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First-Order Phase Transitions in Large Entropy Lattice Models

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Abstract. We show the existence of a first-order phase transition in the ν-dimensional Potts model for $\nu \geqq 2$, when the number of states of a single spin is big enough. Low-temperature pure phases are proved to survive up to the critical temperature. Also the existence of a first-order transition in the ν-dimensional Potts gauge model, $\nu \geqq 3$, is obtained if the underlying gauge group is finite but large.

1. Introduction

In [1], Potts introduced a generalization of the Ising model by enlarging the number of values taken by spins residing on lattice sites from 2 to an arbitrary q, and considering a nearest neighbour interaction that is, up to a factor, -1 when the neighbouring spins are alike and 0 when they differ. The problem of the description of the phase structure of the Potts model was later studied by many authors using different methods. We refer the reader to the recent review by Wu [2] for a description of present knowledge about the Potts model. Existent rigorous results are confined mostly to the Potts model on a two-dimensional lattice and are based on its equivalence to an ice-rule vertex model. In particular it seems to have been proven [3, 4] that the two-dimensional Potts model exhibits a first-order transition in temperature for $q > 4$. Essentially nothing was proved for three and higher dimensional lattices though it is believed that the phase transition is first order for $q \geqq 4$ (or even 3) in three dimensions and $q \geqq 3$ in higher dimensions. The conjecture that for large q, the Potts model undergoes a first-order transition is also supported by Pearce and Griffiths [5] who prove that for a lattice of any dimension the mean field approximation of a free energy of the model is exact in the limit $q \to \infty$. At the same time, the suitably rescaled limiting mean field free energy can be easily computed and it turns out that its behaviour is

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consistent with a first-order transition. One aim of the present paper is to understand and prove rigorously the existence of that first-order transition.

Another model we will be interested in is the Potts lattice gauge model. It was introduced by Kogut [6] as a generalization of the Ising lattice gauge model [7, 8]. The gauge version of the Potts model seems to be even more susceptible to developing a first-order transition than its conventional counterpart.

The phase transition is indicated by Monte Carlo simulations that show clearly a first-order transition even for the Ising gauge model on a four dimensional lattice [9], by the large q expansions investigated in both Hamiltonian (continuous time) [6] as well as Lagrangian [10] forms of the model, and by the fact that the mean field approximation (in its fixed gauge version) suggests a first-order transition while being exact in the limit $q \to \infty$ [11].

The present paper is the result of our attempts to understand the nature of the conjectured first-order transitions. In this, a recent article of Dobrushin and Shlosman [12] was very instructive for us. We found that for large q, in both the Potts and Potts gauge models, the existence of a first-order transition can be physically understood and rigorously proved by approximately the same ideas as in [12] where the phase transition was studied in terms of energy-entropy fighting. The critical temperature at which the transition occurs coincides with the point at which energy and entropy factors become equal.

It is easy to see by the usual Peierls arguments that for low temperatures there are q different translation invariant states in Potts model. Our technique implies that those q pure states survive up to critical temperature T_c. At this temperature T_c yet another additional state arises (the state of complete chaos), so there are at least $(q+1)$ different states at T_c. For $T > T_c$ it seems that only one state (the chaotic one) survives. The fact that our estimates can be used to prove that the q states persist up to the critical temperature was originally pointed out to us by S. Pirogov.

Our main tool is the method based on the use of reflection-positivity (RP) and related "multiple reflections" or "chessboard" estimates. It was introduced in [13] and has been systematically studied and applied to phase transition problems in [14–16]. One can also consult [12] where the setting is exactly what we need.

The paper is organized as follows. In Sect. 2 we formulate our main results and indicate the strategy of the proofs. Section 3 contains a simple entropy model which is not covered by [12] (it lacks an energy barrier). This model seems to be of independent interest. After being able to prove first order transition in it, we became convinced that Potts models can be treated in a similar way. Our theorems about the Potts model are proved in Sect. 4. In particular it includes our definition of contours in the Potts model that seems to be new. Namely, we consider contours separating regions with different elementary contributions to energy (there are only two possible values of interaction energy of two neighbouring spins in the Potts model) since we are looking for a phase transition that exhibits itself in a jump of a mean internal energy. This is to be compared with the Peierls argument for the Ising model, where to prove a jump in magnetization, one introduces contours between regions with different elementary magnetizations ($+1$ and -1). The remarkable fact is that our contours, unlike the customary Peierls ones, have small probabilities for all temperatures. Section 4 contains all the central ideas of this work. Finally Sect. 5 is devoted to the proofs for the gauge Potts model. The general technique is approximately the same. A specific feature of our technique in a gauge model case is its inapplicability in two dimensions, as should be true for any method that deals with phase transitions in gauge theories.

2. Statement of the Results and General Strategy of the Proofs

To introduce the q-state Potts model we attach to every lattice site i of a v-dimensional lattice $\mathbb{Z}^v$ a spin variable $\sigma(i)$ that takes values in the set $\{1, \ldots, q\}$ and define the (formal) Hamiltonian by

$$H(\sigma) = -\sum_{(i,j)} \delta_{\sigma(i),\sigma(j)} . \tag{2.1}$$

Here the sum is over pairs of nearest neighbour sites (i,j) and $\delta_{\alpha,\beta}$ is a Kronecker delta. We refer to a pair (i,j) of nearest neighbour sites as a bond $\ell = (i,j)$ and denote by $\mathbf{L}$ the set of all bonds on the underlying lattice $\mathbb{Z}^v$. On the set of all configurations $\sigma = \{\sigma(i), i \in \mathbb{Z}^v\}$ we introduce for each bond $\ell = (i,j)$ the indicator $P_\ell^=(\sigma)$ of the event $\{\sigma \mid \sigma(i) = \sigma(j)\}$ and the indicator $P_\ell^{\neq}(\sigma)$ of the complementary event $P_\ell^{\neq} = 1 - P_\ell^=$.

Theorem 1. *A v-dimensional q-state Potts model undergoes a first-order phase transition in temperature whenever $v \geqq 2$ and q is large enough. Namely, for each $v \geqq 2$ there is a number $q(v)$ such that whenever $q \geqq q(v)$ there exist an inverse temperature $\beta_c = \beta_c(q, v)$ and two different translation invariant Gibbs states $\langle \ \rangle_{\beta_c}^=$; $\langle \ \rangle_{\beta_c}^{\neq}$ of the q-state Potts model at the inverse temperature β_c such that $\langle P_\ell^= \rangle_{\beta_c}^= > \frac{1}{2}$ and $\langle P_\ell^{\neq} \rangle_{\beta_c}^{\neq} > \frac{1}{2}$ for every $\ell \in \mathbf{L}$.*

Even more is true. For q large enough the state $\langle \ \rangle_{\beta_c}^=$ from the above Theorem can be decomposed into at least q pure components. We shall distinguish them using the indicators $P_i^\alpha(\sigma)$ of the events $\{\sigma \mid \sigma(i) = \alpha\}$ which are defined for each $i \in \mathbb{Z}^v$ and $\alpha \in \{1, \ldots, q\}$. It was Professor Pirogov who called our attention to the fact that the technique we used to prove Theorem 1 also proves the following.

Theorem 2. *There is $q_1(v) \geqq q(v)$ such that, for each $v \geqq 2$, $q \geqq q_1(v)$, and $\beta \geqq \beta_c(v, q)$, there is a collection of q translation invariant Gibbs states $\langle \ \rangle_\beta^\alpha$, $\alpha = 1, \ldots, q$, of the q-state Potts model at the inverse temperature β, such that for each $i \in \mathbb{Z}^v$, $\ell \in \mathbf{L}$, and $\alpha \in \{1, \ldots, q\}$ we have $\langle P_i^\alpha \rangle_\beta^\alpha \geqq \frac{3}{2} q^{-1}$, $\langle P_i^\gamma \rangle_\beta^\alpha < q^{-1}$ whenever $\gamma \neq \alpha$, $\langle P_\ell^= \rangle_\beta^\alpha > \frac{1}{2}$, and $\langle P_i^\alpha \rangle_\beta^\alpha \to 1$ as $\beta \to \infty$.*

We should mention that the bounds $q(v)$ and $q_1(v)$ we actually get are ridiculously large and we did not make any attempt to optimize them. We don't think they could be improved to the point that they say anything about the controversy concerning the nature of the transition for $v = 3$, $q = 3$.

Now we turn our attention to the gauge Potts model. We generalize slightly the setting of [6] and [10] by considering as the gauge group $\mathbf{G}$ any finite (not necessarily abelian) group with q elements. By a configuration σ of the model we mean a map $\sigma : \bar{\mathbf{L}} \to \mathbf{G}$ from the set of all oriented bonds (ordered pairs of nearest-neighbour lattice sites) of $\mathbb{Z}^v$ into $\mathbf{G}$ such that $\sigma(i,j) = \sigma(j,i)^{-1}$ for each bond $(i,j) \in \bar{\mathbf{L}}$. With each plaquette $\square$ we associate a plaquette variable $\sigma_\square$ defined by $\sigma_\square$

$=\sigma_{(i_1,i_2)}\sigma_{(i_2,i_3)}\sigma_{(i_3,i_4)}\sigma_{(i_4,i_1)}$, where i_1, i_2, i_3, i_4 are the corners of the plaquette $\square$ ordered along a path that goes around $\square$. Actually $\sigma_\square$ is not well defined – it depends on the orientation and the choice of the starting point of the path around $\square$. But what does not depend on this is whether $\sigma_\square = e$ (e being the unity element of $\mathbf{G}$) (respectively $\sigma_\square \neq e$) – i.e., whether the plaquette $\square$ is nonfrustrated (respectively frustrated) in the configuration σ. Thus the indicator $P_\square^=(\sigma)$ of the set $\{\sigma | \sigma_\square = e\}$ is well defined for each plaquette $\square$. $P_\square^{\neq}$ is defined by $P_\square^{\neq} = 1 - P_\square^=$. The Hamiltonian (or one should rather say euclidean action as we are in the realm of field theory here) of the gauge Potts model is formally

$$H(\sigma) = -\sum_\square \delta_{P_\square^=(\sigma),1}\,. \tag{2.2}$$

Theorem 3. *A ν-dimensional Potts gauge model with a q-element gauge group* $\mathbf{G}$ *undergoes a first-order phase transition in temperature whenever $\nu \geqq 3$ and q is large enough. Namely, for each $\nu \geqq 3$ there is $\bar{q}(\nu) < \infty$ such that whenever $q \geqq \bar{q}(\nu)$ there is an inverse temperature $\beta_c = \beta_c(q,\nu)$ for which there are translation invariant Gibbs states $\langle\ \rangle_{\beta_c}^=$ and $\langle\ \rangle_{\beta_c}^{\neq}$ corresponding to the Hamiltonian* (2.2) *such that $\langle P_\square^= \rangle_{\beta_c}^= > \frac{1}{2}$ and $\langle P_\square^{\neq} \rangle_{\beta_c}^{\neq} > \frac{1}{2}$ for each plaquette $\square$.*

The idea of the proofs is based on a general scheme that was used in [12] to deal with first-order transitions in temperature. Let us recall it briefly. Let H be a Hamiltonian and for each β let $\langle\ \rangle_\beta$ be a Gibbs state that results by thermodynamic limit from finite volume Gibbs states with Hamiltonian H, inverse temperature β and periodic boundary conditions. Let P^1, P^2 be the indicators of two disjoint local events ($P^r(\sigma) = 1$ or 0, $r = 1, 2$, $P^1(\sigma)P^2(\sigma) = 0$). The validity of the following conditions (to be referred to as Hypotheses 1.–3. in the sequel) is sufficient for the occurrence of a first-order transition.

Theorem 4. [12] *Let $A \in [\frac{1}{2}, 1]$, $B \in [0,1]$ be such that $B < \left(\frac{1}{2} - \sqrt{\frac{1-A}{2}}\right)^2$, and let $[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$, $0 < \beta_{\mathrm{I}} < \beta_{\mathrm{II}} < \infty$, be an interval of inverse temperatures such that*

1. *$\langle P^1 + P^2 \rangle_\beta \geqq A$ for each $\beta \in [\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$.*
2. *$\langle P_i^1 P_j^2 \rangle_\beta \leqq B$ for each $\beta \in [\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$, $i, j \in \mathbb{Z}^\nu$.*

Here $P_i(\sigma) = P(T_{-i}\sigma)$, where T_i denotes a lattice translation (so $P_0^r = P^r$, $r = 1, 2$).

3. *$\langle P^1 \rangle_{\beta_{\mathrm{I}}} > \frac{1}{2}$, $\langle P^2 \rangle_{\beta_{\mathrm{I}}} > \frac{1}{2}$.*

Then there exist $\beta_c \in [\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$ such that there are at least two different Gibbs states $\langle\ \rangle_{\beta_c}^{\mathrm{I}}$, $\langle\ \rangle_{\beta_c}^{\mathrm{II}}$ at the temperature β_c and corresponding to the Hamiltonian H; moreover one may choose them in such a way that

$$\langle P^1 \rangle_{\beta_c}^{\mathrm{I}} > \tfrac{1}{2} \quad \textit{and} \quad \langle P^2 \rangle_{\beta_c}^{\mathrm{II}} > \tfrac{1}{2}\,. \tag{2.3}$$

Proof. The proof of this statement is essentially contained in Propositions 3.1–3.3 of [12]. For the convenience of the reader we present here an outline of it.

First of all one defines (non-local) observables

$$\Pi^1 = \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} P_i^1\,,$$

$$\Pi^2 = \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} P_i^2\,;$$

these limits exist in $L^1(\langle\ \rangle_\beta)$ for each β according to the Birkhoff ergodic theorem. Using Hypotheses 1. and 2. one is able to show the existence of two positive numbers a and b such that

(i) The probability $\mathrm{Prob}\{\sigma|\Pi^1(\sigma)+\Pi^2(\sigma)\geqq a, \Pi^1(\sigma)\Pi^2(\sigma)\leqq b\}$ computed in the state $\langle\ \rangle_\beta$ is larger than some positive ε for each $\beta\in[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$.

(ii) The following inclusion holds

$$\{0\leqq x, y\leqq 1, x+y\geqq a, xy\leqq b, x\geqq y\}\subset\{x, y: x>\tfrac{1}{2}+\delta\} \quad \text{for some small } \delta.$$

Indeed, one has

$$\mathrm{Prob}\{\Pi^1+\Pi^2\geqq a\}\geqq\frac{A-a}{1-a}$$

and

$$\mathrm{Prob}\{\Pi^1\Pi^2\leqq b\}\quad\geqq\frac{b-B}{b}$$

for each $a\in[0, A]$, $b\in[B, 1]$. The conditions (i), (ii) hold once

$$\frac{A-a}{1-a}+\frac{b-B}{b}\geqq 1+\varepsilon \tag{2.4}$$

and

$$b<\frac{a^2}{4}, \quad \frac{a}{2}+\sqrt{\frac{a^2}{4}-b}>\frac{1}{2}. \tag{2.5}$$

To fulfill (2.4) and (2.5) we take e.g.

$$a=1-\sqrt{\frac{1-A}{2}} \quad \text{and} \quad b<\frac{a}{2}-\frac{1}{4}.$$

$\left[\text{To verify (2.4) we used the condition } B<\left(\frac{1}{2}-\sqrt{\frac{1-A}{2}}\right)^2.\right]$

Now, from (i) and (ii) we infer that for each $\beta\in[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$ at least one statement

$$\mathrm{Prob}\left\{\sigma|\Pi^1(\sigma)\geqq\frac{1}{2}+\delta\right\}\geqq\frac{\varepsilon}{2}$$

$$\mathrm{Prob}\left\{\sigma|\Pi^2(\sigma)\geqq\frac{1}{2}+\delta\right\}\geqq\frac{\varepsilon}{2}$$

holds true. Let $A^r=\{\sigma|\Pi^r(\sigma)\geqq\frac{1}{2}+\delta\}$, $r=1,2$, and define a mapping $r:[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]\to\{1,2\}$ in such a way that $\mathrm{Prob}(A^{r(\beta)})\geqq\frac{\varepsilon}{2}$ whenever $\beta\in[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$. For each $\beta\in[\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$ we define a new state $\langle\ \rangle_\beta^{\sim}$ by conditioning $\langle\ \rangle_\beta$ with respect to the event $A^{r(\beta)}$:

$$\langle I(B)\rangle_\beta^{\sim}=\frac{\langle I(B)I(A^{r(\beta)})\rangle_\beta}{\langle I(A^{r(\beta)})\rangle_\beta}$$

whenever B is a cylinder set and with $I(\ )$ denoting the indicator function. The state $\langle\ \rangle_\beta^{\sim}$ is again a translation invariant Gibbs state with Hamiltonian H and at inverse temperature β[17]. From its definition it follows that $\langle P^{r(\beta)}\rangle_\beta^\sim > \frac{1}{2}+\delta$ for each $\beta\in[\beta_{\mathrm{I}},\beta_{\mathrm{II}}]$. Now let $\mathbf{M}_1 = r^{-1}(\{1\}) = \{\beta\in[\beta_{\mathrm{I}},\beta_{\mathrm{II}}],\ r(\beta)=1\}$, $\mathbf{M}_2=[\beta_{\mathrm{I}},\beta_{\mathrm{II}}]\backslash\mathbf{M}_1$. Both $\mathbf{M}_1$ and $\mathbf{M}_2$ are nonempty; hence there exist some β_c belonging to the intersection of the closures $\overline{\mathbf{M}_1}\cap\overline{\mathbf{M}_2}$. Finally one shows the existence of states $\langle\ \rangle^{\mathrm{I}}_{\beta_c}$, $\langle\ \rangle^{\mathrm{II}}_{\beta_c}$ with desired properties for the above β_c by compactness arguments.

In Sects. 4 and 5 we shall actually consider complementary events $P^1, P^2 : P^1(\sigma) + P^2(\sigma)=1$. Then Hypothesis 1 of course holds with $A=1$ and in Hypothesis 2 we can choose any $B<\frac{1}{4}$. When referring to Hypothesis 2 in Sects. 4 and 5 we shall have in mind such a choice of B. Also we shall meet situations when $\varliminf_{\beta\to 0}\langle P^1\rangle_\beta > \frac{1}{2}$ and $\varliminf_{\beta\to\infty}\langle P^2\rangle_\beta > \frac{1}{2}$. In such cases we choose $\beta_{\mathrm{I}}=0$, $\beta_{\mathrm{II}}=\infty$, having actually in mind any β_{I} small enough and β_{II} large enough.

Moreover, when proving Theorem 2 we shall use the following refinement of the above statement for complementary P^1 and P^2:

Theorem 4′. *Let $P^1+P^2=1$, and let $B\in[0,1]$, $C\in(\frac{1}{2},1)$, $B<C-C^2$ such that*

2′. $\langle P^1_i P^2_j\rangle \leqq B$ *for each* $\beta\in(0,\infty)$.

3′. $\varliminf_{\beta\to 0}\langle P^1\rangle_\beta > C$, $\varliminf_{\beta\to\infty}\langle P^2\rangle_\beta > C$.

Then there exist $\beta_c\in(0,\infty)$ and two different Gibbs states $\langle\ \rangle^{\mathrm{I}}_{\beta_c}$, $\langle\ \rangle^{\mathrm{II}}_{\beta_c}$ such that $\langle P^1\rangle^{\mathrm{I}}_{\beta_c} > C$ and $\langle P^2\rangle^{\mathrm{II}}_{\beta_c} > C$.

Proof. The proof goes along the lines of the proof above with $a=1$, $b>B$ and (ii) replaced by $\{0\leqq x,y\leqq 1,\ x+y=1,\ xy\leqq b,\ x\geqq y\}\subset\{x,y : x\geqq C+\delta\}$.

When verifying Hypotheses 1–3 for our models we shall use the fact that they fulfill a property known as reflection positivity (RP). This allows us to use a powerful technical tool – an estimate that is referred to as a "multiple reflection" or "chessboard" estimate. The reader should consult [12–16] for a detailed account. We shall mention only the simplest variant of it that may be stated as the following correlation inequality:

Let $\Lambda\subset\mathbb{Z}^\nu$ be a finite box and $\langle\ \rangle$ be a Gibbs state in Λ with periodic boundary conditions and an interaction that is RP with respect to reflections in all planes perpendicular to coordinate axes running mid-way between neighbouring points of Λ. Let $F_i(\sigma(i))$, $i\in\Lambda$ be a collection of one point observables. Then

$$\left|\left\langle\prod_{i\in\Lambda}F_i(\sigma(i))\right\rangle\right| \leqq \prod_{i\in\Lambda}\left\langle\prod_{j\in\Lambda}F_i(\sigma(j))\right\rangle^{\frac{1}{|\Lambda|}}.$$

Actually there are situations where one disseminates a local variable F_i over all Λ using reflections with respect to other planes. Then the power $\frac{1}{|\Lambda|}$ may be replaced by $\frac{k}{|\Lambda|}$ with suitable constant k. We shall not go into further details about it here.

3. "As a Warmup, ..." [16]

Our aim in this section is to construct and investigate a model that would keep some features of the double well model studied in [12] while retaining qualitatively some features of the Potts model. First of all, let us recall the results of [12] concerning the Hamiltonian

$$H(\sigma)=\sum_i U(\sigma_i)+\kappa \sum_{(i,j)} (\sigma_i-\sigma_j)^2, \tag{3.1}$$

where $\sigma_i \in \mathbb{R}^n$, $\kappa>0$, the first sum is over lattice sites and the second sum is over pairs of nearest neighbours; U is a function of a shape shown in Fig. 1 (for $n=1$).

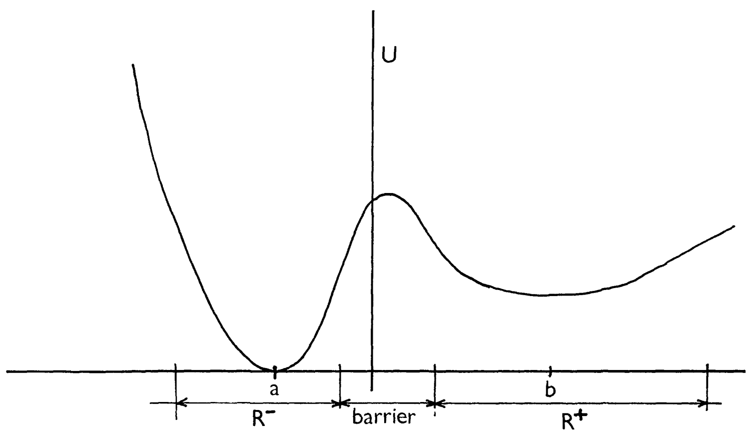

Fig. 1

The main features of the potential U may summarized as follows:

I. At the point a the function U has a sharp global minimum (sharp means here that "mass" or second derivative at a is very large).

II. At the point b the potential U has a mild local minimum (mild means that the mass of it is much smaller).

III. Those two minima are separated by a barrier which is high and wide enough.

IV. The minima are placed far enough from each other (the distance is of the order κ^{-1}).

Let $P_0^+=P_0^+(\sigma)$ be the indicator of the set $\{\sigma|\sigma_0\in\mathbf{R}^+\}$, analogously P_0^-. The main Theorem of [12] states that under conditions of the type I.–IV. there is β_c such that there are at least two different Gibbs states $\langle\ \rangle_{\beta_c}^+$ and $\langle\ \rangle_{\beta_c}^-$ with Hamiltonian (3.1) and at inverse temperature β_c; moreover $\langle P_0^+\rangle_{\beta_c}^+>\frac{1}{2}$, $\langle P_0^-\rangle_{\beta_c}^->\frac{1}{2}$.

What conclusions can be drawn from it for the Potts model? If we try to find some similarity with the Potts model, we cannot proceed too literally since there is no chemical potential in the Potts model and one has to simulate it by blocks of spins. Then one sees something like I. and II. also in the Potts model. Of course, a local minimum [modelled by configurations of zero energy (2.1)] coincides now with the global maximum. But this, after all, is not a contradiction. The more serious difficulty concerns the barrier; one cannot find anything like that in the Potts model. So the question arises whether the theorem above can be proved also in the situation without a barrier. We now address this problem.

Let us consider a model described by the Hamiltonian

$$H(\sigma)=\sum_i U(\sigma_i)+\sum_{(i,j)} V(\sigma_i,\sigma_j), \tag{3.2}$$

where $i\in\mathbb{Z}^\nu$, $\sigma_i\in\mathbb{R}^1$,

$$U(x)=\begin{cases}+\infty & x<-1,\\ 0 & -1\leqq x\leqq 0,\\ 1 & 0<x\leqq N,\\ +\infty & N<x,\end{cases}$$

with $N>0$, and

$$V(x,y)=\begin{cases}(x-y)^2 & xy\leqq 0,\\ 0 & xy>0.\end{cases}$$

Let $P_0^-=P_0^-(\sigma)$ be the indicator of $\{\sigma|\sigma_0\leqq 0\}$, and let $P_0^+=P_0^+(\sigma)$ be the indicator of $\{\sigma|\sigma_0\geqq 0\}$.

Theorem 5. *There is $N(\nu)>0$ such that for each $N\geqq N(\nu)$ the model described by the Hamiltonian* (3.2) *undergoes a first-order phase transition: there exists $\beta_c\geqq 1$ such that there are at least two different Gibbs states $\langle\ \rangle_{\beta_c}^+$, $\langle\ \rangle_{\beta_c}^-$, corresponding to the Hamiltonian* (3.2), *and at inverse temperature β_c. These states may be chosen such that $\langle P_0^-\rangle_{\beta_c}^->\frac{1}{2}$, $\langle P_0^+\rangle_{\beta_c}^+>\frac{1}{2}$.*

Proof. We consider only the case $\nu=2$, when a Gibbs state $\langle\ \rangle_{\beta,\Lambda}$ corresponding to the Hamiltonian (3.2) in a rectangle rotated by 45° is RP with respect to oblique planes $\{i\in\mathbb{Z}^2,\ i_1\pm i_2=n\}$, $n\in\mathbb{Z}^1$. As for the general case one has to use planes perpendicular to the coordinate axes as we will do in Sect. 4. Denote by $P_0^0=P_0^0(\sigma)$ the indicator of $\{\sigma|\sigma_0\in[0,1]\}$, $\tilde{P}_0^+=P_0^+-P_0^0$. We are going to verify Hypotheses 1–3 of Theorem 4 with $P^1=\tilde{P}_0^+$, $P^2=P_0^-$, $\beta_{\mathrm{I}}=1$, $\beta_{\mathrm{II}}=+\infty$, $A=31/32$, $B=1/16$, provided N is large enough:

1. We shall prove that $\langle P_0^0\rangle_\beta\to 0$ uniformly in β as $N\to\infty$. By a chessboard estimate

$$\langle P_0^0\rangle_{\beta,\Lambda}\leqq\langle P_\Lambda^0\rangle_{\beta,\Lambda}^{\frac{2}{|\Lambda|}},$$

with P_Λ^0 an indicator of the event $\{\sigma|\sigma_i\in[0,1]$ *whenever i_1+i_2 is even*$\}$. Now

$$\langle P_\Lambda^0\rangle_{\beta,\Lambda}\leqq Z_\Lambda^{-1}(\beta)e^{-\beta\frac{|\Lambda|}{2}}\left(Ne^{-\beta}+\int_0^1 e^{-4\beta x^2}dx\right)^{\frac{|\Lambda|}{2}}$$

$$\leqq Z_\Lambda^{-1}(\beta)\left[e^{-\beta}\left(Ne^{-\beta}+\frac{\sqrt{\pi}}{4\sqrt{\beta}}\right)\right]^{\frac{|\Lambda|}{2}},$$

where we integrated first over "free spins" σ_i, $i_1+i_2=\text{odd}$, keeping the rest of them fixed, and then over the rest of them. Estimating the partition function $Z_\Lambda(\beta)$ from below by

$$Z_\Lambda(\beta)\geqq\int_{\{\sigma|\sigma_i\leqq 0,\,i\in\Lambda\}}e^{-\beta H(\sigma)}d\sigma+\int_{\{\sigma|\sigma_i\geqq 0,\,i\in\Lambda\}}e^{-\beta H(\sigma)}d\sigma,$$

one has

$$Z_\Lambda(\beta) \geqq 1 + (Ne^{-\beta})^{|\Lambda|},$$

and thus

$$\langle P_0^0 \rangle_{\beta,\Lambda} \leqq e^{-\beta}\left(Ne^{-\beta} + \frac{\sqrt{\pi}}{4\sqrt{\beta}}\right)[1 + (Ne^{-\beta})^{|\Lambda|}]^{-\frac{2}{|\Lambda|}}. \tag{3.3}$$

To conclude, we infer that if β is such that $Ne^{-\beta} \geqq \frac{\sqrt{\pi}}{4\sqrt{\beta}}$, then the right hand side of (3.3) $\leqq e^{-\beta}(2Ne^{-\beta})(Ne^{-\beta})^{-2} = \frac{2}{N}$, and on the other side if $Ne^{-\beta} \leqq \frac{\sqrt{\pi}}{4\sqrt{\beta}}$, then the right hand side of (3.3) $\leqq \frac{\sqrt{\pi}}{4N\sqrt{\beta}}\left(\frac{\sqrt{\pi}}{2\sqrt{\beta}}\right) = \frac{\pi}{8N\beta} \leqq \frac{\pi}{8N}$, since $\beta \in [1, \infty)$.

2. To verify Hypothesis 2 we introduce contours of a configuration in the usual manner as connected components Γ of the set of bonds (i,j) such that $\sigma_i > 1$ and $\sigma_j < 0$ or $\sigma_i < 0$ and $\sigma_j > 1$. (Compare e.g. with [12], Sect. 4 where the setting is exactly the same as we need here.) By a standard chessboard estimate (see [14–16] or [12], Sect. 4 for details) and Hypothesis 1, the validity of Hypothesis 2 is ensured once we show that $\langle P_\Lambda^{+-} \rangle_{\beta,\Lambda}^{\frac{1}{2|\Lambda|}}$ is small enough, where P_Λ^{+-} is the indicator of the "universal contour" $\{\sigma | \sigma_i \leqq 0$ *for* $i_1 + i_2$ *even and* $\sigma_i > 1$ *for* $i_1 + i_2$ *odd*$\}$. Actually

$$\langle P_\Lambda^{+-} \rangle_{\beta,\Lambda} \leqq \frac{e^{-2|\Lambda|\beta}(Ne^{-\beta})^{\frac{|\Lambda|}{2}}}{(Ne^{-\beta})^{|\Lambda|}}.$$

Since $V(x,y) \geqq 1$ if $x \leqq 0$, $y \geqq 1$. Hence

$$\langle P_\Lambda^{+-} \rangle_{\beta,\Lambda}^{\frac{1}{2|\Lambda|}} \leqq \frac{e^{-\frac{3}{4}\beta}}{\sqrt{N}}$$

which is uniformly small for $\beta \in [\beta_{\mathrm{I}}, \beta_{\mathrm{II}}]$ as long as N is big enough.

3. By a chessboard estimate

$$\langle P_0^+ \rangle_{\beta,\Lambda} \leqq \langle P_\Lambda^+ \rangle_{\beta,\Lambda}^{\frac{2}{|\Lambda|}},$$

where P_Λ^+ is the indicator of $\{\sigma | \sigma_i \geqq 0$ *for* $i_1 + i_2$ *even*$\}$. At the same time

$$\langle P_\Lambda^+ \rangle_{\beta,\Lambda} \leqq (1 + Ne^{-\beta})^{\frac{|\Lambda|}{2}} (Ne^{-\beta})^{\frac{|\Lambda|}{2}},$$

since $V(x,y) \geqq 0$ and $Z_\Lambda(\beta) \geqq 1$. Hence $\langle P_0^+ \rangle_{\beta,\Lambda} \leqq (1 + Ne^{-\beta})(Ne^{-\beta})$, which goes to zero as $\beta \to \beta_{\mathrm{II}}$. Again by chessboard $\langle P_0^- \rangle_{\beta,\Lambda} \leqq \langle P_\Lambda^- \rangle_{\beta,\Lambda}^{\frac{2}{|\Lambda|}}$, P_Λ^- being the indicator of $\{\sigma | \sigma_i \leqq 0$ *for* $i_1 + i_2$ *even*$\}$.

$$\langle P_\Lambda^- \rangle_{\beta,\Lambda} \leqq \frac{(1 + Ne^{-\beta})^{\frac{|\Lambda|}{2}}}{(Ne^{-\beta})^{|\Lambda|}},$$

hence

$$\langle P_0^- \rangle_{\beta_1, \Lambda} \leqq \frac{(1+Ne^{-1})}{(N^2 e^{-2})},$$

which is as small as one needs once N is sufficiently large. This completes the proof of Hypothesis 3 and hence of the Theorem.

Remark. Because all estimates were rather rough, there is enough room for different generalizations of Theorem 5. For example, one may perturb slightly (in C^0 sense) the chemical potential, or modify the interaction by introducing e.g.

$$\tilde{V}(x,y) = \begin{cases} (x-y)^2 & xy \geqq 0 \\ \lambda(x-y)^2 & xy \leqq 0 \end{cases}$$

with λ small (of an order N^{-2}). Unfortunately the result seems to fail in the physically interesting case $\lambda = 1$ (though we have no proof). This makes the model somehow artificial.

4. Proof of the Existence of a Phase Transition in the Potts Model

We shall handle in detail only the case $\nu = 2$ and then indicate how to extend the argument to a general $\nu \geqq 2$. Let $\mathbf{L}$ be the set of bonds of the lattice $\mathbb{Z}^2$, and for each $\ell \in \mathbf{L}$ denote $\ell_1, \ell_2 \in \mathbb{Z}^2$ the two endpoints of the bond ℓ. As in Sect. 2 we define $P_\ell^=$ to be the indicator of $\{\sigma | \sigma(\ell_1) = \sigma(\ell_2)\}$ and $P_\ell^{\neq} = 1 - P_\ell^=$. If $P_\ell^{\neq}(\sigma) = 1$ for some σ, we shall call the bond ℓ *excited* in the configuration σ. It will be convenient to picture configurations using a graphical notation: a wavy line for an excited bond and a solid line—for a nonexcited bond. Now we apply the scheme of Theorem 4 with $P^1 = P_\ell^=$ and $P^2 = P_\ell^{\neq}$ where ℓ is any fixed bond. Note that one slight change is necessary in the proof of the statement that Hypotheses 1–3 imply the existence of a phase transition. Namely, instead of defining Π^r by

$$\Pi^r = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} P_i^r, \qquad r = 1, 2,$$

one defines

$$\Pi^r = \lim_{\Lambda \to \infty} \frac{1}{2|\Lambda|} \sum_{\ell \in \mathbf{L}_\Lambda} P_\ell^r,$$

with $\mathbf{L}_\Lambda$ denoting the set of bonds in the volume Λ. Since $\langle P^1 + P^2 \rangle = 1$, we proceed immediately to the Hypothesis 2.

We shall introduce certain contours and transform the necessary estimate into an evaluation of the probability of contours. This will be further transformed by chessboard estimates into the evaluation of the probability of a "universal contour" that will be finally done "by hand." When referring to chessboard estimates we shall use the obvious fact that the Potts Hamiltonian (2.1) is RP under reflections with respect to the planes $\{i \in \mathbb{Z}^\nu;\ i_\alpha = n\}$, $\alpha = 1, \ldots, \nu$, $n \in \mathbb{Z}^1$. The Potts Hamiltonian is for $\nu = 2$ also RP under reflections in oblique lines $\{i \in \mathbb{Z}^2,\ i_1 \pm i_2 = n\}$, $n \in \mathbb{Z}^1$ for a suitable choice of periodic boundary conditions (namely, on a rectangle rotated by 45°). This second RP leads to some improvements of our estimates, but because it cannot be generalized to higher v, we shall not present it here (contrary to Sect. 3 where we were not very interested in generalizations to $v>2$).

We shall define contours in two steps. First we introduce precontours – border lines between excited and nonexcited regions. Given a precontour, the pattern of excitation of the bonds in a close neighbourhood of it may be further specified – defining thus a contour.

Let thus σ be a configuration. We define first an *island* Q (of a pure phase) of the configuration σ as a set of vertices of $\mathbb{Z}^2$ such that there is $\alpha\in\{1,\dots,q\}$ such that Q is a connected component of $\sigma^{-1}(\{\alpha\})=\{i\in\mathbb{Z}^2\,|\,\sigma(i)=\alpha\}$ containing more than one lattice site. Here connected means that $i,j\in Q$ may be joined by a path in Q. Note that an island is maximal in the sense that if $j\in\mathbb{Z}^2$, $j\notin Q$, and $\mathrm{dist}(j,Q)=1$, then $\sigma(j)\neq\sigma(i)$ whenever $i\in Q$. A bond $\ell=(i,j)$ such that both $i,j\in Q$ is called a bond of the island Q and the set of them is denoted $\mathbf{L}(Q)$. A *boundary* of an island Q of the configuration σ is the set $\mathbf{B}(Q)$ of all bonds ℓ such that exactly one vertex of each ℓ belongs to Q. In particular, every bond $\ell\in\mathbf{B}(Q)$ is excited in the configuration σ. To introduce precontours we proceed in precisely the same way as if introducing contours for the Ising model with the Ising configuration, say, $+1$ on all vertices of Q and -1 outside. Namely, we consider a dual lattice $(\mathbb{Z}^2)^*$ and for each $\ell\in\mathbf{B}(Q)$ we draw a dual bond [a dual $(v-1)$-dimensional hypercube in the general v-dimensional case] perpendicular to ℓ and crossing it in the middle. The set of those dual bonds is denoted $\mathbf{B}(Q)^*$. After smoothing corners in a specified way whenever four dual links meet in a point, we get a family of nonintersecting closed curves. Any such curve Γ^* will be called a *precontour.*

To get a hint how to define contours so that their probabilities can be conveniently estimated, let us try naively to evaluate the probability of a precontour Γ^* attached to an island Q. Let $\mathbf{V}(\Gamma^*)$ be the set of vertices contained in $\mathbf{B}(\Gamma^*)=\{\ell\in\mathbf{B}(Q)\,|\,\ell\cap\Gamma^*\neq\emptyset\}$ and denote $\mathbf{V}_Q(\Gamma^*)=\mathbf{V}(\Gamma^*)\cap Q$. There are about $q^{|\mathbf{V}(\Gamma^*)|}$ configurations on $\mathbf{V}(\Gamma^*)$ with the property that all bonds in $\mathbf{B}(\Gamma^*)$ are excited, and among them only about $q^{|\mathbf{V}(\Gamma^*)\setminus\mathbf{V}_Q(\Gamma^*)|}$ configurations are constant on all $\mathbf{V}_Q(\Gamma^*)$. Now since $|\mathbf{V}(\Gamma^*)|$ is about $2|\Gamma^*|$ and $|\mathbf{V}_Q(\Gamma^*)|$ about $|\Gamma^*|$ (consider e.g. a rectangular island Q), the probability of contour is small whenever q as well as $|\Gamma^*|$ is large. Note that the above reasoning is founded solely on entropy arguments; no energy loss along the contour was taken in account. This suggests that the probability in question is small for all temperatures. Indeed, the bounds which we shall eventually prove will be small for all β.

The wisdom of the above naive consideration is that the probability of a precontour Γ^* is small not only due to the fact that a certain amount of bonds crossing it are excited, but also because a certain number of nearby bonds (those belonging to the island) are not excited. *We are not going to claim anything about the probability of an excited bond which is not attached to a nonexcited bond.* $\left(\text{Actually, it goes to } 1-\dfrac{1}{q} \text{ as } \beta\to 0.\right)$

To get some control on contributions to the probability from different pieces of a precontour Γ^* surrounding (or surrounded by) an island Q, we classify plaquettes from the set $\mathbf{S}(\Gamma^*)$ of all plaquettes that intersect Γ^*. Namely, we introduce its subsets $\mathbf{S}_a(\Gamma^*)$ ($a=$acceptable) of those plaquettes from $\mathbf{S}(\Gamma^*)$ that contain at least one bond from $\mathbf{L}(Q)$ (graphically $\sqcup$ or $\square$ up to rotations), and $\mathbf{S}_b(\Gamma^*)=\mathbf{S}(\Gamma^*)\backslash\mathbf{S}_a(\Gamma^*)$ (b = bond) the set of those plaquettes from $\mathbf{S}(\Gamma^*)$ that contain no bond from $\mathbf{L}(Q)$. The "bad" plaquettes are the ones we are going to ignore in our estimates by bounding their number when compared with $|\mathbf{S}(\Gamma^*)|$.

Thus we finally define a *contour* Γ as a precontour Γ^* together with an excitation pattern on all bonds from $\mathbf{S}_a(\Gamma^*)$. Since there are at most two ways to complete an excitation pattern of plaquettes from $\mathbf{S}_a(\Gamma^*)$ ($\sqcup \to \square$ or $\square$), it follows that:

to a given Γ^* and $\mathbf{V}_Q(\Gamma^*)$ (i.e., specifying which "side" of Γ^* belongs to an island Q) there correspond at most $2^{|\Gamma^*|}$ contours Γ. (4.1)

Before evaluating the probability of a contour Γ, we estimate the number of bad plaquettes $N_b=|\mathbf{S}_b(\Gamma^*)|$. Note that each plaquette from $\mathbf{S}_b(\Gamma^*)$ has at least one neighbouring plaquette (common bond) from $\mathbf{S}_a(\Gamma^*)$. Indeed, let $s\in\mathbf{S}_b(\Gamma^*)$, then at least one vertex of s is in Q with at least one bond $\ell\in\mathbf{L}(Q)$ attached to it. Shifting s in the direction of ℓ we get a neighbouring plaquette from $\mathbf{S}_a(\Gamma^*)$. From this and the fact that each plaquette from $\mathbf{S}_a(\Gamma^*)$ has exactly four neighbours, we get $N_b\leqq 4N_a$, where we have denoted $N_a=|\mathbf{S}_a(\Gamma^*)|$. Taking into account that $2(N_a+N_b)\geqq|\Gamma^*|$ (as each plaquette may contain at most two units of the length of Γ^*), we get

$$N_a\geqq\tfrac{1}{10}|\Gamma^*|. \tag{4.2}$$

Let us now consider a Gibbs state $\langle\ \rangle_\beta$ that is a limit point of a sequence of $\langle\ \rangle_{\beta,\Lambda}$ of Gibbs states with Potts interaction, at inverse temperature β and in squares Λ with periodic boundary conditions. According to (4.2) one has

$$\begin{aligned}\operatorname{Prob}\{\sigma\,|\,\Gamma \textit{ is a contour of } \sigma\} &\leqq \Big\langle \prod_{s\in\mathbf{S}_a(\Gamma^*)} P_s\Big\rangle_{\beta,\Lambda}\\ &\leqq\Big\{\min_{i=1,2,3}\langle P^i_\Lambda\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}}\Big\}^{\frac{1}{10}|\Gamma^*|},\end{aligned}\tag{4.3}$$

where P_s is the indicator of the event

$$\left\{\sigma\,\middle|\,\begin{array}{l}\textit{the excitation pattern of the plaquette } s \textit{ coincides}\\ \textit{with the excitation pattern of that plaquette in } \Gamma\end{array}\right\},$$

and P^i_Λ are indicators of "universal contours" resulting from reflections of the excitation pattern of the plaquette s. Namely, there are (up to rotations) three possible excitation patterns of $s\in\mathbf{S}_a(\Gamma^*)$ resulting in three patterns of P^i_Λ as shown in Fig. 2.

To estimate the mean values $\langle P^i_\Lambda\rangle_{\beta,\Lambda}$ we list in Table 1: the energy; an upper bound on the number of configurations for which $P^i_\Lambda=1$; and a lower bound on the number of all configurations with the same energy (this will be used in the lower bound of the partition function).

The last column of the Table 1 was computed as follows. The fact that energy of a configuration σ is $(k-2)|\Lambda|$ means that in σ there are $k|\Lambda|$ excited bonds. Consider configurations with $k|\Lambda|$ excited bonds jammed into a square $\tilde{\Lambda}\subset\Lambda$ (i.e.

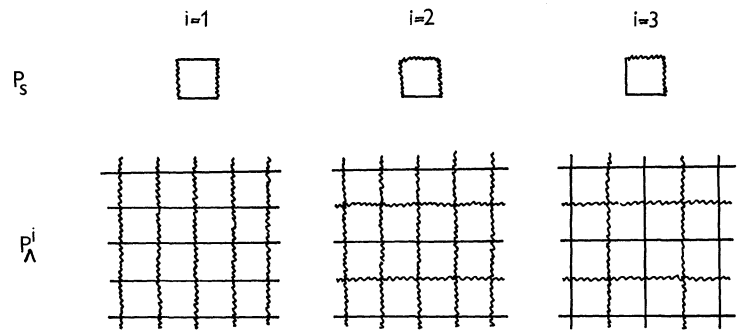

Fig. 2

Table 1

	$H_\Lambda(\sigma)$ for σ fulfilling $P^i_\Lambda(\sigma)=1$	The number of configurations σ with $P^i_\Lambda(\sigma)=1$	The total number of configurations with the same energy
$i=1$	$-\lvert\Lambda\rvert$	$<q^{\sqrt{\lvert\Lambda\rvert}}$	$>[q(q-4)]^{\frac{\lvert\Lambda\rvert}{4}}$
$i=2$	$-\frac{1}{2}\lvert\Lambda\rvert$	$<q^{\frac{\lvert\Lambda\rvert}{2}+\frac{\sqrt{\lvert\Lambda\rvert}}{2}}$	$>[q(q-4)]^{\frac{3\lvert\Lambda\rvert}{8}}$
$i=3$	$-\lvert\Lambda\rvert$	$<q^{1+\frac{\lvert\Lambda\rvert}{4}}$	$>[q(q-4)]^{\frac{\lvert\Lambda\rvert}{4}}$

$2|\tilde{\Lambda}|=k|\Lambda|)$ with all other bonds nonexcited. There are at least $[q(q-4)]^{\frac{|\tilde{\Lambda}|}{2}}$ such configurations, since one may choose arbitrarily the spins on the even sublattice in $\tilde{\Lambda}$, with $q-4$ possibilities remaining at each vertex of the odd sublattice to assure that all four attached bonds are excited.

Using the above table in the Gibbs formula

$$\langle P^i_\Lambda \rangle_{\beta,\Lambda} = \frac{\sum\limits_{P^i_\Lambda(\sigma)=1} e^{-\beta H_\Lambda(\sigma)}}{\sum e^{-\beta H_\Lambda(\sigma)}},$$

we eventually get in the limit $|\Lambda|\to\infty$ that

$$\mathrm{Prob}\{\sigma|\Gamma \textit{ is a contour of } \sigma\} \leqq \left[\frac{q}{(q-4)^3}\right]^{\frac{1}{80}|\Gamma^*|} \text{ uniformly in } \beta. \tag{4.4}$$

To conclude the proof of Hypothesis 2 we consider two arbitrary bonds $\ell^{(1)}, \ell^{(2)} \in \mathbf{L}$, and denoting $P(\Gamma)=\mathrm{Prob}\{\sigma|\Gamma \textit{ is a contour of } \sigma\}$, we evaluate

$$\langle P^{=}_{\ell^{(1)}} P^{\neq}_{\ell^{(2)}} \rangle_{\beta,\Lambda} \leqq \sum_{\Gamma \text{ surrounds } \ell^{(1)}} P(\Gamma) + \sum_{\Gamma \text{ surrounds } \ell^{(2)}} P(\Gamma) + \sum_{\substack{\Gamma \text{ wrapped} \\ \text{around } \Lambda}} P(\Gamma). \tag{4.5}$$

Here by saying that Γ surrounds ℓ we mean that Γ^* either surrounds or crosses ℓ. One estimates in the customary way the number of contours of the length $2k$ surrounding a given bond ℓ by $k\cdot 3^{2k-2}$ (this follows from the "three-way argument" and the fact that each such contour crosses at least once a fixed abscissa of length k containing ℓ). Finally, combining this with (4.1), (4.4), and (4.5) we get in the thermodynamic limit

$$\langle P^{=}_{\ell(1)} P^{\neq}_{\ell(2)} \rangle_\beta \leqq 2 \sum_{k=2}^{\infty} (3.12)^{2k} \cdot 2^k \left[\frac{q}{(q-4)^3}\right]^{\frac{k}{40}}. \tag{4.6}$$

[Here we used also the estimate $k \cdot 3^{2k-2} \leqq (3.12)^{2k}$.] Hypothesis 2 follows since the last estimate may be made less than $\frac{1}{4} - \varepsilon_0$ by taking q sufficiently large.

Now let us turn to Hypothesis 3. Denoting by $P^{\neq}_\Lambda$ the indicator of $\{\sigma | \sigma_i \neq \sigma_j$; *whenever* $i_1 = j_1 =$ *even*, $j_2 = i_2 + 1\}$, we get by a chessboard estimate

$$\langle P^{\neq}_\ell \rangle_{\beta,\Lambda} \leqq \langle P^{\neq}_\Lambda \rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}}.$$

Since $Z_\Lambda(\beta) \geqq q > 1$ for each β, one has

$$\langle P^{\neq}_\Lambda \rangle_{\beta,\Lambda} \leqq e^{-\beta\frac{|\Lambda|}{2}} q^{|\Lambda|}$$

and thus

$$\langle P^{\neq}_\ell \rangle_{\beta,\Lambda} \leqq e^{-\frac{\beta}{2}} q.$$

The first half of Hypothesis 3 follows since $\langle P^{=}_\ell \rangle_\beta \to 1$ as $\beta \to \infty$.

As for the second half,

$$\langle P^{=}_\ell \rangle_{\beta,\Lambda} \leqq \langle P^{=}_\Lambda \rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}}$$

by chessboard estimates with $P^{=}_\Lambda$ the indicator of $\{\sigma | \sigma_i = \sigma_j$ *whenever* $i_1 = j_1 =$ *even*, $j_2 = i_2 + 1\}$. The number of configurations σ with $P^{=}_\Lambda(\sigma) = 1$ is less than $q^{\frac{|\Lambda|}{2} + \frac{\sqrt{|\Lambda|}}{2}}$, hence

$$\langle P^{=}_\Lambda \rangle_{\beta,\Lambda} \leqq \frac{q^{(|\Lambda| + \sqrt{|\Lambda|})/2}}{q^{|\Lambda|} e^{-2\beta\Lambda}}$$

and

$$\langle P^{=}_\ell \rangle_\beta \leqq \frac{e^{2\beta}}{\sqrt{q}},$$

from which one concludes that

$$\langle P^{\neq}_\ell \rangle_\beta > \tfrac{1}{2}$$

as long as β is small and q is large enough. This completes the proof of Theorem 1 for the case $\nu = 2$.

The proof may easily be extended to general $\nu \geqq 2$; Here we only briefly mention a few points that need some care. Thus a precontour Γ^* is generally a surface made of $(\nu-1)$-dimensional cells of the dual lattice that cut the bonds from $\mathbf{B}(\Gamma^*)$. $\mathbf{S}(\Gamma^*)$ is a set of elementary hypercubes. The estimate (4.2) is generalized as

$$N_a \geqq \frac{1}{\nu(2\nu+1)} |\Gamma^*|,$$

where again we used $N_b \leqq 2\nu N_a$ and the fact that a hypercube may contain a piece of Γ^* with surface area at most ν. The subtle point is the analogue of the table that led to the estimate (4.4). There will again be a certain number of universal contours corresponding to excitation patterns that are periodic in all directions with periods at most two. Consider such a universal contour with indicator P^i_Λ. The energy of a configuration with $P^i_\Lambda(\sigma)=1$ is $(k-\nu)|\Lambda|$ with k describing a fraction of excited bonds in σ. The lower bound on the number of *all* configurations with this energy, namely

$$[q(q-2\nu)]^{\frac{k}{2\nu}|\Lambda|}, \tag{4.7}$$

is obtained again by considering the configurations with $k|\Lambda|$ excited bonds jammed into a box of volume $|\tilde{\Lambda}| = \frac{k}{\nu}|\Lambda|$. The upper bound on the number of configurations that contribute to P^i_Λ is

$$q^{\frac{k-2^{-\nu}}{\nu}|\Lambda| + C_i|\Lambda|^{\frac{\nu-1}{\nu}}}. \tag{4.8}$$

Indeed, let us consider the set $\mathbf{V}$ of vertices whose 2ν attached bonds are all excited (or in graphical description of the excitation pattern – the set of vertices that do not lie on any solid line). To estimate the number $|\mathbf{V}|$, let us divide Λ into $\frac{|\Lambda|}{2^\nu}$ disjoint elementary hypercubes (with the same excitation pattern due to periodicity) each of which contains 2^ν vertices and n vertices from $\mathbf{V}$. The number $|\mathbf{V}|$ will be estimated once we estimate the number n. To estimate this we observe that the number of excited bonds $k|\Lambda|$ is at least $(n\nu+1)\frac{|\Lambda|}{2^\nu}$; otherwise all excited bonds would connect only vertices from $\mathbf{V}$ and thus all bonds would be excited, the situation that could not have been created by reflecting an acceptable hypercube $s \in \mathbf{S}_a(\Gamma^*)$. Thus

$$|\mathbf{V}| = n\frac{|\Lambda|}{2^\nu} \leqq \frac{(k-2^{-\nu})}{\nu} \cdot |\Lambda|.$$

These vertices contribute the first factor in (4.8) since we are (almost) free to choose values of spins in them. Moreover, the value of spin in the vertices lying on a solid line should coincide all along the line, while the number of all solid lines is of order $|\Lambda|^{\frac{\nu-1}{\nu}}$, contributing thus the unimportant second factor in (4.8) that is washed out in the limit $|\Lambda| \to \infty$. Since the function

$$\frac{q^{k-2^{-(\nu-1)}}}{(q-2\nu)^k}$$

is growing in k, the estimates (4.7) and (4.8) yield

$$\operatorname{Prob}\{\sigma | \Gamma \textit{ is a contour of } \sigma\} \leqq \left[\frac{q^{\nu-2^{-(\nu-1)}}}{(q-2\nu)^\nu}\right]^{\frac{|\Gamma^*|}{2\nu^2(2\nu+1)}}. \tag{4.4'}$$

This, together with usual estimates on the number of shapes of Γ^*, suffices to complete the proof analogously to the case $\nu=2$.

Before going to the proof of Theorem 2 we would like to compare the Potts model with the Ising model (which is the Potts model with $q=2$) as far as the behaviour of contours is concerned. We learned from (4.4) that contrary to the usual Peierls contours of the Ising model our contours have small probabilities for all temperatures. Exaggerating a bit we could say that the phase transition in temperature in the Ising model takes place because contours become more and more probable with increasing temperature. The role of contours in the Potts model inverts when going from low to high temperatures: for low temperatures the space is filled with some ordered phase with rare contours confining a disorder inside of them, while for high temperatures there is overwhelming chaos everywhere with rare contours around islands of ordered phase ("inversion in contours"). In any case, the fact that contours are improbable for all temperatures prohibits certain observables (e.g. the "density of chaos") from reaching the values in an intermediate region and forces them to jump.

We start the proof of Theorem 2 with several observations. First of all we note that the critical inverse temperature β_c from Theorem 1 is in fact unique. Indeed, denoting $\mathbf{G}(\beta)$ the set of translation-invariant Gibbs states of the Potts model at the inverse temperature β and $\mathbf{R}^+(\beta)$ the set $\{\langle P_\ell^+\rangle | \langle\ \rangle\in\mathbf{G}(\beta)\}$, where ℓ is an arbitrary bond, there is a unique inverse temperature β_c such that $\sup\mathbf{R}^+(\beta) > \frac{1}{2} > \inf\mathbf{R}^+(\beta)$. While the existence is by Theorem 1, the uniqueness follows when noting that $\langle P_\ell^+\rangle$ is in fact a mean energy. Being thus a derivative of a concave function, one infers that the set $\mathbf{R}^+(\beta)$ is a one-point set except for at most a countable set of β's, and moreover that $\beta_1>\beta_2$ implies $\sup\mathbf{R}^+(\beta_1)\leqq\inf\mathbf{R}^+(\beta_2)$.

The next observation concerns a latent heat $L(q)$:

$$L(q)=\sup\mathbf{R}^+(\beta_c)-\inf\mathbf{R}^+(\beta_c)=\max_{\langle\ \rangle_1,\langle\ \rangle_2\in\mathbf{G}(\beta_c)}[\langle P^+\rangle_1-\langle P^+\rangle_2].$$

Lemma 4.1. $\lim\limits_{q\to\infty} L(q)=1$.

Proof. Note first that the bound (4.6) can be made less than any $B>0$ for q large enough. The lemma then follows immediately using Theorem 4′.

Consider further any sequence β_n approaching $\beta_c(q)$ from above. Let $\langle\ \rangle^{=,q}$ be any limit point of the sequence $\langle\ \rangle_{\beta_n}$, where $\langle\ \rangle_{\beta_n}$ are in turn limit points of sequences of Gibbs states at inverse temperatures β_n in finite volumes with periodic boundary conditions. The state $\langle\ \rangle^{=,\alpha}$ fulfills the following:

1. It is a Gibbs state at inverse temperature $\beta_c(q)$.

2. $$\langle P_\ell^+\rangle^{=,q}\leqq 1-L(q) \quad \text{for each} \quad \ell\in\mathbf{L}. \tag{4.9}$$

3. $$\overline{\lim_{q\to\infty}}\,\langle P_{\ell(1)}^{=}P_{\ell(2)}^{+}\rangle^{=,q}=0. \tag{4.10}$$

The first statement is standard (see [17]); the second one follows since the function $\langle P_\ell^+\rangle_\beta$ is decreasing in β; finally, the third one is implied by a similar property for the states $\langle\ \rangle_{\beta_n}$ that was shown to hold when proving Lemma 4.1.

From the state $\langle\ \rangle^{=,q}$ we are going to construct q different phases $\langle\ \rangle^\alpha$ as was promised. Recalling that by P_i^α we denoted the indicators of $\{\sigma|\sigma(i)=\alpha\}$ for each $\alpha=1,\dots,q$ and $i\in\mathbb{Z}^2$, the existence of $\Pi^\alpha=\lim\frac{1}{|\Lambda|}\sum_{i\in\Lambda}P_i^\alpha$ in the sense of $L^1(\langle\ \rangle^=)$

limits follows from the Birkhoff ergodic theorem. When constructing the phases $\langle\ \rangle^{\alpha}$ we shall use the following

Lemma 4.2. *Whenever* $\varepsilon>0$ *there is a* $q(\varepsilon)$ *such that for* $q\geqq q(\varepsilon)$ *it holds* $\left\langle \Pi^{\gamma}\left(\sum_{\alpha\neq\gamma}\Pi^{\alpha}\right)\right\rangle^{=,q}\leqq\frac{\varepsilon}{q}$ *for each* $\gamma=1,\ldots,q$.

Proof. The lemma follows once we know that $\left\langle P_i^{\gamma}\left(\sum_{\alpha\neq\gamma}P_j^{\alpha}\right)\right\rangle^{=,q}\leqq\frac{\varepsilon}{q}$ whenever $q\geqq q(\varepsilon)$, $i,j\in\mathbb{Z}^2$, $i\neq j$. But

$$\begin{aligned}\left\langle P_i^{\gamma}\left(\sum_{\alpha\neq\gamma}P_j^{\alpha}\right)\right\rangle^{=,q} &= \left\langle P_i^{\gamma}(P_\ell^{=}+P_\ell^{\neq})\left(\sum_{\alpha\neq\gamma}P_j^{\alpha}\right)\right\rangle^{=,q}\\ &\leqq \left\langle P_i^{\gamma}P_\ell^{=}\left(\sum_{\alpha\neq\gamma}P_j^{\alpha}\right)\right\rangle^{=,q}+\langle P_i^{\gamma}P_\ell^{\neq}\rangle^{=,q}\\ &\leqq \left\langle P_i^{\gamma}P_\ell^{=}\left(\sum_{\alpha\neq\gamma}P_j^{\alpha}\right)\right\rangle^{=,q}+\frac{1}{q}(1-L(q)),\end{aligned} \tag{4.11}$$

where ℓ is an arbitrary bond, $\ell\in\mathbf{L}$. We used here first the fact that whenever I_1, I_2 are indicators, then $I_1I_2\leqq I_2$; then the last inequality follows from (4.9) by symmetry. In particular, take ℓ to be a bond adjacent to the vertex i. Consider the island of phase γ containing the bond ℓ. This island does not contain the point j; so there must be a contour Γ defined by this island and separating the two vertices $i,j\in\mathbb{Z}^2$. The probability of this contour is estimated by $\frac{1}{q}$ [the right hand side of (4.4) (respectively (4.4'))]; the factor $\frac{1}{q}$ arises since we know not only the location of the contour, but also which phase (γ-phase) defines its island. Thus the first term on the right hand side of (4.11) is bounded by the right hand side of (4.6) times $\frac{1}{q}$. This, together with Lemma 4.1, proves Lemma 4.2.

Now we can complete the proof of Theorem 2. From Lemma 4.2 it follows that in the state $\langle\ \rangle^{=,q}$, $\operatorname{Prob}\left\{\sigma\,\middle|\,\Pi^1(\sigma)\left(\sum_{\alpha\neq1}\Pi^{\alpha}(\sigma)\right)<\frac{1}{10q}\right\}>1-10\varepsilon$ as long as $q\geqq q(\varepsilon)$ is large enough. If $\varepsilon<\frac{1}{10}$, then the probability of at least one event

$$A_1=\left\{\sigma\,\middle|\,\Pi^1(\sigma)\left(\sum_{\alpha\neq1}\Pi^{\alpha}(\sigma)\right)<\frac{1}{10q},\ \Pi^1(\sigma)\geqq\prod_{\alpha\neq1}\Pi^{\alpha}(\sigma)\right\},$$

$$A_2=\left\{\sigma\,\middle|\,\Pi^1(\sigma)\left(\sum_{\alpha\neq1}\Pi^{\alpha}(\sigma)\right)<\frac{1}{10q},\ \Pi^1(\sigma)<\prod_{\alpha\neq1}\Pi^{\alpha}(\sigma)\right\}$$

is larger than $\frac{1}{2}-5\varepsilon$. If $\operatorname{Prob}(A_1)>\frac{1}{2}-5\varepsilon$, then since $\sum_{\alpha=1}^{q}\Pi^{\alpha}(\sigma)\equiv1$, one has $\operatorname{Prob}\{\sigma|\Pi^1(\sigma)>\frac{1}{2}\}>\frac{1}{2}-5\varepsilon$. Similarly if $\operatorname{Prob}(A_2)>\frac{1}{2}-5\varepsilon$, then $\operatorname{Prob}\left\{\sigma\,\middle|\,\Pi^1(\sigma)<\frac{1}{5q}\right\}>\frac{1}{2}-5\varepsilon$. But from this and $\langle\Pi^1\rangle^{=,q}=\frac{1}{q}$ implied by symmetry one infers

$$\operatorname{Prob}\left\{\sigma\,\middle|\,\Pi^1(\sigma)>\frac{3}{2q}\right\}>0. \tag{4.12}$$

Hence (4.12) holds in both cases. Let I_1 be the indicator of the event $\left\{\sigma \,\middle|\, \Pi^1(\sigma) > \frac{3}{2q}\right\}$ and consider the state $\langle\ \rangle^1$ defined by

$$\langle I(B)\rangle^1 = \frac{\langle I_1 I(B)\rangle^{=,q}}{\langle I_1\rangle^{=,q}},$$

whenever $I(B)$ is an indicator of a cylinder set B. The state $\langle\ \rangle^1$ is again a Gibbs state at inverse temperature β_c. It fulfills

$$\langle P^1\rangle^1 \geqq \frac{3}{2q} > \langle P^\alpha\rangle^1, \qquad \alpha \neq 1. \tag{4.13}$$

(The second inequality follows by symmetry.) In the same manner one defines the states $\langle\ \rangle^\alpha$, $\alpha \neq 1$, using Π^α instead of Π^1. Thus, referring to (4.13), we see that the state $\langle\ \rangle^{=,q}$ can be decomposed into at least q ergodic states. To conclude the proof of Theorem 2 we note that similar states exist for each $\beta > \beta_c$; this may be proven in precisely the same way since the function $\langle P^{\neq}\rangle_\beta$ is decreasing in β.

It seems probable that for $\beta < \beta_c$ there is a unique Gibbs state. However, we do not know a proof of this statement.

5. Proof of the Existence of a Phase Transition in the Gauge Potts Model

Again, we shall repeatedly rely on chessboard estimates, using reflections in planes perpendicular to coordinate axes and passing through lattice sites. Observing that reflections act naturally on the set $\bar{\mathbf{L}}$ of oriented bonds and recalling that $P^{=}_{\square}(\sigma)$ does not depend on the choice of starting point and orientation of path around $\square$ used to define $\sigma_\square$, one easily convinces oneself that the Hamiltonian (2.2) is RP.

To apply the strategy of Theorem 4 we take $\beta_{\mathrm{I}} = 0$, $\beta_{\mathrm{II}} = \infty$, $P^1_\square = P^{\neq}_\square$, $P^2_\square = P^{=}_\square$. Again, one has to change the definition

$$\Pi^\alpha = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} P^\alpha_i, \qquad \alpha = 1, 2$$

into

$$\Pi^\alpha = \lim_{\Lambda\to\infty} \frac{1}{\frac{\nu(\nu-1)}{2}|\Lambda|} \sum_{\square\subset\Lambda} P^\alpha_\square .$$

$\left(\text{Here } \frac{\nu(\nu-1)}{2}|\Lambda| \text{ is the number of plaquettes in } \Lambda.\right)$ We shall verify Hypotheses 1–3 explicitly only for the case $\nu = 3$; the generalization is straightforward though cumbersome.

Since $P^1_\square + P^2_\square = 1$, we pass to Hypothesis 2 and define the contours. We call a *path* every sequence $\{\square_i\}_{i=1}^n$ of plaquettes such that the intersection $\square_i \cap \square_{i+1}$ is a bond for each $i = 1, \ldots, n-1$. By an *island* Q of the configuration σ we mean a maximal path-connected set of nonfrustrated plaquettes in σ. A *boundary* of the island Q is then the set $\mathbf{B}(Q)$ of all unit cubes such that each of them contains both a plaquette belonging to Q as well as a plaquette which does not belong to Q. Finally, by a *contour* Γ of the configuration σ we mean a maximal connected component of $\mathbf{B}(Q)$ together with a prescription saying which of the plaquettes belonging to cubes from Γ are frustrated in σ. Here a set of cubes is considered to be connected if the set of their faces is path-connected.

Consider a fixed plaquette $\square$. As usual we shall need an estimate on the number of contours Γ containing n cubes and surrounding $\square$. It is easy to show that there is a constant K such that this number is less than K^n.

Indeed there are no more than $(n+1)(2.18)^n < 40^n$ connected sets consisting of n cubes and surrounding a fixed plaquette. This follows by observing that, given a cube c_1, there are 18 ways to choose a second cube c_2 so that the set $\{c_1, c_2\}$ is connected, and by using the following elementary fact: on every connected graph there is a path that passes through every edge at most twice.

Given a geometrical shape of Γ one estimates the number of possible locations of frustrated plaquettes on Γ as follows. Each cube $c \in \Gamma$ may be endowed with one to four nonfrustrated faces (if there were five of them, then all six would be necessarily nonfrustrated, as it is easy to show, and thus the cube would not belong to Γ). Denoting by K_1 the number of ways of choosing from one to four faces from a standard cube, we finally arrive at our estimate with $K = 40K_1$.

Hypothesis 2 will be verified once we evaluate conveniently $P(\Gamma) \equiv \operatorname{Prob}\{\sigma \mid \Gamma$ *is a contour of* $\sigma\}$. To do it, consider a finite volume Gibbs state $\langle\ \rangle_{\beta,\Lambda}$ with periodic boundary conditions. Applying our reliable chessboard estimate we get

$$P(\Gamma) = \left\langle \prod_{c\in\Gamma} P^c(\sigma) \right\rangle_{\beta,\Lambda} \leqq \prod_{c\in\Gamma} \langle P^c_\Lambda(\sigma)\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}}. \tag{5.1}$$

Here for each cube $c \in \Gamma$ we are given a pattern of frustrations of the faces of c, and P^c is a shorthand notation for the indicator of

$$\left\{ \sigma \,\middle|\, \begin{array}{l} \textit{frustrations of the faces of } c \textit{ in the configurations } \sigma \\ \textit{coincide with the pattern prescribed for the cube } c \end{array} \right\}.$$

P^c_Λ is the indicator of the event defined by disseminating the pattern of frustrations prescribed for the particular cube c by means of reflection all over Λ. So we are left with estimating $\langle P^c_\Lambda\rangle_{\beta,\Lambda}$. There are seven generic patterns of frustrations (up to rotations and symmetries) drawn in Fig. 3:

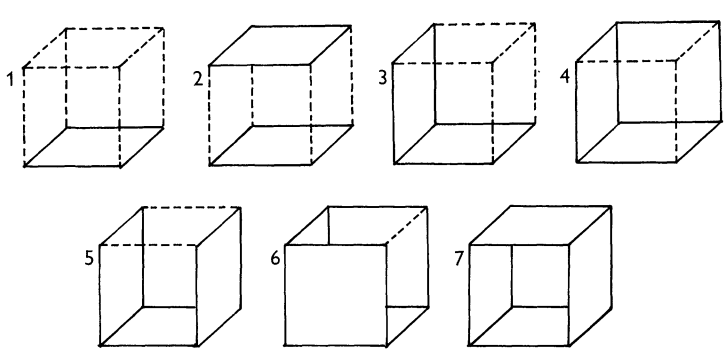

Fig. 3

Here the nonfrustrated plaquettes are shown as opaque and frustrated as transparent. These patterns classify at the same time different indicators P_Λ^c and thus different estimates of $\langle P_\Lambda^c \rangle$ to be done.

Consider Case 1. The corresponding indicator $P_\Lambda^{c_1}$ is the indicator of the following event A_1: the plaquettes in every other (say) horizontal plane are nonfrustrated and the remaining ones are all frustrated. To estimate the probability of this event, we keep in the partition function in the denominator only the sum over the set $\overline{A_1}$ of configurations with the same number of frustrated (and nonfrustrated) plaquettes (hence the same energy) but located in quite another way. Namely, we pack all nonfrustrated plaquettes into a subcube of Λ located in the corner of it. Now the probability of A_1 is estimated by $\frac{|A_1|}{|\overline{A}_1|}$.

Why do we expect that $|\overline{A_1}|$ is much larger than $|A_1|$? There are $\frac{|\Lambda|}{2}$ equations: each of them assures the nonfrustration of one plaquette. (Of course, this gives only an upper bound on $|A_1|$; the system of equations also allows some additional plaquettes outside every other horizontal plane to be nonfrustrated.) An important fact is that these equations are actually independent: if one throws away one of them, then the number of solutions of the remaining system will increase. This follows by observing that on a two-dimensional plane there exist configurations with exactly one frustrated plaquette, the rest being nonfrustrated. On the contrary, when all $\frac{|\Lambda|}{2}$ nonfrustrated plaquettes are packed into a corner of a cube, one has again $\frac{|\Lambda|}{2}$ defining equations, but this time they are highly dependent. In fact one may keep only those equations which correspond to vertical plaquettes and to one layer of horizontal plaquettes without the set of their solutions being changed. Thus $\overline{A_1}$ is effectively described by a much smaller set of equations than A_1.

To transform what was said above into an estimate, we evaluate first the number of configurations in A_1. Denoting q the number of elements of the gauge group $\mathbf{G}$ one has

$$|A_1| \leqq q^{\frac{|\Lambda|}{2}} (q^2)^{\frac{|\Lambda|}{2}} q^{|\Lambda|} C^{o(|\Lambda|)}. \tag{5.2}$$

Indeed, up to spins on the boundary of Λ giving an irrelevant factor $C^{o(|\Lambda|)}$ (hereafter we denote all unimportant constants C though they differ in principle), in each horizontal layer of nonfrustrated plaquettes one can choose arbitrarily the spins on bonds of one horizontal direction while in the neighbouring horizontal layer of frustrated plaquettes the spins in both directions are arbitrary. In addition all vertical spins may be chosen arbitrarily. The remaining spins are then determined from the above mentioned system of equations.

Now let us turn to $\overline{A_1}$. The volume of the subcube into which $\frac{|\Lambda|}{2}$ nonfrustrated plaquettes are packed is $\frac{|\Lambda|}{6}$. Hence

$$|\overline{A_1}| \geqq ((q-3)^3)^{\frac{5|\Lambda|}{6}} q^{\frac{|\Lambda|}{6}} C^{o(|\Lambda|)}, \tag{5.3}$$

since to construct a configuration fully nonfrustrated in the subcube one is free in choosing spins only in one direction (and also spins on one horizontal layer, which yield the factor $C^{o(|\Lambda|)}$), while outside of this subcube the choice is almost unconstrained (some care is needed not to create additional nonfrustrated plaquettes; this is the origin of the factor $q-3$ instead of expected q). Combining (5.2) and (5.3) we get

$$\langle P^{c_1}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \left(\frac{|A_1|}{|\bar{A}_1|}\right)^{\frac{1}{|\Lambda|}} \leqq \frac{q^{5/2}}{(q-3)^{5/2}q^{1/6}}\, C^{\frac{o(|\Lambda|)}{|\Lambda|}}.$$

In fact, the frustration pattern No. 1 is the worst one. To convince the reader, we shall only investigate No. 7 and then present a list of estimates for the other cases.

The indicator $P_\Lambda^{c_7}$ corresponds to the event

$$A_7 = \left\{\sigma \,\middle|\, \begin{array}{l} \textit{all vertical plaquettes are nonfrustrated} \\ \textit{and the rest of them are frustrated in } \sigma \end{array}\right\}.$$

Then

$$|A_7| \leqq q^{|\Lambda|} C^{o(|\Lambda|)},$$

since after fixing spins on vertical bonds, the configuration is determined up to bottom level spins. The number of nonfrustrated plaquettes in a configuration $\sigma \in A_7$ is $2|\Lambda|$; hence they may be packed into a corner of the volume $\frac{2|\Lambda|}{3}$. Thus the bound for A_7 is

$$|\bar{A}_7| \geqq ((q-3)^3)^{|\Lambda|/3} q^{2|\Lambda|/3} C^{o(|\Lambda|)}$$

and we get

$$\langle P_\Lambda^{c_7}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \left(\frac{|A_7|}{|\bar{A}_7|}\right)^{\frac{1}{|\Lambda|}} \leqq \frac{q^{1/3}}{q-3}\, C^{\frac{o(|\Lambda|)}{|\Lambda|}}.$$

The list of remaining estimates $\left(\text{omitting a factor } C^{\frac{o(|\Lambda|)}{|\Lambda|}}\right)$ is

$$\langle P_\Lambda^{c_2}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \frac{q^{5/3}}{(q-3)^2},$$

$$\langle P_\Lambda^{c_3}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \frac{q^{5/3}}{(q-3)^2},$$

$$\langle P_\Lambda^{c_4}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \frac{q^{9/8}}{(q-3)^{3/2}},$$

$$\langle P_\Lambda^{c_5}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \frac{q}{(q-3)^{3/2}},$$

$$\langle P_\Lambda^{c_6}\rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}} \leqq \frac{q^{7/12}}{(q-3)}.$$

Thus, referring to (5.1), one has in all cases

$$\mathrm{Prob}\{\sigma \mid \Gamma \textit{ is a contour of } \sigma\} \leqq \left(\frac{q^{5/2}}{(q-3)^{5/2} q^{1/6}}\right)^{|\Gamma|}$$

and Hypothesis 2 is verified once q is large enough.

Before going to Hypothesis 3, a remark about the case $v=2$. The proof should fail in this case since in two dimensions any gauge model is equivalent to a product of independent one-dimensional models which exhibit no phase transition. The reason why it fails is that conditions for different plaquettes to be nonfrustrated are independent in two dimensions. This leads to the conclusion that any repacking of nonfrustrated plaquettes does not increase the number of configurations with a fixed number of frustrations.

Consider finally Hypothesis 3. For the first half of it we show

$$\langle P_{\square}^{=} \rangle_{\beta} \leqq \frac{q^{5/2}}{(q-3)^3} \quad \text{as} \quad \beta \to 0. \tag{5.4}$$

A handy chessboarding immediately yields

$$\langle P_{\square}^{=} \rangle_{\beta,\Lambda} \leqq \langle P_{A}^{=} \rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}},$$

with $P_A^{=}$ the indicator of the set

$$A = \{\sigma \mid \textit{the plaquettes in every horizontal layer are nonfrustrated}\}.$$

The bound on $|A_1|$ is actually also a bond for $|A|$:

$$|A| \leqq q^{\frac{5|\Lambda|}{2}}.$$

To estimate the partition function $Z_\Lambda(\beta)$ from below we restrict ourselves to completely frustrated configurations. The energy of each of them is precisely $3|\Lambda|$, while their number is certainly larger than $(q-3)^{3|\Lambda|}$. Thus

$$\langle P_{A}^{=} \rangle_{\beta,\Lambda} \leqq \frac{q^{\frac{5|\Lambda|}{2}}}{e^{-3|\Lambda|\beta}(q-3)^{3|\Lambda|}}$$

and (5.4) follows.

To finish the proof of Theorem 3, we show

$$\langle P_{\square}^{\neq} \rangle_{\beta} \to 0 \quad \text{as} \quad \beta \to \infty. \tag{5.5}$$

Applying the chessboard estimate for the last time one has

$$\langle P_{\square}^{\neq} \rangle_{\beta,\Lambda} \leqq \langle P_{A}^{\neq} \rangle_{\beta,\Lambda}^{\frac{1}{|\Lambda|}},$$

where $P_A^{\neq}$ is the indicator of

$$A' = \{\sigma \mid \textit{all plaquettes in every other horizontal layer are frustrated}\}.$$

The energy of every configuration $\sigma \in A'$ is not smaller than $\frac{|\Lambda|}{2}$, while $|A'| < q^{3|\Lambda|}$.

Since the partition function $Z_\Lambda(\beta) \geqq 1$ (this follows from the fact that the energy of ground state is 0), one has

$$\langle P_{\square}^{\pm} \rangle_{\beta,\Lambda} \leqq q^3 e^{-\frac{1}{2}\beta},$$

which proves (5.5).

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