,x+1)$.

Quite a different situation arises if one admits the transitions $x\to x-1, x, x+1$ and adds the symmetry condition $p(x,y)=p(y,x)$ or $w(x,y)=w(y,x)$. This case is

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discussed in the whole series of papers [7–9] and a review article [10]. The main result is that if $E(w(x,x+1))^{-1}<\infty$ and $x(t)$ is a position of the moving particle at the moment t then $Ex^2(t)\sim Dt$ as $t\to\infty$, where D is a constant and the distribution of $x(t)t^{-1/2}$ converges as $t\to\infty$ to a gaussian distributuion with a nonrandom variance. In [10] it is shown that if the condition $E(w(x,x+1))^{-1}<\infty$ is violated then the growth of $x(t)$ can be more slow.

We consider in this paper a symmetric random walk in a random environment for arbitrary $d\geqq 1$ and for the cases of discrete and continuous time. Let a finite subset $\mathfrak{A}^+\subset\mathbb{Z}^d$ be fixed such that

1) $0\notin\mathfrak{A}^+$,
2) $\mathfrak{A}^+\cap(-\mathfrak{A}^+)=\emptyset$,
3) $\mathfrak{A}^+$ generates the whole group $\mathbb{Z}^d$.

Denote $\mathfrak{A}=\mathfrak{A}^+\cup(-\mathfrak{A}^+)\cup 0$ and assume that for each pair $x,y\in\mathbb{Z}^d$, $y-x\in\mathfrak{A}^+$ a random variable $a(x,y)$ is defined. We put $a(x,y)=a(y,x)$ for $y-x\in-\mathfrak{A}^+$, $a(x,x)=-\sum\limits_{\alpha\in\mathfrak{A}\setminus 0}a(x,x+\alpha)$ and $a(x,y)=0$ if $y-x\notin\mathfrak{A}$.

An operator $A=\|a(x,y)\|$ is a linear operator with random matrix elements which is similar in some respects to the Schrödinger operator with random potential. If random variables $a(x,y)\leqq 0$ then $-A$ can be considered as a generator of a Markov semigroup with continuous time. Moreover if $\sum\limits_{\alpha\in\mathfrak{A}^+}a(x,x+\alpha)\geqq-\frac{1}{2}$ then $P=I-A$ is a matrix of transition probabilities of a random walk with discrete time.

Assume that the joint probability distribution of random variables $a(x,y)$ is translationally invariant and put $\bar{A}=\|\bar{a}(x,y)\|=\|Ea(x,y)\|=EA$. Then $-\bar{A}$ is a generator of a Markov semigroup with translationally invariant transition probabilities in the case of continuous time while $\bar{P}=I-\bar{A}$ is a matrix of transition probabilities of an homogeneous Markov chain with discrete time. Let us introduce $Q=\|q(x,y)\|=\bar{A}-A$.

Main assumption

I) Random variables $q(x,y)$, $y-x\in\mathfrak{A}^+$ are mutually independent.
II) $\bar{a}(x,y)\neq 0$ for $y-x\in\mathfrak{A}\setminus 0$.
III) $|q(x,y)|<\delta|\bar{a}(x,y)|$, where $y-x\in\mathfrak{A}$, $\delta<\frac{1}{2}$.

In the case of discrete time we need also

IV) $\bar{a}(x,x)\leqq(1-\delta_0)/(1+\delta)$ for $\delta_0>0$.

The assumption III) means that the random walk defined by the matrix A is a random perturbation of the random walk defined by $\bar{A}$.

Let us fix $r>0$ and a sequence of numbers $r_n\to\infty$, $r_n\sim r\sqrt{n}$ as $n\to\infty$. We denote by T_n the set of r_n^d points of the lattice which are contained in the cube centered at the origin and having the volume r_n^d. T_n may be considered either as a fundamental domain of the subgroup $r_n\mathbb{Z}^d\subset\mathbb{Z}^d$ or as a finite lattice on the torus $\mathrm{Tor}_{r_n}^d=\mathbb{R}^d/r_n\mathbb{R}^d$. For large enough n we replace the sample $\{a(x,y)\}$ by a new sample $\{a_n(x,y)\}$ which coincides with the original one if $x\in T_n$, $y-x\in\mathfrak{A}^+$ and is symmetric and periodic with the period r_n with respect to pairs x,y. New random variables $a_n(x,y)$ can be considered as indexed by points of $T_n\times T_n$. Let $A_n=\|a_n(x,y)\|$, $\bar{A}_n=\|\bar{a}_n(x,y)\|=EA_n$, $Q_n=\|q_n(x,y)\|=\bar{A}_n-A_n$, $x,y\in T_n$. The matrices

A_n, $\bar{A}_n$ determine the random walk on T_n. Trajectories of the random walk are denoted by $\{X_n(\tau), 0 \leqq \tau \leqq n\}$. We want to emphasize that we consider random trajectories on T_n only during n steps. Let us make a contraction of $\mathrm{Tor}^d_{r_n}$ with the scaling coefficient $\varrho_n = r_n/r$. In other words we consider the linear transformation $x \to \varrho_n^{-1} x$ which transforms $\mathrm{Tor}^d_{r_n}$ into Tor^d_r. This contraction is equivalent to the usual renormalization of the random walk, i.e. to the consideration of the trajectories $\{Y_n(\tau) = \varrho_n^{-1} X_n(\tau), 0 \leqq \tau \leqq n\}$, where $Y_n(\tau) \in \mathrm{Tor}^d_r$, because $\varrho_n \sim \sqrt{n}$.

Let $f \in \mathscr{L}^2(\mathrm{Tor}^d_r, \mu)$ be a probability density on Tor^d_r with respect to the normalized Lebesgue measure μ. It defines the initial probability distribution f_n for the random walk on T_n, where $f_n(x) = \int_{\Delta_n(x)} f(y)\, d\mu(y)$, $\Delta_n(x)$ is a d-dimensional cube in Tor^d_r centered at $\varrho_n^{-1} x$, $x \in T_n$, with the side ϱ_n^{-1}. This initial distribution together with the matrix A_n define completely the probability distribution on the set of trajectories $X_n(\tau)$ or $Y_n(\tau)$, $0 \leqq \tau \leqq n$.

Gaussian distribution on the torus Tor^d_r with the covariance matrix a is the probability distribution whose density has the form

$$\theta^r_a(x) = \frac{r^d}{(2\pi)^{d/2} \sqrt{\det a}} \sum_{u \in r\mathbb{Z}^d} \exp\{-\tfrac{1}{2}(a^{-1}(x-u), (x-u))\}.$$

One certainly assumes that a is a nondegenerate positively-defined matrix. Brownian motion on Tor^d_r with the initial probability density f and covariance matrix a is the random process $\{Y(t), 0 \leqq t < \infty\}$ on Tor^d_r with independent increments for which $Y(0)$ is distributed according to f and $Y(t_2) - Y(t_1)$ has gaussian distribution with the covariance matrix $(t_2 - t_1)a$. Now we can formulate the main result of this paper.

Theorem 1. *There exists a nondegenerate positively-definite matrix a not depending on r such that for almost all A finite-dimensional probability distributions of the process* $\{Y_n(t \cdot n), 0 \leqq t \leqq 1\}$ *(continuous time) or* $\{Y_n([t \cdot n]), 0 \leqq t \leqq 1\}$ *(discrete time) converge weakly as* $n \to \infty$ *to the corresponding finite-dimensional probability distribution of the Brownian motion on* Tor^d_r *with the initial density f and covariance matrix a.*

In the case of $\mathfrak{A}^+$ consisting of unit coordinate vectors we have a sharper result. Let $\{X(\tau), 0 \leqq \tau \leqq n\}$ be a random walk on $\mathbb{Z}^d$ which is defined by the original random matrix A and the probability density f defines as before the initial probability distribution on $\mathbb{Z}^d$. Let $Y_n(\tau) = n^{-1/2} X_n(\tau)$. We assume also that f is square-integrable and has a finite support.

Theorem 2. *For almost all A finite-dimensional probability distributions of the process* $\{Y_n(t \cdot n), 0 \leqq t \leqq 1\}$ *(continuous time) or* $\{Y_n([t \cdot n]), 0 \leqq t \leqq 1\}$ *(discrete time) converge weakly to the corresponding probability distributions of the Brownian motion on* $\mathbb{R}^d$ *with the initial probability density f and the covariances matrix a which is the same as in Theorem 1.*

Proof of Theorem 1 is given in Sect. 3. We show that for eigenvalues of the transition operator which are sufficiently close to the boundary of the spectrum the corresponding eigenfunctions are close to $\exp\{2\pi i(\Lambda/r, x)\}$, $\Lambda \in \mathbb{Z}^d$ and thus are nonlocalized. This result is of a more general interest for the theory of random operators. The main part of our arguments concerns the derivation of a more or less explicit expression for the matrix a (see, in particular, Sect. 6).

Another approach to the whole set of problems was developed by Papanicolaou and Varadhan (see [11]) and S. Koslov [12] mainly for the case of solutions of the diffusion equation with random coefficients. In a unpublished paper by Molchanov this method was applied to the case of random walks on the lattice where results which are in some respects stronger than ours were obtained. However, as far as we know this approach does not lead to any explicit formula for the diffusion matrix.

2. Main Lemma

Let us introduce a probability distribution μ_n on T_n putting the measure of each point equal to r_n^{-d}. Denote $H_n = \mathscr{L}^2(T_n, \mu_n)$ and $H_n^{(0)}$ is the subspace of H_n of functions with the mean equal to zero. Also $H = \mathscr{L}^2(\mathrm{Tor}_r^d, \mu)$ and $H^{(0)}$ is the subspace of H of functions whose integral over Tor_r^d with respect to the Lebesgue measure μ is zero. It follows from the symmetry $a(x, y) = a(y, x)$ that A_n, $\bar{A}_n$ are self-adjoint operators in H_n leaving invariant $H_n^{(0)}$ and the one-dimensional subspace of constants. We denote by $A_n^{(0)}$ and $\bar{A}_n^{(0)}$ the restrictions of A_n, $\bar{A}_n$ to the subspace $H_n^{(0)}$. One has a natural orthonormal basis of functions $e_\lambda^{(n)}(x) = \exp\{2\pi i(\lambda, x)\}$, $\lambda = \Lambda/r_n$, $\Lambda \in T_n$ in H_n. We can assume that $\Lambda = \{\Lambda_1, \ldots, \Lambda_d\}$, $-\frac{1}{2}r_n \leqq \Lambda_j < \frac{1}{2}r_n$. In the same way the set of functions $e_\Lambda(x) = \exp\left\{\frac{2\pi i}{r}(\Lambda, x)\right\}$, $\Lambda \in \mathbb{Z}^d$ is an orthonormal basis in H. The functions $e_\lambda^{(n)}$ are eigenfunctions of $\bar{A}_n$ and $\bar{A}_n e_\lambda^{(n)} = \bar{a}_n(\lambda) e_\lambda^{(n)}$. For small λ we have $\bar{a}_n(\lambda) = 2\pi^2(\bar{a}\lambda, \lambda) + o(|\lambda|^2)$, where $\bar{a}$ is a nondegenerate positively-defined matrix (n-d. p-d. m.) which does not depend on n and r. We expect that A_n has also eigenvalues of the form $a_n(\lambda) = 2\pi^2(a\lambda, \lambda) + o(|\lambda|^2)$, where a is a n-d. p-d. m. In order to extract a quadratic part of an $a_n(\lambda)$ and not to deal with the unbounded spectra, we shall pass to the operator $n^{-1}(A_n^{(0)})^{-1}$. It will be seen that this passage has a more deep meaning.

For any n-d. p-d. m. a we introduce the operator A_∞ acting in H via the formula

$$A_\infty e_\Lambda = \frac{2\pi^2}{r^2}(a\Lambda, \Lambda) e_\Lambda, \qquad \Lambda \in \mathbb{Z}^d,$$

$A_\infty^{(0)}$ is its restriction to $H^{(0)}$. We want to show that there exists a such that for almost all A the sequence of operators $n^{-1}(A_n^{(0)})^{-1}$ converges in a proper sense to $(A_\infty^{(0)})^{-1}$. Now our goal is to make this argument more exact. Let us put for any $f \in H_n$ and $y \in \mathrm{Tor}_r^d$

$$(\Pi_n f)(y) = f(x)$$

if $y \in \Delta_n(x)$. Then Π_n is an isometric embedding of H_n into H and $\Pi_n H_n^{(0)} \subset H^{(0)}$. We introduce an orthogonal projection Ψ_n of H onto $\Pi_n H_n$. Then for any $f \in H$ and $x \in T_n$

$$(\Pi_n^{-1} \Psi_n f)(x) = \frac{1}{\mu(\Delta_n(x))} \int_{\Delta_n(x)} f(y)\, d\mu(y).$$

In particular for any $\Lambda \in \mathbb{Z}^d$, $\lambda = \Lambda/r_n$, $\kappa_n(\lambda) = \prod_{j=1}^{d} \frac{\sin \pi\lambda_j}{\pi\lambda_j}$

$$(\Pi_n^{-1}\Psi_n e_\Lambda)(x) = \kappa_n(\lambda) e_\lambda^{(n)}(x).$$

We denote by $\Pi_n^{(0)}$ and $\Psi_n^{(0)}$ the restrictions of Π_n and Ψ_n to $H_n^{(0)}$, $H^{(0)}$ respectively.

Main Lemma. *There exists a n-d. p-d. m. a not depending on r and such that for almost all A and $n \to \infty$*

$$\|n^{-1}\Pi_n^{(0)}(A_n^{(0)})^{-1}(\Pi_n^{(0)})^{-1}\Psi_n^{(0)} - (A_\infty^{(0)})^{-1}\|_{H^{(0)}} \to 0.$$

Let us write

$$(A_n^{(0)})^{-1} = E(A_n^{(0)})^{-1} + (\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2},$$

where $L_n = (\bar{A}_n^{(0)})^{1/2}((A_n^{(0)})^{-1} - E(A_n^{(0)})^{-1})(\bar{A}_n^{(0)})^{1/2}$.

The proof of the main lemma is based upon the following lemmas.

Lemma 1. *The operator $E(A_n^{(0)})^{-1}$ is diagonal in the basis of functions $e_\lambda^{(n)}$, $\lambda = \Lambda/r_n$, $\Lambda \in T_n \backslash 0$ and there exists a n-d. p-d. m. a not depending on r and such that for $n \to \infty$, $\lambda = \Lambda/r_n$, $\Lambda \in \mathbb{Z}^d \backslash 0$ being fixed,*

$$n((E(A_n^{(0)})^{-1})^{-1} e_\lambda^{(n)}, e_\lambda^{(n)})_{H_n^{(0)}} \to \frac{2\pi^2}{r^2}(a\Lambda, \Lambda).$$

Lemma 2. *For almost every A and fixed $\Lambda_1, \Lambda_2 \in \mathbb{Z}^d \backslash 0$*

$$(L_n e_{\lambda_1}^{(n)}, e_{\lambda_2}^{(n)})_{H_n^{(0)}} \to 0, \qquad n \to \infty,$$

where $\lambda_1 = \Lambda_1/r_n$, $\lambda_2 = \Lambda_2/r_n$.

In both cases the scalar product is taken in the subspace $H_n^{(0)}$.

Majorizing Lemma. *Let $A = \|a(x,y)\|$, $A' = \|a'(x,y)\|$ be two matrices satisfying the conditions on the p. 450 of the paper. If $a(x,y) \leqq a'(x,y)$ for $x \neq y$ then $A \geqq A' \geqq 0$ and $A_n \geqq A_n' \geqq 0$.*

Corollaries. 1. $(1-\delta)\bar{A}_n \leqq A_n \leqq (1+\delta)\bar{A}_n$.

2. *Let $D_n = (\bar{A}_n^{(0)})^{-1/2}(\bar{A}_n^{(0)} - A_n^{(0)})(\bar{A}_n^{(0)})^{-1/2}$. Then $\|D_n\|_{H_n^{(0)}} \leqq \delta$.*

3. *There exists a n-d. p-d. m. b such that if one puts $B_n e_\lambda^{(n)} = (b\Lambda, \Lambda) e_\lambda^{(n)}$, $\lambda = \Lambda/r_n$, $\Lambda \in T_n$ then $nA_n \geqq B_n$.*

4. *In the case of discrete time $\|A_n\|_{H_n} \leqq 2(1-\delta_0)$.*

Majorizing Lemma and its corollaries will be proven in Appendix 1.

Proof of the Main Lemma. Let us introduce for any n-d. p-d. m. a a linear operator $G_n^{(0)}: H_n^{(0)} \to H_n^{(0)}$, where $G_n^{(0)} e_\lambda^{(n)} = \frac{2\pi^2}{r^2}(a\Lambda, \Lambda) e_\lambda^{(n)}$, $\lambda = \Lambda/r_n$, $\Lambda \in T_n \backslash 0$. We have

$$(A_n^{(0)})^{-1} = E(A_n^{(0)})^{-1} + (\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2}.$$

Let us consider the following three statements.

1. $n^{-1}\|(\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2}\|_{H_n^{(0)}} \to 0$ as $n \to \infty$.
2. $\|n^{-1}E(A_n^{(0)})^{-1} - (G_n^{(0)})^{-1}\|_{H_n^{(0)}} \to 0$ as $n \to \infty$.
3. $\|\Pi_n^{(0)}(G_n^{(0)})^{-1}(\Pi_n^{(0)})^{-1}\Psi_n^{(0)} - (A_\infty^{(0)})^{-1}\|_{H^{(0)}} \to 0$ as $n \to \infty$.

Because the map $B \to \Pi_n^{(0)} B (\Pi_n^{(0)})^{-1} \Psi^{(0)}$ is an isometric embedding of the algebra of linear operators of $H_n^{(0)}$ into the algebra of bounded linear operators of $H^{(0)}$ the assertion of the main lemma is an immediate consequence of these three statements.

In view of Corollary 3 one can find $b>0$ such that if one puts $\lambda = \Lambda/r_n$, $\Lambda \in T_n \backslash 0$, $B_n^{(0)} e_\lambda^{(n)} = b(\Lambda, \Lambda) e_\lambda^{(n)}$ then

$$nA_n^{(0)} \geqq B_n^{(0)}, \qquad n\bar{A}_n^{(0)} \geqq B_n^{(0)}, \qquad G_n^{(0)} \geqq B_n^{(0)}.$$

Let E_R be an orthogonal projection of $H_n^{(0)}$ onto the subspace generated by the vectors $e_\lambda^{(n)}$, $|\Lambda| \leqq R$ and $E_R^\perp = I - E_R$. Then

$$\begin{aligned} n^{-1}(\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2} &= n^{-1}(\bar{A}_n^{(0)})^{-1/2} E_R L_n E_R (\bar{A}_n^{(0)})^{-1/2} \\ &+ n^{-1} E_R (\bar{A}_n^{(0)})^{-1/2} L_n E_R^\perp (\bar{A}_n^{(0)})^{-1/2} E_R^\perp + n^{-1} E_R^\perp (\bar{A}_n^{(0)})^{-1/2} E_R^\perp L_n (\bar{A}_n^{(0)})^{-1/2} E_R \\ &+ n^{-1} E_R^\perp (\bar{A}_n^{(0)})^{-1/2} E_R^\perp L_n E_R^\perp (\bar{A}_n^{(0)})^{-1/2} E_R^\perp . \end{aligned}$$

We have $\|L_n\|_{H_n^{(0)}} \leqq C$ where C does not depend on n, and

$$\|n^{-1}(\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2}\|_{H_n^{(0)}} \leqq b^{-1} \|E_R L_n E_R\|_{H_n^{(0)}} + 2b^{-1} C/R + b^{-1} C/R^2 .$$

It follows from Lemma 2 that for $n \to \infty$

$$b^{-1} \|E_R L_n E_R\|_{H_n^{(0)}} \leqq \varepsilon_R(n) \to 0 .$$

Let us take $\varepsilon > 0$ and large enough R_0 such that $2b^{-1}C/R_0 + b^{-1}C/R_0^2 < \varepsilon/2$. Then we choose n_0 in such a way that $\varepsilon_{R_0}(n) < \varepsilon/2$ as $n > n_0$. We get for $n > n_0$

$$\|n^{-1}(\bar{A}_n^{(0)})^{-1/2} L_n (\bar{A}_n^{(0)})^{-1/2}\|_{H_n^{(0)}} < \varepsilon .$$

Thus Statement 1 is proven. Statement 2 is proven in an analogous way. Indeed, from Lemma 2 and fixed R

$$\|E_R (n^{-1} E (A_n^{(0)})^{-1} - (G_n^{(0)})^{-1}) E_R\|_{H_n^{(0)}} \to 0, \qquad n \to \infty$$

and

$$\|E_R^\perp (n^{-1} E (A_n^{(0)})^{-1} - (G_n^{(0)})^{-1}) E_R^\perp\|_{H_n^{(0)}} \leqq 2b^{-1} R^{-2} .$$

Now we proceed to the proof of Statement 3. Let us remark that

$$(\Pi_n^{(0)})^{-1} \Psi_n^{(0)} e_\Lambda = \begin{cases} \kappa_n(\lambda) e_\lambda^{(n)}, & \lambda = \Lambda/r_n,\ \Lambda \in T_n \backslash 0 \\ 0, & \lambda = \Lambda/r_n,\ \Lambda \in (\mathbb{Z}^d \backslash T_n) \backslash 0 . \end{cases}$$

where $\kappa_n(\lambda) = \prod_{j=1}^{d} \frac{\sin \pi\lambda_j}{\pi\lambda_j} \to 1$ for fixed Λ, $n \to \infty$ and

$$\|\Pi_n^{(0)} e_\lambda^{(n)} - e_\Lambda\|_{H^{(0)}} \to 0, \qquad \lambda = \Lambda/r_n,\ \Lambda \in \mathbb{Z}^d \backslash 0,\ n \to \infty .$$

This gives

$$\|E_R (\Pi_n^{(0)} (G_n^{(0)})^{-1} (\Pi_n^{(0)})^{-1} \Psi_n^{(0)} - (A_\infty^{(0)})^{-1})\|_{H^{(0)}} \to 0 ,$$

$$\|(\Pi_n^{(0)} (G_n^{(0)})^{-1} (\Pi_n^{(0)})^{-1} \Psi_n^{(0)} - (A_\infty^{(0)})^{-1}) E_R\|_{H^{(0)}} \to 0 ,$$

for fixed R and $n \to \infty$. Moreover

$$\|E_R^\perp \Pi_n^{(0)} (G_n^{(0)})^{-1} (\Pi_n^{(0)})^{-1} \Psi_n^{(0)} E_R^\perp\|_{H^{(0)}} \leqq b^{-1} R^{-2} ,$$

$$\|E_R^\perp (A_\infty^{(0)})^{-1} E_R^\perp\|_{H^{(0)}} \leqq b^{-1} R^{-2} .$$

Another application of the same arguments as in the proof of Statement 1 gives Statement 3.

3. Proof of Theorem 1

We shall use two general theorems of the perturbation theory of linear operators.

Theorem A. *In the Hilbert space H, consider a given sequence of bounded self-adjoint operators S_n and $\|S_n - S\| \to 0$ as $n \to \infty$, where S is a bounded self-adjoint operator. Let ω be an isolated eigenvalue of S, $\mathscr{E}$, and E being the corresponding subspace and the orthogonal projection. Then for all large enough n one can find a subspace $\mathscr{E}_n$ invariant under S_n and the corresponding projection E_n such that for $n \to \infty$*

$$1.\ \|E_n - E\| \to 0\,; \qquad 2.\ \|E_n(S_n - \omega I)E_n\| \to 0\,.$$

Theorem B. *Let φ_n be a sequence of measurable functions defined on $\mathbb{R}^+$, $\varphi_n(0)=0$ and φ_n are uniformly continuous at 0. Assume also that φ_n converge uniformly on any compact subset of $\mathbb{R}^+$ to a continuous function φ. If S_n is a sequence of bounded non-negative self-adjoint operators converging to a non-negative compact self-adjoint operator S in the topology of norm-convergence, then $\varphi_n(S_n) \to \varphi(S)$ as $n \to \infty$ in the same topology.*

Proof of Theorems A and B is given in the Appendices.

Let us take $t>0$ and put $P_n(t) = \Pi_n \exp\{-tnA_n\} \Pi_n^{-1} \Psi_n$, $P_n^{(0)}(t) = \Pi_n^{(0)} \exp\{-tnA_n^{(0)}\} (\Pi_n^{(0)})^{-1} \Psi_n^{(0)}$ in the case of continuous time and $P_n(t) = \Pi_n(I - A_n)^{[nt]} \Pi_n^{-1} \Psi_n$, $P_n^{(0)}(t) = \Pi_n^{(0)}(I - A_n^{(0)})^{[nt]}\ (\Pi_n^{(0)})^{-1} \Psi_n^{(0)}$ in the case of discrete time. The sequence φ_n, where

$$\varphi_n(\omega) = \begin{cases} e^{-t/\omega}, & \omega > 0 \\ 0, & \omega = 0 \end{cases}$$

in the case of continuous time, while in the case of discrete time

$$\varphi_n(\omega) = \begin{cases} \left(1 - \dfrac{1}{n\omega}\right)^{[nt]}, & \omega \geqq \dfrac{1}{2n(1-\delta_0)} \\ 0, & 0 \leqq \omega < \dfrac{1}{2n(1-\delta_0)} \end{cases}$$

satisfies all conditions of Theorem B with $\varphi(\omega) = \exp\{-t/\omega\}$. We consider the operator $S = (A_\infty^{(0)})^{-1}$ and the sequence of the operators $S_n = n^{-1} \Pi_n^{(0)} (A_n^{(0)})^{-1} \cdot (\Pi_n^{(0)})^{-1} \Psi_n^{(0)}$. It follows from the Main Lemma that they also satisfy the conditions of Theorem B. Also $\varphi_n(S_n) = P_n^{(0)}(t)$, $\varphi(S) = \exp\{-tA_\infty^{(0)}\}$ which follows from Corollary 4 in the case of discrete time. Thus we obtain

$$P_n^{(0)}(t) \to P_\infty^{(0)}(t) = \exp\{-tA_\infty^{(0)}\}$$

in the topology of norm-convergence. We have also

$$P_n(t) \to P_\infty(t) = \exp\{-tA_\infty\} \tag{1}$$

in the same topology.

Let us take a finite set of numbers $0<t_1<t_2<\ldots<t_m$ and a set of functions $f\in H$, $g_1, g_2, \ldots, g_m \in \mathscr{L}^\infty(\mathrm{Tor}_r^d)$. We denote by T_g an operator of multiplication on g. The statement of Theorem 1 means that for $n\to\infty$

$$\lim_{n\to\infty} \int_{\mathrm{Tor}_r^d} (T_{g_m}P_n(t_m-t_{m-1})T_{g_{m-1}}P_n(t_{m-1}-t_{m-2})\ldots T_{g_2}P_n(t_2-t_1)T_{g_1}P_n(t_1)f)(x)\,d\mu(x)$$
$$= \int_{\mathrm{Tor}_r^d} (T_{g_m}P_\infty(t_m-t_{m-1})T_{g_{m-1}}P_\infty(t_{m-1}-t_{m-2})\ldots T_{g_2}P_\infty(t_2-t_1)$$
$$\cdot T_{g_1}P_\infty(t_1)f)(x)\,d\mu(x).$$

But this equality follows immediately from (1).

4. Proof of Lemmas 1 and 2

For the operator $D_n=(\bar{A}_n^{(0)})^{-1/2}(\bar{A}_n^{(0)}-A_n^{(0)})(\bar{A}_n^{(0)})^{-1/2}$ we have from the Corollary 2 $\|D_n\|_{H_n^{(0)}}\leqq\delta<\frac{1}{2}$. The first statement of the lemma is obvious because the operators $(\bar{A}_n^{(0)})^{-1/2}$ commute with the translations and the probability distribution of D_n is translationally-invariant. Therefore the operator $E(I-D_n)^{-1}$ also commutes with the translations.

From the estimation $\|D_n\|_{H_n^{(0)}}\leqq\delta<1/2$ it follows that $\|ED_n^k\|_{H_n^{(0)}}\leqq\delta^k$. Thus $\|I-E(I-D_n)^{-1}\|_{H_n^{(0)}}\leqq\sum_{k=1}^{\infty}\|ED_n^k\|_{H_n^{(0)}}\leqq\delta/(1-\delta)<1$ and the operator $E(I-D_n)^{-1}$ is invertible. The operators $\bar{A}_n^{(0)}, F_k^{(n)}=ED_n^k$ commute with translations, and so in the Fourier representation are multiplications by the functions $\bar{a}_n(\lambda)=(\bar{A}_n^{(0)}e_\lambda^{(n)}, e_\lambda^{(n)})_{H_n^{(0)}}$ and $f_k^{(n)}(\lambda)=(F_k^{(n)}e_\lambda^{(n)}, e_\lambda^{(n)})_{H_n^{(0)}}$, where $\lambda=\Lambda/r_n$, $\Lambda\in T_n\backslash 0$. Let us put $C_n=(E(I-D_n)^{-1})^{-1}$. Then C_n is in the Fourier representation the multiplication to the function $c_n(\lambda)=\left(1+\sum_{k=1}^{\infty}f_k^{(n)}(\lambda)\right)^{-1}$. We shall prove now that for each $\Lambda\in\mathbb{Z}^d\backslash 0$ there exist $\lim_{n\to\infty}c_n(\lambda)=c(\Lambda)$, $\lambda=\Lambda/r_n$ and a n-d. p-d. m. a, for which $(\bar{a}\Lambda,\Lambda)c(\Lambda)$ $=(a\Lambda,\Lambda)$ where $(\bar{a}\Lambda,\Lambda)=r^2/(2\pi^2)\lim_{n\to\infty}n\bar{a}_n(\lambda)$. We shall start by proving the existence of $\lim_{n\to\infty}c_n(\lambda)$. Let us write down the explicit expression for $f_k^{(n)}(\lambda)$:

$$f_k^{(n)}(\lambda)=(-1)^k r_n^{-d}\sum_{(z_1,\alpha_1),\ldots,(z_k,\alpha_k)} E(q_{\alpha_1}^{(n)}(z_1)\ldots q_{\alpha_k}^{(n)}(z_k))h_{\alpha_1}^{(n)}(\lambda)\overline{h_{\alpha_k}^{(n)}(\lambda)}$$
$$\cdot\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2-z_1)\Gamma_{\alpha_2\alpha_3}^{(n)}(z_3-z_2)\ldots\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})e^{2\pi i(\lambda,z_1-z_k)},\ z_j\in T_n,\ \alpha_j\in\mathfrak{A}^+ \quad (2)$$
$$q_\alpha^{(n)}(z)=q_n(z,z+\alpha),\qquad h_\alpha^{(n)}(\lambda)=(e^{2\pi i(\lambda,\alpha)}-1)(\bar{\alpha}_n(\lambda))^{-1/2},$$
$$\Gamma_{\alpha\beta}^{(n)}(z)=r_n^{-d}\sum_{\substack{\lambda:\lambda=\Lambda/r_n\\ \Lambda\in T_n\backslash 0}}\overline{h_\alpha^{(n)}(\lambda)}\,h_\beta^{(n)}(\lambda)e^{2\pi i(\lambda,z)}.$$

One can easily check the following properties of the function $\Gamma_{\alpha\beta}^{(n)}(z)$:

1) for each $z\in\mathbb{Z}^d$ there exists

$$\lim_{n\to\infty}\Gamma_{\alpha\beta}^{(n)}(z)=\Gamma_{\alpha\beta}(z)=-\int_{\mathrm{Tor}_1^d}\frac{(e^{-2\pi i(\lambda,\alpha)}-1)(e^{2\pi i(\lambda,\beta)}-1)}{4\sum_{\gamma\in\mathfrak{A}^+}\bar{a}(0,\gamma)\sin^2(\pi(\lambda,\gamma))}e^{2\pi i(\lambda,z)}\,d\lambda,$$

2) for a constant C

$$|\Gamma^{(n)}_{\alpha\beta}(z)| \leqq C((d_n(z,0))^d+1)^{-1},$$

where d_n is the euclidean metric on T_n.

We remark that $\Gamma_{\alpha\beta}(z)$ is not absolutely integrable and this generates the main analytical difficulties of the problem.

In view of $Eq^{(n)}_\alpha(z)=0$ and the independence of different $q^{(n)}_\alpha(z)$ the nonzeroth contribution to (2) is from the terms where each pair (z,α) enters in the sequence of pairs $\{(z_1,\alpha_1),\ldots,(z_k,\alpha_k)\}$ more then once. It gives in particular $f^{(n)}_1(\lambda)=0$. It is convenient to imagine each $\{(z_1,\alpha_1),\ldots,(z_k,\alpha_k)\}=\gamma$ as a path. In terms of the paths we should consider only those paths which pass through each of its points not less than twice. We shall consider the limit of $c_n(\lambda)=\left(1+\sum_{k=1}^{\infty} f^{(n)}_k(\lambda)\right)^{-1}$, $\lambda=\Lambda/r_n$. Let us rewrite it as follows:

$$c_n(\lambda)=1+\sum_{k=1}^{\infty}\bar{f}^{(n)}_k(\lambda),$$

where

$$\bar{f}^{(n)}_k(\lambda)=\sum_{m=1}^{k}(-1)^m\sum_{l_1+\ldots+l_m=k} f^{(n)}_{l_1}(\lambda)\,f^{(n)}_{l_2}(\lambda)\ldots f^{(n)}_{l_m}(\lambda).$$

Let $k=l_1+l_2+\ldots+l_m$. We shall use the equality:

$$f^{(n)}_{l_1}(\lambda)\ldots f^{(n)}_{l_m}(\lambda)=(-1)^k r_n^{-d}\sum_\gamma E(q^{(n)}_{\alpha_1}(z_1)\ldots q^{(n)}_{\alpha_{l_1}}(z_{l_1}))\,E(q^{(n)}_{\alpha_{l_1+1}}(z_{l_1+1})\ldots q^{(n)}_{\alpha_{l_1+l_2}}(z_{l_1+l_2}))\ldots$$
$$E(q^{(n)}_{\alpha_{l_1+l_2+\ldots+l_{m-1}+1}}(z_{l_1+l_2+\ldots+l_{m-1}+1})\ldots q^{(n)}_{\alpha_k}(z_k))\,h^{(n)}_{\alpha_1}(\lambda)\,\overline{h^{(n)}_{\alpha_k}(\lambda)}\,\Gamma^{(n)}_{\alpha_1\alpha_2}(z_2-z_1)\ldots$$
$$\Gamma^{(n)}_{\alpha_{k-1}\alpha_k}(z_k-z_{k-1})\,e^{2\pi i(\lambda,z_1-z_k)}.$$

We can rewrite the expression for

$$\bar{f}^{(n)}_k(\lambda)=(-1)^k r_n^{-d}\sum_\gamma E(\gamma)\,h^{(n)}_{\alpha_1}(\lambda)\,h^{(n)}_{\alpha_k}(\lambda)\,\Gamma^{(n)}_{\alpha_1\alpha_2}(z_2-z_1)\ldots\Gamma^{(n)}_{\alpha_{k-1}\alpha_k}(z_k-z_{k-1})\,e^{2\pi i(\lambda,z_1-z_k)},$$

$$E(\gamma)=\sum_{m=1}^{k}(-1)^m\sum_{l_1+\ldots+l_m=k} E(q^{(n)}_{\alpha_1}(z_1)\ldots q^{(n)}_{\alpha_{l_1}}(z_{l_1}))\ldots$$
$$E(q^{(n)}_{\alpha_{l_1+l_2+\ldots+l_{m-1}+1}}(z_{l_1+l_2+\ldots+l_{m-1}+1})\ldots q^{(n)}_{\alpha_k}(z_k)).$$

Assertion I. *Let a path* $\gamma=\{(z_1,\alpha_1),\ldots,(z_k,\alpha_k)\}$ *be decomposed onto two paths* $\gamma_1=\{(z_1,\alpha_1),\ldots,(z_j,\alpha_j)\}$, $\gamma_2=\{(z_{j+1},\alpha_{j+1}),\ldots,(z_k,\alpha_k)\}$ *in such a way that* $\gamma=\gamma_1\cup\gamma_2$, $\gamma_1\cap\gamma_2=\emptyset$. *Then* $E(\gamma)=0$.

Proof. We consider sets of natural numbers $l_1,l_2,\ldots,l_m$ such that $l_1+l_2+\ldots+l_m=k$. They can be of two types.

1) $l_1+l_2+\ldots+l_s=j$ for some s, $1\leqq s\leqq m$;

2) $l_1+l_2+\ldots+l_s\neq j$ for all s, $1\leqq s\leqq m$.

One can correspond to each set of the second type a set of the first type in the following way. Let us take s_0 for which $l_1+l_2+\ldots+l_{s_0}<j$, $l_1+l_2+\ldots+l_{s_0+1}>j$ and construct a set of the first type $l'_1,l'_2,\ldots,l'_{m+1}$, where $l'_s=l_s$ for $s\leqq s_0$, $l'_{s_0+1}=j-(l_1+l_2+\ldots+l_{s_0})$, $l'_{s_0+2}=(l_1+l_2+\ldots+l_{s_0+1})-j$, $l'_s=l_{s-1}$ for $s>s_0+2$. This correspondence is one-to-one. We have

$$\begin{aligned} &E(q^{(n)}_{\alpha_{l_1+\ldots+l_{s_0}+1}}(z_{l_1+\ldots+l_{s_0}+1})\ldots q^{(n)}_{\alpha_{l_1+\ldots+l_{s_0+1}}}(z_{l_1+\ldots+l_{s_0+1}}))\\ &\quad=E(q^{(n)}_{\alpha_{l'_1+\ldots+l'_{s_0}+1}}(z_{l'_1+\ldots+l'_{s_0}+1})\ldots q^{(n)}_{\alpha_{l'_1+\ldots+l'_{s_0+1}}}(z_{l'_1+\ldots+l'_{s_0+1}}))\\ &\qquad\cdot E(q^{(n)}_{\alpha_{l'_1+\ldots+l'_{s_0+1}+1}}(z_{l'_1+\ldots+l'_{s_0+1}+1})\ldots q^{(n)}_{\alpha_{l'_1+\ldots+l'_{s_0+2}}}(z_{l'_1+\ldots+l'_{s_0+2}})),\end{aligned}$$

which is equal to the product entering into the sum for $E(\gamma)$ for the corresponding set of the second type. Our assertion follows from the fact that the corresponding terms in $E(\gamma)$ have different signs. Q.E.D.

It is obvious also that $E(\gamma)=0$ if the path γ passes through a point only once. We can write now

$$\bar{f}_k^{(n)}(\lambda)=(-1)^k r_n^{-d}\sum_{\gamma}{}^{(1)} E(\gamma)\,h_{\alpha_1}^{(n)}(\lambda)\,\overline{h_{\alpha_k}^{(n)}(\lambda)}\,\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2-z_1)\ldots\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})e^{2\pi i(\lambda,z_1-z_k)},\tag{3}$$

where $\sum^{(1)}$ means the summation over nondecomposing paths passing through each of its points at least twice. It turns out that the summation only over these paths leads to an effective increasing of the power of decay of $\Gamma_{\alpha\beta}^{(n)}$. We shall prove the following assertion.

Assertion II. *For each k the series*

$$\sum_{\gamma=\{(0,\alpha_1),\ldots,(z_k,\alpha_k)\}}^{(1)} E(\gamma)\,h_{\alpha_1}^{(n)}(\lambda)\,\overline{h_{\alpha_k}^{(n)}(\lambda)}\,\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2)\,\Gamma_{\alpha_2\alpha_3}^{(n)}(z_3-z_2)\ldots\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})e^{-2\pi i(\lambda,z_k)}$$

converges absolutely and uniformly with respect to n.

Proof. We have to estimate

$$\sum_{\gamma=\{(0,\alpha),(z_2,\alpha_2),\ldots,(z_{k-1},\alpha_{k-1}),(z_k,\beta)\}}^{(1)} |\Gamma_{\alpha\alpha_2}^{(n)}(z_2)|\cdot|\Gamma_{\alpha_2\alpha_3}^{(n)}(z_3-z_2)|\cdot\ldots\cdot|\Gamma_{\alpha_{k-1}\beta}(z_k-z_{k-1})|.$$

In view of the finiteness of $\mathfrak{A}^+$ and the properties of $\Gamma_{\alpha\beta}^{(n)}(z)$ it is sufficient to establish the convergence of the series

$$W=\sum_{\gamma'}{}^{(1)}(1+|z_2|^d)^{-1}(1+|z_3-z_2|^d)^{-1}\ldots(1+|z_k-z_{k-1}|^d)^{-1},\tag{4}$$

where the sum is taken over such paths $\gamma'=\{0,z_2,\ldots,z_k\}$, $z_j\in\mathbb{Z}^d$ that each of its points is visited not less than twice and the path γ' is nondecomposable.

We denote by V_k the set $\{1,2,\ldots,k\}$ and ξ is a partition of V_k such that there does not exist j, $1<j<k$, for which $\{1,2,\ldots,j\}$, $\{j+1,\ldots,k\}$ are unions of elements of ξ. It is an abstract description of the nondecomposability. $\mathfrak{N}(\xi)$ is a set of such γ' that $z_i=z_j$ iff i and j belong to the same element of ξ, $\mathfrak{M}(\xi)$ is a set of such γ' that $z_i=z_j$ if i and j belong to the same element of ξ. Then

$$\begin{aligned}W&=\sum_{\xi}\sum_{\gamma'\in\mathfrak{N}(\xi)}(1+|z_2|^d)^{-1}(1+|z_3-z_2|^d)^{-1}\ldots(1+|z_k-z_{k-1}|^d)^{-1}\\ &\leqq\sum_{\xi}\sum_{\gamma'\in\mathfrak{M}(\xi)}(1+|z_2|^d)^{-1}(1+|z_3-z_2|^d)^{-1}\ldots(1+|z_k-z_{k-1}|^d)^{-1}.\end{aligned}$$

The first step is to show that one can restrict himself by even k and such ξ for which each element of ξ consists of even number of points. In order to do this we shall correspond to any partition ξ of V_k a nondecomposable partition ξ' of $V_{k+k'}$, where k' is the number of elements of ξ having an odd number of points. The partition ξ' arises if one adds to each element of ξ with odd number of points one extra point which follows exactly after the last point of this element. It is easy to see that for a constant C_k depending only on k

$$W \leqq C_k \sum_{\xi'} \sum_{\gamma' \in \mathfrak{M}(\xi')} (1+|z_2|^d)^{-1}(1+|z_3-z_2|^d)^{-1} \dots (1+|z_k-z_{k-1}|^d)^{-1}.$$

The next step is to reduce the whole sum only to the sum over the partitions where each element contains only two points. In order to do this we construct for each ξ' a new partition ξ'' in the following manner. If an element of ξ' consists of two points then it coincides with an element of ξ''. If it consists of an even number of points then we decompose it on subsets having two elements in such a way that each subset consists of points which are equally distant of its ends. It is clear that $\mathfrak{M}(\xi') \subseteqq \mathfrak{M}(\xi'')$. Now it is sufficient to estimate the sum

$$\sum_{\gamma' \in \mathfrak{M}(\xi'')} (1+|z_2|^d)^{-1}(1+|z_3-z_2|^d)^{-1} \dots (1+|z_{2k}-z_{2k-1}|^d)^{-1}.$$

Lemma. *For every partition ξ'' of V_{2k} one can delete an element of ξ'' not containing* 1 *in such a way that the induced partition of V_{2k-2} will be nondecomposable.*

Proof. If l belongs to the same element as $2k$ and there is an element of ξ'' between l and $2k$ then we delete this element. If such an element does not exist then $l \neq 1$ and we delete the pair $\{l, 2k\}$. Q.E.D.

Let us take a partition ξ'' of V_{2k} and find in accordance with the lemma the element $\{i,j\}$ of ξ''. We assume $i<j$. There are three possibilities.

I. $i+1 \neq j, j \neq 2k$.

II. $i+1 = j, j \neq 2k$.

III. $i+1 \neq j, j = 2k$.

Denote $z = z_i = z_j$ and select a part of the product containing z. We shall have in these three cases:

I. $(1+|z-z_{i-1}|^d)^{-1}(1+|z_{i+1}-z|^d)^{-1}(1+|z-z_{j-1}|^d)^{-1}(1+|z_{j+1}-z|^d)^{-1}$.

II. $(1+|z-z_{i-1}|^d)^{-1}(1+|z_{j+1}-z|^d)^{-1}$.

III. $(1+|z-z_{i-1}|^d)^{-1}(1+|z_{i+1}-z|^d)^{-1}(1+|z-z_{2k-1}|^d)^{-1}$.

Let us make the summation over z. In the first case with the help of the Cauchy inequality

$$\begin{aligned}
&\sum_z (1+|z-z_{i-1}|^d)^{-1}(1+|z_{i+1}-z|^d)^{-1}(1+|z-z_{j-1}|^d)^{-1}(1+|z_{j+1}-z|^d)^{-1} \\
&\leqq \left(\sum_z (1+|z-z_{i-1}|^d)^{-2}(1+|z_{i+1}-z|^d)^{-2}\right)^{1/2} \\
&\quad \cdot \left(\sum_z (1+|z-z_{j-1}|^d)^{-2}(1+|z_{j+1}-z|^d)^{-2}\right)^{1/2} \\
&\leqq \mathrm{const} \cdot (1+|z_{i+1}-z_{i-1}|^d)^{-1}(1+|z_{j+1}-z_{j-1}|^d)^{-1}.
\end{aligned}$$

Thus the summation gives the same product with $2k-2$ factors. It is easy to see that in the second case

$$\sum_z (1+|z-z_{i-1}|^d)^{-1}(1+|z_{j+1}-z|^d)^{-1} \leqq \text{const}\cdot\ln(2+|z_{i+2}-z_{i-1}|)\,(1+|z_{i+2}-z_{i-1}|^d)^{-1}.$$

Analogously in the third case

$$\begin{aligned}&\sum_z (1+|z-z_{i-1}|^d)^{-1}(1+|z_{i+1}-z|^d)^{-1}(1+|z-z_{2k-1}|^d)^{-1}\\ &\leqq \text{const}\cdot\sum_z (1+|z-z_{i-1}|^d)^{-1}(1+|z_{i+1}-z|^d)^{-1}\\ &\leqq \text{const}\cdot\ln(2+|z_{i+1}-z_{i-1}|)(1+|z_{i+1}-z_{i-1}|^d)^{-1}.\end{aligned}$$

Thus the summation over z gives an analogous product of $2k-2$ factors, where instead of $(1+|z_{s+1}-z_s|^d)^{-1}$ one has $\ln(2+|z_{s+1}-z_s|)(1+|z_{s+1}-z_s|^d)^{-1}$.

We can apply the same arguments and get a product of $2k-4$ factors of the form

$$(\ln(2+|z_{s+1}-z_s|))^3(1+|z_{s+1}-z_s|^d)^{-1}$$

and so on. Finally we shall have a product of four factors of the form

$$(\ln(2+|z_{s+1}-z_s|))^{2^{k-2}-1}(1+|z_{s+1}-z_s|^d)^{-1}.$$

The condition of nondecomposability implies that we should consider only two partitions $\{(1,4),(2,3)\}$ and $\{(1,3),(2,4)\}$. The part of the sum for each of them can be estimated by the expression

$$\text{const}\sum_z ((\ln(2+|z|))^{2^{k-2}-1})^3(1+|z|^d)^{-2}<\infty.$$

Thus Assertion II is proven.

Now we can complete the proof of Lemma 1. We have from (3)

$$\bar{a}_n(\lambda)\,\bar{f}_k^{(n)}(\lambda)=(-1)^k\sum_{\alpha,\beta\in\mathfrak{A}^+}(e^{2\pi i(\lambda,\alpha)}-1)(e^{-2\pi i(\lambda,\beta)}-1)\,D_{\alpha\beta}^{k,n}(\lambda),$$

where

$$D_{\alpha\beta}^{k,n}(\lambda)=\sum_{\substack{\gamma=\{(0,\alpha),(z_2,\alpha_2),\ldots,(z_{k-1},\alpha_{k-1}),(z,\beta)\}\\ z_j\in T_n}}^{(1)}\Gamma_{\alpha\alpha_2}^{(n)}(z_2)\,\Gamma_{\alpha_2\alpha_3}^{(n)}(z_3-z_2)\ldots\Gamma_{\alpha_{k-1}\beta}^{(n)}(z-z_{k-1})\cdot e^{-2\pi i(\lambda,z)}E(\gamma).$$

Let us put

$$D_{\alpha\beta}^k=\sum_{\gamma=\{(0,\alpha),\ldots,(z,\beta)\},\,z_j\in\mathbb{Z}^d}\Gamma_{\alpha\alpha_2}(z_2)\,\Gamma_{\alpha_2\alpha_3}(z_3-z_2)\ldots\Gamma_{\alpha_{k-1}\beta}(z-z_{k-1})\,E(\gamma).$$

Assertion II gives for every $\Lambda\in\mathbb{Z}^d\backslash 0$

$$\lim_{n\to\infty}D_{\alpha\beta}^{k,n}(\Lambda/r_n)=D_{\alpha\beta}^k.$$

Therefore

$$\lim_{n\to\infty} n\bar{a}_n(\Lambda/r_n)\,\bar{f}_k^{(n)}(\Lambda/r_n)$$
$$=(-1)^k \sum_{\alpha,\beta\in\mathfrak{A}^+} \lim_{n\to\infty} n(e^{2\pi i(\Lambda/r_n,\alpha)}-1)(e^{-2\pi i(\Lambda/r_n,\beta)}-1)\,D_{\alpha\beta}^{k,n}(\Lambda/r_n)$$
$$=(-1)^k\frac{(2\pi)^2}{r^2}\sum_{\alpha,\beta\in\mathfrak{A}^+}(\Lambda,\alpha)(\Lambda,\beta)\,D_{\alpha\beta}^k=-\frac{2\pi^2}{r^2}\sum_{j,m=1}^{d}(a^{(k)})_{jm}\Lambda_j\Lambda_m,$$

where $(a^{(k)})_{jm}=2(-1)^{k-1}\sum_{\alpha,\beta\in\mathfrak{A}^+}\alpha_j\beta_m D_{\alpha\beta}^k$. The estimation $\|ED_n^k\|\leqq\delta^k$, $\delta<1/2$ gives the uniform over n convergence of the series $c_n(\lambda)=1+\sum_{k=2}^{\infty}\bar{f}_k^{(n)}(\lambda)$. Therefore the limit $c(\Lambda)=\lim_{n\to\infty}c_n(\Lambda/r_n)$ exists and $(\bar{a}\Lambda,\Lambda)\,c(\Lambda)=(a\Lambda,\Lambda)$, $a=\bar{a}-\sum_{k=2}^{\infty}a^{(k)}$.

The fact that a is a n-d. p-d. m. follows from the Majorizing Lemma. Q.E.D.

Proof of Lemma 2. We shall assume that $d>1$. The case $d=1$ is simpler and can be treated in another way. We shall estimate matrix elements of the operator $L_n=\sum_{k=1}^{\infty}(D_n^k-ED_n^k)$. Let $(D_n^k e_\lambda^{(n)},e_{\lambda'}^{(n)})=D_n^k(\lambda,\lambda')$, $\lambda=\Lambda/r_n$, $\lambda'=\Lambda'/r_n$, and $\Lambda,\Lambda'\in T_n\backslash 0$. Then $(L_n e_\lambda^{(n)},e_{\lambda'}^{(n)})=\sum_{k=1}^{\infty}(D_n^k(\lambda,\lambda')-ED_n^k(\lambda,\lambda'))$. We shall estimate $E|D_n^k(\lambda,\lambda')-ED_n^k(\lambda,\lambda')|^2=E|D_n^k(\lambda,\lambda')|^2-|ED_n^k(\lambda,\lambda')|^2$. We have

$$D_n^k(\lambda,\lambda')=(-1)^k r_n^{-d}\sum q_{\alpha_1}^{(n)}(z_1)\dots q_{\alpha_k}^{(n)}(z_k)\,h_{\alpha_1}^{(n)}(\lambda)\,\overline{h_{\alpha_k}^{(n)}(\lambda')}\,\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2-z_1)\dots$$
$$\dots\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})\,e^{2\pi i((\lambda,z_1)-(\lambda',z_k))},$$
$$E|D_n^k(\lambda,\lambda')|^2=r_n^{-2d}\sum E(q_{\alpha_1}^{(n)}(z_1)\dots q_{\alpha_k}^{(n)}(z_k)\,q_{\alpha_1'}(z_1')\dots q_{\alpha_k'}(z_k'))$$
$$\cdot h_{\alpha_1}^{(n)}(\lambda)\,\overline{h_{\alpha_k}^{(n)}(\lambda')}\,\overline{h_{\alpha_1'}^{(n)}(\lambda)}\,h_{\alpha_k'}^{(n)}(\lambda')\,\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2-z_1)\dots\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})\,\overline{\Gamma_{\alpha_1'\alpha_2'}^{(n)}(z_2'-z_1')}\dots$$
$$\overline{\Gamma_{\alpha_{k-1}'\alpha_k'}^{(n)}(z_k'-z_{k-1}')}\,e^{2\pi i((\lambda,z_1-z_1')-(\lambda',z_k-z_k'))}.$$

Let a variable of the first group be equal to a variable of the second group, for example, $z_j=z_m'$. We denote the corresponding part of the sum by $(E|D_n^k(\lambda,\lambda')|^2)_{jm}$. From the inequality $|\Gamma_{\alpha\beta}^{(n)}(z)|\leqq\mathrm{const}(1+(d_n(0,z))^d)^{-1}$, where const does not depend on n, we get $\sum_{z\in T_n}|\Gamma_{\alpha\beta}^{(n)}(z)|\leqq\mathrm{const}\cdot\ln r_n$. Making the summation over all variables except $z_j=z_m'=z$ we get

$$|(E|D_n^k(\lambda,\lambda')|^2)_{jm}|\leqq\delta_1^{2k}r_n^{-2d}\sum_{\alpha_1,\dots,\alpha_k\in\mathfrak{A}^+}\ \sum_{\alpha_1',\dots,\alpha_k'\in\mathfrak{A}^+}\ \sum_{z\in T_n}(c\ln r_n)^{2k-2}$$
$$\sup_{\alpha\in\mathfrak{A}^+,\,\lambda\in\mathrm{Tor}_1^d\backslash 0}|h_\alpha^{(n)}(\lambda)|^4\leqq c_1^k(\ln r_n)^{2k-2}r_n^{-d}.$$

Therefore

$$\sum_{j,m=1}^{k}|(E|D_n^k(\lambda,\lambda')|^2)_{jm}|\leqq k^2c_1^k(\ln r_n)^{2k-2}r_n^{-d}.$$

We have

$$|ED_n^k(\lambda,\lambda')|^2 = r_n^{-2d} \sum E(q_{\alpha_1}^{(n)}(z_1)\dots q_{\alpha_k}^{(n)}(z_k))\, E(q_{\alpha'_1}^{(n)}(z'_1)\dots q_{\alpha'_k}^{(n)}(z'_k))$$
$$\cdot h_{\alpha_1}^{(n)}(\lambda)\,\overline{h_{\alpha_k}^{(n)}(\lambda')}\,\overline{h_{\alpha'_1}^{(n)}(\lambda)}\,h_{\alpha'_k}^{(n)}(\lambda')\,\Gamma_{\alpha_1\alpha_2}^{(n)}(z_2-z_1)\dots$$
$$\Gamma_{\alpha_{k-1}\alpha_k}^{(n)}(z_k-z_{k-1})\,\overline{\Gamma_{\alpha'_1\alpha'_2}^{(n)}(z'_2-z'_1)}\dots\overline{\Gamma_{\alpha'_{k-1}\alpha'_k}^{(n)}(z'_k-z'_{k-1})}\,e^{2\pi i((\lambda,z_1-z'_1)-(\lambda',z_k-z'_k))}.$$

Let again $z_j = z'_m$ and the corresponding part of the sum is $|ED_n^k(\lambda,\lambda')|^2_{jm}$. Then as before

$$\big|\,|ED_n^k(\lambda,\lambda')|^2_{jm}\big| \leqq c_1^k(\ln r_n)^{2k-2} r_n^{-d},$$
$$\sum_{j,m=1}^{k} \big|\,|ED_n^k(\lambda,\lambda')|^2_{jm}\big| \leqq k^2 c_1^k(\ln r_n)^{2k-2} r_n^{-d}.$$

If none of the variables of the first group coincide with a variable of the second group then

$$E(q_{\alpha_1}^{(n)}(z_1)\dots q_{\alpha_k}^{(n)}(z_k)\, q_{\alpha'_1}^{(n)}(z'_1)\dots q_{\alpha'_k}^{(n)}(z'_k))$$
$$= E(q_{\alpha_1}^{(n)}(z_1)\dots q_{\alpha_k}^{(n)}(z_k))\, E(q_{\alpha'_1}^{(n)}(z'_1)\dots q_{\alpha'_k}^{(n)}(z'_k)).$$

Therefore

$$\big|E|D_n^k(\lambda,\lambda')|^2 - |ED_n^k(\lambda,\lambda')|^2\big| \leqq \sum_{j,m=1}^{k} \big(|(E|D_n^k(\lambda,\lambda')|^2)_{jm}|$$
$$+ \big|\,|ED_n^k(\lambda,\lambda')|^2_{jm}\big|\big) \leqq 2k^2 c_1^k(\ln r_n)^{2k-2} r_n^{-d}.$$

We take a sequence of natural numbers $K_n \uparrow \infty$. In view of Chebyshev's inequality

$$P\{|D_n^k(\lambda,\lambda') - ED_n^k(\lambda,\lambda')| > K_n^{-2}\} \leqq 2k^2 c_1^k(\ln r_n)^{2k-2} r_n^{-d} K_n^4.$$

Thus

$$P\left\{\sum_{k=1}^{K_n} |D_n^k(\lambda,\lambda') - ED_n^k(\lambda,\lambda')| > K_n^{-1}\right\} \leqq 2K_n^2 c_1^{K_n}(\ln r_n)^{2K_n-2} r_n^{-d} K_n^5.$$

Assume that K_n increases so slowly that $2K_n^7 c_1^{K_n}(\ln r_n)^{2K_n-2} < r_n^{1/2}$. Then

$$P\left\{\sum_{k=1}^{K_n} |D_n^k(\lambda,\lambda') - ED_n^k(\lambda,\lambda')| > K_n^{-1}\right\} \leqq r_n^{-d+1/2}.$$

Let us take a sequence n_j such that $r_{n_j} = j$. We have

$$P\left\{\sum_{k=1}^{K_{n_j}} |D_{n_j}^k(\lambda,\lambda') - ED_{n_j}^k(\lambda,\lambda')| > K_{n_j}^{-1}\right\} \leqq j^{-d+1/2}.$$

For $d > 1$ the series $\sum_{j=1}^{\infty} j^{-d+1/2} < \infty$ and with probability 1

$$\sum_{k=1}^{K_{n_j}} |D_{n_j}^k(\lambda,\lambda') - ED_{n_j}^k(\lambda,\lambda')| \to 0 \quad \text{as} \quad j \to \infty.$$

Moreover

$$\sum_{k=K_{n_j}+1}^{\infty} |D^k_{n_j}(\lambda,\lambda') - ED^k_{n_j}(\lambda,\lambda')| \leqq 2 \sum_{k=K_{n_j}+1}^{\infty} \delta^k = 2(1-\delta)^{-1}\delta^{K_{n_j}+1} \to 0 \quad \text{as} \quad j\to\infty .$$

The sequence T_{n_j} contains all the tori of the sequence T_n. Therefore with probability 1 $L_n(\lambda,\lambda')\to 0$ as $n\to\infty$, where $\lambda = \Lambda/r_n$, $\lambda' = \Lambda'/r_n$, and $\Lambda, \Lambda' \in \mathbb{Z}^d\backslash 0$ and are fixed. The set of pairs (Λ,Λ') is countable and one can find an universal set of full measure for which $L_n(\lambda,\lambda')\to 0$ for all $\Lambda,\Lambda'\in\mathbb{Z}^d\backslash 0$. Q.E.D.

5. Proof of Theorem 2

In order to prove the limit theorem for the random walk on the whole lattice $\mathbb{Z}^d$ we need an estimation of the probability of exit of the random trajectory from the cube with the side equal to $r\sqrt{n}$ during the first n steps as $n\to\infty$ and r is large but fixed. We shall consider the random walk with absorbing boundary conditions. Let us give more exact formulations.

We shall consider the random walk in the cube $M_n = \{x\in\mathbb{Z}^d,\ -r_n \leqq x_\alpha \leqq r_n,\ 1\leqq\alpha\leqq d\}$ with the absorption at the boundary of the cube, $r_n \sim r\sqrt{n}$ as $n\to\infty$. Under conditions of Theorem 2 the moving point either jumps to a neighboring point or stays at the same place. This means that $\mathfrak{A}^+$ consists of unit positive coordinate vectors. Let $f\in\mathscr{L}^2(J_r)$ be a probability density, $J_r = [-r,r]^d$. We construct an initial probability distribution for our random walk with the help of the formula $f_n(x) = \int_{\Delta_n(x)} f(y)\,dy$, where $\Delta_n(x)$ is the d-dimensional cube with the centrum at $x\varrho_n^{-1}$ and the side ϱ_n^{-1}, $\varrho_n = r_n/r$.

Theorem 1′. *Let $X_n^{(1)}(t)$ be a position at the moment t of the randomly moving point with the absorption at the boundary of M_n. Then for almost all A*

$$P\{X_n^{(1)}(tn)\in\partial M_n\} \to 1 - \int_{J_r\times J_r} f(x)\,\varphi_a^r(t,x,y)\,dx\,dy, \qquad n\to\infty,$$

where

$$\varphi_a^r(t,x,y) = \sum_{g\in G} (-1)^{k(g)}\,\Phi_a(t, x-gy),$$

$$\Phi_a(t,x) = (2\pi t)^{-d/2}(\det a)^{-1/2}\exp\{-(2t)^{-1}(a^{-1}x,x)\}$$

and G is the group generated by reflections of the boundaries of J_r, $k(g)$ is the parity of g.

One can easily check that $\varphi_a^r(t,x,y)$ is a fundamental solution of heat equation with zeroth boundary conditions. In fact a theorem stating the convergence of finite-dimensional probability distributions of the random processes $\{Y_n^{(1)}(t) = \varrho_n^{-1}X_n^{(1)}(tn),\ 0\leqq t\leqq 1\}$ to the finite-dimensional probability distributions of the Brownian motion on J_r with zero boundary conditions and initial probability density f is valid. But we need only the statement of Theorem 1′. We shall prove Theorem 1′ later and now we shall finish the proof of Theorem 2. For concreteness we shall consider the case of discrete time.

Let $\bar{X}_n(t)$ be a position of the randomly moving point on the whole lattice with the initial distribution f_n constructed in the same way as above with the help of the density f, Δ_i, $1 \leqq i \leqq k$, are arbitrary bounded measurable subsets of $\mathbb{R}^d$. We choose r in such a manner that $(-r, r)^d$-contains all Δ_i and the support of f. By $X_n^{(1)}(t)$ we denote a position of the randomly moving point on the cube M_n with absorbing boundary conditions and initial distribution $f_n(x)$, $X_n^{(2)}(t)$ is a position of the randomly moving point on the torus T_n with the initial distribution f_n. Let us put

$$P_n = P\{n^{-1/2}\bar{X}_n([nt_i]) \in \Delta_i, \quad i=1,2,\ldots,k\},$$

$$P_n^{(1)}(r) = P\{X_n^{(1)}(n) \in \partial M_n\},$$

$$P_n^{(2)}(r) = P\{n^{-1/2}X_n^{(2)}([nt_i]) \in \Delta_i, \quad i=1,2,\ldots,k\}.$$

Then

$$|P_n - P_n^{(2)}(r)| \leqq 2P_n^{(1)}(r).$$

In view of Theorems 1, 1′ for almost all A

$$\lim_{n\to\infty} P_n^{(2)}(r) \equiv P^{(2)}(r) = r_n^{-kd} \int_{\mathbb{R}^d} dx \int_{\Delta_1} \ldots \int_{\Delta_k} f(x)\, \theta^r_{t_1 a}(y_1 - x)\, \theta^r_{(t_2-t_1)a}(y_2 - y_1) \ldots$$
$$\theta^r_{(t_k - t_{k-1})a}(y_k - y_{k-1})\, dy_1 \ldots dy_k,$$

$$\lim_{n\to\infty} P_n^{(1)}(r) \equiv P^{(1)}(r) = 1 - \int\int_{J_r \times J_r} f(x)\, \varphi^r_a(1, x, y)\, dx\, dy.$$

From the other side for $r \to \infty$

$$P^{(2)}(r) \to P = \int_{\mathbb{R}^d} dx \int_{\Delta_1} \ldots \int_{\Delta_k} f(x)\, \Phi_a(t_1, y_1 - x)\, \Phi_a(t_2 - t_1, y_2 - y_1) \ldots$$
$$\Phi_a(t_k - t_{k-1}, y_k - y_{k-1})\, dy_1\, dy_2 \ldots dy_k.$$

$$P^{(1)}(r) \to 0.$$

Let us take $\varepsilon > 0$ and choose r_0 such that $|P^{(2)}(r_0) - P| \leqq \varepsilon$, $|P^{(1)}(r_0)| \leqq \varepsilon$. For chosen r_0 take N such that for $n > N$

$$|P_n^{(1)}(r_0) - P^{(1)}(r_0)| \leqq \varepsilon, \quad |P_n^{(2)}(r_0) - P^{(2)}(r_0)| \leqq \varepsilon.$$

For almost all A and $n > N$

$$|P_n - P| \leqq |P_n - P_n^{(2)}(r_0)| + |P_n^{(2)}(r_0) - P^{(2)}(r_0)| + |P^{(2)}(r_0) - P|$$
$$\leqq |2P_n^{(1)}(r_0)| + 2\varepsilon \leqq 2\varepsilon + 2|P_n^{(1)}(r_0) - P^{(1)}(r_0)| + 2|P^{(1)}(r_0)| \leqq 6\varepsilon.$$

This gives $P_n \to P$ as $n \to \infty$ in view of arbitrary smallness of ε. Q.E.D.

Now we proceed to the proof of Theorem 1′. It goes mainly in the same way as the proof of Theorem 1. A difference concerns Lemma 1′ which is a generalization of Lemma 1 in the periodic case.

For simplicity we shall consider the cube $W_n = \{x \in \mathbb{Z}^d : 0 < x_\alpha < r_n, 1 \leqq \alpha \leqq d\}$. Let us introduce the uniform measure ν_n on W_n and $H'_n = \mathscr{L}^2(W_n, \nu_n)$. The matrices A'_n, $\bar{A}'_n$ are restrictions of A, $\bar{A}$ to $W_n \times W_n$. One has an orthonormal basis in H'_n consisting of functions $v_\lambda^{(n)}(x) = 2^{d/2} \prod_{\alpha=1}^{d} \sin(\pi \lambda_\alpha x_\alpha)$, $\lambda = \Lambda / r_n$, $\Lambda \in W_n$.

Lemma 1′. *For $n\to\infty$, $\lambda=\Lambda/r_n$, $\mu=M/r_n$, $\Lambda, M\in\mathbb{N}^d$ being fixed*

$$n^{-1}(E(A'_n)^{-1}v_\lambda^{(n)}, v_\mu^{(n)})\to 2r^2\pi^{-2}(a\Lambda,\Lambda)^{-1}\delta(\Lambda-M),$$

where a is the same matrix as in Theorem 1.

Proof. As in the periodic case we put $Q'_n=\bar{A}'_n-A'_n$, $D'_n=(\bar{A}'_n)^{-1/2}Q'_n(\bar{A}'_n)^{-1/2}$. The operator $\bar{A}'_n$ is invertible in H'_n and $E(A'_n)^{-1}=(\bar{A}'_n)^{-1/2}E(I-D'_n)^{-1}(\bar{A}'_n)^{-1/2}$. The operator $\bar{A}'_n$ is diagonal and there exists the limit

$$n(\bar{A}'_n v_\lambda^{(n)}, v_\lambda^{(n)})\equiv n\bar{a}'_n(\lambda)\to\pi^2/2r^2(\bar{a}\Lambda,\Lambda),\ \lambda=\Lambda/r_n,\ \Lambda\in\mathbb{N}^d.$$

We have to show that

$$(E(I-D'_n)^{-1}v_\lambda^{(n)}, v_\mu^{(n)})\to(\bar{a}\Lambda,\Lambda)/(a\Lambda,\Lambda)\,\delta(\Lambda-M),\qquad n\to\infty. \tag{5}$$

As in the periodic case $\|D'_n\|_{H'_n}\leqq\delta<1/2$ and therefore the operator $E(I-D'_n)^{-1}=I+\sum_{k=1}^{\infty}E(D'_n)^k$ exists. However now the boundary conditions spoil some properties of $E(D'_n)^k$. For example, they are not diagonal now. Let us write down the explicit expression for matrix elements

$$(E(D'_n)^k v_\lambda^{(n)}, v_\mu^{(n)})=(-1)^k r_n^{-d}\sum_{(z_1,\alpha_1),\ldots,(z_k,\alpha_k)}E(q_{\alpha_1}(z_1)\ldots q_{\alpha_k}(z_k))\,h_{\alpha_1}^{(n)}(\lambda)\,h_{\alpha_k}^{(n)}(\mu)$$
$$\cdot\Phi_{\alpha_1\alpha_2}^{(n)}(z_1,z_2)\,\Phi_{\alpha_2\alpha_3}^{(n)}(z_2,z_3)\ldots\Phi_{\alpha_{k-1}\alpha_k}^{(n)}(z_{k-1},z_k)\,F_{\alpha_1}^{(n)}(\lambda,z_1)\,F_{\alpha_k}^{(n)}(\mu,z_k),$$

where $z_j\in\{x\in\mathbb{Z}^d: 0\leqq x_\alpha<r_n,\ 1\leqq\alpha\leqq d\}$, $\alpha_j=1,2,\ldots,d$,

$$h_\alpha^{(n)}(\lambda)=2\sin(\pi\lambda_\alpha/2)\,(\bar{a}'_n(\lambda))^{-1/2},\quad F_\alpha^{(n)}(\lambda,z)=\frac{\cos\pi(\lambda_\alpha z_\alpha+\lambda_\alpha/2)}{\sin\pi\lambda_\alpha z_\alpha}v_\lambda^{(n)}(z)$$

$$\Phi_{\alpha\beta}^{(n)}(z_1,z_2)=r_n^{-d}\sum_{\lambda:\lambda=\Lambda/r_n,\Lambda\in W_n}h_\alpha^{(n)}(\lambda)\,h_\beta^{(n)}(\lambda)\,F_\alpha^{(n)}(\lambda,z_1)\,F_\beta^{(n)}(\lambda,z_2).$$

The kernel $\Phi_{\alpha\beta}^{(n)}(z_1,z_2)$ does not depend on differences z_1-z_2 now but it satisfies the uniform estimation

$$|\Phi_{\alpha\beta}^{(n)}(z_1,z_2)|\leqq\text{const}(1+|z_1-z_2|^d)^{-1},\ z_1,z_2\in W_n.$$

We have

$$(E(I-D'_n)^{-1})^{-1}=I+\sum_{k=1}^{\infty}\bar{D}_n^{(k)},$$

where

$$\bar{D}_n^{(k)}=\sum_{m=1}^{k}(-1)^m\sum_{l_1+l_2+\ldots+l_m=k}E(D'_n)^{l_1}\ldots E(D'_n)^{l_m}.$$

As in the periodic case

$$(\bar{D}_n^{(k)}v_\lambda^{(n)}, v_\mu^{(n)})=(-1)^k r_n^{-d}\sum_\gamma E(\gamma)\,h_{\alpha_1}^{(n)}(\lambda)\,h_{\alpha_k}^{(n)}(\mu)$$
$$\cdot\Phi_{\alpha_1\alpha_2}^{(n)}(z_1,z_2)\ldots\Phi_{\alpha_{k-1}\alpha_k}^{(n)}(z_{k-1},z_k)\,F_{\alpha_1}^{(n)}(\lambda,z_1)\,F_{\alpha_k}^{(n)}(\mu,z_k)$$

and the sum is taken over nondecomposable paths. Let us define the functions $F'_{\alpha\beta}(\lambda, z)$, $z \in \mathbb{Z}^d$, $1 \leqq \alpha, \beta \leqq d$, where for $\alpha \neq \beta$

$$F'_{\alpha\beta}(\lambda, z) = -\prod_{\gamma=1}^{d} \cos \pi\lambda_\gamma z_\gamma \frac{\sin \pi(\lambda_\alpha z_\alpha + \lambda_\alpha/2)}{\cos \pi\lambda_\alpha z_\alpha} \cdot \frac{\sin \pi(\lambda_\beta z_\beta - \lambda_\beta/2)}{\cos \pi\lambda_\beta z_\beta}$$

and for $\alpha = \beta$

$$F'_{\alpha\alpha}(\lambda, z) = \prod_{\gamma=1}^{d} \cos \pi\lambda_\gamma z_\gamma .$$

Now we construct the kernel

$$\Phi'_{\alpha\beta}(z) = \int_{[0,1]^d} h_\alpha^{(n)}(\lambda)\, h_\beta^{(n)}(\lambda)\, F'_{\alpha\beta}(\lambda, z)\, d\lambda .$$

The functions $h_\alpha^{(n)}(\lambda)$ do not in fact depend on n because $\bar{a}'_n(\lambda) = \sum_{\alpha=1}^{d} 4p_\alpha \sin^2 \pi\lambda_\alpha/2$, where $p_\alpha = -\bar{a}(0, \alpha)$. It is easy to show that

$$\begin{aligned}(\bar{D}_n^{(k)} v_\lambda^{(n)}, v_\mu^{(n)}) &\to (-1)^k\, 2(\bar{a}\Lambda, \Lambda)^{-1}\, \delta(\Lambda - M) \\ &\cdot \sum_\gamma E(\gamma) \Lambda_\alpha^2 \Phi'_{\alpha\alpha_2}(-z_2)\, \Phi'_{\alpha_2\alpha_3}(z_2 - z_3) \dots \Phi'_{\alpha_{k-1}\alpha\alpha_k}(z_{k-1} - z_k), \quad n \to \infty, \end{aligned} \tag{6}$$

where $\gamma = \{(0, \alpha), (z_2, \alpha_2), \dots, (z_{k-1}, \alpha_{k-1}), (z_k, \alpha)\}$, $z_j \in \mathbb{Z}^d$, $1 \leqq \alpha_j \leqq d$. Also

$$((E(I - D'_n)^{-1})^{-1} v_\lambda^{(n)}, v_\mu^{(n)}) \to (a\Lambda, \Lambda)/(\bar{a}\Lambda, \Lambda)\, \delta(\Lambda - M), \qquad n \to \infty .$$

The matrix a is now diagonal. The weak convergence of $(E(I - D'_n)^{-1})^{-1}$ does not imply the weak convergence of the inverse $E(I - D'_n)^{-1}$. However in our case the following estimate is valid

$$|(\bar{D}_n^{(k)} v_\lambda^{(n)}, v_\mu^{(n)})| \leqq \text{const} \cdot n^{-d + \tau(\lambda, \mu)} \cdot (\ln n)^{\text{const}}, \tag{7}$$

where $\tau(\lambda, \mu) = \sum_{\alpha=1}^{d} \delta(\lambda_\alpha - \mu_\alpha)$ and const depends on k. In view of

$$E(D'_n)^k = \sum_{m=1}^{k} (-1)^m \sum_{l_1 + \dots + l_m = k} \bar{D}_n^{(l_1)} \dots \bar{D}_n^{(l_m)}$$

the operator $E(D'_n)^k$ also satisfies (7). Now (7) and (6) give (5) which leads to the statement of Lemma 1′.

6. Limit Covariance Matrix

In this section we write down the explicit expression for the limit covariance matrix. It follows from Lemma 1 that it is equal to

$$a = \bar{a} - \sum_{k=2}^{\infty} a^{(k)},$$

where

$$\bar{a}_{jm}=2\sum_{\alpha\in\mathfrak{A}^+}\bar{p}(0,\alpha)\alpha_j\alpha_m,\ \bar{p}(0,\alpha)=-\bar{a}(0,\alpha),\ a_{jm}^{(k)}=2(-1)^{k-1}\sum_{\alpha,\beta\in\mathfrak{A}^+}D_{\alpha\beta}^k\alpha_j\beta_m,$$

$$D_{\alpha\beta}^k=\sum_{\substack{\gamma=\{(0,\alpha),(z_2,\alpha_2),\ldots,(z_{k-1},\alpha_{k-1}),(z,\beta)\}\\ z_j\in\mathbb{Z}^d,\,\alpha_j\in\mathfrak{A}^+}}E(\gamma)\,\Gamma_{\alpha\alpha_2}(z_2)\,\Gamma_{\alpha_2\alpha_3}(z_3-z_2)\ldots\Gamma_{\alpha_{k-1}\beta}(z-z_{k-1}),$$

$$q_\alpha(z)=q(z,z+\alpha),$$

$$E(\gamma)=\sum_{m=1}^{k}(-1)^m\sum_{l_1+\ldots+l_m=k}E(q_\alpha(0)\,q_{\alpha_2}(z_2)\ldots q_{\alpha_{l_1}}(z_{l_1}))\ldots E(q_{\alpha_{k-l_m+1}}(z_{k-l_m+1})\ldots q_\beta(z)),$$

$$\Gamma_{\alpha\beta}(z)=\int_{\mathrm{Tor}_1^d}\frac{(e^{-2\pi i(\lambda,\alpha)}-1)(e^{2\pi i(\lambda,\beta)}-1)}{4\sum_{\gamma\in\mathfrak{A}^+}\bar{p}(0,\alpha)\sin^2\pi(\lambda,\gamma)}\,e^{2\pi i(\lambda,z)}\,d\lambda.$$

Theorem 3. $(aA,A)\leqq(\bar{a}A,A)$ *for every* $A\in\mathbb{R}^d$.

The statement of the theorem follows from the Main Lemma and the following general statement.

General Statement. *Let A be a non-negatively definite invertible random matrix. Then* $(EA^{-1})^{-1}\leqq EA$.

Proof. Let $\bar{A}=EA$, $Q=\bar{A}-A$. We can write

$$A^{-1}=\bar{A}^{-1}+\bar{A}^{-1}QA^{-1}=\bar{A}^{-1}+A^{-1}Q\bar{A}^{-1}.$$

Using the first half of the equality and then the second one we get

$$A^{-1}=\bar{A}^{-1}+\bar{A}^{-1}Q\bar{A}^{-1}+\bar{A}^{-1}QA^{-1}Q\bar{A}^{-1}.$$

Using $EQ=0$ and $\bar{A}^{-1}QA^{-1}Q\bar{A}^{-1}\geqq 0$ we have

$$EA^{-1}=\bar{A}^{-1}+E(\bar{A}^{-1}QA^{-1}Q\bar{A}^{-1})\geqq\bar{A}^{-1}$$

and thus $(EA^{-1})^{-1}\leqq\bar{A}$. Q.E.D.

Appendix 1

Proof of the Majorizing Lemma and its Corollaries. We shall give the proof for A_n and A_n'. For A and A' the proof is similar. Let $f\in H_n$. Then

$$\begin{aligned} r_n(A_nf,f)_{H_n}&=\sum_{x,y\in T_n}a_n(x,y)\,\overline{f(x)}\,f(y)\\ &=-\sum_{x\in T_n}\sum_{\alpha\in\mathfrak{A}^+}a_n(x,x+\alpha)\,|f(x+\alpha)-f(x)|^2\\ &\geqq-\sum_{x\in T_n}\sum_{\alpha\in\mathfrak{A}^+}a_n'(x,x+\alpha)\,|f(x+\alpha)-f(x)|^2=r_n(A_n'f,f)_{H_n}\geqq 0.\quad\text{Q.E.D.}\end{aligned}$$

Corollary 1 follows from the condition

$$|\bar{a}_n(x,y)-a_n(x,y)|\leqq\delta\bar{a}_n(x,y),\qquad y-x\in\mathfrak{A}^+.$$

Corollary 3 for $\bar{A}_n$ follows from the assumptions that $\bar{a}(x,y)\neq 0$ for $y-x\in\mathfrak{A}^+$ and $\mathfrak{A}^+$ contains d linear independent vectors. Therefore it is valid for A_n also. Corollary 4 follows from the inequalities

$$\|A_n\|_{H_n}\leqq(1+\delta)\,\|\bar{A}_n\|_{H_n}\leqq(1+\delta)\frac{2(1-\delta_0)}{1+\delta}=2(1-\delta_0).$$

Appendix 2

Proof of Theorem A

The method of the proof of Theorem A is taken from [13]. We can assume that $\omega=0$. Otherwise we replace S and S_n by $S-\omega I$ and $S_n-\omega I$. Let us look for the subspace $\mathscr{E}_n$ and a projection P_n onto $\mathscr{E}_n$ not necessarily orthogonal such that $\mathscr{E}_n$ is invariant under S_n and $P_nE=P_n$, $EP_n=E$. In other words we have to solve the system of equations

$$S_nP_n=P_nS_nP_n, \tag{8}$$

$$P_nE=P_n,\qquad EP_n=E. \tag{9}$$

We write $S_n=S+V_n$. In view of $SE=0$ and $P_nSP_n=P_nESP_n=P_nSEP_n=0$, (8) can be rewritten as

$$S(P_n-E)=-(I-P_n)V_nP_n.$$

The spectrum of S on $H\ominus\mathscr{E}$ is separated from zero. Therefore there exists a bounded operator X for which $XS=SX=I-E$. Namely, we can take $X=0$ on $\mathscr{E}$ and $X=S^{-1}$ on $H\ominus\mathscr{E}$. Multiplying from the left to X and taking into account $E(P_n-E)=E-E=0$ we get

$$P_n-E=-X(I-P_n)V_nP_n.$$

Let us denote $P_n-E=Q_n$. Equation (8) takes the form

$$Q_n=-X(I-E-Q_n)\,V_n(E+Q_n) \tag{10}$$

or $Q=f(Q)$, where $f(Q)=-X(I-E-Q)\,V_n(E+Q)$. Let $Q^{(0)}=0$, $Q^{(k+1)}=f(Q^{(k)})$. We shall show that $Q^{(k)}$ converges in norm to a solution of (10).

One can find $\beta>0$ for which $\|X\|\leqq\beta$ because $\omega=0$ is an isolated eigenvalue of S and β^{-1} is the gap in the spectrum. If $\|Q\|\leqq q$, then $\|f(Q)\|\leqq\beta(1+q)^2\,\|V_n\|$. If $\|Q\|\leqq q$, $\|Q'\|\leqq q$, then

$$\|f(Q)-f(Q')\|\leqq 2\beta(1+q)\,\|V_n\|\cdot\|Q-Q'\|.$$

We put $\theta=2\beta(1+q)\,\|V_n\|$ and fix q such that $0<q<1/2$. Then we take N so large that $\|V_n\|\leqq q\beta^{-1}(1+q)^{-2}$ for $n>N$. Now if $\|Q\|\leqq q$ then $\|f(Q)\|\leqq q$ and $\theta\leqq 2q(1+q)^{-1}<1$. Therefore $\lim\limits_{k\to\infty}Q^{(k)}=Q_n$ exists and is a solution of (10). For this solution $\|Q_n\|\leqq q$. But we can take q arbitrarily small and get $\|Q_n\|\to 0$ for $n\to\infty$. Equation (9) takes the form

$$Q_nE=Q_n,\qquad EQ_n=0. \tag{11}$$

We see that $Q^{(0)}$ satisfies (11). If $Q^{(k)}$ satisfies (11) too then $Q^{(k+1)}=f(Q^{(k)})$ also satisfies (11), because $f(Q)E=f(Q)$, $Ef(Q)=0$. Therefore $Q_n=\lim_{k\to\infty} Q^{(k)}$ satisfies (11).

Now we have a solution of Eqs. (8) and (9) for large enough n for which $\|P_n-E\|\to 0$ as $n\to\infty$. Let $\mathscr{E}_n=P_nH$ and E_n be the orthogonal projection to $\mathscr{E}_n$. We shall show that $\|E_n-E\|\to 0$ as $n\to\infty$.

Let us consider $P_n^*P_n$. The subspace $\mathscr{E}$ is invariant under $P_n^*P_n$. Indeed, for $\xi\in\mathscr{E}$ we have

$$P_n^*P_n\xi=EP_n^*P_n\xi\in\mathscr{E}.$$

Moreover, for $\xi\in\mathscr{E}$ and large enough n

$$\begin{aligned}(P_n^*P_n\xi,\xi)&=(P_n\xi,P_n\xi)=((E+Q_n)\xi,(E+Q_n)\xi)\\&=(\xi,\xi)+(Q_n\xi,\xi)+(\xi,Q_n\xi)+(Q_n\xi,Q_n\xi)\geqq(1-\varepsilon)\|\xi\|^2.\end{aligned}$$

Thus $P_n^*P_n$ is invertible on $\mathscr{E}$ and its inverse $(P_n^*P_n)^{-1}$ is uniformly bounded. We put $E_n=P_n(P_n^*P_n)^{-1}P_n^*$. Then E_n is an orthogonal projection onto $\mathscr{E}_n$ and $\|E_n-E\|\to 0$ because E_n is hermitean, $E_n^2=E_n$ and $E_n\xi\in\mathscr{E}_n$ for every $\xi\in H$. At last

$$\begin{aligned}\|E_nS_nE_n\|=\|S_nE_n\|&\leqq\|S(E+(E_n-E))\|+\|V_nE_n\|\\&\leqq\|SE\|+\|S\|\cdot\|E_n-E\|+\|V_n\|\to 0,\end{aligned}$$

as $n\to\infty$.

Appendix 3

Proof of Theorem B

Let $\omega_1,\omega_2,\omega_3,\ldots$ be eigenvalues of S in decreasing order, $S=\sum_{j=1}^{\infty}\omega_jE_j$ is a spectral decomposition of S. We put $S^{(N)}=\sum_{j=1}^{N}\omega_jE_j$, $\tilde{S}^{(N)}=S-S^{(N)}$. Then

$$\|\tilde{S}^{(N)}\|\to 0. \tag{12}$$

It follows from Theorem A that for large enough n there exist orthogonal projections E_{jn}, $j=1,2,\ldots,N$ commuting with S_n and such that for $n\to\infty$

$$\|E_{jn}-E_j\|\to 0,\qquad \|E_{jn}(S_n-\omega_jI)E_{jn}\|\to 0.$$

Let $E_n^{(N)}=\sum_{j=1}^{N}E_{jn}$, $S_n^{(N)}=E_n^{(N)}S_nE_n^{(N)}$, $\tilde{S}_n^{(N)}=S_n-S_n^{(N)}$. Then

$$\|S_n^{(N)}-S^{(N)}\|\to 0\quad\text{as}\quad n\to\infty. \tag{13}$$

From the uniform convergence on compacts of functions φ_n to the continuous function φ we get

$$\|\varphi_n(S_n^{(N)})-\varphi(S^{(N)})\|\to 0,\qquad n\to\infty. \tag{14}$$

Let us take $\varepsilon > 0$. The sequence φ_n being uniformly continuous at 0 one can find $\xi > 0$ such that for $0 \leqq \omega < \xi$ we shall have $|\varphi_n(\omega)| = |\varphi_n(\omega) - \varphi_n(0)| < \frac{\varepsilon}{3}$. Assume that for $n > n_1(\varepsilon)$

$$\|S - S_n\| < \xi/3. \tag{15}$$

It follows from (12) that there exists $N = N(\varepsilon)$ such that

$$\|\tilde{S}^{(N)}\| < \xi/3. \tag{16}$$

Then from (13) we can find $n_2(\varepsilon)$ for which

$$\|S_n^{(N)} - S^{(N)}\| < \xi/3 \tag{17}$$

if $n > n_2(\varepsilon)$. Now from (15)–(17) for $n > \max(n_1, n_2)$

$$\|\tilde{S}_n^{(N)}\| \leqq \|S - S_n\| + \|S_n^{(N)} - S^{(N)}\| + \|\tilde{S}^{(N)}\| < \xi. \tag{18}$$

The estimations (16) and (18) give

$$\|\varphi_n(\tilde{S}_n^{(N)})\| < \varepsilon/3, \qquad \|\varphi(\tilde{S}^{(N)})\| < \varepsilon/3. \tag{19}$$

In view of (14) one can find $n_3 = n_3(\varepsilon)$ for which for $n > n_3$

$$\|\varphi(S^{(N)}) - \varphi_n(S_n^{(N)})\| < \varepsilon/3. \tag{20}$$

From (19), (20) for $n > \max(n_1, n_2, n_3)$ we have

$$\|\varphi_n(S_n) - \varphi(S)\| \leqq \|\varphi_n(S_n^{(N)}) - \varphi(S^{(N)})\| + \|\varphi_n(\tilde{S}_n^{(N)})\| + \|\varphi(\tilde{S}^{(N)})\| < \varepsilon. \quad \text{Q.E.D.}$$

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References

1. Kesten, H., Kozlov, M., Spitzer, F.: Compositio Math. **30**, 145–168 (1975)
2. Ritter, G.A.: Random walk in a random environment. Thesis, Cornell University, 1976
3. Chernov, A.A.: Usp. Phys. Nauk (in Russian) **100**, 277–328 (1970)
4. Temkin, D.E.: J. Crystal Growth **5**, 193–202 (1969)
5. Solomon, F.: Ann. Prob. **3**, 1–31 (1975)
6. Sinai, Ya.G.: Theor. Prob. Appl. (in press)
7. Anshelevich, V.V., Vologodskii, A.V.: J. Stat. Phys. **25**, 419–430 (1981)
8. Alexander, S., Bernasconi, J., Orbach, R.: Phys. Rev. B **17**, 4311–4314 (1978)
9. Bernasconi, J., Beyeler, H.U., Strässler, S.: Phys. Rev. Lett. **42**, 819–822 (1979)
10. Alexander, S., Bernasconi, J., Schneider, W.R., Orbach, R.: Rev. Mod. Phys. **53**, 175–198 (1981)
11. Papanicolaou, G., Varadhan, F.R.S.: Boundary value problems with rapidly oscillating random coefficients. In: Colloquia mathematica societatis Janos Bolyai 27, Random fields, Vol. 2, pp. 835–873. Amsterdam, Oxford, New York: North-Holland Publishing Company 1981
12. Kozlov, S.: Mat. Sb. (in Russian) **113**, 302–328 (1980)
13. Friedrichs, K.O.: Perturbation of spectra in Hilbert space. Providence, R.I.: Math. Soc. 1965

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