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Analytic Structure and Explicit Solution of an Important Implicit Equation

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Abstract. The equation $z = 2G(z) - \exp G(z) + 1$ (and similar ones obtained from it by substitutions) appears in connection with a variety of problems ranging from pure mathematics (combinatorics; some first order, nonlinear differential equations) over statistical thermodynamics to renormalization theory. It is therefore of interest to solve this equation for $G(z)$ explicitly. It turns out, after study of the complex structure of the z and G planes, that an explicit integral representation of $G(z)$ can be given, which may be directly used for numerical calculations of high precision.

1. Introduction

The equation to be studied in this paper [called the "bootstrap equation" (BE)], namely

$$z = 2G(z) - \exp G(z) + 1 \tag{1.1}$$

can, by substitutions, be brought into various forms. Take, for instance, the substitution

$$\begin{aligned} z &= f(w), \\ G(z) &= A + B \cdot H(w), \end{aligned} \tag{1.2}$$

which yields

$$\begin{aligned} f(w) &= 2B \cdot H(w) - C \exp [B \cdot H(w)] + D, \\ C &:= e^A; \ D := 2A + 1. \end{aligned} \tag{1.3}$$

Substituting further $H(w) = \ln J(\mathrm{w})$ or any other function which can be explicitly inverted, one arrives at a large variety of equations which are equivalent to Eq. (1.1). We discuss therefore, without loss of generality, the solution of Eq. (1.1) as a representative of a whole class of equations.

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Equation (1.1) is well over 100 years old. To our knowledge it first appeared in a combinatorial problem formulated by Schröder [1]: how many different ways exist of placing brackets in an algebraic expression consisting of n elements which may be arbitrarily permuted to obtain all possible combinations? Schröder finds that the number of possible ways of placing these brackets is

$$S_n = n!\, c_n, \tag{1.4}$$

where c_n is the coefficient in the power series expansion of the solution of Eq. (1.1):

$$G(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{1.5}$$

Schröder uses Eq. (1.1) as a generating equation for the unknown coefficients.

As far as we know, the equation was almost forgotten for about 100 years and rediscovered by Yellin [2] in the context of the statistical bootstrap model [3]. There Eq. (1.1) appears in the following way (for the sake of brevity we over-simplify here): let $\varrho(E)$ be a density of states of a system of energy E, that is, $\varrho(E)\,dE$ is the number of energy levels between E and $E + dE$. There are situations in physics where the system with energy E may be considered as being composed of an unspecified number of subsystems, which themselves have the same composite structure – and so on until one arrives at a basic constituent. Examples are hadronic clusters (nuclei, resonances) in strong interaction physics or droplets in a gas near condensation. If clusters then consist of clusters, which consist of clusters, etc., the density of states $\varrho(E)$ will obey an equation of the type

$$\varrho(E) = \delta(E - m_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta\left(E - \sum_{i=1}^{n} E_i\right) \prod_{i=1}^{n} \varrho(E_i)\, dE_i. \tag{1.6}$$

In words: the cluster, described by its level density $\varrho(E)$, is either an elementary object of mass = energy m_0 or it is composed of any number $\geqq 2$ of subclusters having level densities $\varrho(E_i)$ and adding up to total energy E. Introducing the Laplace transforms

$$\phi(\beta) := \int_0^{\infty} \varrho(E)\, e^{-\beta E}\, dE,$$

$$\varphi(\beta) := \int_0^{\infty} \delta(E - m_0)\, e^{-\beta E}\, dE = \exp(-\beta m_0), \tag{1.7}$$

and Laplace transforming Eq. (1.6) immediately gives

$$\phi(\beta) = \varphi(\beta) + e^{\phi(\beta)} - \phi(\beta) - 1, \tag{1.8}$$

which is of the type (1.3) with $f(w) = \phi(\beta) = e^{-\beta m_0}$. The solution of Eq. (1.8) is equivalent to having solved the "bootstrap equation" (1.6). Assume, for instance, that the power series expansion (1.5) is known, then

$$\begin{aligned}\phi(\beta) &= \int \varrho(E)\, e^{-\beta E}\, dE = \sum c_n \varphi(\beta)^n = \sum c_n \exp(-\beta n m_0) \\ &= \int \sum c_n \delta(E - n m_0)\, e^{-\beta E}\, dE,\end{aligned} \tag{1.9}$$

whence

$$\varrho(E) = \sum_{n=0}^{\infty} c_n \, \delta(E - n m_0). \tag{1.10}$$

For $E \gg m_0$ the sum may be replaced by an integral; then

$$\varrho(E) \underset{E \gg m_0}{\Rightarrow} \frac{1}{m_0} c_{[E/m_0]}. \tag{1.11}$$

Thus the coefficients c_n are themselves close to the solution of the bootstrap equation.

In realistic models, Eq. (1.6) is replaced by a more complicated one, where the energy becomes a four momentum and where, apart from four momentum, other quantities (Abelian or non-Abelian charges) are conserved and where the input term may consist of a more complicated function of several variables. Independent of all this, we always obtain Eq. (1.8) with the (here irrelevant) difference that a number of variables (chemical potentials) equal to the number of conserved quantities is added to β; the problem is always to solve Eq. (1.1). That one invariably comes back to this equation, whether one starts from a Lorentz invariant model or not and whether there are Abelian or non-Abelian symmetries or none at all, clearly shows that the basic problem is Schröder's old combinatorial one, which at his time must have appeared rather academic.

Another context in which Schröder's equation appears is in renormalization theory. Here the Gell-Mann-Low function $\beta(g)$ appearing in the renormalization group equations plays the central rôle. Pursuing a particular aspect, Khuri and McBryan [4] are led to consider the differential equation

$$\frac{dG}{dg} = \frac{a}{\beta(g)} (G^2 + bG^3) \tag{1.12}$$

for a function $G(g)$. This equation has the implicit solution

$$b \ln\left(\frac{1}{G(g)} + b\right) - \left(\frac{1}{G(g)} + b\right) = f(g, a), \tag{1.13}$$

where $f(g, a)$ is a known function of g and a. Putting

$$H(g) := \ln\left(\frac{1}{G(g)} + b\right) \tag{1.14}$$

gives

$$f(g, a) = bH(g) - e^{H(g)}, \tag{1.15}$$

which is of type (1.3). Solving Eq. (1.1) for $G(z)$ is then equivalent to an explicit solution of the nonlinear differential equation (1.12) and of a specific problem of renormalization theory.

The interest of obtaining and understanding the solution of Eq. (1.1) is obvious. While the inverse function $z(G)$ as defined by Eq. (1.1) is relatively simple, it will be seen in the following sections that understanding the analytical structure as well as constructing an explicit solution of Eq. (1.1) is a nontrivial task.

In the next section we shall study the power series solutions on the principal Riemann sheet; in Sect. 3 we explore the analytic properties of the map $z \leftrightarrow G$ and in Sect. 4 the integral representations of the solution $G(z)$ are worked out.

2. Simple Real Solutions of Equation (1.1)

2.1. Graphical Solution

The function $z(G)$ can be easily drawn (Fig. 1a) and by interchanging $z \leftrightarrow G$ a graphical solution $G(z)$ is found (Fig. 1b).

For later use we note that $z(G)$ has a maximum at z_0 with value G_0:

$$\begin{aligned} z_0 &= 2\ln 2 - 1 = 0.3863 \ldots \equiv x_0, \\ G_0 &= \ln 2 \qquad = 0.6931 \ldots \end{aligned} \tag{2.1}$$

and that the second derivative $d^2 z/dG^2 \neq 0$; the function $G(z)$ has at z_0 a square root singularity. For $x > x_0$ no real G exists.

2.2 Power Series Expansion

By differentiating Eq. (1.1) and using (1.1) to eliminate exp G, we obtain

$$1 = \frac{dG}{dz} \cdot (1 + z - 2G) \tag{2.2}$$

with the ansatz

$$G(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{2.3}$$

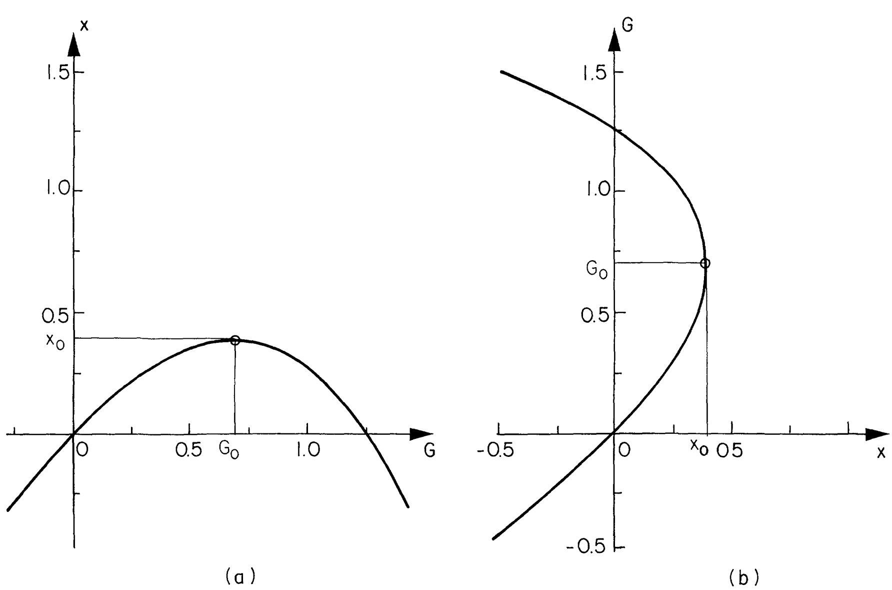

Fig. 1. a Equation (1.1) along the real G axis. **b** Graphical solution of Eq. (1.1) for real $x \leqq x_0$

and comparing coefficients we obtain the following recursion relation [5]

$$\begin{aligned} c_1 &= 1 \\ c_n &= -\frac{n-1}{n}\, c_{n-1} + \sum_{k=1}^{n-1} c_k\, c_{n-k}. \end{aligned} \tag{2.4}$$

A closed formula (less convenient however than the recursion relation), and an asymptotic formula for large n has been given earlier [6]: starting from the slightly generalized BE

$$z = (1+\alpha)\, G - \alpha e^{G} + \alpha, \tag{2.5}$$

one now obtains α dependent coefficients

$$c_l(\alpha) = \frac{\alpha^{1-l}}{l!\,(1+\alpha)} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\alpha k}{1+\alpha}\right)^{k+l-1} \cdot \exp\left(-\frac{\alpha k}{1+\alpha}\right). \tag{2.6}$$

For $\alpha = 1$ this expansion was already known to Schröder [1].

$$c_l(\alpha) \underset{l\to\infty}{\to} z_0^{-l}\, l^{-3/2} \sqrt{\frac{z_0}{2\pi(1+\alpha)}}. \tag{2.7}$$

The coefficients $c_l(\alpha)$ obey a recursive differential equation [7]

$$c_l(\alpha) = \frac{\alpha}{l(1+\alpha)^{l-2}} \frac{d}{d\alpha} \left[(1+\alpha)^{l-1}\, c_{l-1}(\alpha)\right]. \tag{2.8}$$

Our c_l are obtained with $\alpha = 1$.

The coefficients c_l are most easily calculated using the recursion relation (2.4). One finds

$$\begin{array}{lll} c_1 = 1 & & c_2 = 0.5 \\ c_3 = 0.6666\ldots & & c_4 = 1.0833\ldots \\ c_5 = 1.9666\ldots & & c_6 = 3.8222\ldots \\ & \vdots & \\ c_{10} = 77.75 & \text{and} & c_{10}^{\text{asympt.}} = 75 \\ c_{40} = 2.33\times 10^{13} & \text{and} & c_{40}^{\text{asympt.}} = 2.31\times 10^{13}. \end{array} \tag{2.9}$$

The c_l grow very fast [see (2.7)] and therefore the power series is useful only for very small $|z|$; indeed one sees immediately from Eq. (2.7) that z_0 is the convergence radius [2] (d'Alembert's criterion) as we know already from Fig. 1b.

2.3. *Expansion at the singularity*

Knowing that the singularity at z_0 is of the square root type, we can make another ansatz:

$$G(z) = G_0 - \sum_{n=1}^{\infty} s_n \sqrt{z_0 - z}^{\,n}. \tag{2.10}$$

Using again the method of Sect. 2.2 we obtain

$$s_1 = 1,$$
$$s_n = \frac{1}{2}\left[\frac{n-1}{n+1} s_{n-1} - \sum_{k=2}^{n-1} s_k s_{n+1-k}\right]. \tag{2.11}$$

The first few coefficients are

$$G_0 = \ln 2 = 0.6931 \ldots$$
$$s_1 = 1 \qquad s_2 = 0.1666 \ldots$$
$$s_3 = 0.2777 \ldots \times 10^{-1} \qquad s_4 = 0.3704 \ldots \times 10^{-2}$$
$$s_5 = 0.2315 \ldots \times 10^{-3} \qquad s_6 = 0.5870 \ldots \times 10^{-4}.$$
$$\vdots$$

The series (2.10) converges rapidly in the physically interesting region $0 \leqq x \leqq x_0$ and has been used extensively in our numerical work [8].

3. The Analytic Structure of the Bootstrap Equation

In some applications the bootstrap equation (1.1) results from Laplace transforming another (microcanonical) version [Eq. (1.6)] of it. To obtain the solution of the original problem, one must do an inverse Laplace transform on $G(z)$. For this, the singularities and the Riemann sheet structure must be known.

In this section we discuss the analytic structure of the bootstrap equation. Its knowledge will allow us to write down an integral representation for $G(z)$ in Sect. 4.

3.1. Singularities

We already know [see (2.10)] that $z_0 = 2\ln 2 - 1$ is a square root branch point from which a cut connecting two Riemann sheets must start; we take this cut along the real z axis from z_0 to ∞.

This is not the only cut, because the substitution $G \to G + 2\pi i$ in Eq. (1.1) leads immediately to

$$G(z + 4\pi i) = G(z) + 2\pi i, \tag{3.1}$$

which implies that the square root cut along the real axis is repeated by parallel cuts at distances $\pm n \cdot 4\pi i$; $n = 1, 2, \ldots, \infty$. Equation (3.1) implies further that the mapping $z \leftrightarrow G$ has some strip structure which will be explored later.

The discontinuity along all cuts is the same as that along the real axis cut: from Eq. (3.1) we read off

$$\operatorname{disc} G(z + n \cdot 4\pi i) = \operatorname{disc} G(z), \tag{3.2}$$

so that we need only calculate the discontinuity of the real cut: put

$$G =: g + i\gamma \tag{3.3a}$$

$$z =: x + iy, \tag{3.3b}$$

then Eq. (1.1) gives

$$x = 2g - e^g \cos\gamma + 1, \tag{3.4a}$$

$$y = 2\gamma - e^g \sin\gamma. \tag{3.4b}$$

Noting that

$$\operatorname{disc} G(x) \underset{\varepsilon\to 0}{=} G(x+i\varepsilon) - G(x-i\varepsilon);\ x \geqq x_0, \tag{3.5}$$

we now consider

$$x = x_0 + \xi; \qquad x_0 = 2\ln 2 - 1, \tag{3.6a}$$

$$y = \varepsilon\,(\to 0). \tag{3.6b}$$

We obtain from Eq. (3.4)

$$e^g = \frac{2\gamma - \varepsilon}{\sin\gamma}\ [> 0 \text{ by definition}]. \tag{3.7a}$$

Inserting into Eq. (3.6) we find

$$x_0 + \xi = 2\ln\frac{2\gamma-\varepsilon}{\sin\gamma} - \frac{2\gamma-\varepsilon}{\sin\gamma}\cos\gamma + 1. \tag{3.7b}$$

With $x_0 = 2\ln 2 - 1$ we obtain

$$\xi = 2\left[\ln\frac{\gamma-\varepsilon/2}{\sin\gamma} + 1 - \frac{\gamma-\varepsilon/2}{\sin\gamma}\cos\gamma\right]_{\varepsilon\to 0}. \tag{3.8}$$

At $\xi = \varepsilon = 0$ we must have $\gamma = 0$, which fixes the solution of (3.8) to be chosen. Letting γ vary from 0 to π, ξ goes from 0 to ∞ along the entire cut. As Eq. (3.8) is invariant under $\gamma \to -\gamma$, $\pm\gamma$ correspond to the same position on the cut. To see which sign of γ corresponds to the upper lip of the cut, we use Eq. (3.7a) with $\gamma \to 0^+$ to find $\gamma - \varepsilon/2 = \gamma e^g/2 > 0$, hence $\gamma > \varepsilon/2$; thus $(\varepsilon > 0) \to (\gamma > 0)$.

The discontinuity (3.5) becomes

$$\begin{aligned}\operatorname{disc} G(x) &= g(x+i\varepsilon) - g(x-i\varepsilon) + i\,[\gamma(x+i\varepsilon) - \gamma(x-i\varepsilon)]\\ &= 2i\gamma(x+i\varepsilon)_{\varepsilon\to 0}.\end{aligned} \tag{3.9}$$

Equation (3.7a) ensures that the real part of the discontinuity vanishes.

Recalling Eq. (3.2) we have finally for all cuts:

$$\boxed{\begin{aligned}&\operatorname{disc} G(x_0 \pm n\cdot 4\pi i + \xi) = 2i\gamma(\xi),\\ &\gamma(\xi) = \text{principal roof of: } \xi = 2\left[\ln\frac{\gamma}{\sin\gamma} + 1 - \frac{\gamma}{\sin\gamma}\cos\gamma\right],\\ &\gamma(\xi\to 0^+) = 0^+; \qquad \gamma(\xi\to\infty) = +\pi.\end{aligned}} \tag{3.10}$$

Apart from the branch points $x_0 \pm n \cdot 4\pi i$ no other singularities exist for $|z| < \infty$. In fact from Eq. (1.1)

$$\frac{dG}{dz} = \frac{1}{2 - e^G}, \tag{3.11}$$

which exists everywhere except at $G = \ln 2 \pm 2\pi i n$.

We find for $\operatorname{Re} G \to \pm \infty$

$$\operatorname{Re} G \to \infty \quad : G \sim \ln(-z), \tag{3.12a}$$

$$\operatorname{Re} G \to -\infty : G \sim z/2. \tag{3.12b}$$

Hence $z = \infty$ is a logarithmic winding point.

$z_n := 2\ln 2 - 1 \pm n \cdot 2\pi i$ (square root branch points) and $z = \infty$ (logarithmic branch point) are the only singularities of $G(z)$.	(3.13)

3.2. *The Map $z \leftrightarrow G$ and the Riemann Sheet Structure*

The image of the cut is already partly known from Eq. (3.8): for $\xi \to \infty$ we have $\gamma \to \pm \pi$; at $\xi = 0$ we have $\gamma = 0$. The way γ moves away from there, is found by expanding Eqs. (3.4a) and (3.4b) near the branch point. Put

$$\begin{aligned} x &= x_0 + \xi, \\ y &= \varepsilon, \\ g &= G_0 + \tau, \end{aligned} \tag{3.14}$$

then Eqs. (3.4a) and (3.4b) become

$$\xi = 2[\tau - e^{\tau} \cos\gamma], \tag{3.15a}$$

$$\varepsilon = 2[\gamma - e^{\tau} \sin\gamma]. \tag{3.15b}$$

For $\xi, \varepsilon, \tau \ll 1$ and expanding up to γ^3 one finds from the condition $\varepsilon \to 0$

$$\tau = \gamma^2/6; \quad \gamma = \pm \sqrt{6\tau}, \tag{3.16a}$$

$$\xi = \gamma^2 = 6\tau. \tag{3.16b}$$

Therefore the image of the cut opens up to the right like a square root and asymptotically approaches $\pm \pi$. This is shown in Fig. 2 together with the image in z of the imaginary axis $\operatorname{Re} G = 0$. Shaded regions are mapped correspondingly and the same pattern repeats itself in parallel strips of width 2π in the G plane and 4π in the z plane. Indeed, as seen from Eq. (3.4b), the parallel straight lines $\gamma = \pm n\pi$ are mapped onto straight lines $y = \pm 2n\pi$.

The interior of the cigar-shaped region (enclosed by the image of the cut) goes to the second sheet of the z plane. This is seen by taking a vertical straight line in the G

plane from the lower to the upper border of the cigar. Let $g \gg 1$. Then

$$z = 2G - e^G + 1 \approx -e^g e^{i\gamma}. \tag{3.17}$$

Hence the image of this straight line is almost a complete circle of radius $R = e^g$, starting at the lower lip of the cut, going once around in the second sheet – without meeting any singularity! – and coming up at the upper lip. In Fig. 3 we illustrate this with a few numerically computed line mappings.

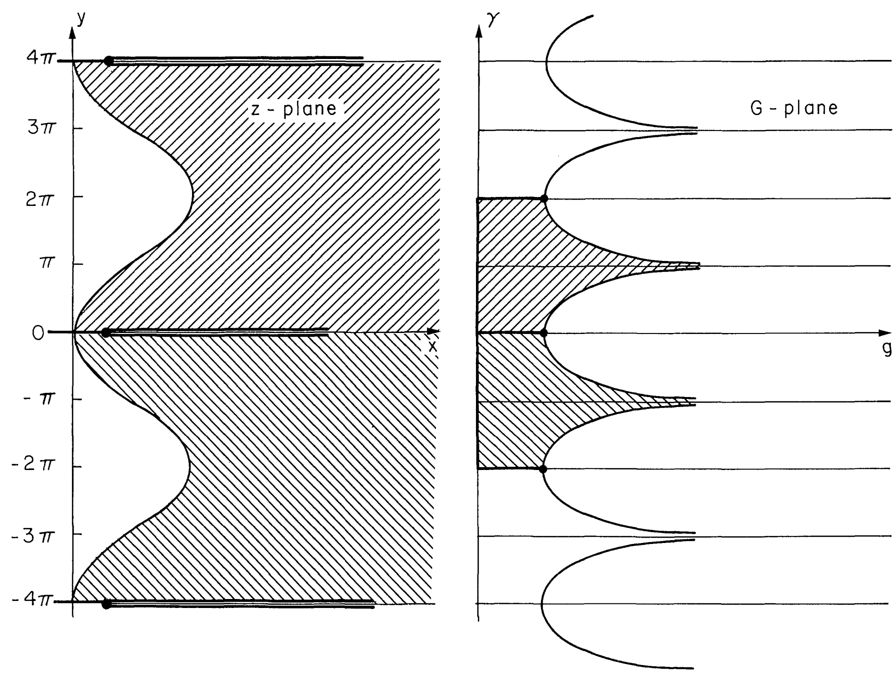

Fig. 2. The map $z \leftrightarrow G$ illustrated in a particular region. Shaded areas are mapped onto each other. The curves enclosing the cigar-shaped areas of the G plane map onto the cuts in the z plane; the sinusoidal line in z is the image of the imaginary axis in G

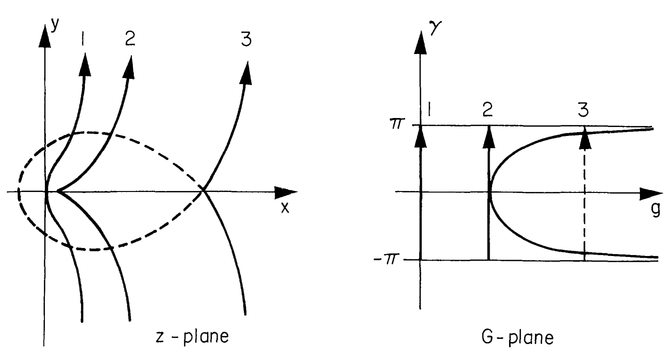

Fig. 3. Mapping of a few selected curves. The broken straight line in the cigar (G plane) goes into a loop in the next lower sheet of the z plane

If the straight line is continued through the next cigar, the same happens except that now we dive into a third sheet. And so it goes on: each cut connects the principal sheet (the one on which all cuts lie) with exactly one of the other sheets, as required by the fact that the points $x_0 + 4\pi i n$ are square root branch points. There are infinitely many other sheets connected to the principal one in this way, each of them having exactly one cut in common with the principal sheet. Figure 4 shows how a straight line extending through several cigars is mapped onto the consecutive z sheets and Fig. 5 shows the topology of the Riemann z sheets in a perspective view.

As a last question illustrating the mapping $G \leftrightarrow z$ we ask: which points $G_k \to z = 0$? The principal solution is $G = 0 \leftrightarrow z = 0$ but for each z sheet another $G_k \neq 0$ goes to $z = 0$; we find them by putting [see Eqs. (3.4)]

$$x = 0 = 2g + 1 - e^g \cos\gamma, \tag{3.18a}$$

$$y = 0 = 2\gamma - e^g \sin\gamma. \tag{3.18b}$$

Excluding the trivial solution $g = \gamma = 0$, G_k must lie inside the k^{th} cigar, hence $g > g_0 = \ln 2$. Furthermore, Eq. (3.18a) implies $\cos\gamma \geqq 0$, hence

$$\left[-\frac{\pi}{2} \leqq \gamma_k \leqq \frac{\pi}{2}\right] \bmod 2\pi. \tag{3.19}$$

If γ is a solution of Eqs. (3.18), then $-\gamma$ is also a solution; we restrict the discussion to $\gamma > 0$. Inserting g from Eq. (3.18b) into (3.18a) we obtain

$$(2\ln 2 + 1 + 2\ln\gamma - 2\ln\sin\gamma)\sin\gamma = 2\gamma\cos\gamma. \tag{3.20}$$

As we know that γ_k jumps by steps of roughly 2π, we put

$$\gamma_k = \gamma_k^{(0)} + k \cdot 2\pi, \tag{3.21}$$

then

$$g_k = \ln \frac{\gamma_k^{(0)} + k \cdot 2\pi}{\sin\gamma_k^{(0)}}. \tag{3.22}$$

With (3.21) we obtain, using $2\ln 2 + 1 = 2\ln(2\sqrt{e})$,

$$\frac{\sin\gamma_k^{(0)}}{\gamma_k} \cdot \ln \frac{2\sqrt{e}\,\gamma_k}{\sin\gamma_k^{(0)}} = \cos\gamma_k^{(0)}. \tag{3.23}$$

Letting now $k \to \infty$, the left-hand side goes to 0^+; hence

$$\gamma_k^{(0)} \underset{k\to\infty}{\to} \left(\frac{\pi}{2}\right)^-, \tag{3.24}$$

so that

$$\gamma_k \underset{k\to\infty}{\to} \left[\frac{\pi}{2}(1 + 4k)\right]^-, \tag{3.25a}$$

$$g_k \underset{k\to\infty}{\to} \ln[\pi(1 + 4k)], \tag{3.25b}$$

$$\gamma_k \underset{k\to\infty}{\to} \exp(g_k/2). \tag{3.25c}$$

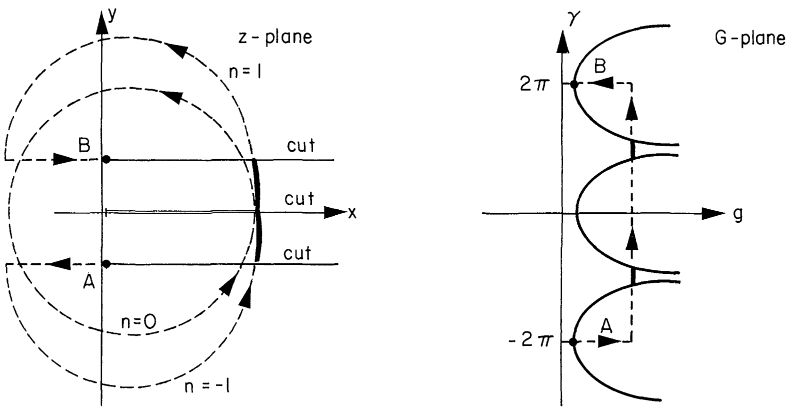

Fig. 4. Mapping of a rectangle in the G plane onto the z plane. The broken parts within the cigars map onto (almost) circular loops in different z sheets, from where they emerge at the corresponding cuts to continue (fat full curves) in the principal sheet to the next cut, where they dive to another sheet for another loop

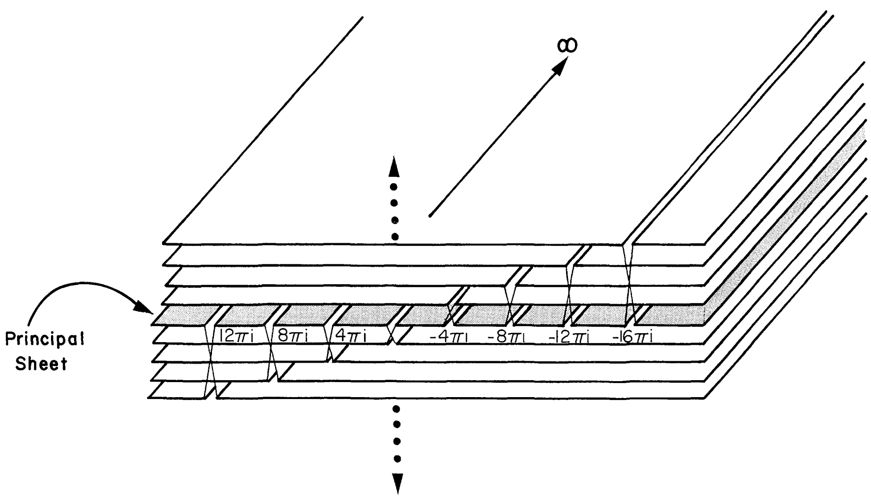

Fig. 5. Perspective view of the z plane cut open along a line parallel to the imaginary axis ($\mathrm{Re}\, z > x_0$)

Thus the points $G_k \to (z=0)$ lie on an exponential curve in the G plane, one in each cigar. The exact solution of Eq. (3.23) requires numerical computation; it turns out that the asymptotic expressions (3.25) are already good at $k=3$.

4. Integral Representation of $G(z)$

4.1. On the Principal Sheet

For convergence reasons we use a once subtracted Cauchy integral:

$$\Delta G := G(z) - G(z_1) = \frac{1}{2\pi i} \int_c dt\, G(t) \left[\frac{1}{t-z} - \frac{1}{t-z_1} \right], \tag{4.1}$$

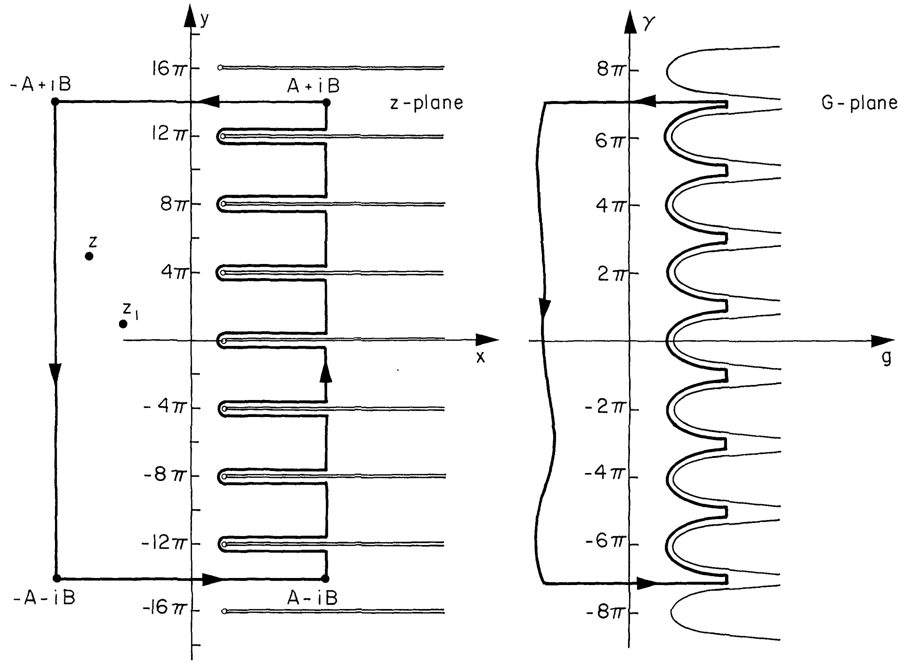

Fig. 6. The integration path for Eqs. (4.1) and (4.2), shown in the G and z planes (principal z sheet). This path is pushed to ∞ in all directions in the order first $B\to\infty$, then $A\to\infty$. Apart from the cuts, the vertical boundary lines give a finite contribution

where the path of integration is indicated in Fig. 6: a large rectangle contouring the cuts, the sides of which we push to infinity. Thus

$$2\pi i\,\Delta G=\underbrace{\int_{A+iB}^{-A+iB}+\int_{-A+iB}^{-A-iB}+\int_{-A-iB}^{A-iB}+\int_{A-iB}^{A+iB}}_{\text{"boundary terms"}}+\int_{x_0}^{A}\sum_{-iB}^{iB}\operatorname{disc}G(x)\,dx. \tag{4.2}$$

We shall sum over the discontinuities under the last integral, because the sum can, for $B\to\infty$, be evaluated. This implies that in all other integrals the limits must be taken in this same order: first B, then A to ∞. All individual integrals indeed depend on this order, but the final result of course does not. From (3.10)

$$\Delta G_{\text{cut}}=\frac{1}{2\pi i}\int_0^\infty d\xi\,2i\gamma(\xi)\sum_{n=-\infty}^{\infty}\left[\frac{1}{\xi+4\pi i n-z}-\frac{1}{\xi+4\pi i n-z_1}\right]. \tag{4.3}$$

With

$$\frac{1}{\xi+4\pi i n-z}=\frac{\xi-z-4\pi i n}{(\xi-z)^2+(4\pi n)^2}, \tag{4.4}$$

we see that the imaginary part cancels in the sum. With $\xi-z=t$ we obtain [9]

$$\sum_{n=-\infty}^{\infty}\frac{t}{t^2+(4\pi n)^2}=\frac{1}{4}\coth\frac{\xi-z}{4}, \tag{4.5}$$

so that

$$\Delta G_{\text{cut}} = \frac{1}{4\pi} \int_0^\infty d\xi\, \gamma(\xi) \left[\coth \frac{\xi - z}{4} - \coth \frac{\xi - z_1}{4} \right]. \tag{4.6}$$

ΔG_{cut} is not yet in an explicit form since $\gamma(\xi)$ is the principal root of the transcendent Eq. (3.10). We therefore introduce γ as integration variable. From (3.10)

$$\gamma(\xi)\, d\xi = \gamma \frac{d\xi}{d\gamma} d\gamma = 2 \left[1 - 2\gamma \frac{\cos\gamma}{\sin\gamma} + \frac{\gamma^2}{\sin^2\gamma} \right] d\gamma, \tag{4.7}$$

where γ goes from 0 to π when ξ goes from 0 to ∞. With this ΔG_{cut} and an additional term $(z - z_1)/2$, which comes from the boundaries of the integration and which will be derived below, we obtain on the principal sheet the integral representation

$$\begin{aligned} G(z) &= G(z_1) + \frac{z - z_1}{2} + \frac{1}{2\pi} \int_0^\pi d\gamma \left(1 - \frac{2\gamma \cos\gamma}{\sin\gamma} + \frac{\gamma^2}{\sin^2\gamma} \right) \\ &\quad \cdot \left[\coth \frac{\xi(\gamma) - z}{4} - \coth \frac{\xi(\gamma) - z_1}{4} \right], \\ \xi(\gamma) &:= 2 \left[\ln \frac{\gamma}{\sin\gamma} + 1 - \gamma \frac{\cos\gamma}{\sin\gamma} \right]; \quad \text{valid on principal sheet.} \end{aligned} \tag{4.8}$$

For numerical calculations it is simplest to put $z_1 = 0$, since $G(0) = 0$. By numerical inspection one finds that the integrand is a rather well-behaved function as long as z does not come near to the branch points. The use of 51 points and Simpson integration leads to results within drawing accuracy for real z, as one easily checks by inserting the numerical result of the integration into Eq. (1.1).

4.1.1. The Contribution of the Boundaries. Consider first

$$\frac{1}{2\pi i} \int_{-A-iB}^{A-iB} = \frac{z - z_1}{2\pi i} \int_{-A}^{A} dx\, G(x - iB) \cdot \frac{1}{(x - iB - z)(x - iB - z_1)}. \tag{4.9}$$

Choose $B = (2n+1) \cdot 2\pi$, then $|\gamma| = (2n+1) \cdot \pi$ while $x = 2g + 1 + e^g$ so that $|g|$ is bounded of order A. With $|g|$ bounded and $|\gamma| \approx B/2$ the integrand vanishes like $1/B$ for $B \to \infty$. As we are obliged to let first $B \to \infty$, the integral is zero for all A and remains zero for $A \to \infty$ (the limits taken in reverse order lead to a result $\neq 0$).

In the same way the integral $(A + iB) \to (-A + iB)$ vanishes. Thus

$$\lim_{A\to\infty} \lim_{B\to\infty} \left[\int_{-A-iB}^{A-iB} + \int_{A+iB}^{-A+iB} \right] = 0. \tag{4.10}$$

Next comes the integral $-A + iB \to -A - iB$:

$$-\frac{1}{2\pi i} \int_{-A-iB}^{-A+iB} = -\frac{z - z_1}{2\pi i} \int_{-B}^{B} dy\, G(-A + iy) \cdot \frac{1}{(-A + iy - z)(-A + iy - z_1)}. \tag{4.11}$$

From Eq. (3.12b) we know that for $\operatorname{Re} z \to -\infty$, $G \approx z/2$, hence

$$\lim_{A\to\infty}\lim_{B\to\infty} -\frac{1}{2\pi i}\int_{-A-iB}^{-A+iB} = -\frac{z-z_1}{4\pi}\int_{-\infty}^{\infty}\frac{dy}{-A+iy} = \frac{z-z_1}{4}. \tag{4.12}$$

This is half of the boundary term included above in Eq. (4.8). (Note that the reverse order of limits would have given zero.)

The last boundary integral, $A-iB \to A+iB$, goes over the strips between the cuts: in the G plane over the infinitely narrow intervals [containing the y values $(2n+1)\pi$] between the cigars (see Fig. 2), where, for A very large

$$G(A+iy) \approx \ln A + (2n+1)\, i\pi,$$
$$2n\,2\pi \leqq y \leqq (2n+2)\,2\pi. \tag{4.13}$$

Therefore the n^{th} contribution is, for $A \to \infty$,

$$I_n = \frac{z-z_1}{2\pi}\left[\ln A + i\pi(2n+1)\right] \cdot \int_{2n\,2\pi}^{(2n+1)\,2\pi} dy \left(\frac{1}{A+iy}\right)^2. \tag{4.14}$$

To sum over n, we define the saw-teeth curve:

$$\varepsilon(y) = 2\pi\,(2n+1) - y \;[\text{for } 2n\,2\pi \leqq y \leqq 2\pi\,(2n+2)],$$
$$\varepsilon(4\pi n) = 0 \quad \text{and continued periodically for } -\infty < n < \infty, \tag{4.15}$$

which is an odd function of y. With its help the factor

$$(2n+1)\,\pi = \frac{1}{2}(y + \varepsilon(y)) \tag{4.16}$$

can be taken under the integral. The sum over n becomes then an integration over y from $-\infty$ to ∞:

$$\lim_{A\to\infty}\lim_{B\to\infty} \frac{1}{2\pi i}\int_{A-iB}^{A+iB} = \frac{z-z_1}{2\pi}\ln A \int_{-\infty}^{\infty} dy \frac{1}{(A+iy)^2}$$
$$+ \frac{z-z_1}{2\pi}\cdot\frac{i}{2}\int_{-\infty}^{+\infty}[y+\varepsilon(y)]\frac{dy}{(A+iy)^2}. \tag{4.17}$$

These integrals can be evaluated by standard techniques with the result

$$\lim_{A\to\infty}\lim_{B\to\infty} \frac{1}{2\pi i}\int_{A-iB}^{A+iB} = \frac{z-z_1}{4},$$

which gives the other half of the boundary term in Eq. (4.8).

4.2. On the n^{th} Sheet

We call "n^{th} sheet" the sheet connected to the principal sheet by the cut starting at $x_0 + 4\pi i n$. Thus the sheet connected to the principal one by the cut along the real axis is the 0^{th} sheet.

The generalization being trivial, we consider first the 0^{th} sheet. The path of integration is indicated in Fig. 7. Note that the lower lip of the cut in the n^{th} sheet is

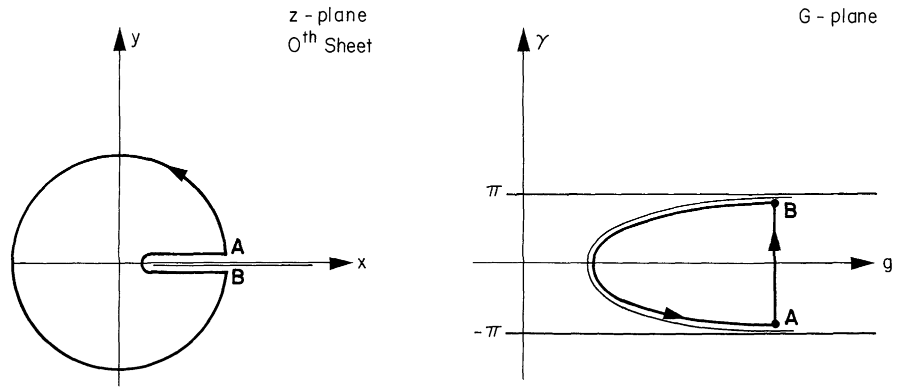

Fig. 7. The integration path for Eq. (4.19) in the zeroth z sheet (below the principal one) and in the G plane. The contribution of the circle vanishes when its radius goes to ∞

connected to the upper lip of the n^{th} cut of the principal sheet; hence

$$\operatorname{disc} G(x^{(n)}) = -2i\gamma(x). \tag{4.18}$$

In the present case we do not need to sum over infinitely many cuts because there is only one. No problem of sequence of limits occurs. We now have (see Fig. 7):

$$G(z) - G(z_1) = \frac{1}{2\pi i}\left[\int_{\text{cut}} + \int_{\text{circle}}\right], \tag{4.19}$$

where the circle corresponds to the straight line $A \to B$ crossing the cigar in the G plane. We then have from Eq. (4.1)

$$\frac{1}{2\pi i}\int_{\text{circle}} dt\, G(t)\frac{z-z_1}{(t-z)(t-z_1)} = \frac{z-z_1}{2\pi i}\int_0^{2\pi} d\varphi\, iRe^{i\varphi}\frac{\ln R + i\varphi - i\pi}{R^2 e^{2i\varphi} + \dots}, \tag{4.20}$$

where we have used the fact that for $\operatorname{Re} G \to \infty$ the circle mapped onto the straight line has radius $R \cong \exp[\operatorname{Re} G]$ as shown in Sect. 3. For $R \to \infty$ the integral (4.20) vanishes. The same is true for the n^{th} sheet.

The cut contribution can be written down immediately for the n^{th} sheet:

$$[G(z) - G(z_1)]^{(n)} = -\frac{1}{2\pi i}\int_0^{\infty} d\xi\, 2i\gamma(\xi)\left[\frac{1}{\xi + 4\pi i n - z} - \frac{1}{\xi + 4\pi i n - z_1}\right]. \tag{4.21}$$

Changing the integration variable as in Eq. (4.7) we obtain finally

$$\begin{aligned} &[G(z) - G(z_1)]^{(n)} \\ &= -2\frac{z-z_1}{\pi}\int_0^{\pi} d\gamma\left(1 - \frac{2\gamma\cos\gamma}{\sin\gamma} + \frac{\gamma^2}{\sin^2\gamma}\right)\cdot\frac{1}{(\xi + 4\pi i n - z)(\xi + 4\pi i n - z_1)} \\ &\xi = \xi(\gamma) = 2\left[\ln\frac{\gamma}{\sin\gamma} + 1 - \frac{\gamma\cos\gamma}{\sin\gamma}\right]; \quad \text{valid on } n^{\text{th}} \text{ sheet with branche point at } x_0 + 4\pi i n. \end{aligned} \tag{4.22}$$

Again a numerical check with 51 points and Simpson integration gives (for real x) a precision of $\Delta x/x \lesssim 10^{-4}$. This concludes the inversion problem posed by Eq. (1.1).

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