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Decay of Correlations under Dobrushin's Uniqueness Condition and its Applications

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Abstract. An estimate on the correlation of functionals of Gibbs fields satisfying Dobrushin's uniqueness condition is given. As a consequence a result of Gross saying that the truncated pair correlation function decays in the same weighted summability sense as the potential can be extended to the whole Dobrushin uniqueness region. Applications to the central limit theorem and the second derivative of the pressure are also given.

0. Introduction

The well-known uniqueness theorem of Dobrushin [3] states that there is only one Gibbs state if the interaction is weak which means that the temperature is high or the activity small. This theorem has the advantage of being very general. No condition like finite range, pair interactions, finiteness of the single spin space or translation invariance is needed. Despite its generality the condition is surprisingly sharp as shown by Simon [12]. Moreover one gets not only uniqueness from it, but also properties of the Gibbs state: Dobrushin [4] showed that it is uniformly mixing, and Gross [6], [7] proved results on the decay of correlation and on the differentiability of the pressure. However one of his results, Theorem 2 in [6], was not proved in the whole Dobrushin uniqueness region, and his expression for the second derivative of the pressure in [7] is different from the ususal covariance series. In our paper here we close these two gaps.

In Sect. 2 we recall results from Dobrushin [4] in the form we will need them later. In Sect. 3 we state then our main result on the decay of correlation (Theorem 3.2). As corollaries we get the results of Gross [6] in the whole Dobrushin uniqueness region. In Sect. 4 we apply our results to check known conditions for the central limit theorem for functionals of Gibbs fields, and in Sect. 5 we show that the second derivative of the pressure is equal to the usual covariance series. The main theorems are proved in Sect. 6 by an extension of Dobrushin's uniqueness proof in [4]. We do not construct a dynamics which has the Gibbs state as an invariant measure like in Vasershtein [13] and Gross [6]. Finally in Sect. 7 we

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formulate our results for a general Vasershtein distance which covers also interesting non-compact examples.

1. Notations and Definitions

Let L be a countable set of sites, X a compact metric space, the individual spin space, and $\Omega = X^L$ the configuration space containing all functions $s: L \to X$. For $M \subset L$ we denote by $\mathscr{F}_M$ the σ-field generated by the maps $s \to s_a, a \in M$. Instead of $\mathscr{F}_L$ we write simply $\mathscr{F}$. The set of all continuous real-valued functions on Ω (with respect to the product topology) is denoted by $C(\Omega)$. For $f \in C(\Omega)$ and $a \in L$ put

$$\rho_a(f) = \sup\{|f(s) - f(t)|, s = t \text{ except at } a\}. \tag{1.1}$$

For such f the following holds for all $s, t \in \Omega$:

$$|f(t) - f(s)| \leqq \sum_{a \in L} \rho_a(f). \tag{1.2}$$

Let $\mathscr{V}$ be the class of finite non-empty subsets V of L and let $(p^V)_{V \in \mathscr{V}}$ be a *specification* on Ω. By this we mean that for each V $p^V(A|s)$ $(A \in \mathscr{F}_V, s \in X^{L \setminus V})$ is a probability kernel, i.e. a probability in the first and a measurable function in the second argument. For many purposes it is convenient to introduce the associated kernels $\pi^V(A|s)$ $(A \in \mathscr{F}, s \in \Omega)$ which are uniquely defined by the following properties (compare Preston [10], Chapter 1):

$\pi^V(A|\cdot)$ is $\mathscr{F}_{L \setminus V}$-measurable,
$\pi^V(\cdot|s)$ is a probability on $(\Omega, \mathscr{F})$ which coincides with
the Diract measure δ_s on $\mathscr{F}_{L \setminus V}$ and with $p^V(\cdot|s_{L \setminus V})$ on $\mathscr{F}_V$. (1.3)

We will always assume that the specification is *continuous* in the sense that $\pi^V f \in C(\Omega)$ for all $f \in C(\Omega)$, where

$$\pi^V f(s) = \int \pi^V(dt|s) f(t). \tag{1.4}$$

A *Gibbs state* to a specification $(p^V)_{V \in \mathscr{V}}$ is a probability μ on $(\Omega, \mathscr{F})$ whose conditional distribution with respect to $\mathscr{F}_{L \setminus V}$ are given by π^V, i.e.

$$\mu[A|\mathscr{F}_{L \setminus V}](\cdot) = \pi^V(A|\cdot) \ \mu\text{-a.s.} \quad (A \in \mathscr{F}, V \in \mathscr{V}), \tag{1.5}$$

or equivalently

$$\mu(\pi^V f) = \mu(f) \ \ (f \in C(\Omega), V \in \mathscr{V}). \tag{1.6}$$

The set of all Gibbs states is denoted by $\mathscr{G}(p)$. $\mathscr{G}(p)$ is convex and compact in the weak topology if $(p^V)_{V \in \mathscr{V}}$ is continuous. In order to prove the existence of Gibbs states we must assume that the specification is *consistent* which means that

$$\pi^V(\pi^W f) = \pi^V f \quad (f \in C(\Omega), W \subset V \in \mathscr{V}). \tag{1.7}$$

By compactness arguments the weak limit of $(\pi^{V_n}(\cdot|s))_{n \in \mathbb{N}}$ exists for $s \in \Omega$ and $V_n \uparrow L$

(at least if we choose a suitable subsequence), and using (1.7) it is easy to see that the limit is in $\mathscr{G}(p)$.

By a *potential* we mean a family of continuous $\mathscr{F}_V$-measurable functions $\varphi_V: \Omega \to \mathbb{R}$, $V \in \mathscr{V}$, satisfying

$$\sum_{V \ni a} |V| \sup \{ |\varphi_V(s)|, s \in \Omega \} < \infty \quad (a \in L). \tag{1.8}$$

Let ν be a finite measure on X, the *a priori* single-spin measure. The case of main interest is when all $p^V(dt|s)$ are absolutely continuous with respect to the product measure $\prod_{a \in V} \nu(dt_a)$ and the density is given by a potential:

$$p^V(dt|s) = Z_V(s)^{-1} \exp\left(- \sum_{W \cap V \neq \phi} \varphi_W(ts) \right) \prod_{a \in V} \nu(dt_a). \tag{1.9}$$

Here $Z_V(s)$ is a normalizing constant and ts is the configuration which is equal to t in V and equal to s in $L \backslash V$. Such specifications are always continuous and consistent. However the potentials will be used only in Sect. 5.

If X is not compact, then all the results which follow can be proved in the same way: we only have to take instead of $C(\Omega)$ an appropriate class of functions. However the hypotheses we will make are never satisfied in the interesting examples with a non-compact X. In order to deal with such examples it is useful to introduce a general Vasershtein distance as it was done in Dobrushin [4]. In Sect. 7 we will briefly state our results in this more general situation.

2. Comparison of Gibbs States and Uniqueness

Let $(p_i^V)_{V \in \mathscr{V}}$, $i = 1, 2$, be two continuous specifications and put

$$\begin{aligned}
\gamma_{ab} &= \tfrac{1}{2} \sup \{ \|p_i^b(\cdot|s) - p_i^b(\cdot|t)\|_{\mathrm{Var}}, s = t \text{ except at } a, i = 1, 2\} (a \neq b \in L), \\
\gamma_{aa} &= 0 \ (a \in L), \\
\beta_a &= \tfrac{1}{2} \sup \{ \|p_1^a(\cdot|s) - p_2^a(\cdot|s)\|_{\mathrm{Var}}, s \in X^{L \backslash a}\} \quad (a \in L).
\end{aligned} \tag{2.1}$$

Let Γ be the infinite matrix $(\gamma_{ab})_{a,b \in L}$ and Γ_V its restriction to $a \in V, b \in V$. We put

$$\begin{aligned}
\chi_{ab}^V &= \sum_{n=0}^{\infty} (\Gamma_V^n)_{ab} \quad (a, b \in V \in \mathscr{V}), \\
\chi_{ab} &= \sum_{n=0}^{\infty} (\Gamma^n)_{ab} \quad (a, b \in L).
\end{aligned} \tag{2.2}$$

$\chi_{ab}(\chi_{ab}^V)$ is nothing else than the Green's function (of the set V) for the random walk on L with transition probabilities γ_{ab} provided $\sum_a \gamma_{ab} \leqq 1$.

The next theorem is essentially contained in Theorem 3 of Dobrushin [4]:

Theorem 2.1. *Suppose* $(p_i^V)_{V \in \mathscr{V}}$, $i = 1, 2$, *are two continuous specifications such that*

$\sum_a \gamma_{ab} \leqq \alpha < 1$ *and* $\mu_i \in \mathscr{G}(p_i)$, $i = 1, 2$. *Then for all* $f \in C(\Omega)$ *we have*

$$|\mu_1(f) - \mu_2(f)| \leqq \sum_{a,b} \beta_a \chi_{ab} \rho_b(f).$$

Remark 2.2 It will become clear in the proof that $\sum_a \gamma_{ab} \leqq \alpha < 1$ can be replaced by the slightly weaker condition $\sum_a \chi_{ab} < \infty (b \in L)$. However we will state all our results with the former condition which is easier to check.

The proof will be given in Sect. 6. We will state now some consequences of Theorem 2.1. Whenever we have only one specification $(p^V)_{V \in \mathscr{V}}$ we will take an analogous definition of γ_{ab} as in (2.1) just omitting all indices i.

Corollary 2.3 (Dobrushin). *Let* $(p^V)_{V \in \mathscr{V}}$ *be a continuous specification with* $\sum_a \gamma_{ab} \leqq \alpha < 1$. *Then* $|\mathscr{G}(p)| \leqq 1$.

Proof. Let μ_1 and μ_2 be in $\mathscr{G}(p)$. Then $\beta_a = 0$ for all $a \in L$, whence $\mu_1(f) = \mu_2(f)$ for all $f \in C(\Omega)$, i.e. $\mu_1 = \mu_2$. □

Corollary 2.4. *Let* $(p^V)_{V \in \mathscr{V}}$ *be a continuous and consistent specification with* $\sum_a \gamma_{ab} \leqq \alpha < 1$. *Then for any sequence* $V_n \uparrow L$ *and any* $s \in \Omega$ $\pi^{V_n}(\cdot|s)$ *converges weakly to the unique Gibbs state* μ, *and the difference* $\mu(f) - \pi^V f(s)$ *can be estimated by formula (2.4) for* f *in* $C(\Omega)$ *with* $\sum_b \rho_b(f) < \infty$.

Proof. Fix $V \in \mathscr{V}$ and $s \in \Omega$. By the consistency condition (1.7) $p^V(\cdot|s)$ is a Gibbs state on $\Omega_V = X^V$ to the specification $(p^W(\cdot|\cdot s))_{W \subset V}$. Therefore by Theorem 2.1 for $f \in C(\Omega)$

$$|\pi^V f(t) - \pi^V f(s)| \leqq |\int f(ut)(p^V(du|t) - p^V(du|s))| + |\int f(ut) - f(us))p^V(du|s)|$$

$$\leqq \sum_{a \in V, b \in V} \left(\sum_{c \notin V} \gamma_{ca} \right) \chi^V_{ab} \rho_b(f) + \sum_{c \notin V} \rho_c(f). \tag{2.3}$$

Therefore also

$$|\mu(f) - \pi^V f(s)| \leqq \sum_{a,b \in V} \left(\sum_{c \notin V} \gamma_{ca} \right) \chi^V_{ab} \rho_b(f) + \sum_{c \notin V} \rho_c(f). \tag{2.4}$$

Now for all $a \in L$ $\sum_{c \notin V} \gamma_{ca}$ tends to zero for $V \uparrow L$, and

$$\sum_{a,b \in V} \left(\sum_{c \notin V} \gamma_{ca} \right) \chi^V_{ab} \rho_b(f) \leqq \alpha \sum_{a,b \in V} \chi^V_{ab} \rho_b(f) \leqq \alpha(1-\alpha)^{-1} \sum_b \rho_b(f).$$

Therefore by dominated convergence $\pi^V f(s) \to \mu(f)$ for f in $C(\Omega)$ with $\sum_b \rho_b(f) < \infty$. Such f generate weak convergence. □

From the estimate (2.3) we can easily deduce an estimate of the uniform mixing rate of the Gibbs state μ:

Proposition 2.5. (Dobrushin). *Let μ be the unique Gibbs state to a consistent and continuous specification $(p^V)_{V\in\mathscr{V}}$ with $\sum_a \gamma_{ab} \leqq \alpha < 1$. Then for all $W \subset V \in \mathscr{V}$ we have*

$$\varphi(W, V) = \sup\{|\mu(A|B) - \mu(A)|,\ A\in\mathscr{F}_W, B\in\mathscr{F}_{L\setminus V}, \mu(B) \neq 0\} \leqq \sum_{b\in W}\left(\sum_{c\notin V, a\in V} \gamma_{ca}\chi_{ab}\right)$$

which tends to zero for W fixed and $V\uparrow L$.

Proof. It is sufficient to prove the inequality for open sets $A\in\mathscr{F}_W$. Approximating $f = 1_A$ by a suitable sequence of continuous functions we find from (2.3)

$$|p^V(A|s) - p^V(A|t)| \leqq \sum_{a\in V, b\in W}\left(\sum_{c\notin V}\gamma_{ca}\right)\chi^V_{ab}. \tag{2.5}$$

So for $B\in\mathscr{F}_{L\setminus V}$ we get

$$|\mu(A\cap B) - \mu(A)\mu(B)| = |\textstyle\int 1_B(s)p^V(A|s)\mu(ds) - \int\int 1_B(s)p^V(A|t)\mu(dt)\mu(ds)|$$

$$\leqq \mu(B)\sum_{a\in V, b\in W}\left(\sum_{c\notin V}\gamma_{ca}\right)\chi^V_{ab}. \tag{2.6}$$

Because $\chi^V_{ab} \leqq \chi_{ab}$ the inequality is proved. The convergence to zero follows from dominated convergence: For fixed $a\in L$ $\sum_{c\notin V}\gamma_{ca}$ goes to zero for $V\uparrow L$, and for fixed $b\in W$

$$\sum_{a\in V}\sum_{c\notin V}\gamma_{ca}\chi_{ab} \leqq \alpha\sum_a \chi_{ab} \leqq \alpha(1-\alpha)^{-1}. \quad \square$$

3. Decay of Correlation

It is well known that the uniform mixing coefficients $\varphi(W, V)$ which were estimated in Proposition 2.5 tell us something about the correlation of functions localized in disjoint domains. Namely if f and g are in $C(\Omega)$ and f is $\mathscr{F}_W$-measurable and g $\mathscr{F}_{L\setminus V}$-measurable, then we have

$$|\mathrm{Cov}(f,g)_\mu| \leqq 2\varphi(W, V)\|f\|_\infty \|g\|_\infty. \tag{3.1}$$

In order to put this into a more intuitive appealing form it is convenient to consider a semimetric d on L, i.e. $d(\cdot,\cdot)$ is a nonnegative symmetric function on $L\times L$ for which the triangle inequality holds.

Proposition 3.1. *Let μ be the unique Gibbs state to a consistent and continuous specification $(p^V)_{V\in\mathscr{V}}$ satisfying $\sum_a \gamma_{ab}e^{d(a,b)} \leqq \alpha < 1$ for some semimetric d. Then for f and g in $C(\Omega)$, f $\mathscr{F}_W$-measurable and g $\mathscr{F}_V$-measurable with $d(W, V) = \inf\{d(a,b),\ a\in W, b\in V\} > 0$*

$$|\mathrm{Cov}(f,g)_\mu| \leqq e^{-d(W,V)}|W| 2\alpha(1-\alpha)^{-1}\|f\|_\infty \|g\|_\infty.$$

Proof. Let $\tilde V$ be the set $\{a\in L, d(W,a) < d(W,V)\}$. Then using (3.1) and Proposition 2.5 with $\tilde V$ instead of V we have

$$|\mathrm{Cov}(f,g)_\mu| \leqq 2\sum_{b\in W}\left(\sum_{c\notin\tilde V, a\in\tilde V}\gamma_{ca}\chi_{ab}\right)\|f\|_\infty \|g\|_\infty.$$

Now for $b \in W, a \in \tilde{V}, c \notin \tilde{V}: e^{d(W,V)} \leqq e^{d(b,c)} \leqq e^{d(b,a)} e^{d(a,c)}$. Therefore

$$|\mathrm{Cov}(f,g)_\mu| \leqq 2e^{-d(W,V)} \sum_{b \in W} \left(\sum_{c \notin \tilde{V}, a \in \tilde{V}} \gamma_{ca} e^{d(c,a)} \chi_{ab} e^{d(a,b)} \right) \|f\|_\infty \|g\|_\infty$$

$$\leqq 2e^{-d(W,V)} |W| \alpha \sup \left\{ \sum_a \chi_{ab} e^{d(a,b)}, b \in L \right\} \|f\|_\infty \|g\|_\infty.$$

But $\sum_a \gamma_{ab} e^{d(a,b)} \leqq \alpha < 1$ implies by the triangle inequality

$$\sum_a \chi_{ab} e^{d(a,b)} \leqq (1-\alpha)^{-1}. \quad \square \tag{3.2}$$

The next theorem which is our main result improves Proposition 3.1 in two ways: First it applies also to functions not necessarily localized in disjoint domains, and second we will get a decay of correlation at the same speed as the coefficients γ_{ab}. Let us define

$$\gamma^* = \sup \{ (2\chi_{aa} - 1)^{-1}, a \in L \} \leqq 1. \tag{3.3}$$

Theorem 3.2. *Let μ be the unique Gibbs state to a continuous and consistent specification $(p^V)_{V \in \mathscr{V}}$ with $\sum_a \gamma_{ab} \leqq \alpha < 1$. Then for any two functions f and g in $C(\Omega)$*

$$|\mathrm{Cov}(f,g)_\mu| \leqq \gamma^* \sum_{a,b} \left(\sum_c \chi_{ca} \chi_{cb} \right) \rho_a(f) \rho_b(g).$$

We will give the proof together with the proof of Theorem 2.1 in Sect. 6. Let us give now two corollaries of it which extend results of Gross [6] (see also the Remark 3.5i) below).

Corollary 3.3. *Let μ be the unique Gibbs state to a continuous and consistent specification $(p^V)_{V \in \mathscr{V}}$ satisfying $\sum_a \gamma_{ab} e^{d(a,b)} \leqq \alpha < 1$ for some semimetric d on L. Then for any two functions f and g in $C(\Omega)$ and any points $a, b \in L$*

$$|\mathrm{Cov}(f,g)_\mu| \leqq e^{-d(a,b)} \gamma^* (1-\alpha)^{-2} \sum_c e^{d(a,c)} \rho_c(f) \sum_c e^{d(b,c)} \rho_c(g).$$

Proof. By the triangle inequality we find

$$e^{d(a,b)} \sum_{a',b'} \left(\sum_c \chi_{ca'} \chi_{cb'} \right) \rho_{a'}(f) \rho_{b'}(g) \leqq \sum_{a',b'} \left(\sum_c \chi_{ca'} e^{d(c,a')} \chi_{cb'} e^{d(c,b')} \right)$$
$$\cdot e^{d(a,a')} \rho_{a'}(f) e^{d(b,b')} \rho_{b'}(g). \tag{3.4}$$

Furthermore by (3.2) $\sum_c \chi_{ca'} e^{d(c,a')} \chi_{cb'} e^{d(c,b')} \leqq \sum_c \chi_{ca'} e^{d(c,a')} \cdot \sum_c \chi_{cb'} e^{d(c,b')} \leqq (1-\alpha)^{-2}$. So the corollary follows immediately from Theorem 3.2. $\square$

Corollary 3.4. *Assume $L = \mathbb{Z}^\nu$. Let μ be the unique Gibbs state to a continuous and consistent specification $(p^V)_{V \in \mathscr{V}}$ satisfying $\sum_a \gamma_{ab} e^{d(a,b)} \leqq \alpha < 1$ and $\sum_b \gamma_{ab} e^{d(a,b)} \leqq \gamma < 1$*

for some translation invariant semimetric d on $\mathbb{Z}^\nu$. *Then for any functions f and g in* $C(\Omega)$ *and any point* $b \in \mathbb{Z}^\nu$

$$\sum_a |\mathrm{Cov}(f, g\circ\theta_a)_\mu| e^{d(b,a)} \leqq \gamma^*(1-\alpha)^{-1}(1-\gamma)^{-1} \sum_a e^{d(b,a)}\rho_a(f)\cdot \sum_a e^{d(0,a)}\rho_a(g).$$

Here $\theta_a: \Omega \to \Omega$ *is the shift operator defined by* $(\theta_a s)_b = s_{a+b}$.

Proof. Using Theorem 3.2 and formula (3.4) we get

$$\sum_a |\mathrm{Cov}(f, g\circ\theta_a)_\mu| e^{d(b,a)}\gamma^{*-1} \leqq \sum_{a,a',b'} \left(\sum_c \chi_{ca'} e^{d(c,a')}\chi_{cb'} e^{d(c,b')}\right)$$
$$\cdot e^{d(b,a')}\rho_{a'}(f) e^{d(a,b')}\rho_{b'}(g\circ\theta_a).$$

But since $d(a,b') = d(0, b'-a)$ and $\rho_{b'}(g\circ\theta_a) = \rho_{b'-a}(g)$, we can first take the sum over a and then the corollary follows immediately using (3.2). □

Remarks 3.5. i) In the Corollaries 3.3 and 3.4 we have smaller constants than Gross [6] in his Theorems 1 and 2, and—what is more important—Corollary 3.4 holds under a weaker condition: We request only $\alpha < 1$ and $\gamma < 1$ while Gross needs $\alpha < 1$ and $\gamma(1+\alpha) < 1$.

ii) In Corollary 3.4 the case $d \equiv 0$ is also of great interest. In the applications in Sect. 4 and 5 we will use only this case.

iii) If $L = \mathbb{Z}^\nu$ and $(p^V)_{V\in\mathscr{V}}$ is of finite range, i.e. $p^a(A|s)$ depends only on $s_{a+b}, b \in N$, for some $N \in \mathscr{V}$, then $\sum_a \gamma_{ab} \leqq \alpha < 1$ implies $\sum_a \gamma_{ab} \exp(\varepsilon|a-b|) \leqq \alpha' < 1$ if ε is small enough.

(iv) Assume $L = \mathbb{Z}^\nu$ and the specification $(p^V)_{V\in\mathscr{V}}$ is translation invariant. Then $\gamma_{ab} = \gamma_{a-b}$, so in particular $\alpha = \gamma$. Assume moreover that $\sum_a \gamma_a < 1$ and $\sum_a \gamma_a |a|^\beta < \infty$ for some $\beta > 0$. We then consider the metric $d_{\varepsilon,\beta}(a,b) = \min(\varepsilon|a-b|, \beta\log(1+|a-b|))$. It is not hard to see that for ε small enough $\sum_a \gamma_a \exp(d_{\varepsilon,\beta}(a,0)) < 1$. Therefore we have in Proposition 3.1 at least asymptotically a decay like $\mathrm{dist}(W,V)^{-\beta} = \inf\{|a-b|, a\in W, b\in V\}^{-\beta}$, and from Corollary 3.4 it follows that $\sum_a |\mathrm{Cov}(f, g\circ\theta_a)_\mu| |a|^\beta < \infty$ for f and g in $C(\Omega)$ *with* $\sum_a \rho_a(f)|a|^\beta < \infty$ and $\sum_a \rho_a(g)|a|^\beta < \infty$.

4. The Central Limit Theorem

In this section we always assume that $L = \mathbb{Z}^\nu$ and the specification $(p^V)_{V\in\mathscr{V}}$ is translation invariant. Then $\gamma_{ab} = \gamma_{a-b}$ and $\alpha = \gamma = \sum_a \gamma_a$. Moreover if there is only on Gibbs state, all its finite dimensional distributions are also translation invariant.

Let us denote the cube $\{a = (a_1, \ldots, a_\nu) \in \mathbb{Z}^\nu, -n \leqq a_i \leqq n, i = 1, \ldots, \nu\}$ by V_n. We investigate here when the central limit theorem holds, i.e. for which functions f

and probabilities μ

$$S_n^*(f)=(2n)^{-\nu/2}\sum_{a\in V_n}(f\circ\theta_a-\mu(f)) \tag{4.1}$$

converge weakly to a normal law. This is known to be true if f is $\mathscr{F}_V$-measurable for some $V\in\mathscr{V}$ and μ is a Gibbs state to a specification with weak interaction, see Dobrushin–Tirozzi [5], Sect. 1.3. We give here precise conditions on the specification without assuming finite range, and we allow also for general $f\in C(\Omega)$.

Theorem 4.1. *Let μ be the unique Gibbs state to a translation invariant, continuous and consistent specification $(p^V)_{V\in\mathscr{V}}$ with $\sum_a \gamma_a<1$ and let f be in $C(\Omega)$. If $\sum_a \gamma_a|a|^{\nu+\delta}<\infty$ for a $\delta>0$ and $\sum_a \rho_a(f)|a|^\nu<\infty$, then $S_n^*(f)$ converges weakly to a Gaussian random variable with mean zero and variance $\sum_a \operatorname{Cov}(f,f\circ\theta_a)_\mu$.*

Proof. If f is $\mathscr{F}_V$-measurable for some $V\in\mathscr{V}$, then Proposition 3.1 and Remark 3.5 iv) imply that the conditions of Bolthausen [2] for the central limit theorem are satisfied. Equivalently we can also use Proposition 2.5 and adapt Nahapetian's result [8] to the case of bounded variables. For a general $f\in C(\Omega)$ we fix $s\in\Omega$ and approximate f by f^m, where $f^m(t)=f(t_{V_m}s_{\mathbb{Z}^\nu\setminus V_m})$. Because f is continuous, $\operatorname{Cov}(f^m,f^m\circ\theta_a)_\mu$ converges to $\operatorname{Cov}(f,f\circ\theta_a)_\mu$ for $m\to\infty$ and fixed $a\in\mathbb{Z}^\nu$, and since $\rho_a(f^m)\leqq\rho_a(f)$, we have by Theorem 3.2 that $|\operatorname{Cov}(f^m,f^m\circ\theta_a)|\leqq\sum_{c,a',b}\chi_{c-a'}\chi_{c-b}\,\rho_{a'}(f)\rho_{b-a}(f)$. Therefore by dominated convergence the normal law $\mathscr{N}(0,\sum_a \operatorname{Cov}(f^m,f^m\circ\theta_a)_\mu)$ converges weakly to the normal law $\mathscr{N}(0,\sum_a \operatorname{Cov}(f,f\circ\theta_a)_\mu)$.

Furthermore by Corollary 3.4 $\lim_n \operatorname{Var}(S_n^*(f)-S_n^*(f^m))_\mu=\left|\sum_a \operatorname{Cov}((f-f^m),(f-f^m)\circ\theta_a)_\mu\right|\leqq\gamma^*(1-\alpha)^{-2}\left(\sum_a \rho_a(f-f^m)\right)^2$. But for $a\notin V_m$, $\rho_a(f-f^m)=\rho_a(f)$ and for $a\in V_m$, $\rho_a(f-f^m)\leqq 2\sup\{|f(t)-f(u)|, t=u \text{ in } V_m\}\leqq 2\sum_{a\notin V_m}\rho_a(f)$. Since by assumption $m^\nu\sum_{a\notin V_m}\rho_a(f)\leqq\sum_{a\notin V_m}\rho_a(f)|a|^\nu$ converges to zero for $m\to\infty$, we have $\lim_m\lim_n \operatorname{Var}(S_n^*(f)-S_n^*(f^m))_\mu=0$. So the theorem follows by a standard argument, see e.g. Billingsley [1], Theorem 4.2. □

We can weaken the assumption on the decay of γ_a and $\rho_a(f)$ if we assume instead that the real-valued random field $(f\circ\theta_a)_{a\in\mathbb{Z}^\nu}$ satisfies the FKG *inequalities*: For all $V\in\mathscr{V}$ and all increasing functions $F,G\colon\mathbb{R}^V\to\mathbb{R}$ is $\operatorname{Cov}(F(f\circ\theta_a,a\in V),G(f\circ\theta_a,a\in V))_\mu\geqq 0$. Then the following result is an immediate consequence of Corollary 3.4 and the work of Newman [9].

Theorem 4.2. *Let μ be the unique Gibbs state to a translation invariant, continuous and consistent specification $(p^V)_{V\in\mathscr{V}}$ with $\sum_a\gamma_a<1$, and let f be in $C(\Omega)$ with $\sum_a\rho_a(f)<\infty$. If $(f\circ\theta_a)_{a\in\mathbb{Z}^\nu}$ satisfies the* FKG *inequalities, then $S_n^*(f)$ converges weakly to a Gaussian law with mean zero and variance $\sum_a\operatorname{Cov}(f,f\circ\theta_a)_\mu$.*

5. Second Derivative of the Pressure

In this whole section we take $L = \mathbb{Z}^\nu$ and we consider specifications which are given as in (1.9) with the help of a translation invariant potential $\varphi = (\varphi_V)_{V \in \mathscr{V}}$ satisfying (1.8). The set of all such potentials is denoted by $\mathscr{P}$. Taking as norm $\|\varphi\| = \sum_{V \ni 0} |V| \sup\{|\varphi_V(s)|, s \in \Omega\}$ $\mathscr{P}$ turns into a Banach space. The Dobrushin uniqueness region $\mathscr{D} = \{\varphi \in \mathscr{P},\ \alpha(\varphi) = \sum_a \gamma_a(\varphi) < 1\}$ is then a non-empty open subset of $\mathscr{P}$ (see Gross [7], Proposition 2).

The *pressure* is defined as usual by

$$P_\varphi = \lim_{V \uparrow \mathbb{Z}^\nu} |V|^{-1} \log \int \exp\left(- \sum_{W \subset V} \varphi_W(s) \right) \prod_{a \in V} \nu(ds_a). \tag{5.1}$$

This limit exists if $V \uparrow \mathbb{Z}^\nu$ is suitably defined, see Ruelle [11]. The main result of this section is the following.

Theorem 5.1. *The pressure P is twice continuously differentiable on $\mathscr{D}$. Specifically the second derivative*

$$P''_\varphi(\psi^1, \psi^2) = \left. \frac{\partial^2 P(\varphi + u\psi^1 + v\psi^2)}{\partial u \partial v} \right|_{u=v=0}$$

exists for $\varphi \in \mathscr{D}$, ψ^1 and $\psi^2 \in \mathscr{P}$, and it is equal to

$$\sum_a \mathrm{Cov}(f_{\psi^1}, f_{\psi^2} \circ \theta_a)_\mu,$$

where $f_{\psi^i}(t) = - \sum_{V \ni 0} |V|^{-1} \psi^i_V(t)$ $(i = 1, 2)$ and μ is the unique Gibbs state in $\mathscr{G}(p(\varphi))$.

Remark 5.2. The existence of the second derivative was already proved by Gross [7], but the identification of the limit as the above covariance series in new. This series converges absolutely by Corollary 3.4 because $\alpha = \gamma$ and $\sum_a \rho_a(f_\psi) < \infty$, see Gross [7], formula (4.24).

The proof of Theorem 5.1 is based on the following result.

Proposition 5.3. *For $\varphi \in \mathscr{D}$ let μ_φ be the unique Gibbs state to the potential φ and let g be in $C(\Omega)$ with $\sum_a \rho_a(g) < \infty$. The map $\varphi \to \mu_\varphi(g)$ is then once continuously differentiable on $\mathscr{D}$. Specifically the derivative $\left. \frac{\partial}{\partial u} \mu_{\varphi + u\psi}(g) \right|_{u=0}$ exists for $\varphi \in \mathscr{D}, \psi \in \mathscr{P}$, and it is equal to $\sum_a \mathrm{Cov}(g, f_\psi \circ \theta_a)_{\mu_\varphi}$ where f_ψ is as in Theorem 5.1.*

Proof. For $\varepsilon > 0$ we put $\gamma_a = \sup\{\gamma_a(\varphi + u\psi), |u| < \varepsilon\}$, and we choose ε so small that $\sum_a \gamma_a < 1$. This is always possible, see the proof of Proposition 2 in Gross [7]. We fix $s \in \Omega$ and use for the measure $\pi^V_{\varphi + u\psi}(\cdot|s)$ the shorter notation π^V_u. Similarly we write μ_u instead of $\mu_{\varphi + u\psi}$. For $|u| < \varepsilon$ we have by Corollary 2.4 that $\mu_u(g) = \lim_{V \uparrow \mathbb{Z}^\nu} \pi^V_u(g)$. By a simple calculation we get $\frac{\partial}{\partial u} \pi^V_u(g) = \mathrm{Cov}(g, -$

$\sum_{W \cap V \neq \phi} \psi_W | \pi_u^V)$. Therefore it is sufficient to show that this covariance converges to $\sum_a \mathrm{Cov}(g, f_\psi \circ \theta_a | \mu_u)$ uniformly for $|u| < \varepsilon$.

In a first step we replace $- \sum_{W \cap V \neq \phi} \psi_W$ by $\sum_{a \in V} f_\psi \circ \theta_a$ which is equal to $- \sum_{W \cap V \neq \phi} \psi_W |W \cap V| / |W|$. Because $p_u^V(\cdot | s)$ is a Gibbs state on $\Omega_V = X^V$ to the specification $(p_u^W(\cdot | \cdot s))_{W \subset V}$, we get from Theorem 3.2 that the error in this step is bounded by

$$\sum_{a,b \in V} \left(\sum_{c \in V} \chi_{ca}^V \chi_{cb}^V \right) \rho_a(g) \rho_b \left(\sum_{W \cap V \neq \phi} (1 - |W \cap V| / |W|) \psi_W \right)$$

$$\leqq 2 \sum_{a,b \in V} \left(\sum_{c \in V} \chi_{ca}^V \chi_{cb}^V \right) \rho_a(g) \sum_{\substack{W \cap V^c \neq \phi \\ W \ni b}} \sup \{ |\psi_W(t)|, t \in \Omega \}.$$

Now the sum over W in the last expression converges to zero for fixed b and $V \uparrow \mathbb{Z}^\nu$. Moreover it is surely bounded by $\|\psi\|$ and we can use dominated convergence as in previous proofs in order to see that the error in this first step converges to zero for $V \uparrow \mathbb{Z}^\nu$ uniformly for $|u| < \varepsilon$.

In a second step we replace the expectations with respect to π_u^V by expectations with respect to μ_u. For any $V_0 \subset V$ we split the error into four terms, namely

$$\mu_u(g \cdot \sum_{a \in V_0} f_\psi \circ \theta_a) - \pi_u^V(g \cdot \sum_{a \in V_0} f_\psi \circ \theta_a), \tag{5.2}$$

$$\mu_u(g) \mu_u \left(\sum_{a \in V_0} f_\psi \circ \theta_a \right) - \pi_u^V(g) \pi_u^V \left(\sum_{a \in V_0} f_\psi \circ \theta_a \right), \tag{5.3}$$

$$\sum_{a \notin V_0} \mathrm{Cov}(g, f_\psi \circ \theta_a | \mu_u) \tag{5.4}$$

and

$$\sum_{a \in V \setminus V_0} \mathrm{Cov}(g, f_\psi \circ \theta_a | \pi_u^V). \tag{5.5}$$

Using Corollary 2.4, respectively formula (2.4), and similar arguments as before, the terms (5.2) and (5.3) can be shown to be arbitrarily small for fixed V_0 and V big enough (uniformly for $|u| < \varepsilon$). For the terms (5.4) and (5.5) we use Theorem 3.2: they are arbitrarily small if V_0 is big enough (again uniformly for $|u| < \varepsilon$, the term (5.5) also uniformly in $V \supset V_0$). So the proof is completed by choosing first a suitable V_0 and then a suitable V. □

Proof of Theorem 5.1. This is now straightforward, see also Gross [7], p. 70. If there is only one Gibbs state μ_φ, then there is only one tangent functional to the pressure P at φ (see e.g. Preston [10], Theorem 8.3), and we have $\left. \frac{\partial}{\partial u} P_{\varphi + u\psi^1} \right|_{u=0} = \mu_\varphi(f_{\psi^1})$. So it is sufficient to apply Proposition 5.2 with $g = f_{\psi^1}$. □

6. Proofs of the Main Theorems

We first give a proof of Theorem 2.1 though it will be essentially the same as in

Dobrushin [4], but we will use an analogous argument for Theorem 3.2 which can be understood more easily in the simpler case of Theorem 2.1. We call $(\alpha_a)_{a\in L}$ an estimate for μ_1 and μ_2, $\mu_i \in \mathcal{G}(p_i)$, if

$$|\mu_1(f) - \mu_2(f)| \leqq \sum_{a\in L} \alpha_a \rho_a(f) \qquad (f\in C(\Omega)). \tag{6.1}$$

For any $(\alpha_a)_{a\in L}$ and $b\in L$ we define $(\tilde{\alpha}_a(b))_{a\in L}$ by

$$\tilde{\alpha}_a(b) = \begin{cases} \alpha_a & \text{if } a \neq b \\ \beta_b + \sum_{c\neq b} \alpha_c \gamma_{cb} & \text{if } a = b \end{cases}. \tag{6.2}$$

The clue for the proof is the following lemma.

Lemma 6.1 *If $(\alpha_a)_{a\in L}$ is an estimate for μ_1 and μ_2, then for any $b\in L$ $(\tilde{\alpha}_a(b))_{a\in L}$ is also an estimate for μ_1 and μ_2.*

Proof. Using formula (3.5) of Gross [6] we find

$$\begin{aligned} |\mu_1(f) - \mu_2(f)| &\leqq |\mu_1(\pi_1^b f) - \mu_1(\pi_2^b f)| + |\mu_1(\pi_2^b f) - \mu_2(\pi_2^b f)| \\ &\leqq \beta_b \rho_b(f) + \sum_{a\in L} \alpha_a \rho_a(\pi_2^b f) \leqq \beta_b \rho_b(f) + \sum_{a\neq b} \alpha_a \rho_a(f) \\ &\quad + \sum_{a\neq b} \alpha_a \gamma_{ab} \rho_b(f). \quad \square \end{aligned}$$

So starting with an arbitrary estimate, e.g. $\alpha_a \equiv 1$, one can apply (6.2) repeatedly for different $b\in L$ and hopefully one will reach $\left(\sum_c \beta_c \chi_{ca}\right)_{a\in L}$ which is not changed by (6.2). However we do not show such a convergence, our proof is indirect: if the best possible estimate is bigger than $\left(\sum_c \beta_c \chi_{ca}\right)_{a\in L}$, we can always make it smaller by (6.2) which gives a contradiction.

Proof of Theorem 1.2. We fix $V\in\mathcal{V}$ and consider

$$\alpha_a^S = \begin{cases} 1 & \text{if } a\notin V, \\ S\sum_{c\in V} \beta_c \chi_{ca}^V + S \sum_{b\notin V, c\in V} \gamma_{bc}\chi_{ca}^V & \text{if } a\in V. \end{cases} \tag{6.3}$$

By the definition of χ_{ab}^V we have

$$\sum_{a\in V} \chi_{ca}^V \gamma_{ab} = \chi_{cb}^V - \delta_{cb} \qquad (c\in V, b\in V), \tag{6.4}$$

which implies for $b\in V$

$$\begin{aligned} \sum_{a\in L} \alpha_a^S \gamma_{ab} &= \sum_{a\notin V} \gamma_{ab} + S\sum_{c\in V} \beta_c \chi_{cb}^V - S\beta_b + S \sum_{d\notin V, c\in V} \gamma_{dc}\chi_{cb}^V \\ &- S\sum_{d\notin V} \gamma_{db} = \alpha_b^S - S\beta_b - (S-1)\sum_{d\notin V} \gamma_{db}. \end{aligned} \tag{6.5}$$

Let us assume that $\beta_a > 0$ for all $a\in L$. This is no essential restriction since we can always consider first $\tilde{\beta}_a = \beta_a + \varepsilon$ and then let ε tend to zero. Under this assumption

$(\alpha_a^S)_{a\in L}$ is an estimate for μ_1 and μ_2 if $S \geqq \max(\beta_a^{-1}, a\in V)$. We put $\bar{S} = \inf\{S, (\alpha_a^S)_{a\in L}$ is an estimate for μ_1 and $\mu_2\} < \infty$. We suppose that $\bar{S} > 1$ and show that this leads to a contradiction.

To any estimate $(\alpha_a)_{a\in L}$ for μ_1 and μ_2 satisfying $\alpha_a \leqq \alpha_a^{\bar{S}(1+\delta)}(a\in L)$ for some $\delta > 0$ there is by the definition of $\bar{S}$ a point $b\in V$ such that $\alpha_b > \alpha_b^{\bar{S}(1-\delta)}$. To any such estimate we consider the estimate $(\tilde{\alpha}_a(b))_{a\in L}$ defined by (6.2). Then we get from (6.5) $\tilde{\alpha}_b(b) \leqq \beta_b + \sum_{a\in L} \alpha_a^{\bar{S}(1+\delta)}\gamma_{ab} \leqq \alpha_b^{\bar{S}(1-\delta)} + 2\bar{S}\delta\alpha_b^1 - (\bar{S}(1+\delta)-1)(\beta_b + \sum_{d\notin V}\gamma_{db}) \leqq \alpha_b^{\bar{S}(1-\delta)}$ if δ is small enough because $\bar{S} > 1$. So by repeated application of (6.2) we can find an estimate satisfying $\alpha_a \leqq \alpha_a^{\bar{S}(1-\delta)}$ for all $a\in V$ and therefore also for all $a\in L$ which contradicts the definition of $\bar{S}$. Hence $\bar{S} \leqq 1$.

Finally we want to expand V. This can be done easily because χ_{ab}^V increases to χ_{ab} for $V\uparrow L$ and for fixed $a\in L$ $\sum_{b\notin V, c\in V} \gamma_{bc}\chi_{ca}^V$ tends to zero, see the argument in the proof of Proposition 1.4. □

Turning now to the proof of Theorem 3.2 we call $(\alpha_{ab})_{a,b\in L}$ with $\alpha_{ab} = \alpha_{ba}$ a covariance estimate for $\mu\in\mathscr{G}(p)$ if

$$|\mathrm{Cov}(f,g)_\mu| \leqq \sum_{a,b\in L} \alpha_{ab}\rho_a(f)\rho_b(g) \qquad (f,g\in C(\Omega)). \tag{6.6}$$

$\alpha_{ab} \equiv 1$ is always a covariance estimate for any μ because $\inf f(s) \leqq \mu(f) \leqq \sup f(s)$. The analogue of (6.2) is

$$\tilde{\alpha}_{ab}(c) = \begin{cases} \alpha_{ab} & \text{if } a \neq c \text{ and } b \neq c, \\ \sum_{a'} \alpha_{aa'}\gamma_{a'c} & \text{if } a \neq c \text{ and } b = c, \\ \sum_{a'} \alpha_{ba'}\gamma_{a'c} & \text{if } a = c \text{ and } b \neq c, \\ \sum_{a',b'} \alpha_{a'b'}\gamma_{a'c}\gamma_{b'c} + 1 & \text{if } a = b = c. \end{cases} \tag{6.7}$$

and the following lemma corresponds to Lemma 6.1:

Lemma 6.2. *If $(\alpha_{ab})_{a,b\in L}$ is a covariance estimate for μ, then for any $c\in L$ $(\tilde{\alpha}_{ab}(c))_{a,b\in L}$ is also a covariance estimate for μ.*

Proof. $|\mathrm{Cov}(f,g)_\mu| \leqq |\mu(\pi^c(fg)) - \mu(\pi^c f \pi^c g)| + |\mu(\pi^c f \pi^c g) - \mu(\pi^c f)\mu(\pi^c g)| \leqq \rho_c(f)\rho_c(g)$ $+ \sum_{a,b} \alpha_{ab}\rho_a(\pi^c f)\rho_b(\pi^c g) \leqq \rho_c(f)\rho_c(g) + \sum_{a\neq c, b\neq c} \alpha_{ab}\rho_a(f)\rho_b(g) + \sum_{a\neq c, b} \alpha_{ab}\gamma_{bc}\rho_a(f)\rho_c(g),$ $+ \sum_{b\neq c, a} \alpha_{ab}\gamma_{ac}\rho_c(f)\rho_b(g) + \sum_{a,b} \alpha_{ab}\gamma_{bc}\gamma_{ac}\rho_c(f)\rho_c(g)$, and we note that $\alpha_{ab} = \alpha_{ba}$. □

The proof of Theorem 3.2 follows now the same lines as the previous proof. The only difference is that $\gamma^*\left(\sum_c \chi_{ca}\chi_{cb}\right)_{a,b\in L}$ with γ^* defined by (3.3) does not remain unchanged by (6.7), but it is at least not increased by (6.7) which is sufficient for us (see also Remark 6.3 below).

Proof of Theorem 3.2. We fix V, put $M = 1/\gamma^*$ and consider

$$\alpha_{ab}^S = \begin{cases} 1 & \text{if } a \notin V \text{ or } b \notin V \\ S \sum_{c \in V} \chi_{ca}^V \chi_{cb}^V + M \cdot S \sum_{c \notin V, d \in V} \gamma_{cd}(\chi_{da}^V + \chi_{db}^V) & \text{if } a \text{ and } b \in V. \end{cases} \tag{6.8}$$

Then using (6.4) we find for $a \in V, b \in V$

$$\begin{aligned} \sum_c \alpha_{ac}^S \gamma_{cb} &= \sum_{c \notin V} \gamma_{cb} + S \sum_{c' \in V} \chi_{c'a}^V \chi_{c'b}^V - S \cdot \chi_{ba}^V + M \cdot S \sum_{c' \notin V, d \in V} \gamma_{c'd} \chi_{da}^V \cdot \sum_{c \in V} \gamma_{cb} \\ &+ M \cdot S \sum_{c' \in V, d \in V} \gamma_{c'd} \chi_{db}^V - M \cdot S \sum_{c' \notin V} \gamma_{c'b} \leqq \alpha_{ab}^S - S \cdot \chi_{ba}^V \\ &- (M \cdot S - 1) \sum_{c \notin V} \gamma_{cb}, \end{aligned} \tag{6.9}$$

since $\sum_{c \in V} \gamma_{cb} < 1$. From (6.9) it follows that for $b \in V$

$$\begin{aligned} \sum_{a,c} \alpha_{ac}^S \gamma_{ab} \gamma_{cb} &\leqq \sum_{a \notin V} \gamma_{ab} \sum_c \gamma_{cb} + \sum_{a \in V} \alpha_{ab}^S \gamma_{ab} - S \sum_{a \in V} \chi_{ba}^V \gamma_{ab} - (M \cdot S - 1) \\ &\cdot \sum_{c \notin V} \gamma_{cb} \sum_{a \in V} \gamma_{ab} \leqq \sum_{a \notin V} \gamma_{ab} + \alpha_{bb}^S - S \chi_{bb}^V - M \cdot S \sum_{c \notin V} \gamma_{cb} \\ &- S \chi_{bb}^V + S - (M \cdot S - 1) \sum_{c \notin V} \gamma_{cb} = \alpha_{bb}^S - 1 - (S(2\chi_{bb}^V - 1) - 1) \\ &- 2(M \cdot S - 1) \sum_{c \notin V} \gamma_{cb} \end{aligned} \tag{6.10}$$

(we have used $\alpha_{ab}^S = \alpha_{ba}^S$ and $\sum_c \gamma_{cb} < 1$). The important point is that for $S > \gamma^*$ the final bound in (6.9) and (6.10) is less than α_{ab}^S respectively $\alpha_{bb}^S - 1$ if we assume that $\chi_{ab}^V > 0$ for all $a \in V, b \in V$. If this assumption does not hold we can take first $\tilde{\gamma}_{ab} = \gamma_{ab} + \varepsilon \beta_{ab}$ for a suitable (β_{ab}) and then let ε go to zero.

We put $\bar{S} = \inf\{S, (\alpha_{ab}^S)_{a,b \in L}$ is a covariance estimate for $\mu\} < \infty$. Suppose that $\bar{S} > \gamma^*$ and let $(\alpha_{ab})_{a,b \in L}$ be any covariance estimate for μ satisfying $\alpha_{ab} \leqq \alpha_{ab}^{\bar{S}(1+\delta)}$ $(a, b \in L)$ for some $\delta > 0$. By definition of $\bar{S}$ there is then a pair $a_0 \in V, b_0 \in V$ such that $\alpha_{a_0 b_0} > \alpha_{a_0 b_0}^{\bar{S}(1-\delta)}$. To such an estimate we consider the estimate $(\tilde{\alpha}_{ab}(a_0))$ defined by (6.7). From (6.9), (6.10) and $\sum_c \gamma_{cb} < 1$ we get then $\tilde{\alpha}_{a_0 b}(a_0) = \tilde{\alpha}_{b a_0}(a_0) \leqq \alpha_{a_0 b}^{\bar{S}(1-\delta)} (b \in L)$ for δ small enough. So by repeated application of (6.7) we come to a contradiction like in the previous proof.

Finally for letting V tend to L we argue as before. □

Remark 6.3. It is a little disturbing that the estimate $\left(\gamma^* \sum_c \chi_{ca} \chi_{cb}\right)_{a,b \in L}$ is not optimal because it can still be made smaller by (6.7). In the case $\gamma_{ab} = \gamma_{ba} (a, b \in L)$, e.g. if we have pair interactions, $(\chi_{ab})_{a,b \in L}$ is unchanged by (6.7), and the same proof as before shows that then $(\chi_{ab})_{a,b \in L}$ is also a covariance estimate for μ. It can be proved that always $\chi_{ab} \leqq \gamma^* \sum_c \chi_{ca} \chi_{cb} (a, b \in L)$, however this new estimate gives in the Corollaries 3.3 and 3.4 only the smaller constant $(1-\alpha)^{-1}$ instead of $\gamma^*(1-\alpha)^{-2}$, but no substantial improvement.

7. Generalization to the Non-Compact case

Let $(X,\mathscr{B})$ be a measurable space with a metric $r(\cdot,\cdot)$ which is a measurable function on $(X,\mathscr{B})\times(X,\mathscr{B})$. For two probabilities q_1 and q_2 on $(X,\mathscr{B})$ the *Vasershtein distance* is defined by

$$R(q_1,q_2)=\inf\int r(s_1,s_2)Q(ds_1,ds_2), \tag{7.1}$$

where the infimum is taken over all probabilities Q on $(X,\mathscr{B})\times(X,\mathscr{B})$ with projections q_1 and q_2. For the metric $r(s_1,s_2)=0$ if $s_1=s_2$, $r(s_1,s_2)=1$ otherwise, we have $R(q_1,q_2)=\frac{1}{2}\|q_1-q_2\|_{\text{Var}}$. For two specifications $(p_i^V)_{V\in\mathscr{V}}$, $i=1,2$, on $\Omega=X^L$ we define

$$\begin{aligned}\gamma_{ab}&=\sup\{R(p_i^b(\cdot|s),p_i^b(\cdot|t))/r(s_a,t_a),\ s=t \text{ expect at } a,\ i=1,2\}\\ \beta_a&=\sup\{R(p_1^a(\cdot|s),p_2^a(\cdot|s)),s\in\Omega\},\end{aligned} \tag{7.2}$$

The role of $C(\Omega)$ is taken over by the "Lipschitz continuous" functions: For $f:\Omega\to\mathbb{R}$ we put

$$\rho_a(f)=\sup\{|f(s)-f(t)|/r(s_a,t_a), s=t \text{ expect at } a\} \tag{7.3}$$

and let $LC(\Omega)$ be the set of functions for which $\rho_a(f)<\infty$

$$(a\in L) \text{ and } |f(s)-f(t)|\leqq\sum_a\rho_a(f)r(s_a,t_a)\quad(s\in\Omega,t\in\Omega).$$

Like Theorem 2.1 the next result is essentially in Dobrushin [4].

Theorem 7.1. *Suppose $(p_i^V)_{V\in\mathscr{V}}$, $i=1,2$, are two specifications such that $\sum_a\gamma_{ab}\leqq\alpha<1$ and $\pi_i^V(LC(\Omega))\subset LC(\Omega)$. Let μ_i be in $\mathscr{G}(p_i)$ such that for some $u\in\Omega$ $\int r(s_a,u_a)\mu_i(ds)\leqq C<\infty$ $(i=1,2)$. Then we have for all $f\in LC(\Omega)$*

$$|\mu_1(f)-\mu_2(f)|\leqq\sum_{a,b}\beta_b\chi_{ba}\rho_a(f).$$

Proof. By the definition of $LC(\Omega)$ we have for $f\in LC(\Omega)$:

$$\begin{aligned}|\mu_1(f)-\mu_2(f)|&\leqq\int\int|f(s)-f(t)|\mu_1(ds)\mu_2(dt)\\&\leqq\sum_a\rho_a(f)\left(\int r(s_a,u_a)\mu_1(ds)+\int r(u_a,t_a)\mu_2(dt)\right)\\&\leqq 2C\sum_a\rho_a(f),\end{aligned}$$

so there exists a uniformly bounded estimate for μ_1 and μ_2. The rest of the proof, in particular Lemma 6.1, is the same. For details see Dobrushin [4]. □

For the generalization of Theorem 3.2 we need one more definition. Let

$$\sigma_a^2=\sup\{\inf\{\int r(u,m(s))^2p^a(du|s), m:X^{L\setminus a}\to X\},\quad s\in X^{L\setminus a}\}. \tag{7.4}$$

Theorem 7.2. *Suppose* $(p^V)_{V\in\mathscr{V}}$ *is a specification such that* $\sum_a \gamma_{ab} \leqq \alpha < 1$, $\sigma_a^2 \leqq \sigma^2 < \infty$ *and* $\pi^V(LC(\Omega)) \subset LC(\Omega)$. *Let* μ *be in* $\mathscr{G}(p)$ *such that for some* $u\in\Omega$ $\int r(s_a, u_a)^2 \mu(ds) \leqq C < \infty$. *Then we have for all* f *and* g *in* $LC(\Omega)$

$$|\mathrm{Cov}(f,g)_\mu| \leqq 4\sigma^2\gamma^* \sum_{a,b,c} \chi_{ca}\chi_{cb}\rho_a(f)\rho_b(g).$$

Proof. First we observe that

$$\begin{aligned}|\mathrm{Cov}(f,g)_\mu| &\leqq \iiint |f(s)-f(t)||g(s)-g(v)|\mu(ds)\mu(dt)\mu(dv)\\ &\leqq \sum_{a,b} \rho_a(f)\rho_b(g) \iiint (r(s_a,u_a)+r(u_a,t_a))(r(s_b,u_b)+r(u_b,v_b))\\ &\quad \cdot \mu(ds)\mu(dt)\mu(dv) \leqq 4C \sum_{a,b} \rho_a(f)\rho_b(g),\end{aligned}$$

so there exists a uniformly bounded covariance estimate for μ. Furthermore

$$\begin{aligned}|\mu(\pi^c(fg)) - \mu(\pi^c f \pi^c g)| &\leqq \rho_c(f)\rho_c(g) \sup_s \iiint r(u,t) r(u,v) p^c(du|s)p^c(dt|s)p^c(dv|s)\\ &\leqq 4\sigma_c^2 \rho_c(f)\rho_c(g),\end{aligned}$$

so Lemma 6.2 will be correct if we define $\tilde{\alpha}_{cc}(c)$ in (6.7) by $\sum_{a',b'} \alpha_{a'b'}\gamma_{a'c}\gamma_{b'c} + 4\sigma^2$. The rest of the proof is the same. □

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