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Construction of Euclidean $(\mathrm{QED})_2$ via Lattice Gauge Theory

Boundary Conditions and Volume Dependence*

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Abstract. Let $\mathscr{V}=\det_{\mathrm{ren}}(1+K_g)$ be the renormalized Matthews-Salam determinant of $(\mathrm{QED})_2$, where $K_g=ieS\not{A}_g$, $S=(\sum\gamma_\mu\partial_\mu+m)^{-1}$ is euclidean fermion propagator of one of the following boundary conditions: (1) free, (2) periodic at $\partial\Lambda$, $\Lambda=[-L/2;L/2]^2$, (3) anti-periodic at $\partial\Lambda$, and $\not{A}_g(x)=(\sum\gamma_\mu A_\mu(x))g(x)$. Here $g(x)=1$ if $x\in\Lambda_0=[-r/2,r/2]^2\subset\Lambda$ and 0 otherwise. Then we show

(i) $\mathscr{V}\in L^p(d\mu(A))$, $p>0$. Further we prove a new determinant inequality which holds for the QED, QCD-type models containing fermions. This enables us to prove:

(ii) $Z(\Lambda_0)=\int\mathscr{V}d\mu(A)\leqq\exp[c|\Lambda_0|]$. Similar volume dependence is shown for the Schwinger functions.

1. Introduction

Several years ago, the author tried to construct $(\mathrm{QED})_2$ taking a basis on a Hamiltonian formalism of QED, where an indefinite metric is explicitly used to ensure the renormalizability. Because of the indefinite matric, however, there are difficulties: for example it is difficult to prove the existence of the vacuum vector [2].

Recently Weingarten [10] proved the integrability of the renormalized Matthews-Salam determinant $\mathscr{V}=\det_{\mathrm{ren}}(1+K_\Lambda)$, where $K_\Lambda=ieS\not{A}$, $S=(\sum\gamma_\mu\partial_\mu+m)^{-1}$ the euclidean fermion propagator which satisfies anti-periodic boundary conditions at $\partial\Lambda$, $\Lambda=[-L/2,L/2]^2$, $\not{A}(x)=\sum\gamma_\mu A_\mu(x)$ and $\{A_\mu(x)\}$ are the euclidean vector fields which satisfy the periodic boundary conditions at $\partial\Lambda$. The anti-periodic boundary condition of S comes from the use of the transfer matrix to prove the diamagnetic inequality. In this work we show the integrability of $\mathscr{V}$ for any one of the following boundary conditions of S and A_μ:

S; free, periodic, anti-periodic boundary conditions,

A_μ; free, periodic, anti-periodic, boundary conditions.

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Moreover we obtain a new determinant inequality by applying Hölder's inequality to the transfer matrices, which clarifies the volume dependence of the Schwinger functions.

Let $d\mu(A)$ be a Gaussian probability measure with mean zero and covariance $C_{\mu\nu}(x,y)$:

$$\int A_\mu(x)A_\nu(y)d\mu(A)=C_{\mu\nu}(x-y)=\int\frac{d^2p}{(2\pi)^2}e^{ip(x-y)}(\delta_{\mu\nu}+\text{gauge}+\text{term})\frac{1}{p^2+\mu^2}. \tag{1.1}$$

Here $\mu>0$ denotes the mass and the gauge term takes a form $-c(k^2)k_\mu k_\nu$ with $|c|\leqq\text{const}|k^2|^{-1}$. Let $\Lambda=[-L/2,L/2]^2$ and let $\Lambda_0=[-r/2,r/2]^2$ with $r\leqq L$. Further let

$$A_{\mu,g}(x)\equiv A_\mu(x)g(x),$$

where $g(x)\geqq 0$ and $\text{supp}\,g(x)\subset\Lambda_0$ are assumed. We take $g=\chi_{\Lambda_0}$ or as $g\in C_0^\infty(\Lambda_0)$ for convenience. Let

$$\Lambda_N=\{a(n_0,n_1);n_\mu=-N,-N+1,\dots,N-1\},$$

$$a=\frac{L}{2N}:\text{ lattice width},$$

$$\tilde{\Lambda}=\frac{2\pi}{L}Z^2,$$

$$\tilde{\Lambda}_N=\left\{\delta(n_0,n_1);\delta=\frac{2\pi}{L},n_\mu=-N,-N+1,\dots,N-1\right\},$$

and let

$$\mathscr{H}_N=\left\{f(x),x\in\Lambda_N;\|f\|^2_{\mathscr{H}_N}\equiv a^2\sum_{x\in\Lambda_N}|f(x)|^2\right\}.$$

Any $f\in\mathscr{H}=L^2(\Lambda:d^2x)$ can be mapped into $\mathscr{H}_N$ by the Q-identification [1]:

$$f_a(x)\equiv(Qf)(x)=a^{-2}\int_{-a/2}^{a/2}\int_{a/2}^{a/2}f(x+\eta)d^2\eta,\qquad x\in\Lambda_N. \tag{1.2}$$

Further $\mathscr{H}_N$ can be embedded in $\mathscr{H}$ via Q^*:

$$(Q^*f_a)(y)=f_a(x),\qquad y\in\left[x_0-\frac{a}{2},x_0+\frac{a}{2}\right)\otimes\left[x_1-\frac{a}{2},x_1+\frac{a}{2}\right). \tag{1.3}$$

Let

$$\tilde{f}(k)=\int_\Lambda f(x)e^{ikx}d^2x,\qquad k\in\tilde{\Lambda},$$

$$\tilde{f}_a(k)=a^2\sum_{x\in\Lambda_N}e^{ikx}f_a(x),\qquad k\in\tilde{\Lambda}_N$$

Then

$$\tilde{f}_a(k) = \eta(ak)\tilde{f}(k), \tag{1.4}$$

where

$$\eta(x) = \prod_{\mu=0}^{1} \frac{\sin 1/2x_\mu}{1/2x_\mu}.$$

Now we define

$$\begin{aligned} A_{\mu,g,a}(x) &\equiv (QA_{\mu,g})(x), \qquad x \in \Lambda_N^\mu \equiv \Lambda_N + 1/2e_\mu, \\ e_0 &= (a,0), \qquad e_1 = (0,a), \end{aligned} \tag{1.5}$$

and let [9, 11]

$$\begin{aligned} B_N(x,y) &= (2a^{-3} + ma^{-2})\delta_{x,y} - a^{-3}\gamma(x,y), \\ \Gamma_N(x,y) &= -a^{-3}[U(x,y) - 1]\gamma(x,y), \end{aligned} \tag{1.6}$$

where $x, y \in \Lambda_N$,

$$\gamma(x,y) = \begin{cases} 1/2(1 \mp \gamma_\mu) & y = x \pm e_\mu, \\ 0 & \text{otherwise}, \end{cases} \tag{1.7}$$

$$U(x,y) = \begin{cases} \exp[\pm ieaA_{\mu,g,a}(x \pm 1/2e_\mu)] & y = x \pm e_\mu, \\ 0 & \text{otherwise}, \end{cases} \tag{1.8}$$

and $\{\gamma_\mu^* = \gamma_\mu\}_{\mu=0,1}$ are two dimensional euclidean Dirac matrices:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}1_2.$$

Thus one formally finds:

$$\begin{aligned} (B_N f)(x) &\equiv a^2 \sum_{y \in \Lambda_N} B_N(x,y) f(y) \to (\not{\partial} + m) f(x), \\ (\Gamma_N f)(x) &\equiv a^2 \sum_{y \in \Lambda_N} \Gamma_N(x,y) f(y) \to ie\not{A}_g(x) f(x), \end{aligned}$$

for suitable $f(x) \in \mathscr{H}_N^{\otimes 2}$ as $N \to \infty$ or as $a \to 0$. We may apply the same approximation for

$$(g, [\not{\partial} + m + ie\not{A}_g]^{-1} h),$$

where

$$g, h \in \mathscr{H}_{-1/2}(R^2, d^2x) \otimes C^2.$$

Let

$$\begin{aligned} S_N &= (B_N)^{-1} = P_N^2 U_N, \\ P_N &> 0, \qquad U_N^+ = U_N^{-1}, \end{aligned} \tag{1.9}$$

and let

$$K_N = P_N U_N \Gamma_N P_N. \tag{1.10}$$

As usual we are to consider:

$$S_N(f_1, \ldots, f_m; g_1, \ldots, g_n; h_1, \ldots, h_n)$$
$$= Z_N^{-1} \int d\mu \prod A(f_i) \cdot \det_{jk}^{n \times n}[(g_j[B_N + \Gamma_N]^{-1} h_k)] \cdot \det_{\text{ren}}(1 + K_N), \tag{1.11}$$

$$Z_N = Z_N(g) = \int \det_{\text{ren}}(1 + K_N) d\mu(A), \tag{1.12}$$

$$\det_{\text{ren}}(1 + K_N) = \det_{(4)}(1 + K_N) \exp[- :T_2^N:], \tag{1.13}$$

where

$$\det_{(1+p)}(1 + K) = \det \left[(1 + K) \exp \left[\sum_{n=1}^{p} \frac{(-K)^n}{n} \right] \right]$$
$$T_2^N = \operatorname{Tr}[-K_N + \tfrac{1}{2}(K_N)^2 - \tfrac{1}{3}(K_N)^3],$$

and

$$:T_2^N: = T_2^N - \int T_2^N d\mu(A).$$

2. Convergences of K_N and $\det_{\text{ren}}(1+K_N)$

Let

$$K_g = iePU\not{A}_g P, \tag{2.1}$$

where $S = P^2 U$, $P > 0$, $U^* = U^{-1}$, and $S = (\not{\partial} + m)^{-1}$ is the euclidean fermion Green's function which satisfies periodic or anti-periodic boundary conditions at $\partial \Lambda$ and or free boundary conditions. Our B_N and Γ_N in Sect. 1 correspond to the periodic boundary conditions at $\partial \Lambda$ since we identify the points $\{a(-N, n_1)\}$ with the points $\{a(N, n_1)\}$ and the points $\{a(n_0, -N)\}$ with the points $\{a(n_0, N)\}$, respectively. The anti-periodic B_N and Γ_N are obtained from periodic B_N and Γ_N by a slight modification which does not change our estimates at all. Thus we will not discuss anti-periodic cases (see Sect. 4).

In the case of periodic S, we sometimes assume that the width of the rectangle Λ, namely L, depends on N so that $\Lambda \nearrow R^2$ and $a = \frac{L}{2N} \searrow 0$ sufficiently rapidly as $N \to \infty$. One possible choice is

$$L = L_N = L_0 N^{1/2}. \tag{2.2}$$

Then $a = a_N = L_0 / 2N^{1/2}$. Then it is necessary to clarify the L-dependence in our estimates. (We choose $L \geqq 1$ or N is sufficiently large so that $\operatorname{supp} g \subset [-L_N/2, L_N/2]^2$.)

Theorem I. *Let S_Λ be the euclidean fermion propagator which satisfies periodic or anti-periodic boundary conditions at $\partial \Lambda$. Then*

$$\lim_{\Lambda \uparrow R^2} K_\Lambda = K \text{ in } C_4 \tag{2.3}$$

a.e. with respect to $d\mu(A)$, where $K = K_g$,

$$K_\Lambda = ieP_\Lambda U_\Lambda \not{A}_g P_\Lambda, \tag{2.4}$$

$$C_p = \{x \in \mathscr{B}(\mathscr{H}) : \|x\|_p = (\operatorname{Tr}|x|^p)^{1/p} < \infty\}. \tag{2.5}$$

Lemma I-1. *Let $\Lambda'=[-L'/2, L'/2]^2$, $L'\geqq L$. Then there exists a polynomial Q of $A_{\mu,g}$ of order 4 such that*

$$\|K_\Lambda - K_{\Lambda'}\|_4^4 \leqq Q, \qquad \int Q d\mu(A) \leqq dL^{-\varepsilon},$$

where d and $\varepsilon>0$ are independent of $L(\geqq 1)$.

Lemma I-2 (Hypercontractive Inequality [8, 9]). *Let Q be a polynomial of $\{A_\mu(x)\}$ of order p and let $\int |Q|^2 d\mu \leqq \sigma^2$. Then*

$$\int |Q|^{2n} d\mu \leqq (2n-1)^{np}\sigma^{2n}.$$

Lemma I-3 [9]. *Let $\{Q_N \geqq 0\}$ be a sequence of polynomials of $\{A_\mu\}$ of order p and let $\int Q_N d\mu(A) \leqq dN^{-\varepsilon}$ $(d, \varepsilon>0$ independent of $N)$. Then*

$$\mu\{A_\mu; \lim Q_N(A_\mu) \neq 0\} = 0.$$

Then Theorem 1 obviously follows from Lemma I-1:

Proof of Lemma I-1. Let

$$P(x,y) = P(x-y) = (-\partial_\mu^2 + m^2)^{-1/4}(x,y)$$

and

$$(PU)(x,y) = (PU)(x-y) = (-\not{\partial}_x + m)(-\partial_\mu^2 + m^2)^{-3/4}(x,y)$$

be the Green's functions which satisfy free boundary conditions. Then the periodic Green's functions are given by

$$P_\Lambda(x-y) = \sum_{n\in Z^2} P(x-y+L_n),$$

$$(P_\Lambda U_\Lambda)(x-y) = \sum_{n\in Z^2} (PU)(x-y+L_n),$$

where $L_n=(n_0, n_1)L$, and the anti-periodic ones are obtained by replacing $\sum_n$ by $\sum_n (-1)^{n_0+n_1}$. Then if $y\in\Lambda_0 = \operatorname{supp} g$ and Λ is large enough so that $\operatorname{dist}(\Lambda_0, \partial\Lambda) \geqq cL (c>0)$,

$$\begin{aligned}|\chi_{\Lambda'}(x)P_{\Lambda'}(x,y) - \chi_\Lambda(x)P_\Lambda(x,y)| &\leqq K_0 \exp[-m_0 L - \tilde{m}|x-y|] \\ &\leqq KL^{-\varepsilon}e^{-\tilde{m}|x-y|},\end{aligned}$$

where the positive constants $m_0, \tilde{m}, k_0, K$, and ε are independent of L, L', x, and y. This follows from the exponential decay property of P. The same upper bound again holds for PU:

$$|\{\chi_{\Lambda'}(P_{\Lambda'}U_{\Lambda'})(x,y) - \chi_\Lambda(P_\Lambda U_\Lambda)(x,y)\}_{ij}| \leqq KL^{-\varepsilon}e^{-\tilde{m}|x-y|},$$

where $i, j = 1, 2$ (spinor indices). These also hold for the anti-periodic ones.

Then using the fact that $\operatorname{supp} g \subset \Lambda \subseteqq \Lambda'$,

$$\begin{aligned}\|K_{\Lambda'} - K_\Lambda\|_4 &\leqq |e|\,\|(\chi_{\Lambda'}P_{\Lambda'}U_{\Lambda'} - \chi_\Lambda P_\Lambda U_\Lambda)\not{A}_g P_{\Lambda'}\chi_{\Lambda'} \\ &\quad + \chi_\Lambda P_\Lambda U_\Lambda \not{A}_g(\chi_{\Lambda'}P_{\Lambda'}\chi_{\Lambda'} - \chi_\Lambda P_\Lambda \chi_\Lambda)\|_4 \\ &\leqq |e|(Q_1^{1/4} + Q_2^{1/4}) \leqq 2|e|(Q_1+Q_2)^{1/4},\end{aligned}$$

where

$$Q_1 = \|(\chi_{\Lambda'} P_{\Lambda'} U_{\Lambda'} - \chi_\Lambda P_\Lambda U_\Lambda) A\!\!\!/_g P_{\Lambda'} U_{\Lambda'}\|_4^4$$
$$= \|P_{\Lambda'} A\!\!\!/_g F_1 A\!\!\!/_g P_{\Lambda'}\|_2^2,$$
$$Q_2 = \|\chi_\Lambda P_\Lambda U_\Lambda A\!\!\!/_g (\chi_{\Lambda'} P_{\Lambda'} \chi_{\Lambda'} - \chi_\Lambda P_\Lambda \chi_\Lambda)\|_4^4,$$
$$= \|P_\Lambda A\!\!\!/_g F_2 A\!\!\!/_g P_\Lambda\|_2^2,$$

are polynomials of $\{A_\mu\}$ of order 4. Here

$$(F_1)_{ij} = \{|\chi_{\Lambda'} P_{\Lambda'} U^*_{\Lambda'} \chi_{\Lambda'} - \chi_\Lambda P_\Lambda U^*_\Lambda \chi_\Lambda|^2\}_{ij},$$
$$(F_2)_{ij} = \{|\chi_{\Lambda'} P_{\Lambda'} \chi_{\Lambda'} - \chi_\Lambda P_\Lambda \chi_\Lambda|^2\}_{ij}$$

are dominated by $\tilde{K} L^{-\tilde{\varepsilon}} e^{-\tilde{m}|x-y|} (\tilde{K}, \tilde{\varepsilon}, \tilde{m} > 0)$.

Then

$$\int Q_i d\mu(A) \leqq \text{const}\, L^{-2\varepsilon}. \quad \square$$

2.1. $\lim K_N = K_\Lambda$ *or* K_g

We may assume that the size of box Λ depends on N:

$$L = L_N = L_0 N^\varepsilon, \qquad 0 \leqq \varepsilon < 1. \tag{2.6}$$

The key point is that the lattice spacing $a = L/2N$ tends to zero like $N^{-\delta} (\delta > 0)$ as $N \to \infty$.

Theorem II. *Let Λ be chosen as*

(1) $[-L/2, L/2]^2$, *L fixed, or as*

(2) $[-L_N/2, L_N/2]^2$, $L = L_N = L_0 N^{1/2}$,

and let $\operatorname{supp} g = \Lambda_0 \subset \Lambda$. *Then there exists a polynomial Q of A_μ of order 8 such that*

$$\|K_\Lambda - K_N\|_4^4 \leqq Q, \qquad \int Q d\mu \leqq d N^{-\varepsilon},$$

where positive constants d and ε may depend on g but not on $N (\geqq 1)$ and $L (\geqq 1)$.

If $L = L_0 N^{1/2}$, since

$$\|K_M - K_N\|_4 \leqq \|K_{\Lambda'} - K_\Lambda\|_4 + \|K_{\Lambda'} - K_M\|_4 + \|K_\Lambda - K_N\|_4$$

with $L' = L_0 M^{1/2} \geqq L = L_0 N^{1/2}$, one has:

Theorem II'. *If L is fixed,*

$$\lim K_N = K \textit{ in } C_4 \textit{ a.e. with respect to } d\mu.$$

If $L = L_0 N^{1/2}$, then

$$\lim K_N = K_g \textit{ in } C_4 \textit{ a.e. with respect to } d\mu.$$

Remarks (2). Theorem II was essentially proved in [9]. But the main different points are:

(i) We must show the L-dependence explicitly, for example, as

$$\frac{1}{L^2} \sum_{k \in \tilde{\Lambda}_N} \tilde{f}(k).$$

Then this is uniformly bounded in $L (\geqq 1)$ if $\tilde{f}$ is a bounded L^1 function.

(ii) Let

$$\tilde{C}_{\mu\nu,a}(k,k')=\tilde{C}_{\mu\nu,g,a}(k,k')=\int \tilde{A}_{\mu,g,a}(k)\tilde{A}_{\nu,g,a}(-k')d\mu$$
$$=\eta(ak)\eta(ak')\tilde{C}_{\mu\nu}(k,k'), \tag{2.7}$$

where

$$\tilde{C}_{\mu\nu}(k,k')=\tilde{C}_{\mu\nu,g}(k,k')=\int \tilde{A}_{\mu,g}(k)\tilde{A}_{\nu,g}(-k')d\mu, \tag{2.8}$$

$k,\ k'\in\tilde{\Lambda}_N$, and $\operatorname{supp} g\subset\Lambda$ is assumed. Since $\tilde{C}_{\mu\nu}$ is not diagonalized, a slightly complicated calculation may be required. The following bounds for $\tilde{C}_{\mu\nu,g}$ are sufficient for our estimates (see Appendix):

$$|\tilde{C}_{\mu\nu,g}(k,k')|\leqq K_1 J(k_0,k'_0)J(k_1,k'_1), \tag{2.9a}$$

$$J(x,y)=\ln(2+|x|)\ln(2+|y|)\ln(2+|x-y|)$$
$$\cdot\frac{1}{1+|x-y|}\left(\frac{1}{1+|x|}+\frac{1}{1+|y|}\right), \tag{2.9b}$$

$$|\tilde{C}_{\mu\nu,g}(k,k)|\leqq K_2\frac{1}{1+k^2}, \tag{2.9c}$$

where the constants K_1 and K_2 may depend on g $(=\chi_{\Lambda_0}$ or $\in C_0^\infty(\Lambda_0))$ but are independent of L or $N(\geqq 1)$.

Let

$$\Gamma_N=\Gamma_N^{(1)}+\Gamma_N^{(2),r}, \tag{2.10a}$$

and let

$$K_N^{(1)}=P_N U_N\Gamma_N^{(1)}P_N \tag{2.10b}$$
$$K_N^{(2),r}=P_N U_N\Gamma_N^{(2),r}P_N, \tag{2.10c}$$

where

$$\Gamma_N^{(1)}(x,y)=-\frac{ie}{a^2}\sum_\mu\left\{A_{\mu,g,a}(x+1/2e_\mu)\frac{1-\gamma_\mu}{2}\delta_{y,x+e_\mu}\right.$$
$$\left.-A_{\mu,g,a}(x-1/2e_\mu)\frac{1+\gamma_\mu}{2}\delta_{y,x-e_\mu}\right\}, \tag{2.11}$$

and $\Gamma_N^{(2),r}$ denotes the remaining term. It is convenient to consider the problem in momentum space. Let

$$\tilde{P}_N(k)=\left[\left(m+\frac{1}{a}(2-\sum\cos ak_\mu)\right)^2+\frac{1}{a^2}\sum\sin^2 ak_\mu\right]^{-1/4}, \tag{2.12a}$$

$$\tilde{U}_N(k)=\left\{m+\frac{1}{a}(2-\sum\cos ak_\mu)+\frac{i}{a}\sum\gamma_\mu\sin ak_\mu\right\}\tilde{P}_N^2(k). \tag{2.12b}$$

Then

$$\tilde{P}_N(k,k')\equiv(a^2)^2\sum_{x,y\in\Lambda_N}e^{ikx-ik'y}P_N(x,y)$$
$$=L^2\delta_{k,k'}\tilde{P}_N(k)1_2, \tag{2.13a}$$

and similarly one has

$$\tilde{U}_N(k,k')=L^2\delta_{k,k'}\tilde{U}_N(k), \tag{2.13b}$$

$$\tilde{\Gamma}_N^{(1)}(k,k')=ie\sum_\mu \tilde{A}_{\mu,g,a}(k-k')\tilde{\Gamma}_{N,\mu}(k+k'), \tag{2.13c}$$

where

$$\tilde{\Gamma}_{N,\mu}(K)=\gamma_\mu \cos\frac{a}{2}K_\mu + i\sin\frac{a}{2}K_\mu, \tag{2.13d}$$

$k,k'\in\tilde{\Lambda}_N$ and we have assumed $\tilde{A}_{\mu,g,a}(p)=\tilde{A}_{\mu,g,a}(q)$ if $p=q \bmod \frac{4\pi N}{L}$. Further define

$$\tilde{P}_\Lambda(k)=[m^2+k^2]^{-1/4}, \tag{2.14a}$$

$$\tilde{U}_\Lambda(k)=(m+i\not{k})/(m^2+k^2)^{1/2}, \tag{2.14b}$$

$$\tilde{\Gamma}_\Lambda(k,k')=ie\sum_\mu \gamma_\mu\tilde{A}_{\mu,g}(k-k'), \tag{2.14c}$$

which correspond to the continuum limit. Let χ_N be the projection operator from $\mathscr{H}_\Lambda=L^2(\Lambda;d^2x)$ onto $\mathscr{H}_N=\{\mathrm{f}\in\mathscr{H}_\Lambda;\tilde{\mathrm{f}}(\mathrm{k})=0,\mathrm{k}\notin\tilde{\Lambda}_N\}$ which commutes with P_Λ and U_Λ, and let:

$$P_N=\chi_N P_\Lambda \chi_N+\delta P_N, \qquad U_N=\chi_N U_\Lambda \chi_N+\delta U_N,$$
$$\Gamma_N^{(1)}=ie\chi_N \not{A}_g \chi_N+\delta\Gamma_N^{(1)}.$$

Now:

$$\begin{aligned}\|K_\Lambda-K_N\|_4 &\leqq \|K_\Lambda-K_N^{(1)}\|_4+\|K_N^{(2),r}\|_4\\ &\leqq \|K_\Lambda-ie\chi_N P_\Lambda U_\Lambda \not{A}_g P_\Lambda \chi_N\|_4+\|\delta K_N^{(1)}\|_4+\|K_N^{(2),r}\|_4,\end{aligned}$$

where

$$\delta K_N^{(1)}=K_N^{(1)}-ie\chi_N P_\Lambda U_\Lambda \not{A}_g P_\Lambda \chi_N.$$

Lemma II-1. *There exist polynomials $Q_N^{(1)}$, $Q_N^{(2)}$, and $Q_N^{(3)}$ of A_μ of order 4 such that*

$$\|K_\Lambda-ie\chi_N P_\Lambda U_\Lambda \not{A}_g P_\Lambda \chi_N\|_4^4 \leqq Q_N^{(1)},$$
$$\|\delta K_N^{(1)}\|_4^4\leqq Q_N^{(2)}, \qquad \|K_N^{(2),r}\|_4^2\leqq Q_N^{(3)},$$

where

$$\int Q_N^{(i)}d\mu \leqq d_i N^{-\varepsilon_i}, \qquad i=1,2,3,$$

and $\{d_i,\varepsilon_i>0\}$ are independent of $L(\geqq 1)$ or $N(\geqq 1)$.

Theorem II follows from this lemma. We sketch the proof, with our Remarks (2) in mind. As for $Q_N^{(1)}$, since U_Λ is unitary,

$$\begin{aligned}\|K_\Lambda-\chi_N K_\Lambda \chi_N\|_4^4 &= |e|^4\|P_\Lambda \not{A}_g P_\Lambda-\chi_N P_\Lambda \not{A}_g P_\Lambda \chi_N\|_4^4\\ &\leqq 3^4|e|^4\|(1-\chi_N)P_\Lambda \not{A}_g P_\Lambda\|_4^4 \equiv Q_N^{(1)}.\end{aligned}$$

Except for the trivial constant, $Q_N^{(1)}$ is proportional to:

$$\frac{1}{(L^2)^3}\sum_{\substack{k_i\in\tilde{\Lambda}_N\\ i=1,2,3}}\mathscr{P}(\tilde{A}_g(k_1),\dots,\tilde{A}_g(k_4))T_N(k_1,k_2,k_3),$$

where $\mathscr{P}$ is a sum of $\tilde{A}_{\mu_1,g}(k_1)\times\dots\times\tilde{A}_{\mu_4,g}(k_4)$ with their coefficients ± 2 or 0, $k_4=-\sum_1^3 k_i$ and

$$T_N=\frac{1}{L^2}\sum_{\substack{p\in\tilde{\Lambda}\backslash\tilde{\Lambda}_N\\ p-k_1-k_2\in\tilde{\Lambda}\backslash\tilde{\Lambda}_N}}\tilde{P}^2(p)\tilde{P}^2(p-k_1)\tilde{P}^2(p-k_1-k_2)\tilde{P}^2(p-k_1-k_2-k_3).$$

Since $\tilde{P}^2(p)\leqq\mathrm{const}\,N^{-\alpha}(p^2+m^2)^{-\beta}$ with $\alpha>0$, $0<\beta<1/2$, whenever $p\in\tilde{\Lambda}\backslash\tilde{\Lambda}_N$, Hölder's inequality together with $(p^2+m^2)((p+k)^2+m^2)\geqq m^2(m^2+1/4k^2)$ shows:

$$T_N\leqq CN^{-\alpha}(k_1^2+m^2)^{-\beta/3}((k_1+k_2)^2+m^2)^{-\beta/3}((k_1+k_2+k_3)^2+m^2)^{-\beta/3},$$

where C, α, β are independent of $L(\geqq 1)$ or $N(\geqq 1)$. Therefore Eq. (2.9a), (2.9b) show

$$\int Q_N^{(1)}d\mu\leqq\mathrm{const}\,N^{-\alpha}$$

again by repeating usage of Hölder's inequality.

$Q_N^{(2)}$ arises from the terms which contain at least one of $\{\delta P_N,\delta U_N,\delta\Gamma_N^{(1)}\}$. Since

$$\begin{aligned}&\delta\tilde{P}_N(k)\leqq C_1\tilde{P}(k),\\ &|\delta\tilde{P}_N(k)|\leqq C_2\tilde{P}(k)f(ak),\quad f(x)\leqq|x|,\\ &|\delta\hat{U}_N(k)_{i,j}|\leqq C_3 f(ak),\quad f(x)\leqq|x|,\\ &\delta\tilde{\Gamma}_N^{(1)}(k,k')=iC_4\sum_\mu g_\mu(ak,ak')\tilde{A}_{\mu,g}(k-k')\\ &\cdot|g_\mu(x,y)|\leqq|x+y|,\end{aligned}$$

whenever $k,k'\in\tilde{\Lambda}_N$, where $\{C_i\}$ are constants independent of $L(\geqq 1)$ and $N(\geqq 1)$, one finds

$$\int Q_N^{(2)}d\mu\leqq\mathrm{const}\,N^{-\alpha}$$

again by the same method.

As for the $Q_N^{(3)}$, use the following facts:

$$|\Gamma_N^{(2),r}(x,x\pm e_\mu)|\leqq\frac{1}{a}e^2A^2_{\mu,g,a}\left(x\pm\frac{e_\mu}{2}\right),$$

$$Q_N^{(3)}=\|K_N^{(2),r}\|_4^2\leqq\|K_N^{(2),r}\|_2^2\leqq\mathrm{Tr}\,P_N^2\Gamma_N^{(2),r}P_N^2\Gamma_N^{(2),r*}.$$

Let R_N be defined by $\tilde{R}_N(k)=\left[m^2+\frac{2}{a^2}(2-\sum\cos ak_\mu)\right]^{-1/4}$. Then [9] $R_N(x,y)\geqq 0$ and $\|P_NR_N^{-1}\|_\infty\leqq t$ (independent of $L\geqq 1$ and $N\geqq 1$). Thus

$$Q_N^{(3)}\leqq t^4a^{-2}e^4\,\mathrm{Tr}\,R_N^2A_{g,a}^2R_N^2A_{g,a}^2,$$

which shows

$$\int Q_N^{(3)}d\mu\leqq\mathrm{const}\,N^{-\alpha}.\quad\square$$

2.2. Convergence of $\det_{\mathrm{ren}}(1+K_N)$

We have just proved K_N converges to K_Λ or to K_g in C_4 a.e. as $N\to\infty$. After rewriting $\det(1+K_N)$ as $\det_{(4)}(1+K_N)\exp([-T_N]$, where $T_N=\mathrm{Tr}\{-K_N+1/2K_N^2-1/3K_N^3\}$, one therefore finds $\det_{(4)}(1+K_N)$ converges to $\det_{(4)}(1+K_\Lambda)$ or to $\det_{(4)}(1+K_g)$ which are a.e. finite.

Theorem III. *Let T_N be as above, and let $C_N\equiv\int T_N d\mu(A)$. Then T_N-C_N converges to $:T:\in L^p(d\mu)$, $p\geqq 1$ a.e. as $N\to\infty$ and $|C_N|\leqq c\ln(2+N)$, where $c>0$ is independent of $L(\geqq 1)$ and $N(\geqq 1)$ and*

$$T=\frac{e^2}{2\pi}\left(\frac{2\pi}{L}\right)^2\sum_{k\in\tilde{\Lambda}}\tilde{T}_{\mu\nu}(k)\tilde{A}_{\mu,g}(k)\tilde{A}_{\nu,g}(-k)\,.$$

Here

$$\tilde{T}_{\mu\nu}=\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\tilde{T}(k)\,. \tag{2.15a}$$

$$\tilde{T}=1-\frac{4m^2}{k\sqrt{4m^2+k^2}}\mathrm{Tanh}^{-1}\left(\frac{k}{\sqrt{4m^2+k^2}}\right)+E_L(k)\,, \tag{2.15b}$$

$$\begin{aligned}&|E_L(k)|\leqq \mathrm{const}(1+k^2)^{-1}\log(2+k^2)\,,\\ &|E_L(k)|\leqq C(p)L^{-p}\,,\qquad \text{for any } p>0\,.\end{aligned} \tag{2.15c}$$

Remarks 3. (1) If $L_N=L$ depends on N like $L_0N^{1/2}$, then $E(k)\equiv 0$ and $\left(\frac{2\pi}{L}\right)^2\sum_{k\in\tilde{\Lambda}}$ should be replaced by $\int d^2k$. (2) $L_N=L$ can depend on N highly arbitrarily as far as $a=L_N/2N$ tends to zero as $N\to\infty$.

This is also essentially proved in [9]. We sketch the proof since it is much simplified compared to [9].

Proof. (Step 1). Let

$$K_N=K_N^{(1)}+K_N^{(2),r}$$

as before, and let

$$\Gamma_N=\Gamma_N^{(1)}+\Gamma_N^{(2)}+\Gamma_N^{(3)}+\Gamma_N^{(4),r}\,,$$

corresponding to the expansion of Γ_N in terms of $aA_{\mu,g,a}$. Thus

$$\begin{aligned}\mathrm{Tr}(K_N)^3&=\mathrm{Tr}(K_N^{(1)})^3+\mathrm{Tr}[3K_N^{(1)}(K_N^{(2),r})^2+3(K_N^{(1)})^2K_N^{(2),r}+(K_N^{(2),r})^3]\,,\\ \mathrm{Tr}(K_N)^2&=\mathrm{Tr}(K_N^{(1)})^2+\mathrm{Tr}[2K_N^{(1)}K_N^{(2),r}+(K_N^{(2),r})^2]\,,\\ \mathrm{Tr}K_N&=\mathrm{Tr}S_N[\Gamma_N^{(1)}+\Gamma_N^{(2)}+\Gamma_N^{(3)}+\Gamma_N^{(4),r}]\,.\end{aligned}$$

Note that $\gamma_5\Gamma_N\gamma_5=\Gamma_N^*$ and $\gamma_5B_N\gamma_5=B_N^*$, where $\Gamma_N=\Gamma_N$ or $\Gamma_N^{(\ell)}$ and $\gamma_5=\gamma_5^*=\gamma_5^{-1}=i\gamma_0\gamma_1$. Then $\mathrm{Tr}(K_N^{(1)})^n$ and $\mathrm{Tr}S_N\Gamma_N^{(\ell)}$ are real. Since $\Gamma_N^{(\ell)}(-ie)=(-1)^\ell\Gamma_N^{(\ell)}(ie)$ and there exists a complex conjugacy operator C such that $C\gamma_\mu C=\gamma_\mu$, one finds:

$$\mathrm{Tr}(K_N^{(1)})^3=\mathrm{Tr}S_N\Gamma_N^{(1)}=\mathrm{Tr}S_N\Gamma_N^{(3)}=0\,.$$

As for other terms containing $K_N^{(2),r}$, use Hölder's inequality and a trivial inequality $\|A\|_p \leqq \|A\|_{p'}$ $(p \geqq p')$, to show each of them is dominated by a factor of the form Q_N^r where $r=1$ or $1/2$ and Q_N is a polynomial of A_μ such that

$$\int Q_N d\mu \leqq cN^{-\alpha}, \qquad c>0.$$

Then these terms converge to zero a.e. with respect to $d\mu$. For example:

$$\begin{aligned}\tilde{Q}_N &\equiv |\mathrm{Tr}(K_N^{(1)})^2 K_N^{(2),r}| \leqq \|K_N^{(1)}\|_4^2 \|K_N^{(2),r}\|_2 \\ &\leqq \|K_N^{(1)}\|_2^2 \|K_N^{(2),r}\|_2,\end{aligned}$$

where

$$\|K_N^{(1)}\|_2^2 \leqq C\log\left(1+\frac{1}{am}\right)\frac{1}{L^2}\sum_{k\in\tilde{\Lambda}_N}|\tilde{A}_{\mu,g,a}(k)|^2$$

[C: independent of $L(\geqq 1)$ and N], and use Lemma II-1 (see the proof to replace $\| \ \|_4$ by $\| \ \|_2$) to see that $\|K_N^{(2),r}\|_2^2$ is dominated by a polynomial of $A_{\mu,g}$ of order 4 which converges to zero rather rapidly. Hölder's and the hypercontractive inequality mean

$$\begin{aligned}\int \tilde{Q}_N^2 d\mu &\leqq \int \|K_N^{(1)}\|_2^4 \|K_N^{(2),r}\|_2^2 d\mu \\ &\leqq \{\int \|K_N^{(1)}\|_2^8 d\mu\}^{1/2}\{\int \|K_N^{(2),r}\|_2^4 d\mu\}^{1/2} \\ &\leqq cN^{-\alpha}.\end{aligned}$$

As for $\mathrm{Tr}\, S_N \Gamma_N^{(4),r}$, one explicitly finds:

$$\begin{aligned}|\mathrm{Tr}\, S_N \Gamma_N^{(4),r}| &\leqq (a^2)^2 \sum_{ij}\sum_{x,y\in\Lambda_N}|S_N(x-y)_{ij}||\Gamma_N^{(4),r}(y,x)_{ji}| \\ &\leqq (a^2)^2 \sum_{ij,\mu}\sum_{x\in\Lambda_N}|S_N(e_\mu)_{ij}|\, e^4\, \frac{1}{a^3}\left(aA_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right)\right)^4 \\ &\leqq \mathrm{const}\, e^4 (a^2)^2 \sum_{x\in\Lambda_N} A_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right)^4,\end{aligned}$$

where const is independent of $L(\geqq 1)$ and $N(\geqq 1)$, and we have used

$$\left|e^{ix}-\left(1+ix-\frac{x^2}{2}-\frac{i}{6}x^3\right)\right| \leqq x^4$$

and

$$|S_N(e_\mu)_{ij}| \leqq \frac{1}{L^2}\sum_{k\in\tilde{\Lambda}_N}\tilde{P}_N^2(k) \leqq \mathrm{const}\,\frac{1}{a}.$$

Since $\int A_{\mu,g,a}^4(x)d\mu(A) \leqq \mathrm{const}(g_a(x))^4 \log^2\left(2+\frac{1}{a\mu}\right)$, this converges to zero.

(Step 2). It remains to consider

$$\mathrm{Tr}[-S_N\Gamma_N^{(2)} + 1/2(K_N^{(1)})^2] = e^2 \frac{1}{L^2}\sum_{k\in\tilde{\Lambda}_N}\tilde{A}_{\mu,g,a}(k)\tilde{A}_{\nu,g,a}(-k)\tilde{T}_{\mu\nu}(k). \qquad (2.16)$$

The gauge invariance requires [1, 9]

$$\sum_\mu \sin\frac{ak_\mu}{2}\tilde{T}_{\mu\nu}(k)=\sum_\nu \sin\frac{ak_\nu}{2}\tilde{T}_{\mu\nu}(k)=0,$$

which means

$$\tilde{T}_{\mu\nu}(k)=\left[\delta_{\mu\nu}-\frac{\sin\frac{ak_\mu}{2}\sin\frac{ak_\nu}{2}}{\sum\sin^2\frac{ak_\mu}{2}}\right]\tilde{T}(k). \tag{2.17}$$

Then

$$\begin{aligned}
\tilde{T}(k)&=\sum_\mu \tilde{T}_{\mu\mu}(k)\\
&=1/2\frac{a^2}{L^2}\sum_{q\in\tilde{\Lambda}_N}\frac{\sum\sin^2 aq_\mu-(2-\sum\cos aq_\mu)(\sum\cos aq_\mu)+am\sum\cos aq_\mu}{a^2\Delta(q)}\\
&\quad-1/2\frac{1}{L^2}\sum_{q\in\tilde{\Lambda}_N}\frac{1}{\Delta(k+q)\Delta(q)}\\
&\quad\cdot\left\{\frac{2}{a^2}[\sin aq_0\sin a(k+q)_0-\sin aq_1\sin a(k+q)_1]\right.\\
&\quad\cdot\left[\cos^2 a\left(k+\frac{q}{2}\right)_0-\cos^2 a\left(k+\frac{q}{2}\right)_1\right]\\
&\quad+\frac{2}{a^2}\left[\sum_\mu \sin aq_\mu\sin a(k+q)_\mu\right]\left[\sum\sin^2 a\left(k+\frac{q}{2}\right)_\mu\right]\\
&\quad+[m+1/a\sum(1-\cos a(k+q)_\mu)]\frac{2}{a}[\sum\sin aq_\mu\sin a(2k+q)_\mu]\\
&\quad+[m+1/a\sum(1-\cos aq_\mu)]\frac{2}{a}[\sum\sin a(k+q)_\mu\sin a(2k+q)_\mu]\\
&\quad\left.+2[m+1/a\sum(1-\cos aq_\mu)][m+1/a\sum(1-\cos a(k+q)_\mu)][\sum\cos a(2k+q)_\mu]\right\}
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&=-1/2\frac{1}{L^2}\sum_{q\in\tilde{\Lambda}_N}\frac{4m^2}{\Delta(k+q)\Delta(q)}\\
&\quad-1/2\frac{a^2}{L^2}\sum_{q\in\tilde{\Lambda}_N}\left\{\frac{1}{a^2\Delta(q)}[\sum\sin^2 aq_\mu-(2-\sum\cos aq_\mu)(\sum\cos aq_\mu)]\right.\\
&\quad+\frac{1}{a^4\Delta(q)\Delta(k+q)}\left\{2[\sin aq_0\sin a(k+q)_0-\sin aq_1\sin a(k+q)_1]\right.\\
&\quad\cdot\left[\sin^2 a\left(k+\frac{q}{2}\right)_0-\sin^2 a\left(k+\frac{q}{2}\right)_1\right]\\
&\quad+2[\sum\sin aq_\mu\sin a(k+q)_\mu]\left[\sum\sin^2 a\left(k+\frac{q}{2}\right)_\mu\right]
\end{aligned}$$

$$+2[\sum(1-\cos a(k+q)_\alpha)]\,[\sum \sin aq_\alpha \sin a(2k+q)_\alpha]$$
$$+2[\sum(1-\cos aq_\mu)]\,[\sum \sin aq_\mu \cdot \sin a(2k+q)_\mu]$$
$$+2[\sum(1-\cos aq_\mu)]\,[\sum(1-\cos a(k+q)_\mu)]\,[\sum \cos a(2k+q)_\mu]\Big\}\Big\}$$
$$+C\,, \tag{2.19}$$

where $|C| \leqq \text{const}\, am \log\left[2+\frac{1}{am}\right]$ uniformly in $L \geqq 1$ as $am \to 0$, and $\Delta(k)=\tilde{P}_N(k)^{-4}$.

The first term is written as:

$$-\frac{m^2}{2\pi^2}\int d^2q \frac{1}{[(k+q)^2+m^2]\,[q^2+m^2]}$$
$$+\frac{m^2}{2\pi^2}\left[\frac{4\pi^2}{L^2}\sum_{q\in\frac{2\pi}{L}Z^2} - \int d^2q\right]\frac{1}{[(k+q)^2+m^2]\,[q^2+m^2]}$$
$$+\frac{m^2}{2\pi^2}\left[\frac{4\pi^2}{L^2}\sum_{q\in\tilde{\Lambda}_N}\frac{1}{\Delta(k+q)\Delta(q)}\right.$$
$$\left.-\frac{4\pi^2}{L^2}\sum_{q\in\frac{2\pi}{L}Z^2}\frac{1}{[(k+q)^2+m^2]\,[q^2+m^2]}\right]$$
$$\equiv -\frac{1}{2\pi}\frac{4m^2}{k\sqrt{4m^2+k^2}}\,\mathrm{Tanh}^{-1}\left(\frac{k}{\sqrt{4m^2+k^2}}\right)$$
$$+E_L(k)+C_N(k)$$

in this order. Obviously

$$|E_L(k)| \leqq \text{const}\log[2+k]\,(1+k^2)^{-1}$$

uniformly in $L \geqq 1$, and $E_L(k) \leqq \text{const}\,L^{-p}$ $(p>0)$ uniformly in k and $L \geqq 1$. P can be chosen arbitrarily large [9]. Further

$$|C_N(k)| \leqq \text{const}\, a^\delta (1+k^2)^{-\varepsilon}$$

with some positive constants δ and ε, uniformly in $L \geqq 1$.

As for the second term, let $x_\mu = aq_\mu \in \left\{\frac{\pi}{N}n_\mu\,;\, n_\mu = -N, -N+1, \ldots, N-1\right\}$ and note that $a^2/L^2 = \frac{1}{4\pi^2}\left(\frac{\pi}{N}\right)^2$. Thus this converges to a k-independent constant which can be calculated by a contour integral (see also [9]), and is equal to $\frac{1}{2\pi}$.

The remaining statements of the theorem are now rather trivial. □

3. Transfer Matrix and Determinant Inequalities

3.1. Transfer Matrix and Diamagnetic Inequality

Let

$$R_N^p(A_\mu) = B_N^p + \Gamma_N^p\,,$$

where B_N^p and Γ_N^p are given in (1.6) and "p" means periodic. Let $R_N^A(A_\mu) \equiv R_N^p(A_\mu + \delta A_\mu)$, where

$$\delta A_\mu(x) = \frac{\pi}{eL}$$

for all $x \in \Lambda_N^\mu$. Thus $R_N^A = B_N^A + \Gamma_N^A$ with

$$\begin{aligned} B_N^A(x,y) &= (ma^{-2} + 2a^{-3})\delta_{x,y} - a^{-3}\gamma(x,y)\, V(x,y), \\ \Gamma_N^A(x,y) &= -a^{-3}[U(x,y) - 1]\,\gamma(x,y)\, V(x,y), \end{aligned} \tag{3.1}$$

where

$$V(x,y) = \begin{cases} \exp\left[\pm \dfrac{i\pi}{2N}\right] & y = x \pm e_\mu \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

and we define

$$S_N^A = (B_N^A)^{-1} = (P_N^A)^2 U_N^A, \; K_N^A = U_N^A P_N^A \Gamma_N^A P_N^A \tag{3.3}$$

with $P_N^A \geqq 0$, $U_N^{A*} = U_N^{A-1}$ as before. Though this changes the periodic boundary conditions into the anti-periodic ones, this does not change our previous theorems and lemmas at all. In fact $\tilde{P}_N^A(k) = \tilde{P}_N^p(k-\delta)$, $\tilde{\Gamma}_N^A(k,k') = \tilde{\Gamma}_N^p(k-\delta, k'-\delta)$, etc., with

$$\delta = \frac{\pi}{L}(1,1) \tag{3.4}$$

mean our Feynman diagram estimates do not change at all, and one can easily confirm that the Furry theorem again holds for this boundary condition.

This choice of boundary condition is indispensible for the introduction of the transfer matrix [4, 10] or for proving the OS positivity [1].

Theorem IV [4, 10].

$$\det[R_N^A(A_N)] = \mathrm{Tr}\, T_{-N} U_{-N} \dots T_{N-1} U_{N-1}, \tag{3.5}$$

where $\{T_\ell, U_\ell\}$ are operators on a 2^{4N} dimensional Hilbert space spanned by operating the fermion creation operators $\{a^+(n), b^+(n)\}_{n=-N}^{N-1}$ on a cyclic vacuum vector Ω, and satisfy:

(1) *T_ℓ depends only on $\{A_{1,g,a}[a\ell, a(n+\frac{1}{2})]\}_{n=-N}^{N-1}$ and $T_\ell > 0$ if $e \in R$. T_ℓ is analytic in e in a neighbourhood of $e=0$.*

(2) *$U_\ell^* = U_\ell^{-1}$ if $e \in R$,*

$$U_\ell = \exp\left\{iae \sum_{n=-N}^{N-1} A_{0,g,a}[a(n+\tfrac{1}{2}), a\ell]\,[a^+(n)\,a(n) - b^+(n)\,b(n)]\right\}.$$

See [4, 10] for the proof. It is sufficient to replace A_0 by A_1 and A_1 by A_0 to introduce the transfer matrix for $\mu = 1$ direction.

Theorem V.

(1)
$$0<\det[1+K_N^A]\leqq 1\,. \tag{3.6a}$$

(2) *Let* $m>0$. *Then*

$$0<\det[1+K_N^p]\leqq C\,, \tag{3.6b}$$

uniformly in $L(\geqq 1)$ *and* $N(\geqq 1)$.

Proof. Since K_N^p (respectively K_N^A) is unitarily equivalent to K_N^{p*} (respectively K_N^{A*}) with the unitary $\gamma_5 U_N$ (respectively $\gamma_5 U_N^A$), the determinants are real. Thus the positivity of the determinants follows from $(-\infty,0]\cap\mathrm{spec}(R_N)=\emptyset$ [9]. Applying the Hölder inequality to (3.5) and the unitary of U_ℓ, one has:

$$\det[R_N^A(A_\mu)]\leqq \prod_{\ell=-N}^{N-1}\{\mathrm{Tr}(T_\ell)^{2N}\}^{1/2N}\,,$$

namely all A_0 are set at zero in the right hand side. Next apply the same discussion for each $\mathrm{Tr}(T_\ell)^{2N}$ after introducing the transfer matrix for the $\mu=1$ direction. This means

$$\det[R_N^A(A_\mu)]\leqq\det[R_N^A(A_\mu=0)]\,, \tag{3.7}$$

and then (3.6a) follows.

Finally since $R_N^A(A_\mu)=R_N^p(A_\mu+\delta A_\mu)$,

$$\det(1+K_N^p)=\frac{\det[R_N^p(A_\mu)]}{\det[R_N^p(A_\mu=0)]}\leqq\frac{\det[R_N^A(0)]}{\det[R_N^p(0)]}\equiv R\,. \tag{3.8}$$

Then $R\geqq 1$ by the definition and

$$R=\prod_{k\in\tilde{\Lambda}_N}\frac{\hat{P}_N(k)^4}{\tilde{P}_N(k-\delta)^4}=\prod_{k\in\tilde{\Lambda}_N}\frac{\Delta(a;k+\delta)}{\Delta(a;k)}\,,$$

$$\Delta(a;k)=\left[m+\frac{1}{a}\sum(1-\cos ak_\mu)\right]^2+\frac{1}{a^2}\sum\sin^2 ak_\mu\,.$$

The upperbound for R follows from next lemma. □

Lemma V-1. *Let* $\zeta=(\zeta_0,\zeta_1)$, $|\zeta_\mu|\leqq\frac{\pi}{L}$ *be given, and let*

$$R=\prod_{k\in\tilde{\Lambda}_N}\frac{\Delta(a;k+\zeta)}{\Delta(a;k)}\,, \tag{3.9}$$

with $L\geqq 1$ *and* $N\geqq 1$. *Then*

$$0<c_1\leqq R\leqq c_2<\infty \tag{3.10}$$

uniformly in $L\geqq 1$ *and* $N\geqq 1$.

Proof. Note that

$$R=\prod_{k\in\tilde{\Lambda}_N}(1+f_1+f_2+f_3+\delta f)\,,$$

where

$$f_1=\frac{1}{\Delta}\left\{\frac{1}{a^2}\sum_\mu \sin a\zeta_\mu \sin 2ak_\mu\right\},$$

$$f_2=\frac{1}{\Delta}\left\{\frac{1}{a^2}\sum \sin^2 a\zeta_\mu \cos^2 ak_\mu\right\},$$

$$f_3=\frac{1}{\Delta}\left(\frac{2}{a}\sum \sin a\zeta_\mu\cdot \sin ak_\mu\right)\left(m+\frac{1}{a}\sum(1-\cos ak_\mu)\right),$$

and δf is the remaining term which is defined in the obvious way. Use

$$|\sin a\zeta_\mu|\leqq|a\zeta_\mu|\leqq a\frac{\pi}{L},$$

$$(k^2+m^2)/\Delta(a;k)\leqq C$$

uniformly in $k\in\tilde{\Lambda}_N$ and $a\geqq 0$ to show

$$\sum_{k\in\tilde{\Lambda}_N}|\delta f(k)|\leqq C_1$$

uniformly in $L(\geqq 1)$ and $N(\geqq 1)$.

Next use $k\leftrightarrow -k$ symmetry of $\tilde{\Lambda}_N$ to see

$$R^2=\prod_k(1+f_1+f_2+f_3+\delta f)(1-f_1+f_2-f_3+\delta\tilde{f})$$
$$=\prod_k(1+2f_2-f_1^2+\delta f'),$$

where $\delta\tilde{f}=\delta f(-k)$ and $\delta f'$ is defined in the obvious way. It is not difficult to see

$$\sum_{k\in\Lambda_N}|\delta f'|\leqq C_2$$

uniformly in $L(\geqq 1)$ and $N(\geqq 1)$ just by the same method.

As for $g\equiv 2f_2-f_1^2$, rewrite this as

$$g_1+g_2+\delta g,$$

where

$$g_1=\frac{1}{\Delta^2}\frac{2}{a^4}(\sin^2 ak_0-\sin^2 ak_1)(\sin^2 a\zeta_1-\sin^2 a\zeta_0),$$

$$g_2=-\frac{1}{\Delta^2}\frac{2}{a^4}\sin 2a\zeta_0 \sin 2a\zeta_1 \sin 2ak_0 \sin 2ak_1,$$

and δg is the remaining term. It is easy to see

$$\sum_k|\delta g|\leqq C_3$$

uniformly in $L(\geqq 1)$ and $N(\geqq 1)$. As for g_1, g_2, use a symmetry $(k_0, k_1)\to(-k_1, k_0)$ which changes the signs of $\{g_i\}_{1,2}$. Then letting $\delta g' = \delta g + \delta f'$,

$$R^4 = \prod_{k\in\tilde{\Lambda}_N} (1+g_1+g_2+\delta g')(1-g_1-g_2+\delta\tilde{g}')$$
$$= \prod_{k\in\tilde{\Lambda}_N} (1-(g_1+g_2)^2+\delta g''),$$

where $\delta\tilde{g}'(k_0,k_1)=\delta g'(-k_1,k_0)$ and $\delta g''$ is defined in the obvious way. Obviously

$$\sum_{k\in\tilde{\Lambda}_N} (g_1+g_2)^2 \leqq C_4,$$
$$\sum_{k\in\Lambda_N} |\delta g''| \leqq C_5,$$

uniformly in $L(\geqq 1)$ and $N(\geqq 1)$.

Finally use $|\prod(1+z_i)|\leqq \exp[\sum|z_i|]$. As for the lower bound, remember

$$0 < C_6 \leqq \frac{\Delta(k+\zeta)}{\Delta(k)}$$

for all $k\in\tilde{\Lambda}_N$ and $a\geqq 0$, provided $m>0$. Then use $\exp[-|\log\alpha/\alpha|\sum|z_i|]\leqq\prod(1+z_i)$ if $1+z_i\geqq\alpha>0$, $1\geqq\alpha$. □

Corollary V-1. *Let* $L=L_N=L_0N^{\delta}$ $(0\leqq\delta<1)$ *and let* $K_N=K_N^A$ *or* K_N^p. *Then*

$$0<\det_{\mathrm{ren}}(1+K_N)\leqq\exp[d_1+d_2\log N], \tag{3.11}$$

where $\{d_i\}$ *are independent of* $N(\geqq 1)$ *and* $L\geqq 1$.

3.2. Determinant Inequalities

In order to study the volume dependence (Λ_0 or g-dependence) of the Schwinger functions, we need a determinant inequality which decomposes the Matthews-Salam determinant. For this purpose, for the moment, assume

$$L=2n,\ n \text{ fixed positive integer},$$
$$N=2nM,\ M \text{ positive integers},$$
$$g=\chi_\Lambda$$

for simplicity. Thus $a=L/2N=1/2M$ is the lattice width which tends to zero as $M=N/2n\to\infty$. Now

$$\begin{aligned}\det[R_N(A)] &= \mathrm{Tr}\{T_{-N}U_{-N}\cdots T_{-N+2M-1}U_{-N+2M-1}\}\{T_{-N+2M}U_{-N+2M}\cdots\}\\ &\quad\cdot\ldots\cdot\{T_{N-2M}U_{N-2M}\cdots T_{N-1}U_{N-1}\}\\ &\leqq\{\mathrm{Tr}|T_{-N}U_{-N}\cdots T_{-N+2M-1}U_{-N+2M-1}|^{2n}\}^{1/2n}\\ &\quad\cdot\ldots\cdot\{\mathrm{Tr}|T_{N-2M}U_{N-2M}\cdots T_{N-1}U_{N-1}|^2\}^{1/2n}\\ &\leqq\{\mathrm{Tr}\,U^*_{-N+2M-1}T_{-N+2M-1}\cdots U^*_{-N}T_{-N}T_{-N}U_{-N}\cdots\\ &\quad\cdot T_{-N+2M-1}U_{-N+2M-1}\}^{1/2}\\ &\quad\cdot\ldots\cdot\{\mathrm{Tr}\,U^*_{N-1}T_{N-1}\cdots U^*_{N-2M}T_{N-2M}T_{N-2M}\cdots T_{N-1}U_{N-1}\}^{1/2},\end{aligned} \tag{3.12}$$

by Hölder's inequality and by a trivial inequality $\|A\|_{2n} \leqq \|A\|_2$ $(n \geqq 1)$. We repeat the same discussion for each of the terms in the right hand side after introducing the transfer matrix to the other direction to find

$$\det[R_N(A)] \leqq \prod_{i_0, i_1 = -n}^{n-1} \det{}^{1/4}[R_{2M}(B^{(i)})],$$

where $i=(i_0, i_1)$ and $R_{2M}(B^{(i)})$ is the R_N function defined by the region $\lambda=[-1,1]^2$, lattice width $a=\frac{L}{2N}=\frac{2}{4M}$ and the lattice gauge fields $\{B^{(i)}_{\mu,a}((an_0, an_1)+\frac{1}{2}e_\mu);\ -2M \leqq n_\mu \leqq 2M-1\}$:

(I)

$$\begin{aligned} B^{(i)}_0(a[n_0+\tfrac{1}{2}], an_1) &= A^{(i)}_0(a[n_0+\tfrac{1}{2}], an_1), \quad \text{if} \quad 0 \leqq n_0 \leqq 2M-2, 0 \leqq n_1 \leqq 2M-1, \\ &\qquad 0, \quad \text{if} \quad n_0 = 2M-1, \\ B^{(i)}_1(an_0, a[n_1+\tfrac{1}{2}]) &= A^{(i)}_1(an_0, a[n_1+\tfrac{1}{2}]), \quad \text{if} \quad 0 \leqq n_0 \leqq 2M-1, 0 \leqq n_1 \leqq 2M-2, \\ &\qquad 0, \quad \text{if} \quad n_1 = 2M-1, \end{aligned}$$

(II)

$$\begin{aligned} B^{(i)}_0(a[-n_0-\tfrac{1}{2}], an_1) &= -A_0(a[n_0-\tfrac{1}{2}], an_1), \quad \text{if} \quad 1 \leqq n_0 \leqq 2M-1, 0 \leqq n_1 \leqq 2M-1, \\ &\qquad 0, \quad \text{if} \quad n_0 = 0, \\ B^{(i)}_1(-an_0, a[n_1+\tfrac{1}{2}]) &= A^{(i)}_1(a[n_0-1], an_1+\tfrac{1}{2}), \quad \text{if} \quad 1 \leqq n_0 \leqq 2M, 0 \leqq n_1 \leqq 2M-1, \\ &\qquad 0, \quad \text{if} \quad n_1 = 2M-1, \end{aligned} \tag{3.14}$$

and so on, where $A^i_\mu(x)=A_\mu(x-i)$ and we have omitted the subscripts g and a. Approximately

$$B^{(i)}_{\mu,a}(x_0, x_1) = \operatorname{sgn}(x_\mu)\, A^{(i)}_{\mu,a}(|x_0|, |x_1|). \tag{3.14'}$$

In fact one finds:

$$\begin{aligned} \tilde{B}^{(\mathrm{II})}_0(k_0, k_1) &= -\tilde{B}^{(\mathrm{I})}_0(-k_0, k_1)e^{-iak_0} = -\tilde{A}_0(-k_0, k_1)e^{-iak_0}, \\ \hat{B}^{(\mathrm{II})}_1(k_0, k_1) &= \tilde{B}^{(\mathrm{I})}_1(-k_0, k_1)e^{-iak_0} = \tilde{A}_1(-k_0, k_1)e^{-iak_0}, \quad \text{etc.} \end{aligned} \tag{3.15}$$

Now let

$$\Xi(n; N) = \frac{\det^{n^2}[R_{2M}(0)]}{\det[R_N(0)]}. \tag{3.16}$$

Then

$$\det[1+K_N(A)] \leqq \Xi(n; N) \prod_i \det[1+K_{2M}(B^{(i)})]^{1/4}, \tag{3.17}$$

where we have omitted A for simplicity.

Lemma VI-1. *There exist constants α_1 and α_2 uniformly in $M \geqq 1$ and n such that*

$$\exp[\alpha_1 n^2] \leqq \Xi(n; N) \leqq \exp[\alpha_2 n^2]. \tag{3.18}$$

Proof. One can replace R_{2M} and R_N by the periodic ones by the proof of Theorem V(2). Thus consider

$$\Xi = \frac{\prod_{k\in\tilde{\lambda}_{2M}} \Delta(a;k)^{n^2}}{\prod_{k\in\tilde{\Lambda}_N} \Delta(a;k)},$$

where $\Delta(a;k)=\tilde{P}_N(k)^{-4}$. Now $\tilde{\lambda}_{2M}=\left\{\frac{2\pi}{2}(j_0,j_1);\ -2M\leqq j_\mu\leqq 2M-1\right\}$. Thwn let

$$\Omega=\left\{\frac{2\pi}{2n}(j_0,j_1);\ -\left[\frac{n}{2}\right]\leqq j_\mu\leqq\left[\frac{n+1}{2}\right]-1\right\}.$$

Then $|\Omega|=n^2$, $|\zeta_\mu|\leqq\frac{\pi}{2}$ if $\zeta\in\Omega$ and

$$\Xi^{-1}=\prod_{\zeta\in\Omega}\prod_{k\in\tilde{\lambda}_{2M}}\frac{\Delta(a;k+\zeta)}{\Delta(a;k)}.$$

Thus the lemma follows from Lemma V-1. □

Lemma VI-2. *Let*

$$C_N=\int \mathrm{Tr}[-K_N(A)+\tfrac{1}{2}K_N(A)^2-\tfrac{1}{3}K_N(A)^3]\,d\mu(A),$$
$$C_M^{(i)}=\tfrac{1}{4}\int \mathrm{Tr}[-K_{2M}(B^i)+\tfrac{1}{2}K_{2M}(B^i)^2-\tfrac{1}{3}K_{2M}(B^i)^3]\,d\mu(A).$$

There exists a constant C uniformly in $M\geqq 1$ such that

$$|C_N-\sum C_M^{(i)}|\leqq C(2n)^2=C|\Lambda|. \tag{3.19}$$

Proof. It is sufficient to consider

$$\tilde{C}_N=\int \mathrm{Tr}[-S_N\Gamma_N^{(2)}(A)+\tfrac{1}{2}(S_N\Gamma_N^{(1)}(A))^2]\,d\mu(A),$$
$$\tilde{C}_M^i=\tfrac{1}{4}\int \mathrm{Tr}[-S_{2M}\Gamma_{2M}^{(2)}(B^i)+\tfrac{1}{2}(S_{2M}\Gamma_{2M}^{(1)}(B^i))^2]\,d\mu(A).$$

First consider the contributions from $\Gamma_N^{(2)}$ and $\Gamma_{2M}^{(2)}$:

$$\mathrm{Tr}\,S_N\Gamma_N^{(2)}(A)=v_M\left(a^2\sum_\mu\sum_{x\in\Lambda_N^\mu}A_{\mu,g,a}^2(x)\right),$$
$$\tfrac{1}{4}\mathrm{Tr}\,S_{2M}\Gamma_{2M}^{(2)}(B^i)=\tfrac{1}{4}v_{2M}a^2\sum_\mu\sum_{x\in\lambda_{2M}^\mu}B_{\mu,g,a}^i(x)^2$$
$$=v_{2M}\left(a^2\sum_\mu\sum_{x\in\Delta_{2M}^\mu}A_{\mu,g,a}^2(x-(i))\right),$$

where $\Delta=[0,1)^2$ and the forms of v_N and v_{2M} are essentially given in the proof of Theorem III. [The first term in (2.18) with q replaced by $q+\delta$.] Thus

$$\int\left|\mathrm{Tr}\,S_N\Gamma_N^{(2)}(A)-\tfrac{1}{4}\sum_i \mathrm{Tr}\,S_{2M}\Gamma_{2M}^{(2)}(B^i)\right|d\mu(A)$$
$$\leqq|v_N-v_{2M}|(2n)^2K_0\log\left(2+\frac{1}{a\mu}\right)+\text{boundary term} \tag{3.20}$$

uniformly as $a\to 0$. Since $|v_N - v_{2M}| \leqq \text{const}\, a^\varepsilon$, $\varepsilon>0$, as $N=4nM\to\infty$, this difference uniformly tends to zero as $a\to 0$. [Boundary term $\leqq \text{const}(2n)^2 a^\varepsilon$.]

In order to consider the other terms, let $\gamma_\mu(\pm)=\gamma_\mu^N(\pm)=\frac{1}{2}(\gamma_\mu \pm 1)\exp\left[\pm\frac{i\pi}{2N}\right]$, and let

$$\begin{aligned}\Pi_{\mu\nu}^N(x,y) = \mathrm{Tr}\{&\gamma_\mu(+)S_N(x+e_\mu, y)\gamma_\nu(+)S_N(y+e_\nu, x)\\ &+\gamma_\mu(-)S_N(x,y)\gamma_\nu(+)S_N(y+e_\nu, x+e_\mu)\\ &+\gamma_\mu(+)S_N(x+e_\mu, y+e_\nu)\gamma_\nu(-)S_N(y,x)\\ &+\gamma_\mu(-)S_N(x,y+e_\nu)\gamma_\nu(-)S_N(y,x+e_\mu)\}\,. \end{aligned} \tag{3.21}$$

Then

$$\begin{aligned}\mathrm{Tr}\, S_N\Gamma_N^{(1)}(A)\, S_N\Gamma_N^{(1)}(A) &= -e^2a^4\sum_{\mu,\nu}\sum_{\substack{x\in\Lambda_N\\ y\in\Lambda_N}}\left\{A_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right)A_{\nu,g,a}\left(y+\frac{e_\nu}{2}\right)\times\Pi_{\mu\nu}^N(x,y)\right\}\\ &= -e^2a^4\sum_i\sum_{\mu\nu}\sum_{x,y\in\Delta_M^{(i)}}\left\{A_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right)A_{\nu,g,a}\left(y+\frac{e_\mu}{2}\right)\times\Pi_{\mu\nu}^N(x,y)\right\}\\ &\quad -e^2a^4\sum_{i\neq j}\sum_{\mu\nu}\sum_{x\in\Delta_M^{(i)},\, y\in\Delta_M^{(j)}}\{\quad\}, \end{aligned} \tag{3.22}$$

where $\Delta^{(i)}=[i_0, i_0+1)\otimes[i_1, i_1+1)$. One also has

$$\begin{aligned}&\frac{1}{4}\sum_i \mathrm{Tr}\, S_{2M}\Gamma_{2M}^{(1)}(B^{(i)})\, S_{2M}\Gamma_{2M}^{(1)}(B^{(i)})\\ &\quad= -\frac{1}{4}e^2a^4\sum_i\sum_{\mu,\nu}\left\{\sum_j\sum_{x,y\in\Delta_M^{(j)}}\left[B_{\mu,g,a}^{(i)}\left(x+\frac{e_\mu}{2}\right)B_{\nu,g,a}^{(i)}\left(y+\frac{e_\nu}{2}\right)\Pi_{\mu\nu}^{2M}(x,y)\right]\right\}\\ &\quad\quad -\frac{1}{4}e^2a^4\sum_i\sum_{\mu,\nu}\left\{\sum_{j\neq k}\sum_{\substack{x\in\Delta_M^{(j)}\\ y\in\Delta_M^{(k)}}}[\quad]\right\}, \end{aligned} \tag{3.23}$$

where $j, k=(-1,-1), (-1,0), (0,-1), (0,0)$.

Let $S_a(x)$ be the euclidean free fermion propagator on aZ^2. Then one finds

$$|S_a(x)| \leqq K_0\frac{1}{a+|x_0|+|x_1|}\exp[-m_0(|x_0|+|x_1|)] \tag{3.24}$$

with positive constants K_0, m_0 uniformly in $a\leqq 1$. Since

$$S_N(x)=\sum_{\alpha\in Z^2}(-1)^{\alpha_0+\alpha_1}S_a(x+2Na\cdot\alpha)\,, \tag{3.24a}$$

$$S_{2M}(x)=\sum_{\alpha\in Z^2}(-1)^{\alpha_0+\alpha_1}S_a(x+4Ma\cdot\alpha) \tag{3.24b}$$

(note that $2Na=L=2n$, $4Ma=2$), the contributions from the second terms in Eqs. (3.22) and (3.23) are dominated by $\text{const}(2n)^2$ uniformly in $M\geqq 1$ after integrating by $d\mu(A)$.

Thus we consider

$$\sum_i \int \left\{ \frac{a^4}{4} \sum_{\mu,\nu} \sum_j \sum_{x,y\in\Lambda_M^{(j)}} B^{(i)}_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right) B^{(i)}_{\nu,g,a}\left(y+\frac{e_\nu}{2}\right) \Pi^{2M}_{\mu\nu}(x,y) \right\} d\mu(A). \quad (3.25)$$

Since $\Pi_{\mu\nu}$ is translationally invariant, set $\Pi^{2M}_{\mu\nu}(x-y)=\Pi^{2M}_{\mu\nu}(x,y)$. As is easily seen by the proof of Theorem III, one approximately has [like Eq. (3.14′)]

$$\Pi_{\mu\mu}(x_0,x_1)=\Pi_{\mu\mu}(-x_0,-x_1)= \quad \Pi_{\mu\mu}(-x_0,x_1)=\Pi_{\mu\mu}(x_0,-x_1),$$

$$\Pi_{\mu\nu}(x_0,x_1)=-\Pi_{\mu\nu}(-x_0,x_1)=-\Pi_{\mu\nu}(x_0,-x_1)=\Pi_{\mu\nu}(-x_0,-x_1),$$

if $\mu\neq\nu$ (this also holds for anti-periodic conditions). In fact analysis due to Fourier transform shows that (3.25) equals

$$\int \left\{ a^4 \sum_i \sum_{\mu,\nu} \sum_{x,y\in\Lambda_M^{(i)}} A_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right) A_{\nu,g,a}\left(y+\frac{e_\nu}{2}\right) \Pi^{2M}_{\mu\nu}(x,y) \right\} d\mu(A)+C(a), \quad (3.25')$$

where $|C(a)|\leqq \mathrm{const}(2n)^2$. Then it suffices to estimate

$$\int \left\{ a^4 \sum_i \sum_{\mu\nu} \sum_{x,y\in\Lambda_M^{(i)}} A_{\mu,g,a}\left(x+\frac{e_\mu}{2}\right) A_{\nu,g,a}\left(x+\frac{e_\nu}{2}\right) [\Pi^{N}_{\mu\nu}(x,y)-\Pi^{2M}_{\mu\nu}(x,y)] \right\} d\mu(A). \quad (3.26)$$

Since $|\gamma^N_\mu(\pm)-\gamma^{2M}_\mu(\pm)|\leqq \mathrm{const}\,a$ and because of the bound (3.24) and expressions (3.24a) and (3.24b), one finds:

$$|\Pi^N_{\mu\nu}(x)-\Pi^{2M}_{\mu\nu}(x)|\leqq K \sum_{\substack{\alpha\in Z^2\\ |\alpha|\leqq 2}} \frac{1}{|x+2\alpha|+a}$$

with constant K uniformly in $a\leqq 1$, whenever $x\in aZ^2$, $|x_\mu|\leqq 2$. This completes the proof. □

Therefore

Theorem VI.

$$\det{}_{\mathrm{ren}}(1+K_N(A))\leqq \exp[K|\Lambda_0|]\prod_i \det{}^{1/4}_{\mathrm{ren}}(1+K_{2M}(B^{(i)})) \quad (3.27)$$

with some constant K uniformly in $a\leqq 1$.

Remark 4. In this theorem, we have assumed the length of box L is fixed. However it may be possible to extend this for $L=L_N=L_0N^{1/2}$, provided that $\mathrm{supp}\,A_\mu=\Lambda_0$ is bounded, rectangular.

3.3. Volume Dependence of the Schwinger Functions

Theorem VII (Weingarten [10]).

(1) *Let $K_N(A)=K^A_N(A)$ or $K^P_N(A)$, and let $L=L_N=L_0N^\delta$ with $0\leqq\delta<1$. Then $v_N=\det_{\mathrm{ren}}(1+K_N(A))$ converges to $v(A)=\det_{\mathrm{ren}}(1+K(A))$ in $L^p(d\mu)$, $p>0$. $v>0$ a.e. with respect to $d\mu(A)$.*

(2) *Let $K_N(B)=K^A_N(B)$, L be fixed, and let B_μ be defined as before. Then $w_N=\det_{\mathrm{ren}}(1+K_N(B))$ converges to $w(B)=\det_{\mathrm{ren}}(1+K(B))$ in $L^p(d\mu)$, $p>0$.*

Theorem VII (1) is proved by showing $\mathrm{prob}[\upsilon \geqq x] \leqq \exp[-\alpha x^{\varepsilon}]$ with $\alpha, \varepsilon > 0$. Since

$$\begin{aligned}\mathrm{prob}[\upsilon \geqq x] &= \mathrm{prob}[\log((\upsilon/\upsilon_N)\upsilon_N) \geqq \log x] \\ &\leqq \mathrm{prob}[\log(\upsilon/\upsilon_N) \geqq \log x - c\log(N+1)],\end{aligned}$$

with N arbitrary, by Corollary V-1, it suffices to show that there exists a polynomial Q_N of A_μ of order $p < \infty$ such that

$$Q_N \geqq \log(\upsilon/\upsilon_N),$$
$$\int |Q_N|^2 \, d\mu \leqq C N^{-\varepsilon}, \qquad \varepsilon > 0.$$

See [10] for the detail. Our previous estimations are now used to prove this with rather trivial modifications. The part (2) is also similar. [Convergence of $K_N(B)$ to $K(B)$, etc. are almost trivial though the covariance of $\{B_\mu\}$ is slightly singular compared to that of $\{A_\mu\}$.]

Now let

$$Z_N(\Lambda_0) = \int \upsilon_N(A) \, d\mu(A) \tag{3.28}$$

and let $Z(\Lambda_0) = \lim_{N \to \infty} Z_N(\Lambda_0)$. By applying the checkerboard estimate (Theorem III-2 of [8]) for Theorem VI, one finds:

$$\begin{aligned} Z_N(\Lambda_0) &\leqq \exp K|\Lambda_0| \prod_i \{\textstyle\int w_{2M}(B^i)^{\beta^2/4} \, d\mu(A)\}^{1/\beta^2} \\ &= \exp\left[\left(K + \frac{\kappa}{\beta^2}\right)|\Lambda_0|\right], \end{aligned} \tag{3.29}$$

where $\beta = 2(1 - e^{-\mu})^{-1}$, and

$$\exp \kappa = \int w_{2M}(B^i)^{\beta^2/4} \, d\mu(A). \tag{3.30}$$

Theorem VIII. *There exists constant K such that*

$$Z(\Lambda_0) \leqq \exp[K|\Lambda_0|]. \tag{3.31}$$

This theorem can be extended to the Schwinger functions. Let

$$f_i \in \mathscr{H}_{-1}, \qquad \mathrm{supp}\, f_i \subset \Delta^\alpha \quad \text{for some} \quad \alpha \in Z^2,$$

and let $g_j, h_j \in \mathscr{H} \otimes C^2$. Let

$$\begin{aligned} &(Z(\Lambda_0)S)(f_1, \ldots, f_m; g_1, \ldots, g_n; h_1, \ldots, h_n) \\ &\quad = \int d\mu(A) \left[\prod_{i=1}^m A_{\mu_i}(f_i)\right] \det{}^n_{jk}[(\bar{g}_j, [\not{\partial} + m + ie\not{A}_g]^{-1} h_k)] \det{}_{\mathrm{ren}}(1 + K_g). \end{aligned} \tag{3.32}$$

Theorem IX. *There exist constants C_1 and C_2 such that*

$$|ZS| \leqq \exp[C_1|\Lambda| + (m+n)C_2] \prod_{\alpha \in Z^2} (n_\alpha!)^{1/2} \prod_{i=1}^m \|f_i\|_{-1} \prod_{j=1}^n \|g_j\| \, \|h_j\|, \tag{3.33}$$

where $n_\alpha = \#\{i; \mathrm{supp}\, f_i \subset \Delta^\alpha\}$.

The proof is essentially equal to that in [5] except $\| \ \|_{-1/2}$ is replaced by $\| \ \|$ here. The reason is that we have used a rather trivial inequality [9]

$$\|(\not\partial + m + ie\not A_g)^{-1}\| \leqq \frac{1}{m}.$$

As is investigated in [5–7], it may be possible to replace $\| \ \|$ by $\| \ \|_{-1/2}$, but this may be possible only when we succeed in the study of the kernal K_g.

One may be able to obtain a lower bound for $Z(\Lambda_0)$. But the discussion in [5] cannot be applied directly since an indefinite metric appears when one considers the Hamiltonian and its counterterms [2]. This problem together with the problem of the thermodynamic limit will be considered in a forthcoming paper.

Appendix

Proof of the Bounds (2.9b), (2.9c). First we show the bound for $g = \chi_{\Lambda_0}$. It suffices to show the bound for

$$\tilde{C}_g(k,k') = \int d^2p \left(\prod_\mu \frac{\sin r(p_\mu + k_\mu) \sin r(p_\mu + k'_\mu)}{(p_\mu + k_\mu)(p_\mu + k'_\mu)} \right) \frac{1}{p^2+1}.$$

Since $|\sin x/x| \leqq 2(1+|x|)^{-1}$ and $(p^2+1)^{-1} \leqq K_1 \times \pi(1+|p_\pi|)^{-1}$ with some constant K_1, one finds:

$$|\tilde{C}_g| \leqq K_2 J(k_0, k'_0)\, J(k_1, k'_1),$$

with some constant K_2, where

$$J(k,k') = \int dp \frac{1}{(1+|p+k|)(1+|p+k'|)(1+|p|)}.$$

An easy estimation after the direct integral shows (2.7b).

When $k = k'$, note $(1+|x|)^{-2} \leqq (1+x^2)^{-1}$. Then

$$|\tilde{C}_g| \leqq \int d^2p \frac{1}{1+p_0^2} \frac{1}{1+p_1^2} \frac{1}{(p-k)^2+1}.$$

Let $R_0 = \{p; |p-k| \leqq |k|/2\}$, and let $R_1 = R^2 \backslash R_0$. Without loss of generality, assume $k_0 \geqq k_1 \geqq 0$. If $p \in R_0$ then $[(p-k)^2+1]^{-1} \leqq [\frac{1}{4}k^2+1]^{-1}$, and if $p \in R_1$ then $(1+p_0)^{-2} \leqq \left[1 + \left(\frac{\sqrt{2}-1}{2}\right)^2 k^2\right]^{-1}$. Then (2.9c) is proved. If $g \in C_0^\infty$, then $|\tilde{g}(k)| \leqq \frac{C}{1+|k^2|}$ since $\tilde{g} \in \mathscr{S}$. Since

$$\tilde{C}_g(k,k') = \int d^2p \tilde{g}(k+p) \overline{\tilde{g}(k'+p)}\, \tilde{C}(p)$$

the bounds are obvious for $g \in C_0^\infty$. □

Proof of the Bound (3.24). Remember

$$S_a(x) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d^2\theta \frac{m + \frac{1}{a}\sum(1-\cos\theta_i) + \frac{i}{a}\sum \gamma_i \sin\theta_i}{[am+2-\sum\cos\theta_i]^2 + \sum \sin^2\theta_i} \exp[in_1\theta_1 + in_2\theta_2],$$

where $(x_0 = an_1,\ x_1 = an_2) \in aZ^2$ and assume $n_1, n_2 \geqq 1$. Since it suffices to consider the term which is proportional to $\frac{i}{a}\sin\theta_1$, let

$$S'_a = \int \frac{1}{B(\theta_2)} e^{in_2\theta_2} \int \frac{i}{a} \sin\theta_1 \frac{e^{in_1\theta_1}}{1-2A(\theta_2)\cos\theta_1} d\theta_1 ,$$

where

$$B(\theta) = 2 + (am+2)^2 - 2(am+2)\cos\theta > 2 ,$$

$$A(\theta) = \frac{1}{B(\theta)}(am+2-\cos\theta) = \tfrac{1}{2} - \zeta(\theta) < \tfrac{1}{2} ,$$

$$\zeta(\theta) = \frac{1}{2B(\theta)}[a^2m^2 + 2(1+am)(1-\cos\theta)] > 0 .$$

This is also written as

$$\int \frac{1}{B(\theta_1)} \frac{i}{a} \sin\theta_1 e^{in_1\theta_1} d\theta_1 \int \frac{e^{in_2\theta_2}}{1-2A(\theta_1)\cos\theta_2} d\theta_2 .$$

Contour integrals give

$$\begin{aligned}
\frac{1}{a}\left|\int \sin\theta_1 \frac{e^{in_1\theta_1}}{1-2A\cos\theta_1} d\theta_1\right| &= \frac{\pi}{a}\frac{1}{A}\left(\frac{2A}{1+2\sqrt{\zeta(1-\zeta)}}\right)^{n_1} \\
&\leqq \frac{2\pi}{a}\left[1 + \frac{1}{\sqrt{2}}\zeta^{1/2}\right]^{-n_1} \\
&\leqq \frac{2\pi}{a}\exp[-K_1 amn_1 - K_2|\theta_2|n_1] ,
\end{aligned}$$

$$\begin{aligned}
\left|\int \frac{e^{in_2\theta_2}}{1-2A\cos\theta_2} d\theta_2\right| &= \frac{2\pi}{\sqrt{1-4A^2}}\left(\frac{2A}{1+2\sqrt{\zeta(1-\zeta)}}\right)^{n_2} \\
&\leqq \frac{2\sqrt{2}\pi}{\sqrt{\zeta}}\left[1 + \frac{1}{\sqrt{2}}\zeta^{1/2}\right]^{-n_2} \\
&\leqq \frac{2\sqrt{2}\pi}{\sqrt{\zeta}}\exp[-K_1 amn_2 - K_2|\theta_1|n_2]
\end{aligned}$$

with positive constants K_1 and K_2, where we have used $2A < 1$, $\zeta < \frac{1}{2}$ and a bound

$$K'_1 am + K'_2|\theta| \leqq \zeta^{1/2}(\theta), \qquad K'_i > 0 ,$$

which holds whenever $|\theta| \leqq \pi$, $am \leqq 1$. Since $|\sin\theta/\sqrt{\zeta}| \leqq \text{const}$, $B > 2$, one finally has

$$\begin{aligned}
|S'_a| &\leqq \min\left\{\frac{K}{an_i + a} e^{-K_1 amn_i}\right\}_{i=1,2} \\
&\leqq K\frac{2}{|x_0| + |x_1| + 2a}\exp[-K_1 \tfrac{1}{2}(|x_0| + |x_1|)m] . \quad \square
\end{aligned}$$

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Note added in proof. Another method to obtain approximative equations below Eq. (3.25) is to operate $\gamma_\beta \ldots \gamma_\beta^{-1}$ ($\beta=0$, 1 or 5) to the inside of the trace in Eq. (3.21). For example $\gamma_5\gamma_\beta\gamma_5^{-1}=-\gamma_\beta$ ($\beta=0$ or 1). Then $\gamma_5 S_N(x)\gamma_5^{-1}=S_N(-x)$ (for periodic and anti-periodic boundary conditions), $\gamma_5\gamma_\beta(\pm)\gamma_5^{-1}=-\gamma_\beta(\mp)$ (for periodic ones) and $\gamma_5\gamma_\beta(\pm)\gamma_5^{-1}=-\gamma_\beta(\mp)^*$ (for anti-periodic ones). Therefore (remarking that $\Pi_{\mu\nu}$ is real) one finds:

$$\Pi_{\mu\nu}^N(x)=\Pi_{\mu\nu}^N(-x+e_\mu-e_\nu),$$

for the both boundary conditions. The other relations are obtained in the same way. Especially it is easy to find: $\Pi_{\mu\nu}^N(x_0,x_1)=\Pi_{\mu\nu}^N(|x_0|,|x_1|)$ if $\mu=\nu$.