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The Rotation Number for Almost Periodic Potentials

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Abstract. We define and analyze the rotation number for the almost periodic Schrödinger operator $L = \frac{-d^2}{dx^2} + q(x)$. We use the rotation number to discuss (i) the spectrum of L; (ii) its relation to the Korteweg-de Vries equation.

1. Introduction

Almost Periodic Potentials

The spectral theory for second order differential operators

$$L\varphi = \left(-\frac{d^2}{dx^2} + q(x)\right)\varphi = \lambda\varphi \tag{1.1}$$

on the x-axis $(-\infty, \infty)$ is well understood, having been developed by H. Weyl in 1910. In particular, if $q(x)$ is bounded (which is the case when $q(x)$ is almost periodic), then one has the limitpoint case at ∞ and at $-\infty$. The nature of the spectrum $\sigma(L)$, however, is not as well understood; it depends rather subtly on the asymptotic behavior of $q(x)$ for large $|x|$. For periodic potentials $q(x)$-in this case one speaks of the "Hill's equation" – it is well known that the spectrum is continuous and consists of finitely or infinitely many intervals, the so-called band spectrum. These facts can be deduced from the Floquet theory, which describes the behavior of the solutions of any system with periodic coefficients.

We are interested in the case of almost periodic potentials in the sense of H. Bohr. In this case there is no such elementary Floquet theory, and little is known about the nature of the spectrum. It certainly can have features which do not occur in the periodic case; for example, one can have point eigenvalues, and one can have nowhere dense spectrum (see [17]).

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Definition of Rotation Number

It is our aim to introduce a function $\alpha = \alpha(\lambda)$ – "the rotation number" – which is motivated by Floquet theory, and which allows the determination of the essential spectrum of L. The definition of the rotation number is simple enough: if $\varphi = \varphi(x, \lambda)$ is any non-zero solution of (1.1), then $\varphi' + i\varphi$ does not vanish for any x, and we can consider its argument $\arg(\varphi' + i\varphi)$, which is defined mod 2π. For any real λ, we define

$$\alpha(\lambda) = \lim_{x \to +\infty} \frac{1}{x} \arg(\varphi'(x; \lambda) + i\varphi(x; \lambda)). \tag{1.2}$$

We will show that this limit exists, is independent of the particular solution chosen, and defines a continuous function on the real λ-axis. Since $\alpha(\lambda)$ measures the average increase of the angle in the φ'-φ-plane, we call $\alpha(\lambda)$ the rotation number. We will also show that, if $\alpha = \alpha(\lambda, q)$ is considered as a functional of q, then α is continuous with respect to uniform variation of q.

There are other ways of introducing the rotation number. For example,

$$\alpha_-(\lambda) = \lim_{x \to -\infty} \frac{1}{x} \arg(\varphi'(x; \lambda) + i\varphi(x; \lambda))$$

agrees with $\alpha(\lambda)$, as we will see. More generally,

$$\alpha(\lambda) = \lim_{|b-a| \to \infty} \frac{1}{b-a} \arg(\varphi' + i\varphi) \,|_{x=a}^{x=b}. \tag{1.3}$$

Also, if $N(a, b; \lambda)$ denotes the number of zeroes of $\varphi(x; \lambda)$ in $[a, b]$, then

$$\frac{\pi N(a, b; \lambda)}{b-a} \to \alpha(\lambda) \text{ as } b - a \to \infty. \tag{1.4}$$

We can relate this last statement to the (regular) eigenvalue problem

$$-\varphi'' + q\varphi = \lambda\varphi \quad \text{on} \quad a \leqq x \leqq b;$$
$$\varphi(a, \lambda) = \varphi(b, \lambda) = 0.$$

In fact, it follows from Sturm's comparison theorem that the number $\nu(a, b; \lambda)$ of eigenvalues $\lambda_j \leqq \lambda$ differs from $N(a, b; \lambda)$ by ± 1, so that

$$\frac{\nu(a, b; \lambda)}{b-a} \to \frac{\alpha(\lambda)}{\pi} \quad \text{as} \quad b - a \to \infty.$$

This limit is often called the "density of states"; we will see that it agrees with $\alpha(\lambda)/\pi$.

Finally, we give a description of α in terms of a *complex* solution $\psi(x; \lambda)$ for *real* λ. In this case, $\bar{\psi}(x; \lambda)$ is also a solution, and the Wronskian

$$[\psi, \bar{\psi}] = \psi\bar{\psi}' - \bar{\psi}\psi',$$

is a constant. If $\operatorname{Im}[\psi, \bar{\psi}] < 0$, then ψ has no zero, and

$$\frac{\arg \psi(x; \lambda) \,|_a^b}{b-a} = \frac{1}{b-a} \operatorname{Im} \int_a^b \frac{\psi'}{\psi} \, dx \to \alpha(\lambda) \text{ as } |b - a| \to \infty. \tag{1.5}$$

It will be easy to show the equivalence of these various definitions of $\alpha(\lambda)$.

Facts about Almost Periodic Functions

With any continuous almost periodic function $q(x)$, one can associate a mean value

$$M_x(q) = \lim_{x\to\infty} \frac{1}{x} \int_0^x q(t)\,dt,$$

and a Fourier series

$$q \sim \sum_{\nu=1}^{\infty} c_\nu e^{i\lambda_\nu x},$$

where the frequencies λ_ν are the (denumerably many) values of λ for which $M_x(qe^{-i\lambda x}) \neq 0$. The *frequency-module* is the set

$$\mathcal{M} = \mathcal{M}(q) = \left\{ \sum_\nu j_\nu \lambda_\nu,\ j_\nu \in Z \right\} \tag{1.6}$$

of finite integer combinations of these frequencies. It will be useful to consider the set $A = A(\mathcal{M})$ of all almost periodic functions with frequency module contained in $\mathcal{M}$. It is closed in the uniform topology, it is separable (in contrast to the space of all almost periodic functions), and it is an algebra: if $f, g \in A(\mathcal{M})$, then $f \cdot g$ is also in $A(\mathcal{M})$.

The Rotation Number for Complex λ

It turns out that $\alpha(\lambda)$ can be extended to a function continuous for $\operatorname{Im}\lambda \geqq 0$ and harmonic for $\operatorname{Im}\lambda > 0$; one thus has a natural notion of rotation number for complex λ. The simplest definition is based on (1.5): we consider any solution $\psi = \psi(x;\lambda)$ of (1.1) for which

$$\operatorname{Im}[\psi, \bar{\psi}] < 0 \qquad \text{for } x = 0.$$

Then one finds that this inequality holds also for $x > 0$ (note that this Wronskian is *not* a constant, since $\bar{\psi}$ is not a solution of (1.1)), and therefore $\psi(x;\lambda)$ has no zero for $x \geqq 0$. Then

$$\lim_{x\to\infty} -\frac{1}{x} \arg \psi(x;\lambda) \tag{1.7}$$

exists, and is the desired rotation number. It is harmonic in the upper half λ-plane, and is there the imaginary part of a holomorphic function $w(\lambda)$ which, it happens, can be defined in an entirely different way.

For this purpose we consider the resolvent

$$R_z = (L - z)^{-1} \qquad (\operatorname{Im} z > 0),$$

where we write z in place of λ; it is an integral operator with kernel (Green's function) $G(x, y; z)$. For fixed z with $\operatorname{Im} z > 0$, the Green's function is never zero, and $G(x, x; z)$ is an almost periodic function with frequency module contained in $\mathcal{M}(q)$ (G. Scharf [23]). The same is true of $G^{-1}(x, x; z)$, and we define

$$w(z) = M_x\left(\frac{-1}{2G(x, x; z)}\right). \tag{1.8}$$

Then $w(z)$ is holomorphic for $\operatorm{Im} z \neq 0$, and $\operatorname{Im} w(z)$ agrees with the limit in (1.7) if $\operatorname{Im} z > 0$. Thus $\alpha(\lambda)$ (λ real) may be viewed as the boundary value of the harmonic function $\operatorname{Im} w(z)$:

$$\operatorname{Im} w(z) \to \alpha(\lambda) \text{ as } z \to \lambda \; (\operatorname{Im} z > 0, \; \lambda \in \mathbb{R}). \tag{1.9}$$

From the standard formula for holomorphic functions with positive imaginary part, we have

$$w(z_1) - w(z_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\lambda)\, d\lambda}{(\lambda - z_1)(\lambda - z_2)}. \tag{1.10}$$

We remark that $\operatorname{Re} w(z)$ has also geometric significance: it measures the exponential rate of decay of solutions of (1.1) which are in $L^2(0, \infty)$. We will not study $\operatorname{Re} w(z)$ in any detail in this paper.

Relation to the Spectral Resolution

The rotation number $\alpha(\lambda)$ has a close relation to the spectral resolution. To clarify it, we consider the resolvent $R_z = (L - z)^{-1}$ and its kernel $G(x, y; z)$. In analogy with the standard definition of a trace (which clearly does not exist, since R_z is not compact), we define

$$\tau(R_z) = M_x(G(x, x, z)), \tag{1.12}$$

which is holomorphic on the resolvent set. Similarly we define the "trace" of the spectral resolution E_λ with kernel $e(x, y; \lambda)$ by

$$\tau(E_\lambda) = M_x(e(x, x; \lambda)). \tag{1.13}$$

(As we will see, care must be taken in interpreting the right-hand side of (1.13)). We will show that

$$\tau(E_\lambda) = \frac{1}{\pi} \alpha(\lambda) \tag{1.14}$$

agrees with the density of states. This implies, in particular, that α is constant in any interval in the complement of the spectrum. Moreover, the monotone function $\alpha(\lambda)$ is constant precisely on the complement of the spectrum, and, in fact, if I is an interval in the resolvent set, then

$$2\alpha(\lambda) \in \mathscr{M}(q) \qquad \text{for} \qquad \lambda \in I. \tag{1.15}$$

That is, the real intervals in the resolvent set (the gaps) can be labelled by the nonnegative values $\sum_\nu j_\nu \lambda_\nu$ in $\mathscr{M}(q)$. This basic fact will be proven in Sect. 4, see Theorem 4.7.

The formulae (1.12), (1.13) can be extended to operators

$$f(L) = \int_{-\infty}^{+\infty} f(\lambda)\, dE_\lambda$$

defined through the usual functional calculus. Here $f(\lambda)$ is continuous and bounded. If even $f(\lambda)(1+\lambda^2)$ is bounded, then (1.13) leads to

$$\tau(f(L)) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\lambda)\, d\alpha(\lambda).$$

For example, the trace of the fundamental solution of the parabolic differential equation $u_t = Lu$ is given by

$$\tau(e^{-tL}) = \frac{1}{\pi} \int e^{-\lambda t}\, d\alpha(\lambda).$$

This is related by the Laplace transform to $w'(z) = \tau(R_z)$.

If $f(z)$ is rational and 0 at $z = \infty$ one can define $f(L)$ via the Cauchy integral

$$f(L) = \frac{1}{2\pi i} \int f(z)\, R_z\, dz,$$

where the integration is taken over an appropriate path. Then (1.12) leads to

$$\tau(f(L) = \frac{1}{2\pi i} \int f(z)\, dw(z)$$

with w defined by (1.8). Both formulae for $\tau(f(L))$ are, of course, equivalent.

Eigenvalue Problem on the Half Line

In addition to the operator L, given by (1.1), on the whole real axis, it is interesting to consider the operators L^+, L^- on the half axis $[0, \infty)$ or $(-\infty, 0]$, respectively, with boundary condition $\varphi(0) = 0$. It is known (see G. Scharf [23]) that their esential spectra agree,

$$\sigma_{\text{ess}}(L^+) = \sigma_{\text{ess}}(L^-) = \sigma_{\text{ess}}(L),$$

and, moreover, that

$$\sigma(L) = \sigma_{\text{ess}}(L),$$

while L^+, L^- may have point eigenvalues in the complement of $\sigma(L)$.

The spectrum of L is generally double but that of L^+, L^- is always simple and is determined from the density function $\varrho_0(\lambda)$ which we introduce now. We consider the eigenvalue problem

$$L^+ \psi(x; z) = z\psi(x; z); \quad \psi(0; z) = 0; \quad \psi \in L^2(0, \infty). \tag{1.16}$$

If $\operatorname{Im} z > 0$, then $L\psi = z\psi$ has a solution $\psi_+(x; z)$ in $L^2(0, \infty)$, unique up to constant multiple (since we are in the limit-point case). Define

$$m_+(x; z) \equiv m(x; z) = \frac{\psi'_+(x; z)}{\psi_+(x; z)}.$$

Then, for fixed x, $m(x; z)$ is holomorphic in $\operatorname{Im} z > 0$, with positive imaginary part. As is well-known, there is a non-decreasing "spectral function" $\varrho_x(t)$ such that

$$\operatorname{Im} m(x; z) = \int_{-\infty}^{\infty} \operatorname{Im} \frac{d\varrho_x(t)}{t - z};$$

to make ϱ_x unique, we require that ϱ_x be right-continuous with $\varrho_x(0)=0$. We will show that

$$w(z) = \lim_{x\to\infty} \frac{1}{x} \int_0^x m(t;z)\,dt \qquad (\operatorname{Im} z > 0). \tag{1.17}$$

We then prove that the measures $\{d\varrho_x \mid x \in \mathbb{R}\}$ form an almost-periodic family of measures in an appropriate sense, and that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x (d\varrho_s)\,ds = \alpha(\lambda)\,d\lambda. \tag{1.18}$$

α and w as Functionals

Actually $\alpha = \alpha(\lambda; q)$ and $w = w(z; q)$ can be viewed as functionals of q; more precisely they depend on $\lambda - q$ and $z - q$, respectively. We will show that, if $\operatorname{Im} z = 0$ and $p(\cdot)$ is almost periodic with $\mathcal{M}(p) \subset \mathcal{M}(q)$, then

$$\frac{d}{d\varepsilon} w(z, q + \varepsilon p)\,|_{\varepsilon=0} = -M_x(G(x,x,z)\,p(x)),$$

which we write also as

$$\frac{\delta w}{\delta q} = -G(x,x;z). \tag{1.19}$$

For $p = -1$ we obtain

$$\frac{dw}{dz} = M_x(G(x,x;z)) = \tau(R_z), \tag{1.20}$$

which shows that $\dfrac{dw}{dz} = w'(z;q)$ is holomorphic on the resolvent set.

However, w is not 1-valued on the resolvent set: if I is a spectral gap and $\lambda \in I$, we have

$$[w] = \lim_{\varepsilon\to 0+} (w(\lambda + i\varepsilon; q) - w(\lambda - i\varepsilon; q)) = 2i\alpha(\lambda).$$

Thus dw can be viewed as a differential on the resolvent set, and the values $2i\alpha = i\sum_\nu j_\nu \lambda_\nu$ as periods of the differential.

Finally, we show that for any two values z_1, z_2, $\operatorname{Im} z_j \neq 0$, one has the identity

$$M_x\left(\frac{\delta w_1}{\delta q}\frac{d}{dx}\frac{\delta w_2}{\delta q}\right) = 0, \tag{1.21}$$

where $w_j = w(z_j; q)$. If one interprets the left-hand side of (1.21) as a Poisson bracket $\{w_1, w_2\}$, as is customary in the recent theory of the Korteweg-de Vries equation, then we can say that the functionals $w_1 = w(z_1; q)$ and $w_2 = w(z_2; q)$ are in involution. Moreover, if q is C^∞, then the asymptotic behavior of $w(z;q)$ for $z \to -\infty$ is given by

$$w(z;q) \sim \sqrt{-z}\left\{1 + \sum_{j=1}^{\infty} z^{-1} w^{(j)}(q)\right\}, \tag{1.22}$$

where

$$w^{(j)} = M_x(P_j(q, q', q'', \ldots)),$$

and the P_j are polynomials in q, q', ... They are the densities in the familiar conservation laws of the Korteweg-de Vries equation. That they are in involution in the above sense follows from (1.21).

We will prove all these statements. The main burden will be the proof of the existence and continuity of the rotation number $\alpha(\lambda)$, which requires tools of ergodic theory. This and the result (1.15) about 2α lying in the frequency module $\mathcal{M}(q)$ for λ in a spectral gap will be proven in Sect. 4. The extension of the rotation number to complex λ is discussed in Sect. 5. In Sect. 6 and 7 we derive properties of the rotation number as functionals of q and establish a connection with the Korteweg-de Vries equation. These ideas are related to the work of Dubrovin-Matveev-Novikov [7], P.D. Lax [13] and McKean-van Moerbeke-Trubowitz [14, 15]. In Sect. 8 we discuss the relation to the work of Sacker and Sell, give an example of a quasi-periodic potential with a point eigenvalue, and mention some open problems.

Many of the properties of the rotation number derived here for almost periodic $q(x)$ are similar in nature as in the case of random potentials. In that theory (see Pastur [11, 19]) one derives "almost everywhere" statements about the existence and continuity of the density of states showing the close analogy to the present work, where exceptional sets of measure zero do not intervene. We also want to point out the close relation to the work of S. Schwartzman [25] on "asymptotic cycles," especially to Theorem 4.7.

2. The Hull of an Almost Periodic Function

In the previous section, we introduced the frequency module $\mathcal{M}(f)$ of an almost periodic function f, and the set $A = A(\mathcal{M})$ of all almost periodic function with frequency module contained in $\mathcal{M}$. It is possible to identify $A(\mathcal{M})$ with the space of continuous functions $C(E)$ on a compact abelian topological group E, where E may be chosen as the "hull" (see below) of the function f.

If f is any almost periodic function, we define the hull $E = E(f) =$ closure $\{f_t(x) = f(x+t) \mid t \in \mathbb{R}\}$, the closure taken in the uniform topology. This is a compact topological space. We will denote its elements by ξ, and the evaluation at a point x_0 by $\xi(x_0)$. Obviously f itself belongs to E and will be denoted by ξ_0. Clearly, if $g \in E = E(f)$ then $E(g) = E(f)$, and E can be generated by any of its elements.

The translation $f(x) \to f(x+t)$ gives rise to a flow $\xi \to \xi \cdot t$ on E; i.e., if $\xi \in E$, x, $t \in \mathbb{R}$, we define $(\xi \cdot t)(x) = \xi(x+t)$. It is well known that we may use this flow to give E the structure of a compact Abelian group with identity ξ_0: if

$$\xi_1 = \lim_{n\to\infty} \xi_0 \cdot t_n, \qquad \xi_2 = \lim_{n\to\infty} \xi_0 \cdot s_n$$

are elements of E, then we define the product

$$\xi_1 \xi_2 = \lim_{n\to\infty} \xi \cdot (t_n + s_n).$$

This limit is well defined and the product obviously commutative. Moreover,

$$\xi_1^{-1} = \lim_{n\to\infty} \xi_0 \cdot (-t_n),$$

and ξ_0 is the identity element of this group. Note that we may view the reals $\mathbb{R}$ as an embedded dense subgroup of E if we identify $t \in \mathbb{R}$ with $\xi_0 \cdot t \in E$.

We need a characterization of almost periodic functions in $A(\mathcal{M})$, due to S. Bochner (see, e.g., [8]).

Proposition 2.1. *Let f have frequency module $\mathcal{M}$. A continuous function g on $\mathbb{R}$ belongs to $A(\mathcal{M})$ provided the following condition holds: For any sequence t_n for which $f(x+t_n) = f_{t_n}(x)$ converges uniformly, also $g(x+t_n) = g_{t_n}(x)$ converges uniformly.*

Remark. It is clear that $\lim_{n\to\infty} g_{t_n} = g$ is determined by $\lim_{n\to\infty} f_{t_n} = f$, and is independent of the choice of the sequence. Otherwise, if t_n' is another sequence with $f_{t_n'} \to f$, but $g_{t_n'} \to h \neq g$, then the mixed sequence $t_1, t_1', t_2, \ldots$ would contradict the hyperthesis.

For the concept of frequency module and Proposition 2.1 see, for example, Fink [8], Theorem 4, 5, p. 61.

Now let f be almost periodic, $\mathcal{M} = \mathcal{M}(f)$ its frequency module and $A = A(\mathcal{M})$ defined as above. We will define a mapping of $A(\mathcal{M})$ into $C(E)$, where $E = E(f)$ is the hull of f. Let $g \in A(\mathcal{M})$ and $\xi \in E$ be given, where ξ is represented by $\lim_{n\to\infty} f_{t_n} = \lim_{n\to\infty} \xi_0 \cdot t_n$. Then $\lim_{t_n\to\infty} g_{t_n} = \tilde{g}$ converges uniformly, according to Proposition 2.1, and we associate with g the continuous function $G \varepsilon C(E)$ by

$$G(\xi) = \tilde{g}(0).$$

In particular, with $g = f$ we associate

$$F(\xi) = \xi(0).$$

Taking $(t_n) = (x)$ we obtain the relation

$$G(\xi_0 \cdot x) = g(x),$$

which shows that g can be recovered from G. One verifies that this mapping $A(\mathcal{M}) \to C(E)$ is actually a homeomorphism with respect to the natural topologies, and therefore they are both function algebras:

Proposition 2.2. *$A(\mathcal{M})$ and $C(E)$ are isomorphic as Banach algebras.*

The Haar measure of the compact Abelian group E will be denoted by μ. Alternately a measure v is defined on E by the mean value of $g(x) = G(\xi_0 \cdot x)$:

$$M_x(g) = \lim_{x\to\infty} \frac{1}{x} \int_0^x g(t)\,dt = \int_E G(\xi)\,dv.$$

These two measures agree, which shows that μ is invariant under the flow $\xi \to \xi \cdot t$ and, moreover, is an ergodic measure. In fact, μ is the only invariant measure under this flow.

In the following we will frequently use the above extension of an almost periodic function f to $F \in C(E)$ on its hull. We assume that q is the given almost periodic potential; $\mathcal{M}=\mathcal{M}(q)$, $E=E(q)$ its frequency module and hull, respectively. According to the above discussion, $q(x)$ will be extended to a function $Q\in C(E)$, and we will often consider the family of differential operators

$$L(\xi)=-\left(\frac{d}{dx}\right)^2+Q(\xi\cdot x), \qquad \xi\in E \tag{2.3}$$

in a dense subspace of $L^2(\mathbb{R})$. For $\xi=\xi_0$ this reduces to the given one. If $\xi=\lim_{n\to\infty}\xi_0\cdot t_n$, we can define $L(\xi)$ also by

$$L(\xi)=\lim_{n\to\infty} T^{t_n}L(\xi_0)\,T^{-t_n}, \tag{2.4}$$

where $T^t f=f(x+t)$ is the translation operator. Since T^t is unitary in $L^2(-\infty,+\infty)$ it follows from this representation that the essential spectrum $\sigma_{\mathrm{ess}}(L(\xi))$ is independent of ξ [23].

We illustrate the construction of the hull E and of $C(E)$ with two simple examples: if $\omega_1,\omega_2\neq 0$ are two real numbers, ω_1/ω_2 irrational and

$$q(x)=c_1\,e^{i\omega_1 x}+c_2\,e^{i\omega_2 x}; \qquad c_1c_2\neq 0,$$

then E is a torus T^2. If $\xi=(\xi_1,\xi_2)$ are taken mod 2π as variables on T^2 we find

$$Q(\xi)=c_1\,e^{i\xi_1}+c_2\,e^{i\xi_2}.$$

This follows from Kronecker's Theorem that the line $(\omega_1 t,\omega_2 t)$ is dense on T^2. For any continuous function $G(\xi_1,\xi_2)$ of period 2π in ξ_1,ξ_2, we find for

$$g(x)=G(\omega_1 x,\omega_2 x)$$

a quasi-periodic function with basic frequencies ω_1,ω_2.

For the example

$$q(x)=\sum_{\nu=0}^{\infty}c_\nu e^{i2^{-\nu}x}, \qquad \sum_{0}^{\infty}|c_\nu|<\infty, \qquad c_\nu\neq 0,$$

the space $E=E(q)$ is more complicated; it is a so-called solenoid.

We need a sharpened version of a theorem by H. Bohr according to which a complex-valued almost periodic function $f(x)\neq 0$ for which $1/f(x)$ is bounded can be represented in the form

$$f(x)=|f(x)|\,e^{i(\beta x+\psi(x))} \tag{2.5}$$

with almost periodic functions $|f(x)|$, $\psi(x)$ and a real constant β given by

$$\beta=\lim_{x\to\infty}\frac{1}{x}\arg\frac{f(x)}{f(0)}=\lim_{x\to\infty}\frac{1}{x}\int_0^x \operatorname{Im}\frac{f'}{f}(t)\,dt. \tag{2.6}$$

In fact this factorization can be made within $A(\mathcal{M})$:

Theorem 2.7. *If $f\in A(\mathcal{M})$, $f^{-1}(x)$ bounded, then f can be represented in the form* (2.5) *where*

$$|f|,\ \psi,\ e^{i\beta x}\in A(\mathcal{M}),$$

hence $\beta\in\mathcal{M}$.

The proof is readily reduced to the case $|f|=1$, and in this case it is found in [8], Lemma 6.7, p. 104.

Corollary. *If $f, f'=\frac{df}{dx}\in A(\mathcal{M})$, and if any $\tilde{f}$ in the hull $E(f)$ has only simple zeroes, then the number $N(x)$ of zeroes of $f(t)$ in $[0,x]$ satisfies*

$$\lim_{x\to\infty}\frac{\pi N(x)}{x}\in\mathcal{M}.$$

Proof. We form the complex-valued function

$$g=f'+if,$$

which also belongs to $A(\mathcal{M})$ *and does not vanish.* The same holds for its extension to $C(E)$. Therefore g^{-1} is bounded, and by Theorem 2.7

$$\frac{\arg g(x)}{x}\to\beta\in\mathcal{M}.$$

On the other hand g is real precisely at the zeroes of f, and at such a zero

$$\frac{d}{dx}\arg g(x)=\operatorname{Im}\frac{g'}{g}=1.$$

Therefore if $\arg g$ increases by π, one has exactly one zero of f, i.e.

$$\left|\arg\frac{g(x)}{g(0)}-\pi N(x)\right|<\pi.$$

This proves the corollary.

3. L^2-Solutions and Green's Function

We assume that q is a real, continuous, almost periodic function with frequency module $\mathcal{M}$, and study the solutions of the differential equation

$$\varphi''=(q(x)-z)\,\varphi \tag{3.1}$$

for z complex or real. According to our discussions in the previous sections we will extend q to a continuous function $Q: E\to\mathbb{R}$ on the hull E of q, and consider the family of differential equations

$$\varphi''=(Q(\xi\cdot x)-z)\,\varphi, \tag{3.2}$$

which for $\xi=\xi_0$ agree with (3.1).

Since q is bounded, these equations possess up to a factor at most one solution in $L^2(0,\infty)$. From the spectral theory of these differential equations it is known that for $\operatorname{Im} z\neq 0$, or even if z belongs to the resolvent set of L, (3.1) has such a solution in $L^2(0,\infty)$ and we denote it by $\psi_+(x,z)\not\equiv 0$. Similarly $\psi_-(x,z)$ stands for a solution of (3.1) in $L^2(-\infty,0)$. These solutions are linearly independent, if z in the resolvent set, and the Green's function $G(x, y; z)$, i.e. the kernel of $(L-z)^{-1}$, is given by

$$G(x,y;z) = G(y,x;z) = \frac{\psi_+(x,z)\,\psi_-(y,z)}{[\psi_+,\psi_-]}; \qquad x \geqq y,$$

where $[\psi_+, \psi_-]$ is the Wronskian.

For $\operatorname{Im} z \neq 0$ the solutions ψ_+, ψ_- have no zeroes. Indeed, if $\psi_+(x_0, z) = 0$, then z would be an eigenvalue of the selfadjoint eigenvalue problem (3.1) in $[x_0, \infty)$ with boundary condition $\varphi = 0$ at $x = x_0$, and z would have to be real.

Since ψ_+, ψ_- are unique only up to a factor, it is natural to consider their logarithmic derivatives

$$m_+(x,z) = \frac{\psi'_+(x,z)}{\psi_+(x,z)}, \qquad m_- = \frac{\psi'_-}{\psi_-},$$

which are uniquely determined.

Proposition 3.3. *$m_+, m_- \in A(\mathcal{M})$ for $\operatorname{Im} z \neq 0$, and their extensions $M_+ = M_+(\xi, z)$, $M_- = M_-(\xi, z)$ to E are given by*

$$M_\pm(\xi,z) = \frac{\psi'_\pm(0;\xi,z)}{\psi_\pm(0;\xi,z)}$$

if $\psi_\pm(x; \xi, z)$ are the L^2-solutions corresponding to (3.2).

Proof. According to Proposition 2.1, we have to show that $m_+(x+t_n)$ converges uniformly for any sequence for which $q(x+t_n)$ does. For this purpose we denote by

$$\psi_n(x) = c_n\,\psi_+(x+t_n)$$

the uniquely determined L^2-solution satisfying

$$\int_0^\infty |\psi_n|^2\,dx = 1; \qquad \psi_n(0) > 0.$$

Using the differential equation which ψ_n satisfies we see that

$$\int_0^\infty (|\psi_n|^2 + |\psi''_n|^2)\,dx \leqq K,$$

and by Rellich's Lemma ψ_n has a subsequence converging in $H^1(0,\infty)$ as well as in $C^1[0,\infty)$. Let ψ^* be the limit of the subsequence and $q^* = \lim_{n\to\infty} q(x+t_n)$; then ψ^* satisfies the differential equation $\psi^{*\prime\prime} = (q^* - z)\,\psi^*$ as well as the normalizations

$$\int_0^\infty |\psi^*|^2\,dx = 1; \qquad \psi^*(0) > 0,$$

by which ψ^* is uniquely fixed. Therefore it was unnecessary to select a subsequence and ψ_n tends uniformly to ψ^*, and

$$m(x+t_n, z) = \frac{\psi'_n}{\psi_n} \to \frac{\psi^{*\prime}}{\psi^*} = m^*_+.$$

With Proposition 2.1 this proves Proposition 3.3.

By the same argument we see that for fixed s also

$$\frac{\psi_+(x+s)}{\psi_+(x)}; \qquad |\psi(x)|^{-2} \int_{x+s}^{\infty} |\psi_+(t)|^2\,dt;$$
$$G(x,x;z) = \frac{\psi_+(x,z)\,\psi_-(x,z)}{[\psi_+,\psi_-]}, \quad \frac{d}{dx}G(x,x;z) \tag{3.4}$$

belong to $A(\mathcal{M})$ and can be extended to $C(E)$. For $G(x,x;z)$ this also holds for real z in the resolvent set. The above argument is due to Scharf [23] who proved $G(x,x;z) \in A(\mathcal{M})$. We will denote the extension of $G(x,x;z)$ to $C(E)$ by $\Gamma(\xi,z)$ so that $G(x,x;z) = \Gamma(\xi_0 \cdot x, z)$.

We need a standard identity for ψ_+, ψ_-. For any solution of (3.1) the Wronskian $[\varphi,\bar\varphi] = \varphi\bar\varphi' - \varphi'\bar\varphi$ satisfies the differential equation

$$\frac{d}{dx}[\varphi,\bar\varphi] = (z-\bar z)\,|\varphi|^2.$$

For $\varphi = \psi_+$ one has $[\psi_+,\bar\psi_+] \to 0$ for $x \to +\infty$, $\operatorname{Im} z \neq 0$, and therefore the identity

$$[\psi_+,\bar\psi_+] = -(z-\bar z)\int_x^{\infty} |\psi_+|^2\,dx.$$

(See, for example, Coddington-Levinson, Chap. 9, Sect. 2.) For $\operatorname{Im} z > 0$ we write this in the form

$$\operatorname{Im} m_+ = -\frac{\operatorname{Im}[\psi_+,\bar\psi_+]}{2|\psi_+|^2} = \operatorname{Im} z\,|\psi_+|^{-2}\int_x^{\infty}|\psi_+|^2\,dx > 0,$$

and similarly \hfill (3.5)

$$\operatorname{Im} m_- = -\operatorname{Im} z\,|\psi_-|^{-2}\int_{-\infty}^{x}|\psi_-|^2\,dx < 0.$$

Moreover, since $\operatorname{Im} m_\pm$ can be extended to $\operatorname{Im} M_\pm$ on the compact hull with

$$\operatorname{Im} M_+ > 0 > \operatorname{Im} M_- \qquad \text{for} \qquad \operatorname{Im} z > 0,$$

we conclude that there exists a positive constant $\delta = \delta(z)$ with

$$\operatorname{Im} M_+, \ -\operatorname{Im} M_- \geqq \delta > 0;$$

in particular, $\operatorname{Im} m_\pm$ are bounded away from 0.

The Green's functions on the diagonal $x = y$ can be expressed in terms of m_+, m_- since

$$\frac{1}{G(x,x;z)} = \frac{[\psi_+,\psi_-]}{\psi_+\psi_-} = m_- - m_+ \in A(\mathcal{M}).$$

Hence

$$G(x,x;z) = (m_- - m_+)^{-1};$$
$$\Gamma(\xi,z) = (M_- - M_+)^{-1}. \tag{3.6}$$

Moreover,

$$\operatorname{Im} G(x, x; z) = \frac{-\operatorname{Im} m_- + \operatorname{Im} m_+}{|m_- - m_+|^2} > 0. \tag{3.6'}$$

In the following the mean values of m_+, m_- will play a basic role:

Proposition 3.7. *For* $\operatorname{Im} z \neq 0$ *we have*

$$M_x(m_+) = -M_x(m_-) = M_x\left(-\frac{1}{2G(x, x; z)}\right),$$

or equivalently

$$\int_E M_+ \, d\mu = -\int_E M_- \, d\mu = -\int_E \frac{1}{2\Gamma} d\mu.$$

This quantity will be denoted by $w = w(z)$.

Proof. By (3.6) we have

$$\frac{1}{G(x, x; z)} = m_- - m_+,$$

hence

$$M_x\left(\frac{1}{G(x, x; z)}\right) = M_x(m_-) - M_x(m_+),$$

and it suffices to show that $M_x(m_-) + M_x(m_+) = 0$. For this purpose note that

$$m_- + m_+ = \frac{(\psi_- \psi_+)'}{\psi_- \psi_+} = \frac{d}{dx} \log(\psi_- \psi_+) = \frac{d}{dx} \log G(x, x; z).$$

Since $G(x, x; z)$ is almost periodic and $\operatorname{Im} G(x, x; z)$ bounded away from zero, so is $\log G(x, x; z)$ bounded and

$$M_x(m_- + m_+) = 0.$$

This proves Proposition 3.7.

The function $w = w(z)$ defined in Proposition 3.7 is holomorphic for $\operatorname{Im} z \neq 0$ and satisfies

$$\frac{\operatorname{Im} w}{\operatorname{Im} z} > 0, \qquad \operatorname{Re} w < 0. \tag{3.8}$$

The first of these inequalities follows from (3.6'), the second from the representation

$$\frac{\psi_+(x, z)}{\psi_+(x, 0)} = \exp\left(\int_0^x m_+(t, z) \, dt\right). \tag{3.9}$$

Since $\psi_+ \in L^2(0, \infty)$, one has

$$\lim_{x \to +\infty} \frac{1}{x} \int_0^x \operatorname{Re} m_+ \, dt = \operatorname{Re} w \leqq 0,$$

and by the maximum principle for harmonic functions $\operatorname{Re} w < 0$.

Every positive harmonic function $h(z)$ in $\operatorname{Im} z > 0$ can be represented in the form

$$h(z) = \operatorname{Im} z \left(\int_{-\infty}^{+\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} + c \right), \tag{3.10}$$

where ϱ is monotone increasing and $c \geqq 0$ a constant. Moreover,

$$\varrho(\lambda_2) - \varrho(\lambda_1) = \lim_{\varepsilon \to +\infty} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} h(\lambda + i\varepsilon)\, d\lambda,$$

$$c = \lim_{t \to \infty} \frac{h(it)}{t},$$

where the first relation holds at all points λ_1, λ_2 of continuity of ϱ.

We apply this representation to the function $\operatorname{Im} M_+(\xi, z)$ and denote the corresponding density and constant by $\varrho_\pm(\xi, t)$, $c_\pm$. From standard asymptotic estimates of the solutions one sees that $c_+ = c_- = 0$ (see [10]).

Proposition 3.11. *Let μ be the normalized Haar measure on E and λ_1, λ_2 two real numbers. Then, for almost all $\xi \in E$, the function $\varrho_+(\xi, \lambda)$ is continuous at λ_1 and λ_2, and the relation*

$$\varrho_+(\xi, \lambda_2) - \varrho_+(\xi, \lambda_1) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} M_+(\xi, \lambda + i\varepsilon)\, d\lambda$$

holds; the convergence is bounded in ξ.

Proof. Write $\varrho = \varrho_+$. Since

$$\operatorname{Im} M_+(\xi, i) = \int_{-\infty}^{+\infty} \frac{d\varrho(\xi, \lambda)}{1 + \lambda^2},$$

and since $\operatorname{Im} M_+(\xi, i)$ is continuous on E, this quantity is bounded on E, hence

$$|\varrho(\xi, \lambda) - \varrho(\xi, 0)| \leqq K(1 + \lambda^2),$$

with K independent of ξ. We note the continuity of $M_+(\xi, \lambda)$ in ξ and λ; using Helly's theorem we see that for any real continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with compact support the mapping

$$\xi \to \int_{-\infty}^{+\infty} f(\lambda)\, d\varrho(\xi, \lambda)$$

is continuous. Taking for $f(\lambda)$ the characteristic function of an interval shrinking to a point λ_0 we see that the mapping

$$H\colon \xi \to \int_{\lambda_0 - 0}^{\lambda_0 + 0} d\varrho(\xi, \lambda) = [\varrho]_{\lambda_0} \geqq 0$$

is a μ measurable and bounded function on E. By Birkhoff's ergodic theorem

$$\frac{1}{x} \int_0^x H(\xi \cdot t)\, dt \to \int_E H\, d\mu$$

for almost all $\xi \in E$. On the other hand, $H(\xi \cdot t)$ vanishes for all $t>0$ except for those t's for which $[\varrho]_{\lambda_0}>0$, that is, when λ_0 is an eigenvalue of the boundary value problem in $[t', \infty)$ with the boundary condition $\varphi(t')=0$. These t' are exactly the zeroes of $\psi_+(t', \lambda_0)=0$, and therefore the left hand side equals zero. We conclude

$$\int_E H\,d\mu = 0,$$

thus $H=0$ for almost all $\xi \in E$.

The second statement follows from the representation

$$\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \operatorname{Im} M_+(\xi, \lambda + i\varepsilon)\,d\lambda = \int_{-\infty}^{+\infty} f_\varepsilon(t)\,d\varrho(\xi, t), \tag{3.12}$$

with

$$f_\varepsilon(t) = \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \operatorname{Im}\left(\frac{1}{t-\lambda-i\varepsilon}\right) d\lambda = \frac{1}{\pi}\left(\tan^{-1}\frac{\lambda_2 - t}{\varepsilon} - \tan^{-1}\frac{\lambda_1 - t}{\varepsilon}\right),$$

and the uniform boundedness of $(1+t^2)\,f_\varepsilon(t)$.

For later purposes we prove the following

Proposition 3.13. *For* $\operatorname{Im} z>0$ *there exists a* $\delta=\delta(z)>0$ such that

$$|\psi_+(x,z)|^{-2}\int_{x+s}^{\infty} |\psi_+(t,z)|^2\,dt \leqq \delta^{-1} e^{-\delta s} \qquad \text{for} \qquad s \geqq 0.$$

Proof. Since $m_+(x,z)$ is almost periodic we can find $\delta=\delta(z)>0$ so that

$$0<\frac{\operatorname{Im} m_+}{\operatorname{Im} z}<\delta^{-1}.$$

From the identity (3.5) we conclude that

$$f(x) = \int_x^{\infty} |\psi_+(t,z)|^2\,dt$$

satisfies the inequality

$$f'(x) = -|\psi_+(t,z)|^2 \leqq -\delta f(x).$$

By integration we find for $s \geqq 0$ the estimate

$$f(x+s) \leqq f(x)\,e^{-\delta s} \leqq -\frac{1}{\delta} f'(x)\,e^{-\delta s},$$

which agrees with the statement.

This estimate yields also a pointwise estimate

$$\psi_+(x,z) = 0\,(e^{-\delta x}).$$

This was shown before, but the main point of the proposition is to show that the quantity in Proposition 3.13 tends to zero uniformly in x. This will be needed in Sect. 6.

4. The Rotation Number on the Real Axis

In this section we show that the rotation number defined in the introduction exists and is a continuous function of λ on the real axis.

With $y_1 = y$, $y_2 = y'$ our differential equation can be written as a first order system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Since this system is linear, it gives rise to a differential equation for the lines through the origin in the y_1-y_2-plane. Writing

$$y_2 + i y_1 = y' + i y = r\, e^{i\theta}, \qquad r \geqq 0,$$

a line is characterized by $\theta \pmod{\pi}$. The differential equation for θ is found to be

$$\frac{d\theta}{dx} = \cos^2\theta - (q(x) - \lambda)\sin^2\theta \tag{4.1}$$

as is also known from the so-called Prüfer transformation.

We will extend the right hand side to a function

$$F(\xi, \theta) = \cos^2\theta - (Q(\xi) - \lambda)\sin^2\theta$$

on the circle bundle B over E, where θ is taken modulo π. We denote the flow obtained on B by $\Phi^x : (\xi_0, \theta_0) \to (\xi_0 \cdot x, \theta(x; \xi_0, \theta_0))$, where $\theta(x; \xi_0, \theta_0)$ is the solution of $\theta' = F(\xi, \theta)$ with initial value $\theta = \theta_0$, $\xi = \xi_0$ for $x = 0$.

According to our definition in the introduction the rotation number $\alpha = \alpha(\lambda)$ is defined as the limit of

$$\frac{\theta(x; \xi_0, \theta_0) - \theta(0; \xi_0, \theta_0)}{x} = \frac{1}{x}\int_0^x F(\Phi^t(\xi_0, \theta_0))\, dt. \tag{4.2}$$

We will show that these time averages converge for *all* $(\xi_0, \theta_0) \in B$ and the convergence is uniform on B.

First we show: If (4.2) converges for (ξ_0, θ_0), then also for (ξ_0, θ_1) and the limit is independent of θ_1. For this purpose let $0 < \theta_1 - \theta_0 < \pi$ and let $\theta_\nu(x) = \theta(x; \xi_0, \theta_\nu)$, $\nu = 0, 1$; then $0 < \theta_1(x) - \theta_0(x) < \pi$ for all x. Indeed, otherwise $\theta_1(x) - \theta_0(x) = 0, \pi$ for some x, and by the uniqueness theorem for all x; this contradicts the assumption for $x = 0$. Hence

$$\theta_1(x) - \theta_0(x)$$

is bounded and our claim is obvious.

According to the theorem by Krylov and Bogoliubov ([3], see also Nemytskii and Stepanov [18]), the flow Φ^x on the compact space B possesses at least one normalized invariant measure, say ν, and by Birkhoff's ergodic theorem we conclude that

$$\frac{1}{x}\int_0^x F(\Phi^t(\xi, \theta))\, dt \to F^*(\xi, \theta)$$

for almost all (ξ, θ) (with respect to ν). More precisely, there is a set $B_0 \subset B$ with $\nu(B - B_0) = 0$ such that convergence holds for all $(\xi, \theta) \in B_0$. Moreover, F^* is ν-integrable,

$$\int_B F^* \, d\nu = \int_B F \, d\nu, \tag{4.3}$$

and F^* is invariant under Φ^x.

According to our preliminary remark, B_0 is of the form $B_0 = E_0 \times S^1$ and F^* independent of θ. Therefore F^* can be considered as a function on E, which is invariant with respect to the flow $\xi \to \xi \cdot x$ on E. Since this flow preserves only the Haar measure μ, which is ergodic, we conclude that $F^*(\xi, \theta)$ agrees with a constant, say α, on a set $B_1 = E_1 \times S^1$, where $\mu(E - E_1) = 0$. By (4.3) this constant agrees with

$$\alpha = \int_B F \, d\nu .$$

This argument holds for any invariant measure ν, and therefore

$$\int_B (F - \alpha) \, d\nu = 0$$

holds for any invariant measure ν on B.

Now the existence of the limit in (4.2) will follow from

Lemma 4.4. *Let G be a continuous function on B such that*

$$\int_B G \, d\nu = 0$$

for all invariant measures ν. Then

$$\frac{1}{b-a} \int_a^b G(\Phi^t \beta) \, dt \to 0, \qquad b - a \to \infty$$

for all $\beta = (\xi, \theta)$, and the convergence is uniform.

Proof. We repeat the classical arguments of Krylov and Bogoliubov. Since $C(B)$ is separable we can find a dense linear subspace D generated by a countable set of functions. We assume the statement to be false for some function $G \in C(B)$. We may choose D so that $G \in D$ and select b_j, a_j, β_j such that $b_j - a_j \to \infty$ and

$$\frac{1}{b_j - a_j} \int_{a_j}^{b_j} G(\Phi^t \beta_j) \, dt \to \delta \neq 0; \qquad b_j - a_j \to \infty .$$

We may assume that $\beta_j \to \beta$. Using the Cantor diagonal process we can pick a subsequence, which we call a_j, b_j, β_j again, such that

$$\frac{1}{b_j - a_j} \int_{a_j}^{b_j} H(\Phi^t \beta_j) \, dt$$

converges for all $H \in D$. This limit defines a linear functional $l = l(H)$, $H \in D$, and since l is bounded (with norm 1) it extends uniquely to a bounded linear functional on $C(B)$. Since $b_j - a_j \to \infty$, one verifies as usual that l is invariant, i.e. $l(H \cdot x) = l(H)$ if $(H \cdot x)(\beta) = H(\Phi^x \beta)$. By the Riesz representation theorem,

l defines an invariant measure ν on B. But by our assumption

$$\int_B G\,d\nu = l(G) = \delta \neq 0,$$

which is a contradiction, proving the lemma.

We apply Lemma 4.4 to $G = F - \alpha$ for which the hypothesis has been verified. Hence

$$\frac{1}{x}\int_0^x (F(\Phi^t \beta) - \alpha)\,dt$$

converges to zero for all $\beta \in B$ and the convergence is uniform. Thus the expression (4.2) converges to a limit α which is independent of ξ_0, θ_0. Moreover, one has

$$\frac{\theta(b) - \theta(a)}{b-a} \to \alpha \qquad \text{for} \qquad b - a \to \infty$$

for any solution of (4.1).

Now we consider $\alpha = \alpha(\lambda)$ as a function of λ and prove its continuity. Suppose for contradiction that α is not continuous at $\lambda_0 \in \mathbb{R}$ and $\lambda_j \to \lambda_0$ is a sequence with $\alpha(\lambda_j) \to \alpha^* \neq \alpha(\lambda_0)$. Let $F_j = F(\theta, \lambda_j) = \cos^2\theta - (Q(\xi) - \lambda_j)\sin^2\theta$ and Φ_j^x the corresponding flow for $\lambda = \lambda_j$. If ν_j is any invariant measure for this flow we have

$$\alpha(\lambda_j) = \int_B F_j\,d\nu_j.$$

We may suppose that $\nu_j \to \nu$ in the weak topology of measures, so that

$$\int_B F_0\,d\nu_j \to \int_B F_0\,d\nu.$$

It is easily seen that ν is invariant under Φ_0^x. Finally, since

$$|F_j - F_0| \leqq |\lambda_j - \lambda_0| \to 0,$$

we conclude that

$$\alpha(\lambda_j) = \int_B F_j\,d\nu_j = O(|\lambda_j - \lambda_0|) + \int_B F_0\,d\nu_j \to \int_B F_0\,d\nu = \alpha(\lambda_0),$$

which is a contradiction.

We summarize our results:

Theorem 4.5. *For real λ the expression* (4.2), *or for short,*

$$\frac{\theta(x) - \theta(0)}{x},$$

converges for $x \to +\infty$ uniformly with respect to initial conditions $(\xi_0, \theta_0) \in B$ to a function $\alpha = \alpha(\lambda)$ which is independent of ξ_0, θ_0, but continuously dependent on λ.

Moreover, $\alpha(\lambda)$ is monotone increasing, equal to zero for $\lambda \leqq \lambda^*$ for some λ^*, and $\alpha(\lambda) \to +\infty$ for $\lambda \to +\infty$.

The additional result follows from the alternate characterization in terms of the zeroes of solutions and Sturm's theory, which we forego. We just mention that a zero of a solution corresponds to a value x for which $\theta = 0 \pmod{\pi}$, and by (4.1) one has $\frac{d\theta}{dx} = 1$ at such a value. This shows that θ increases at such a zero and therefore one has one zero per increase of θ by π. Therefore, if $N(x, \lambda)$ is the number of zeroes in $[0, x]$ of a solution $\varphi(x)$ one has

$$\lim_{x \to \infty} \frac{\pi N(x, \lambda)}{x} = \alpha(\lambda). \tag{4.6}$$

We consider $\alpha(\lambda)$ in a spectral gap of L. Let $\sigma(L)$ be the spectrum of this operator on the whole real line and let $I = (\lambda_1, \lambda_2)$ be an open interval of $R - \sigma(L)$.

Theorem 4.7. *If q is almost periodic with frequency module $\mathcal{M}$ and I an open interval in a spectral gap, then $\alpha(\lambda)$ is constant in I and*

$$2\alpha(\lambda) \in \mathcal{M} \quad \textit{for} \quad \lambda \in I.$$

Proof. We recall that for λ in the resolvent set $\psi_\pm(x, \lambda)$, $G(x, x; \lambda)$ are well defined and

$$G(x, x \colon \lambda), \qquad \frac{d}{dx} G(x, x; \lambda) \in A(\mathcal{M}).$$

If we normalize ψ_+, ψ_- so that $[\psi_+, \psi_-] = 1$ we have

$$G(x, x; \lambda) = \psi_+(x, \lambda)\, \psi_-(x, \lambda),$$

$$\frac{d}{dx} G(x, x; \lambda) = \psi_+ \psi'_- + \psi'_+ \psi_-,$$

and it is clear that at a zero of $G(x, x, \lambda)$ either ψ_+ or ψ_- will vanish, hence at such a zero

$$\frac{d}{dx} G(x, x, \lambda) = \pm (\psi_+ \psi'_- - \psi'_- \psi_-) = \pm 1,$$

the sign depending on whether ψ_+ or ψ_- is zero. In any event $G(x, x; \lambda)$, and similarly $\Gamma(\xi \cdot x, \lambda)$ has only simple zeroes. According to the Corollary to Theorem 2.7,

$$\lim_{x \to \infty} \frac{\pi N_2(x, \lambda)}{x} \in \mathcal{M},$$

if $N_2(x, \lambda)$ denotes the number of zeroes of $G(t, t; \lambda)$ for $t \in [0, x]$. If $N_\pm(x, \lambda)$ are the number of zeroes of ψ_+, ψ_- in $[0, x]$, then

$$N_2(x, \lambda) = N_+(x, \lambda) + N_-(x, \lambda);$$

we conclude

$$\frac{\pi N_2(x, \lambda)}{x} \to 2\alpha \in \mathcal{M}.$$

Since $\alpha(\lambda)$ is continuous it follows, of course, that $\alpha(\lambda)$ is constant in I, which proves the theorem.

Remark. This theorem asserts for any gap I of the spectrum that there exist finitely may integers j_ν such that

$$2\alpha = \sum_\nu j_\nu \lambda_\nu \qquad \text{in } I,$$

where λ_ν are frequencies of q. The integers j_ν have a topological interpretation. If one approximates $G(x,x;\lambda)$ by a quasi-periodic function $G(x,x;\lambda)=\Gamma(\xi_0\cdot x,\lambda)$ whose hull $E=E(G)$ is a finite dimensional torus, then j_ν can be related to the intersection number of the cycle

$$Z=\{\xi\in E \mid \Gamma(\xi,\lambda)=0\}$$

of codimension 1 with the circles generating the one-dimensional homology group of E.

Another interpretation of the j_ν is as follows. For each $x\in\mathbb{R}$, the vector $(\psi_+(x,\lambda), \psi'_+(x,\lambda))$ defines a straight line $\theta(x)$ in $\mathbb{R}^2$, hence a point $(\xi_0\cdot x, \theta(x))$ of the circle bundle B. Then $\theta(x)$ is almost periodic with frequency module $\mathcal{M}$, so $S = cls\{(\xi_0\cdot x,\theta(x)) \mid x\in\mathbb{R}\}\subset B$ covers E once. Approximate q by a quasi-periodic q, so $E=E(q)$ is a torus. Then the j_ν's measure the winding of S in the θ-direction as one traverses the homology generators of E.

5. Extension of the Rotation Number to the Complex Plane

We give a new definition of the rotation number for $\operatorname{Im}\lambda \geqq 0$, which turns out to agree with our original definition when λ is real.

Theorem 5.1. *If* $\operatorname{Im} z\geqq 0$, *let* $\phi=\phi(x,z)$ *be a complex solution of* (1.1) *satisfying*

$$\operatorname{Im}[\phi,\bar\phi] = \operatorname{Im}(\phi\bar\phi' - \phi'\bar\phi) > 0 \qquad \text{for} \qquad x=0. \tag{5.2}$$

Then the limit $h(z)$ *of*

$$h(x,z) = \frac{-1}{x}\arg\frac{\phi(x,z)}{\phi(0,z)} = \frac{-1}{x}\int_0^x \operatorname{Im}\frac{\phi'(t,z)}{\phi(t,z)}\,dt \tag{5.3}$$

for $x\to\infty$ *exists and is independent of the solution chosen (as long as* (5.2) *holds). For real* λ *we have* $h(x,\lambda)\to\alpha(\lambda)$ *as* $x\to\infty$.

Proof. We observe first that

$$\frac{d}{dx}\operatorname{Im}[\phi,\bar\phi] = \operatorname{Im}(\phi\bar\phi'' - \phi''\bar\phi) = 2\operatorname{Im} z\,|\phi|^2 \geqq 0,$$

hence (5.2) holds for all $x\geqq 0$. Therefore, $\phi(x,z)$ does not vanish for $x\geqq 0$, and $h(x,z)$ is well-defined for $\operatorname{Im} z\geqq 0$, $x\geqq 0$. Incidentally, because of $\operatorname{Im}[\phi,\bar\phi] = -|\phi|^2\operatorname{Im}\frac{\phi'}{\phi}$, the condition (5.2) is equivalent to $\operatorname{Im}\frac{\phi'}{\phi}<0$, hence $h(x,z)>0$.

Next we show: if $\tilde\phi$ is a second solution satisfying (5.2) then

$$\left|\int_0^x\left(\operatorname{Im}\frac{\phi'}{\phi} - \operatorname{Im}\frac{\tilde\phi'}{\tilde\phi}\right)dt\right| < \pi. \tag{5.4}$$

This implies that, if the limit $h(z)$ of $h(x,z)$ exists for one solution satisfying (5.2), then it also exists for any other solution, and the limit is independent of the choice of ϕ.

To prove (5.4) we note that neither the statement (5.4) nor the assumption (5.2) is affected if we replace ϕ by $c\phi$ when $c \neq 0$ is a constant. Therefore, we may assume that $\phi(0,z)=1$, $\tilde{\phi}(0,z)=1$, so that because of (5.2), we have $\operatorname{Im}\phi'(0,z)<0$, $\operatorname{Im}\phi'(0,z)<0$. If (5.4) were false, then there would exist an $x^*>0$ for which *arg* $\phi(x^*,z)-\arg\tilde{\phi}(x^*,z)=\pm\pi$, and therefore there would exist a positive number μ so that

$$\phi+\mu\tilde{\phi}=0 \qquad \text{for} \qquad x=x^*.$$

On the other hand, $\phi+\mu\tilde{\phi}$ is a solution for which (5.2) holds, because

$$\operatorname{Im}\frac{(\phi+\mu\tilde{\phi})'}{\phi+\mu\tilde{\phi}}\bigg|_{x=0}=\frac{1}{1+\mu}(\operatorname{Im}\phi'+\mu\operatorname{Im}\tilde{\phi}')\bigg|_{x=0}<0.$$

But as we saw, such a solution has no zeros. This contradiction proves (5.4).

To prove existence of $\lim\limits_{x\to+\infty} h(x,z)$, we treat the real case and $\operatorname{Im}z>0$ separately. For real λ, let $\phi_1(x,\lambda)$, $\phi_2(x,\lambda)$ be real solutions with

$$\begin{pmatrix}\phi_1 & \phi_2\\ \phi_1' & \phi_2'\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{at} \qquad x=0.$$

Then $[\phi_1,\phi_2]=1$, and

$$\phi=\phi_1-i\phi_2 \tag{5.5}$$

satisfies (5.2). We will show that

$$\int_0^x \operatorname{Im}\frac{\phi'(t,\lambda)}{\phi(t,\lambda)}\,dt+\pi N(x,\lambda)=0(1) \tag{5.6}$$

is bounded, where $N(x,\lambda)$ is the number of zeros of $\varphi(x,\lambda)$ in $[0,x]$. Indeed, by the same argument used already in the proof of the corollary of Theorem 2.7 we have

$$\left|\arg\frac{\phi(x,\lambda)}{\phi(x,0)}+\pi N(x,\lambda)\right|<\pi,$$

which proves (5.6).

Combining (5.4), (5.6) and (4.6) we have

$$\lim_{x\to\infty} h(x,\lambda)=\lim_{x\to\infty}\frac{\pi N(x,\lambda)}{x}=\lim_{x\to\infty}\frac{\arg\phi(x,\lambda)}{x}=\alpha(\lambda).$$

This establishes the existence of the limit and its identity with $\alpha(\lambda)$ for real λ.

Now we consider the complex case $\operatorname{Im}z>0$. According to Sect. 3 the solution $\psi_-(x,z)\in L^2(-\infty,0)$ satisfies the identity (3.5) and therefore $\phi=\psi_-$ satisfies (5.2) and it suffices to establish the existence of $\lim\limits_{x\to\infty} h(x,z)$ for ϕ replaced by ψ_-.

To do this, note that by Proposition 3.3

$$m_-=\frac{\psi_-'}{\psi_-}$$

is almost periodic with frequency module in $\mathcal{M}(q)$. Hence

$$w_-(z) = -M_x(m_-) = -\lim_{x\to\infty} \frac{1}{x}\int_0^x \frac{\psi'_-}{\psi_-}\,dt$$

exists, and hence also

$$\operatorname{Im} w_-(z) = -\lim_{x\to\infty} \frac{1}{x}\int_0^x \operatorname{Im}\frac{\psi'_-}{\psi_-}\,dt = \lim_{x\to\infty} h(x,z)$$

exists. This completes the proof of Theorem 5.1.

We have represented the rotation number for $\operatorname{Im} z > 0$ as $\operatorname{Im} w_-(z)$ where $w_- = -M_x(m_-)$. According to Proposition 3.7 it can also be written as

$$\operatorname{Im} M_x(m_+) = \operatorname{Im} M_x\left(-\frac{1}{2G(x,x,z)}\right) = \operatorname{Im} w,$$

with w defined in Proposition 3.7. We now proceed to show that $h(z) = \operatorname{Im} w(z)$ is continuous in the closed upper half plane $\operatorname{Im} z \geqq 0$.

Note first that $G(x,y;z)$ is holomorphic in z if $\operatorname{Im} z > 0$, hence $w(z)$ is holomorphic if $\operatorname{Im} z > 0$, and $\operatorname{Im} w(z)$ is positive and harmonic if $\operatorname{Im} z > 0$. Therefore we may write $\operatorname{Im} w$ in the form (3.10), or

$$\operatorname{Im} w(z) = \operatorname{Im} z \int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{|\lambda - z|^2}, \tag{5.7}$$

where $\sigma(\lambda)$ is monotone increasing. For the monotone function σ, one has

$$\sigma(\lambda_2) - \sigma(\lambda_1) = \frac{1}{\pi} \lim_{\varepsilon\to+0} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} w(\lambda + i\varepsilon)\,d\lambda \tag{5.8}$$

at all continuity points of σ. Also, for any continuous function $f(\lambda)$ with compact support, we have

$$\int_{-\infty}^{\infty} f(\lambda)\,d\sigma(\lambda) = \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(\lambda) \operatorname{Im} w(\lambda + i\varepsilon)\,d\lambda. \tag{5.9}$$

Finally, if $w_\nu(z)$ is a sequence of functions, holomorphic and satisfying $\operatorname{Im} w_\nu(z) > 0$ for $\operatorname{Im} z > 0$, and if $w_\nu(z) \to w(z)$ pointwise, then for the corresponding densities σ_ν, one has

$$\int_{-\infty}^{\infty} f\,d\sigma_\nu \to \int_{-\infty}^{\infty} f\,d\sigma \tag{5.10}$$

for all continuous functions f of compact support. We can prove

Theorem 5.11. *The function* $h(z) = \operatorname{Im} w(z)$ *is continuous in the closed upper half-plane* $\operatorname{Im} z \geqq 0$, *and*

$$\lim_{\varepsilon\to+0} h(\lambda + i\varepsilon) = \alpha(\lambda) \quad (\lambda \in \mathbb{R}).$$

Proof. Define $\phi(x,z) = \phi_1(x,z) - i\phi_2(x,z)$ $(\operatorname{Im} z \geqq 0,\ x \geqq 0)$. Then (5.2) holds, and ϕ is entire with no zeroes if $x \geqq 0$, $\operatorname{Im} z \geqq 0$. Hence the function $h(x,z)$ is harmonic in $\operatorname{Im} z \geqq 0$ for every $x \geqq 0$. If we represent $h(x, z)$ in the form (5.7), the corresponding density which we denote by $\sigma(x, \lambda)$, satisfies by (5.8)

$$\sigma(x, \lambda_2) - \sigma(x, \lambda_1) = \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} h(x, \lambda)\, d\lambda,$$

since h is continuous in $\operatorname{Im} z \geqq \lambda$.

Now, $h(x, z) \to h(z)$ as $x \to \infty$ if $\operatorname{Im} z > 0$. Hence for the density $\sigma(\lambda)$ of $h(z)$, we have

$$\sigma(x, \lambda_2) - \sigma(x, \lambda_1) \to \sigma(\lambda_2) - \sigma(\lambda_1) \qquad \text{as} \qquad x \to \infty$$

at all points of continuity of $\sigma(\lambda)$; i.e.,

$$\sigma(\lambda_2) - \sigma(\lambda_1) = \lim_{x \to \infty} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} h(x, \lambda)\, d\lambda$$

at such points of continuity. On the other hand, by Theorem 5.1 we have $h(x, \lambda) \to \alpha(\lambda)$ as $x \to \infty$, and combining (5.4), (5.6), Lemma 4.4, and Sturm's comparison theorem we see that the convergence is bounded on $[\lambda_1, \lambda_2]$ (in fact, it is *uniform*). At any rate, we can interchange the limit and integral to get

$$\sigma(\lambda_2) - \sigma(\lambda_1) = \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \alpha(\lambda)\, d\lambda.$$

Hence we have

$$h(z) = \frac{\operatorname{Im} z}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\lambda)\, d\lambda}{|\lambda - z|^2}, \qquad \operatorname{Im} z > 0. \tag{5.12}$$

From this representation, and from the continuity of $\alpha(\lambda)$, it follows by standard arguments that $h(\lambda + i\varepsilon) \to \alpha(\lambda)$ as $\varepsilon \to 0$ $(\lambda \in \mathbb{R})$, and that in fact $h(z)$ is continuous in $\operatorname{Im} z \geqq 0$. This proves Theorem 5.11.

Incidentally, from (5.12) one finds the representation

$$\frac{w(z) - w(z_0)}{z - z_0} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\lambda)\, d\lambda}{(\lambda - z_0)(\lambda - z)}, \qquad \operatorname{Im} z, \qquad \operatorname{Im} z_0 > 0,$$

which determines $w(z)$ up to an additive constant if $\alpha(\lambda)$ is known.

One can, of course, define a holomorphic function $w(z)$ for $\operatorname{Im} z < 0$ as well. Using Proposition 3.7 as motivation, we set

$$w(z) = M_x\left(\frac{-1}{2G(x, x, z)}\right) \tag{5.13}$$

for $\operatorname{Im} z \neq 0$. Because of $G(x, y, \bar{z}) = \overline{G(x, y, z)}$, we have

$$w(\bar{z}) = \overline{w(z)}, \qquad \frac{\operatorname{Im} w(z)}{\operatorname{Im} z} > 0 \qquad \text{for} \qquad \operatorname{Im} z \neq 0. \tag{5.14}$$

Moreover, since $\alpha(\lambda) = 0$ for $\lambda < \lambda^* =$ left-most point in the spectrum of $L = \frac{-d^2}{dx^2} + q(x)$, we see from Theorem 5.11 and (5.14) that $w(z)$ is real for real $z = \lambda \leqq \lambda^*$. In other words, $w(z)$ extends analytically through $(-\infty, \lambda^*)$.

However, $w(z)$ can *not* be viewed as a one-valued function on the resolvent set. Indeed, if I is an interval on the real axis outside the spectrum of L, then by Theorem 4.7

$$\operatorname{Im} w(\lambda+i\varepsilon) \to \alpha(\lambda) = \alpha_I \in \tfrac{1}{2}\mathscr{M}(q),$$

where α_I is a constant. By (5.14) we have

$$w(\lambda+i\varepsilon) - w(\lambda-i\varepsilon) = 2i \operatorname{Im} w(\lambda+i\varepsilon) \to 2i\alpha_I \quad \text{if } \lambda \in I \quad \text{and} \quad \varepsilon \to 0;$$

i.e. $w(z)$ suffers a jump of $2i\,\alpha_I$ which is non-zero unless I lies in the lowest gap $(-\infty, \lambda^*)$.

On the other hand, since α_I is a constant, it is clear that $w'(z) = \frac{dw}{dz}$ *is* holomorphic on the resolvent set. We can view dw as a differential on the resolvent set. If γ is a loop crossing the real axis at $\lambda_1 < \lambda^*$ and $\lambda_2 \in I$ and nowhere else, then the period $\int_\gamma dw = 2i\alpha_I \in i\,\mathscr{M}(q)$.

We complete this section by relating $w(z)$ to the half-line spectral functions $\varrho_\xi(t)$ $(\xi \in E)$ introduced in Sect. 3 (see Proposition 3.11). First of all,

$$w(z) = M_x(m_+) = M_x(m_+(x,z)) = \int_E M_+(\xi,z)\,d\mu(\xi)$$

in the notation of Proposition 3.11: $M_+(\xi,z)$ is the continuous extension of $m_+(x,z)$ to E, and μ is the normalized Haar measure on E. (Of course, M_+ and M_x have different meanings). From Proposition 3.11 we have

$$\lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} M_+(\xi, v+i\varepsilon)\,dv \to \varrho_\xi(\lambda_2) - \varrho_\xi(\lambda_1)$$

for μ-a.a. ξ, and the convergence is bounded in ξ; λ_1 and λ_2 are real numbers. Hence, if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous with compact support, we have

$$\begin{aligned} &\lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \int_E \operatorname{Im} M_+(\xi, v+i\varepsilon)\,d\mu(\xi)\,dv \\ &= \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \operatorname{Im} w(v+i\varepsilon)\,dv = \int_E \int_{-\infty}^{\infty} f(\lambda)\,d\varrho_\xi(\lambda)\,d\mu(\xi). \end{aligned}$$

Since $w(v+i\varepsilon) \to \alpha(v)$ as $\varepsilon \to 0$, uniformly on compact v-intervals, we get

$$\int_{-\infty}^{\infty} f(v)\,\alpha(v)\,dv = \int_E \int_{-\infty}^{\infty} f(\lambda)\,d\varrho_\xi(\lambda)\,d\mu(\xi).$$

That is, if we "integrate the measures $d\varrho_\xi$" [5], we get

$$\int_E (d\varrho_\xi)\,d\mu(\xi) = \alpha(\lambda)\,d\lambda, \tag{5.15}$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$.

We record that $w(z)$ has the following properties:

$$\frac{\operatorname{Im} w}{\operatorname{Im} z} > 0, \qquad \frac{\operatorname{Im} w'}{\operatorname{Im} z} > 0, \qquad \operatorname{Re} w < 0 \qquad \text{for} \qquad \operatorname{Im} z \neq 0. \tag{5.16}$$

See (3.8); the middle relation will follow from Theorem 6.4 of the next section.

6. $w(z,q)$ as a Functional of q

The rotation number $\alpha(\lambda)$ and also $w(z)$ depend, of course, also on the potential function $q=q(x)$, and can be viewed as functionals of q. Actually, both functionals depend on $z-q$, $\lambda-q$ only:

$$w(z,q)=w(z-q), \qquad \alpha(\lambda,q)=\alpha(\lambda-q), \tag{6.1}$$

since the differential equation has the form $\phi''=(-z+q)\phi$. We derive some properties of these functionals.

Theorem 6.2. *$\alpha(\lambda,q)$ is continuous in q in the sup-topology*; i.e. *given $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that*

$$|\alpha(\lambda;q)-\alpha(\lambda;q_0)|<\varepsilon \qquad \text{if } \sup_x |q-q_0|<\delta(\varepsilon).$$

Proof. From Sturm's theory it follows that if $q\leqq\tilde{q}$, then the number of zeroes in $[0,x]$ of solutions $\phi(x,\lambda)$, $\tilde{\phi}(x,\lambda)$ normalized, say, by $\phi(0,\lambda)=0$, $\tilde{\phi}(0,\lambda)=0$, satisfy $N(x,\lambda)\geqq\tilde{N}(x,\lambda)$. It follows that $\alpha(\lambda,q)\geqq\alpha(\lambda,\tilde{q})$ if $q(x)\leqq\tilde{q}(x)$ (see 4.6). Therefore from $|q-q_0|<\delta$, we see that $q_0-\delta\leqq q\leqq q_0+\delta$, hence

$$\alpha(\lambda,q_0-\delta)\geqq\alpha(\lambda,q)\geqq\alpha(\lambda,q_0+\delta);$$

and since $\alpha(\lambda,q_0+\delta)=\alpha(\lambda-\delta,q_0)$, we can write this in the form

$$\alpha(\lambda+\delta,q_0)\geqq\alpha(\lambda,q)\geqq\alpha(\lambda-\delta,q_0).$$

Now we know that $\alpha(\lambda,q_0)$ is continuous in λ, so that we can choose $\delta=\delta(\varepsilon)$ with $\alpha(\lambda+\delta,q_0)-\alpha(\lambda-\delta,q_0)<\varepsilon$, which proves this theorem.

We can not expect differentiability properties of α. But for $w(z;q)$ we will determine the functional derivative in the following sense: for any almost periodic function p with $\mathcal{M}(p)\subset\mathcal{M}(q)$, consider the linear functional

$$\frac{d}{d\varepsilon}w(z,q+\varepsilon p)\,|_{\varepsilon=0}=M_x\left(\frac{\delta w}{\delta q}(x)\,p(x)\right) \tag{6.3}$$

through which $\dfrac{\delta w}{\delta q}$ is defined if it exists.

Theorem 6.4. *For* $\operatorname{Im} z\neq 0$, *the functional derivative* $\dfrac{\delta w}{\delta q}$ *exists and is given by*

$$\frac{\delta w}{\delta q}(x)=-G(x,x,z),$$

and the z-derivative is

$$\frac{dw}{dz}=M_x(G(x,x,z)).$$

Proof. If we set $p=-1$ and use (6.1), it is clear that

$$\frac{d}{d\varepsilon}w(z,q-\varepsilon)\,|_{\varepsilon=0}=\frac{d}{d\varepsilon}w(z+\varepsilon,q)\,|_{\varepsilon=0}=\frac{dw}{dz}(z,q), \qquad \frac{dw}{dz}=-M_x\left(\frac{\delta w}{\delta q}\right),$$

and so the second equation follows from the first.

For the computation of $\frac{\delta w}{\delta q}$, we use the formula (5.13) and differentiate the Green's function $G(x,y;z;q)$, which we abbreviate $G(x,y)$. From the resolvent identity, one obtains

$$\delta R_z = -R_z\,\delta q\,R_z,$$

and since $G(x,y)=G(y,x)$, we have

$$\begin{aligned}\delta G(x,x) &= -\int_{-\infty}^{\infty} G(x,y)\,\delta q(y)\,G(y,x)\,dy = -\int_{-\infty}^{\infty} G^2(x,y)\,\delta q(y)\,dy\\ &= \frac{-1}{W^2}\left\{\psi_+^2(x)\int_{-\infty}^{x}\psi_-^2(y)\,\delta q(y)\,dy + \psi_-^2(x)\int_{x}^{\infty}\psi_+^2(y)\,\delta q(y)\,dy\right\},\end{aligned}$$

where $W=[\psi_+,\psi_-]$, and ψ_+,ψ_- are the solutions of (1.1) discussed in Sect. 3 and 5. From this we find

$$\begin{aligned}\delta\frac{1}{2G(x,x)} &= \frac{-1}{2G^2(x,x)}\,\delta G(x,x)\\ &= \frac{1}{2}\left(\psi_-^{-2}(x)\int_{-\infty}^{x}\psi_-^2(y)\,\delta q(y)\,dy + \psi_+^{-2}(x)\int_{x}^{\infty}\psi_+^2(y)\,\delta q(y)\,dy\right).\end{aligned} \tag{6.5}$$

This has to be compared with $G(x,x)\,\delta q$, and we claim

$$\delta\frac{1}{2G(x,x)} - G(x,x)\,\delta q = \frac{d}{dx}H(x), \tag{6.6}$$

where

$$\begin{aligned}H(x) &= \frac{1}{2W}\left\{\frac{-\psi_+}{\psi_-}\int_{-\infty}^{x}\psi_-^2(y)\,\delta q(y)\,dy + \frac{\psi_-}{\psi_+}\int_{x}^{\infty}\psi_+^2(y)\,\delta q(y)\,dy\right\}\\ &= \int_{-\infty}^{+\infty} K(x,y)\,\delta q(y)\,dy,\end{aligned} \tag{6.7}$$

with

$$K(x,y) = \begin{cases} -\dfrac{1}{2}G(x,x)\dfrac{\psi_-^2(y)}{\psi_-^2(x)} & \text{for} \quad x \geqq y\\[2ex] +\dfrac{1}{2}G(x,x)\dfrac{\psi_+^2(y)}{\psi_+^2(x)} & \text{for} \quad x \leqq y.\end{cases}$$

To prove (6.6), one uses the identities

$$\left(\frac{\psi_+}{\psi_-}\right)' = \frac{-W}{\psi_-^2}, \qquad \left(\frac{\psi_-}{\psi_+}\right)' = \frac{W}{\psi_+^2},$$

and rewrites (6.5) in the form

$$\delta\frac{1}{2G(x,x)} = \frac{1}{2W}\left\{-\left(\frac{\psi_+}{\psi_-}\right)'\int_{-\infty}^{x}\psi_-^2(y)\,\delta q(y)\,dy + \left(\frac{\psi_-}{\psi_+}\right)'\int_{x}^{\infty}\psi_+^2(y)\,\delta q(y)\,dy\right\}.$$

Now differentiating the relation (6.7) makes (6.6) clear.

We claim that $H \varepsilon A(\mathscr{M})$. Since $G(x,x)$ has this property it suffices to show that

$$\psi_-^{-2}(x) \int_{-\infty}^{x} \psi_-^2(y)\,\delta q(y)\,dy, \qquad \psi_+^{-2}(x) \int_{x}^{\infty} \psi_+^2(y)\,\delta q(y)\,dy \tag{6.8}$$

belong to $A(\mathscr{M})$. We show this only for the second term which we rewrite as

$$\psi_+^{-2}(x) \int_0^{\infty} \psi_+^2(x+t)\,p(x+t)\,dt; \qquad p(y) = \delta q(y) \in A(\mathscr{M}).$$

According to the argument of Sect. 2

$$\frac{\psi_+^2(x+t)}{\psi_+^2(x)}\,p(x+t) \in A(\mathscr{M})$$

for fixed t. The same holds for

$$\psi_+^{-2}(x) \int_0^{s} \psi_+^2(x+t)\,p(x+t)\,dt,$$

since the integral can be uniformly approximated by finite sums. It remains to show that this integral converges uniformly in x for $s \to +\infty$; but this follows from Proposition 3.13. A similar argument applies to the first term in (6.8).

Hence H is almost periodic; taking the mean value of (6.6) yields the statement of Theorem 6.4.

We write the second formula of Theorem 6.4 as

$$\frac{dw}{dz} = \tau(R_z), \qquad R_z = (L-z)^{-1},$$

where τ denotes an analogue of a trace: if the kernel of an operator A is $a(x,y)$, we define

$$\tau(A) = \lim_{x\to\infty} \frac{1}{X} \int_0^{x} a(t,t)\,dt,$$

if this limit exists. This is natural for almost periodic $a(x,x)$. Now we wish to determine $\tau(E_\lambda)$ where E_λ is the spectral resolution of L, and its kernel is

$$e(x,y;\lambda) = \sum_{i,j=1,2}^{2} \int_{-\infty}^{\lambda} \phi_i(x,\lambda')\,\phi_j(y,\lambda')\,d\varrho_{ij}(\lambda');$$

here $\phi_1(x;\lambda)$, $\phi_2(x;\lambda)$ are normalized solutions, and

$$\begin{pmatrix} d\varrho_{11} & d\varrho_{12} \\ d\varrho_{21} & d\varrho_{22} \end{pmatrix}$$

is a symmetric, monotone increasing density matrix.

It turns out that, in general, $e(x,x;\lambda)$ is not almost periodic; nevertheless we have

Theorem 6.9. $\tau(E_\lambda)=\lim_{x\to\infty}\frac{1}{x}\int_0^x e(t,t;\lambda)\,dt=\frac{1}{\pi}\alpha(\lambda)$ *exists and is continuous in* λ.

From this formula it is evident that $\alpha(\lambda)$ is constant in a spectral gap.

Proof. From the equation $R_z=\int_{-\infty}^{\infty}(\lambda-z)^{-1}\,dE_\lambda$, one finds

$$\operatorname{Im} G(x,x;z)=\int_{-\infty}^{\infty}\operatorname{Im}\frac{1}{\lambda-z}\,d_\lambda e(x,x;\lambda)=\operatorname{Im} z\int_{-\infty}^{\infty}\frac{d_\lambda e(x,x;\lambda)}{|\lambda-z|^2}.$$

Moreover, if we define

$$\tilde{G}(x,z)=\frac{1}{x}\int_0^x G(t,t;z)\,dt,$$

$$\tilde{e}(x,\lambda)=\frac{1}{x}\int_0^x e(t,t;\lambda)\,dt,$$

at all points of continuity of ϱ_{ij}, i.e. of e then we get

$$\operatorname{Im}\tilde{G}(x,z)=\operatorname{Im} z\int_{-\infty}^{\infty}\frac{d_\lambda\tilde{e}(x,\lambda)}{|\lambda-z|^2}.$$

We know from Theorem 6.4 that $\operatorname{Im}\tilde{G}(x,z)$ tends to $\operatorname{Im} w'$ as $x\to\infty$. Now by integration by parts we find from (5.12) with $h(z)=\operatorname{Im} w(z)$

$$\pi\frac{\operatorname{Im} w'(z)}{\operatorname{Im} z}=\frac{d}{dz}\int_{-\infty}^{\infty}\frac{\alpha(\lambda)\,d\lambda}{|\lambda-z|^2}=\int_{-\infty}^{\infty}\frac{d\alpha(\lambda)}{|\lambda-z|^2}.$$

From our remarks on positive harmonic functions in the upper half-plane (Sect. 5), we conclude that

$$\tilde{e}(x,\lambda_2)-\tilde{e}(x,\lambda_1)\to\frac{1}{\pi}(\alpha(\lambda_2)-\alpha(\lambda_1))$$

at all points of continuity of the right-hand side. But α is continuous (Sect. 4), hence the convergence is unrestricted. Moreover, if $\lambda<\lambda^*=$ left end-point of the spectrum of (1.1), then $\tilde{e}(x,\lambda)=0$ and $\alpha(\lambda)=0$. This proves Theorem 6.9.

We remark that there are almost periodic functions possessing point eigenvalues (See Sect. 8). Let $\lambda=\lambda_0$ be such an eigenvalue, and let $\phi_0(x)\in L^2(-\infty,\infty)$ be the corresponding normalized eigenfunction. Then

$$e(x,x;\lambda)\,|_{\lambda_0-0}^{\lambda_0+0}=\phi_0^2(x).$$

This shows that $e(x,x;\lambda)$ need not be almost periodic. However, the discontinuity cancels when we take

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x e(t,t;\lambda)\,dt=\frac{1}{\pi}\alpha(\lambda).$$

There is also a formula involving de analogous to (5.15), which involves spectral densities on $(0,\infty)$. To state it, recall that $G(x,x;z)$ is almost periodic for

$\operatorname{Im} z > 0$ with frequency module in $\mathcal{M} = \mathcal{M}(q)$, hence extends uniquely to a function $\Gamma(\xi, z)$ on the hull E. For each $\xi \in E$, $\Gamma(\xi, \cdot)$ is holomorphic with positive imaginary part in $\operatorname{Im} z > 0$. Let $de_\xi(\lambda)$ be the corresponding measure on $\mathbb{R}$. Then it is easily shown that, if $q(x + c_\nu) \to \xi$ uniformly, then $de(c_\nu, c_\nu; \lambda) \to d\hat{e}_\xi(\lambda)$ in the sense that, if $f: \mathbb{R} \to \mathbb{R}$ is continuous with compact support, then

$$\int_{-\infty}^{\infty} f(\lambda)\, d_\lambda e(c_\nu, c_\nu, \lambda) \to \int_{-\infty}^{\infty} f(\lambda)\, d\hat{e}_\xi(\lambda).$$

Another way of describing $d\hat{e}_\xi(\lambda)$ is as follows. Let $(d\varrho_{ij}^\xi)$, $i, j = 1, 2$, be the spectral density matrix for Eq. (3.2) and write

$$e_\xi(x, y, \lambda) = \sum_{i,j=1}^{2} \int_{-\infty}^{\lambda} \phi_i^\xi(x, \lambda')\, \phi_j^\xi(x, \lambda')\, d\varrho_{ij}^\xi(\lambda'),$$

where ϕ_1^ξ, ϕ_2^ξ are normalized solutions of (3.2); then $d\hat{e}_\xi(\lambda) = de_\xi(0, 0; \lambda)$.

If we now integrate the measures $d\hat{e}_\xi$ with respect to the normalized Haar measure $d\mu(\xi)$ on E, we get

$$\int_E (d\hat{e}_\xi)\, d\mu(\xi) = \frac{1}{\pi}\, d\alpha(\lambda). \tag{6.10}$$

This equation is to be interpreted as saying that

$$\int_E \int_{-\infty}^{\infty} f(\lambda)\, d\hat{e}_\xi(\lambda)\, d\mu(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\lambda)\, d\alpha(\lambda) \tag{6.11}$$

for every continuous $f: \mathbb{R} \to \mathbb{R}$ of compact support.

To prove (6.11) we apply the first formula in the proof of Theorem 6.9 to get

$$\operatorname{Im} \Gamma(\xi, z) = \operatorname{Im} z \int_{-\infty}^{+\infty} \frac{d\hat{e}_\xi(\lambda)}{|\lambda - z|^2}, \qquad \operatorname{Im} z > 0. \tag{6.12}$$

It suffices to prove: Given λ_1, $\lambda_2 \in \mathbb{R}$ then for almost all $\xi \in E$

$$\hat{e}_\xi(\lambda_2) - \hat{e}_\xi(\lambda_1) = \lim_{\varepsilon \to +0} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} \Gamma(\xi, \lambda + i\varepsilon)\, d\lambda, \tag{6.13}$$

and we have bounded convergence in ξ.

This relation is established by following the proof of Proposition 3.11 where Γ, $d\hat{e}_\xi$ play the role of M_+, $d\varrho(\xi, \lambda)$, respectively. For this argument one has to verify that $\hat{e}_\xi(\lambda)$ is continuous at λ_1, λ_2 for almost all $\xi \in E$, or that the function

$$h: \xi \to \int_{\lambda_0 - 0}^{\lambda_0 + 0} d\hat{e}_\xi = [\hat{e}_\xi]_{\lambda_0}$$

satisfies

$$\frac{1}{x} \int_0^x h(\xi \cdot t)\, dt \to 0$$

for almost all ξ. This follows readily, even for all $\xi \in E$, from the continuity of $\alpha(\lambda)$. Indeed, since $\hat{e}_{\xi \cdot t}(\lambda) = e_\xi(t, t; \lambda)$ we obtain from Theorem 6.9

$$\frac{1}{x} \int_0^x h(\xi \cdot t)\, dt \to \frac{1}{\pi}(\alpha(\lambda_0 + 0) - \alpha(\lambda_0 - 0)) = 0.$$

To complete the proof of (6.11) we derive the identity

$$\frac{1}{\pi}\int_E \int_{-\infty}^{+\infty} f(\lambda)\,\mathrm{Im}\,\Gamma(\xi,\lambda+i\varepsilon)\,d\lambda\,d\mu(\xi)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)\,\mathrm{Im}\,w'(\lambda+i\varepsilon)\,d\lambda,$$

which follows from Theorem 6.4 and Fubini's theorem. Now, by (6.13), the left hand side tends to the left hand side of (6.11) as $\varepsilon\to+0$, and because of $\mathrm{Im}\,w(\lambda+i\varepsilon)\to\alpha(\lambda)$ the relation (6.11) follows.

Theorem 6.14. *The support of the measure $d\alpha(\lambda)$ agrees with the spectrum $\sigma(L)$ of* (1.1) *on $L^2(-\infty,+\infty)$.*

Proof. By Theorem 4.7 it is clear that the support of $d\alpha(\lambda)$ is contained in $\sigma(L)$, and it suffices to show: If I is a bounded open interval on which $\alpha(\lambda)$ is constant then

$$\sigma(L)\cap I=\emptyset. \tag{6.15}$$

From (6.11) we obtain

$$\int_E\int_I d\hat{e}_\xi(\lambda)\,d\mu(\xi)=\frac{1}{\pi}\int_I d\alpha(\lambda)=0,$$

hence for almost all $\xi\in E$

$$\int_I d\hat{e}_\xi(\lambda)=0.$$

Let us fix such ξ. Then we see from (6.12) that

$$\mathrm{Im}\,\Gamma(\xi,\lambda+i\varepsilon)\to 0 \quad \text{for} \quad \lambda\in I,\ \varepsilon\to+0.$$

The reflection principle shows that $\Gamma(\xi,z)$ is analytic on I and thus

$$\frac{-1}{\Gamma(\xi,z)}=M_+(\xi,z)-M_-(\xi,z)$$

is meromorphic on I. Let Z denote the set of zeroes of $\Gamma(\xi,z)$ on I. Then it follows from

$$\mathrm{Im}\,M_+(\xi,z)>0>\mathrm{Im}\,M_-(\xi,z) \quad \text{for} \quad \mathrm{Im}\,z>0$$

that

$$\mathrm{Im}\,M_+(\xi,\lambda+i\varepsilon)\to 0 \quad \text{for} \quad \lambda\in I-Z,\ \varepsilon\to+0.$$

Hence also $M_+(\xi,z)$ is meromorphic on I and

$$\lim\frac{1}{\pi}\int_0^\lambda \mathrm{Im}\,M_+(\xi,\lambda'+i\varepsilon)\,d\lambda'=\varrho_\xi(\lambda)$$

is piecewise constant on I, with possible jumps on Z.

Now $\varrho_\xi(\lambda)$ is the spectral function of the operator $L^+=L_\xi^+$ given by (1.16) and therefore $\sigma(L_\xi^+)\cap I$ contains only isolated point eigenvalues. Thus

$$\sigma_{\mathrm{ess}}(L_\xi^+)\cap I=\emptyset.$$

Since (see [23]) the sets $\sigma_{\mathrm{ess}}(L_\xi^+)$, $\sigma(L_\xi)$ agree and are independent of ξ, the assertion (6.15) and hence Theorem 6.14 is proven.

7. Connection with the Korteweg-deVries Equation

In this section we establish a connection of the rotation number with the Korteweg-deVries equation. In this theory one defines for arbitrary smooth functionals $v_j = v_j(q)$, $j = 1, 2$, a Poisson bracket which in our case takes the form

$$\{v_1, v_2\} = M_x\left(\frac{\delta v_1}{\delta q}\frac{d}{dx}\frac{\delta v_2}{\delta q}\right). \tag{7.1}$$

We describe this expression only formally, and mention that under appropriate (boundary) conditions this Poisson bracket is anti-symmetric and satisfies the Jacobi identity.

Our remark is that the functional $w(z, q)$ has the following remarkable property:

Theorem 7.2. *In the space $A(\mathcal{M})$ of continuous almost periodic potentials with frequency module $\mathcal{M}$, one has for* $\operatorname{Im} z_j \neq 0$, $j = 1, 2$:

$$\{w(z_1, q),\ w(z_2, q)\} = 0;$$

i.e., $w(z_1, q)$ *and* $w(z_2, q)$ *are "in involution" in the language of mechanics.*

Proof. By the definition (7.1) and Theorem 6.4, we have to show that

$$G(x, x; z_1)\frac{\partial}{\partial x}G(x, x; z_2)$$

has mean value zero. We write $G_j(x) = G(x, x; z_j)$. It is equivalent to show that

$$\frac{1}{2}\left(G_1\frac{\partial}{\partial x}G_2 - G_2\frac{\partial}{\partial x}G_1\right) = G_1\frac{\partial}{\partial x}G_2 - \frac{1}{2}\frac{\partial}{\partial x}(G_1 G_2)$$

has mean value zero.

If we normalize $\psi_\pm(x, z)$ so that $[\psi_+, \psi_-] = 1$, the Green's function becomes $G(x, x; z) = \psi_+(x, z)\,\psi_-(x, z)$. Let us write $\psi_+(x, z_j) = \alpha_j(x)$, $\psi_-(x, z_j) = \beta_j(x)$, so that $G_j(x) = \alpha_j(x)\,\beta_j(x)$ $(j = 1, 2)$. The proof follows from the identity

$$G_1\frac{\partial}{\partial x}G_2 - G_2\frac{\partial}{\partial x}G_1 = \frac{\partial}{\partial x}\frac{(\alpha_1\alpha_2' - \alpha_2\alpha_1')(\beta_1\beta_2' - \beta_2\beta_1')}{z_1 - z_2}, \tag{7.3}$$

since the right-hand side is again an almost periodic function.

To prove (7.3), we use the differential equation

$$\alpha_j'' = (q - z_j)\alpha_j \quad \text{to derive} \quad (\alpha_1\alpha_2' - \alpha_2\alpha_1')' = \alpha_1\alpha_2'' - \alpha_2\alpha_1'' = (z_1 - z_2)\alpha_1\alpha_2,$$

and a similar relation for β. Therefore, differentiation of the right-hand side of (7.3) gives

$$\begin{aligned}\alpha_1\alpha_2(\beta_1\beta_2' - \beta_2\beta_1') + (\alpha_1\alpha_2' - \alpha_2\alpha_1')\beta_1\beta_2 &= \alpha_1\beta_1(\alpha_2\beta_2' + \alpha_2'\beta_2) - \alpha_2\beta_2(\alpha_1\beta_1' + \alpha_1'\beta_1)\\ &= \alpha_1\beta_1(\alpha_2\beta_2)' - \alpha_2\beta_2(\alpha_1\beta_1)',\end{aligned}$$

which agrees with the left-hand side of (7.3). This proves the theorem.

From the asymptotic theory of Eq. (1.1), one knows that the Green's function admits an asymptotic expansion of the form

$$G(x,x;z) \sim \frac{1}{2\sqrt{-z}} \left\{ 1 + \frac{g_1(x)}{z} + \frac{g_2(x)}{z^2} + \dots \right\}, \tag{7.4}$$

valid in a sector about the negative real axis (we choose $\sqrt{-z} > 0$ for $z < 0$). Since $G(x,x;z)$ satisfies the differential equation

$$2G(G'' - 2(q-z)G) - G'^2 + 1 = 0, \tag{7.5}$$

as one verifies readily, one finds by comparison of coefficients a recursion formula for the g_j. One sees that the g_j are polynomials in $q, q', q'', \dots$. If one differentiates (7.5), one obtains the linear third order differential equation

$$D_3 G = 4zDG, \qquad \text{where} \qquad D_3 = D^3 - 4qD - 2q', \qquad D = \frac{d}{dx}.$$

This leads to the linear recursion formula

$$D_3 g_j = 4Dg_{j+1}, \qquad g_0 = 1, \tag{7.6}$$

which determines Dg_{j+1}, i.e. g_{j+1} only up to a polynominal, while the quadratic equation (7.5) has the advantage of determining g_j completely. It is a problem to show that the recursion (7.6) gives rise to polynomials while for the quadratic recursion (7.5) this is evident.

These formulas can be used to give an asymptotic expression for

$$w(z) = -\sqrt{-z} \left\{ 1 + \frac{w^{(1)}}{z} + \frac{w^{(2)}}{z^2} + \dots \right\}.$$

Indeed, by Theorem 6.4 we get the expansion of $\frac{dw}{dz}$ from (7.4) and therefore by integration

$$w^{(j)} = \frac{1}{1-2j} M_x(g_j).$$

These quantities give the conservation laws of the Korteweg-deVries equation, as follows from (7.6) which is referred to as Lenard's recursion formula in that theory. From Theorem 6.4 it follows that also

$$\{w^{(j)}, w^{(k)}\} = 0; \qquad \{w^{(j)}, w(z,q)\} = 0.$$

This implies that the functional $w = w(z,q)$ is not only translation invariant, but invariant under all higher Korteweg-deVries flows

$$\frac{\partial q}{\partial t} = D \frac{\delta w^{(j)}}{\delta q}.$$

8. Concluding Remarks and an Example

In this paper, we concentrated on studying the rotation number $\operatorname{Im} w$, but it is also of interest to consider $\operatorname{Re} w$, which (if $\operatorname{Im} z > 0$) measures the exponential decay of

$\psi_+(x; z)$ (see 3.9). In contrast to Im w, the real part of w is in general not continuous in Im $z \geqq 0$; an example (based on [16]) will be given in a forthcoming paper by R. Johnson. However, Re w is harmonic on the resolvent set, as noted in [12]; this together with results of this paper shows that the spectrum $F = \sigma(L)$ of (1.1) always has positive logarithmic capacity when q is almost periodic. The boundary value β of Re w on $\mathbb{R}$ is of physical interest; in particular, if $\beta(\lambda) < 0$ except on a subset of $\mathbb{R}$ of Lebesgue measure zero, then the spectrum $\sigma(L(\xi))$ of the Schroedinger operator $L(\xi)$ has no absolutely continuous component for μ–a.a. $\xi \in E$ [19]. See [2] for quasi-periodic *difference* operators with $\beta(\lambda) < 0$ a.e.

The study of Re w leads to another point of view: For Im $z \neq 0$ $\psi_+(\psi_-)$ is exponentially decaying for $x \to \infty$ $(x \to -\infty)$, and one can show that they define a hyperbolic structure in the two-dimensional solution space over E. Flows in a vector bundle with hyperbolic structure have been studied by Selgrade [24] and by Sacker and Sell [20, 21]. They introduced a spectrum for such a flow (which differs from that studied here); it turns out that this spectrum determines the limit values of Re w on the real axis. This approach suggests also generalizations from almost periodic flows on E, as considered here, to minimal or chain recurrent flows on E [12].

We give an example of an almost periodic potential q which has a point eigenvalue. An example of this sort has been given by *A.* Ya. Gordon [9]. One might expect that this is not possible if q is sufficiently smooth, quasi-periodic and the basic frequencies not close to resonance. Therefore we construct a real analytic, quasi-periodic $q(x)$ with two basic frequencies ω_1, ω_2 for which the ratio ω_2/ω_1 is badly approximable by rationals, i.e. satisfies

$$|p\omega_1 - q\omega_2| \geqq c(|p| + |q|)^{-\theta} \tag{8.1}$$

for some constants θ, c and all integers p, q not both zero.

The construction of this example depends on finding an odd quasi-periodic function

$$f(x) = F(\omega_1 x, \omega_2 x)$$

$$F(\xi_1, \xi_2) = \sum_{n=1}^{\infty} a_n \sin(j_n \xi_1 - k_n \xi_2) \tag{8.2}$$

for which

$$g(x) = \int_0^x f(t)\,dt \geqq c|x|^{1-\delta} \qquad \text{for} \qquad |x| \geqq 1, \ c > 0 \tag{8.3}$$

holds with some constant δ in $0 < \delta < 1$. Then the function

$$\phi(x) = e^{-g(x)}$$

belongs to $L^2(-\infty, +\infty)$ and is a solution of $\phi'' - q\phi = 0$ with

$$q = f' + f^2. \tag{8.4}$$

Hence ϕ is an eigenfunction for this q with eigenvalue $\lambda = 0$.

To construct such a function (8.2) we set

$$\omega_1 = e^{-1} = \sum_{m-0}^{\infty} \frac{(-1)^m}{m!}, \qquad \omega_2 = 1;$$

it is well known that e^{-1} satisfies (8.1) for any $\theta > 2$. We set

$$j_n = (n-1)!, \qquad k_n = (n-1)! \sum_{m=0}^{n-1} \frac{(-1)^m}{m!}$$

$$\varepsilon_n = |j_n \omega_1 - k_n \omega_2| = |j_n e^{-1} - k_n| = (n-1)! \left| \sum_{m=n}^{\infty} \frac{(-1)^m}{m!} \right| \tag{8.5}$$

and with δ in $0 < \delta < 1$

$$a_n = (j_n \omega_1 - k_n \omega_2) \varepsilon_n^{\delta}.$$

If we estimate ε_n by retaining the first or the first two terms in the last sum of (8.5) we find readily

$$\frac{1}{n+1} < \varepsilon_n < \frac{1}{n}.$$

Therefore ε_n is a monotone decreasing sequence and

$$|a_n| = \varepsilon_n^{1+\delta} < \frac{1}{n^{1+\delta}}.$$

Thus the sum in (8.2) is absolutely convergent and f is defined. It is even a real analytic function of x since for complex x the sum

$$\sum_{n=1}^{\infty} |a_n| \sinh(\varepsilon_n |\operatorname{Im} x|) < \infty$$

is convergent, therefore also q defined by (8.4) is real analytic, and it remains to prove (8.3).

Integrating f we find

$$g(x) = \sum_{n=1}^{\infty} \varepsilon_n^{\delta} (1 - \cos \varepsilon_n x) \geqq 0. \tag{8.6}$$

We obtain a better estimate by noting that

$$1 - \cos \varepsilon_n x \geqq 1 \qquad \text{for} \qquad \frac{\pi}{2\varepsilon_n} \leqq x \leqq \frac{\pi}{\varepsilon_n}. \tag{8.7}$$

For any $x \geqq 6\pi > \pi \varepsilon_5^{-1}$ we determine $k \geqq 3$ so that

$$\frac{\pi}{\varepsilon_{k-1}} \leqq x < \frac{\pi}{\varepsilon_k} < \pi(k+1). \tag{8.8}$$

Then we claim that (8.7) holds for all n in $k \leqq n \leqq 2k-3$. Indeed, $k \leqq n$ implies

$$x < \frac{\pi}{\varepsilon_k} \leqq \frac{\pi}{\varepsilon_n},$$

and $n \leqq 2k-3$ implies

$$x \geqq \frac{\pi}{\varepsilon_{k-1}} > \frac{1}{2} \frac{\pi}{\varepsilon_n},$$

since $\frac{\varepsilon_{k-1}}{\varepsilon_n} < \frac{n+1}{k-1} \leqq 2$.

Therefore (8.7) holds for $k \leqq n \leqq 2k-3$ and gives

$$g(x) \geqq \sum_{n=k}^{2k-3} (\varepsilon_n)^\delta \geqq (k-2)(\varepsilon_{2k-3})^\delta \geqq C_1 k^{(1-\delta)}$$

if x satisfies (8.8). From $x < \pi(k+1)$ we get $g(x) \geqq C_2 x^{1-\delta}$, as was to be shown.

It should be pointed out that the function $F(\xi)$ on the torus is only continuous and not smooth, even though $f(x)$ is real analytic; hence also the extension $Q(\xi)$ of $q(x)$ is not smooth. One could conjecture that point eigenvalues do not occur if $Q(\xi)$ is smooth and the basic frequencies satisfy

$$\left|\sum_\nu j_\nu \omega_\nu\right| \geqq c\left(\sum_\nu |j_\nu|\right)^{-\tau}$$

for some positive constants c, τ, for all integers j_ν with $\sum_\nu |j_\nu| \geqq 1$. We have not been able to decide this question.

Another interesting question is to decide whether two almost periodic potentials q_1, q_2 for which $w(z, q_1) = w(z, q_2)$ for all z in $\operatorname{Im} z > 0$ give rise to unitarily equivalent operators

$$L_j = -\left(\frac{d^2}{dx}\right) + q_j, \qquad j = 1, 2,$$

at least under appropriate smoothness assumptions. Again this question remains open.

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