Commun. Math. Phys. 83, 469–492 (1982)

Communications in Mathematical Physics

Renormalization Group Study of a Critical Lattice Model

II. The Correlation Functions

K. Gawedzki[1] and A. Kupiainen[2]

1 Department of Mathematical Methods of Physics, University, PL-00-682 Warsaw, Poland
2 Research Institute for Theoretical Physics, University of Helsinki, SF-00170 Helsinki 17, Finland

Abstract. The Renormalization Group is used to study the correlation functions of a nonlocal hierarchical model mimicking the $\lambda(\nabla\phi)^4$ model, dipole gas and the like. It is shown that the infrared behaviour of the correlations is that of the massless gaussian $\frac{1}{2}c(\lambda)(\nabla\phi)^2$.

1. Introduction

In [1] a nonlocal hierarchical model was introduced to mimic the long distance behaviour of the lattice model with Hamiltonian

$$H(\phi)=\sum_x [(\nabla\phi)_x^2+\lambda(\nabla\phi)_x^4].$$

A renormalization group (RG) transformation was defined in finite volume and contractive properties were proved for it uniformly in volume. With the assumption of existence of the thermodynamical limit, the RG was shown to drive the model to a fixed point mimicking the massless lattice field. In this paper we extend the analysis to correlation functions. Using the RG we prove detailed estimates of the long distance behaviour of correlations, showing that in the infrared the model behaves as a massless gaussian lattice field. We also establish the existence of the thermodynamical limit of all correlations and thereby complete the analysis of [1]. In the thermodynamic limit the correlation functions will satisfy convergent (inductive) cluster expansions, generalizations of those working in the massive case now to a massless model. The present paper is selfcontained provided certain results of [1] are taken as given.

Let us briefly recall the model (for motivation, see [1]). We divide the periodic lattice $\Lambda_N=\mathbb{Z}^d_{L^N}$ of volume L^N ($L\in\mathbb{N}$, odd $\geqq 3$) to blocks b^k_y of volume L^{kd} $1\leqq k\leqq N$ centered at yL^k, $y\in\Lambda_{N-k}$ and associate a random variable Z^k_y to the block b^{k+1}_y. Let $\mathscr{A}$ be a function supported on b^1_0 with mean zero, $\mathscr{A}(0)\neq 0$ and nonconstant in $b^1_0\backslash\{0\}$. Denoting

$$z^k_y=\mathscr{A}(y-L[L^{-1}y])Z^k_{[L^{-1}y]} \tag{1}$$

0010-3616/82/0083/0469/$04.80

($[x]$ is the integer part of x), we will consider the following random fields on Λ_N

$$\phi_x = \sum_{k=0}^{N-1} L^{-\frac{dk}{2}} z^k_{[L^{-k}x]}. \tag{2}$$

As explained in [1], ϕ plays the role of the gradient of a massless scalar field.

We define also blockspin fields ϕ^k_x, $x \in \Lambda_{N-k}$

$$\phi^k_x = \sum_{j=k}^{N-1} L^{-\frac{d}{2}(j-k)} z^j_{[L^{-j+k}x]}, \tag{3}$$

which satisfy

$$\phi^k_x = L^{-\frac{d}{2}} \phi^{k+1}_{[L^{-1}x]} + z^k_x. \tag{4}$$

The free model is specified by giving a family of kernels $U = \{U_{2m}(x_1, \ldots, x_{2m})\}_{m=1}^{\infty}$ with $x_j \in \mathbb{Z}^d$, satisfying the following properties:

$$|U|_{LA} \equiv \sup_m \left[\frac{1}{(2m)!} \sum_{(x_2, \ldots, x_{2m})} |U_{2m}(\bar{x})| e^{AL(\underline{x})} \right]^{\frac{1}{2m}} \leqq \kappa, \tag{5}$$

$$U_{2m}(x_1, \ldots, x_{2m}) = U_{2m}(x_1 + d, \ldots, x_{2m} + d) = U_{2m}(x_{\pi(1)}, \ldots, x_{\pi(2m)}), \tag{6}$$

$$U_{2m}(x, x, \ldots, x) = 0. \tag{7}$$

Throughout the paper we denote by $\bar{x}$ the sequence $(x_1, \ldots, x_{2m})$ and by $\underline{x}$ the set $\{x_1, \ldots, x_{2m}\}$. $L(\underline{x})$ is the length of the shortest tree on $\underline{x}$ and possibly other (continuum) points. We measure lengths in the metric $|x| = \sum_{\mu=1}^{d} |x_\mu|$ on the torus Λ. Given U we define a potential $U^{N-k-1}(Z^k)$

$$U^{N-k-1}(Z^k) = \sum_{m=1}^{\infty} \sum_{x_1, \ldots, x_{2m} \in \Lambda_{N-k-1}} U^{N-k-1}_{2m}(\bar{x}) Z^k_{x_1} \ldots Z^k_{x_{2m}}, \tag{8}$$

where $U^{N-k-1}_{2m}(\bar{x})$ is defined as the periodization of U to Λ_{N-k-1}.

The free expectation $\langle - \rangle^N_0$ in volume Λ_N is defined as

$$\langle - \rangle^N_0 = \mathcal{N}^{-1} \int (-) \prod_{k=0}^{N-1} d\nu_k(Z^k), \tag{9}$$

with

$$d\nu_k(Z^k) = e^{-U^k(Z^k)} \prod_y dx(Z^k_y), \tag{10}$$

where X is an even probability measure on $\mathbb{R}$ with compact support. It is easy to see that (8) is well defined provided κ is small enough. In [1] we called the case $U = 0$ the *local case.*

To describe the interacting case, let $\tilde{V} = \{\tilde{V}_{2m}(\bar{x})\}_{m=1}^{\infty}$ satisfy (6) and

$$|\tilde{V}|_A \leqq \eta, \tag{11}$$

$$\tilde{V}_2(x, x) = 0, \tag{12}$$

and again denote by $\tilde{V}^{\ell}_{2m}$ the periodized kernel on $(\Lambda^{\ell})^{2m}$. The interaction potential is

$$V^N(\phi) = V^N_C(\phi) + \tilde{V}^N(\phi), \tag{13}$$

where

$$V^N_C(\phi) = \frac{C}{2} \sum_{x \in \Lambda_N} \phi_x^2, \tag{14}$$

and

$$\tilde{V}^N(\phi) = \sum_{m=1}^{\infty} \sum_{\underline{x} \in \Lambda_N^{2m}} \tilde{V}^N_{2m}(\underline{x}) \phi_{\underline{x}}. \tag{15}$$

We denote

$$\phi_{\underline{x}} \equiv \prod_{i=1}^{2m} \phi_{x_i}. \tag{16}$$

Now the interacting expectation is defined as

$$\langle - \rangle^N_{V^N} = \langle (-) e^{-V^N(\phi)} \rangle^N_0 / \langle e^{-V^N(\phi)} \rangle^N_0. \tag{17}$$

In [1] the case $U=0$, $V^N(\phi) = \sum_x v(\phi_x)$ was called the *local model*. Finally, we defined the RG transformation T_1 in finite volume using (4) to integrate out Z^0:

$$\begin{aligned}(T_1 V^N)(\phi') = & -\log \int \exp\left[-V(L^{-\frac{d}{2}} \phi'_{[L^{-1}\cdot]} + z^0) \right] d\nu_0(Z^0) \\ & + \log \int \exp[-V(z^0)] d\nu_0(Z^0), \end{aligned} \tag{18}$$

and the next RG transformations T_k in an analogous way. The main result of [1] was the following (we denote by $|\tilde{V}^N|_A$ (5) also when the sum is restricted to Λ_N and $L(\underline{x})$ a tree on Λ_N, we also drop the subscript in T since T_k's differ only by the volume).

Proposition 1. *Let V^k be of the form* (13) *with $|\tilde{V}^k|_A < \eta$. For A large enough, κ and η small enough, uniformly in k, TV^k can be written as*

$$TV^k = V^{k-1}_{C'} + \widetilde{TV}^{k-1} \tag{19}$$

with

$$|\widetilde{TV}^{k-1}|_A \leqq \delta\eta, \qquad \delta < 1 \tag{20}$$

and

$$|C - C'| \leqq \alpha\eta. \tag{21}$$

δ and α do not depend on C.

Remark. The $\tilde{V}^k$ in Proposition 1 does not have to be a periodization of any $\tilde{V}$, only $\tilde{V}^N$ is.

Let us now turn to the results of the present paper. The first result deals with the thermodynamical limit.

Theorem 1. *Let* $|\tilde{V}|_A < \eta$ *and let* A, η *and* κ *be as in Proposition* 1. *Let* $V=(c, \tilde{V})$.

(A) *There is an infinite volume state* $\langle - \rangle_V$ *such that*

$$\langle \phi_{\bar{x}} \rangle_{V^N}^N \xrightarrow[N\to\infty]{} \langle \phi_{\bar{x}} \rangle_V \tag{22}$$

for each $\bar{x}=(x_1, \ldots, x_s)$.

(B) *T extends to the thermodynamic limit*, i.e. *there is a* $TV(c', \widetilde{TV})$ *such that for all* $\bar{x}$

$$\langle \phi_{\bar{x}} \rangle_{TV^N}^N \to \langle \phi_{\bar{x}} \rangle_{TV}, \tag{23}$$

$$|\widetilde{TV}|_A \leqq \delta\eta, \qquad |c'-c| \leqq \alpha\eta. \tag{24}$$

(C) *Let* $|\tilde{V}^{(n)} - \tilde{V}|_A \to 0$, $c_n \to c$ *as* $n\to\infty$. *Then*

$$\langle - \rangle_{V^{(n)}} \to \langle - \rangle_V, \qquad V=(c, \tilde{V}) \tag{25}$$

in the sense of uniform convergence of correlations, that is, for all m

$$\left\langle \prod_{i=1}^m \phi_{x_i} \right\rangle_{V^{(n)}} \to \left\langle \prod_{i=1}^m \phi_{x_i} \right\rangle_V$$

uniformly in $\{x_i\}_{i=1}^m$.

Thus the RG drives our interacting model to the line of fixed points.

The following result deals with the long distance behaviour of two point function showing that it agrees with that of the free model.

Theorem 2. *Let V be as in Theorem 1. Then*

(A) $$\sum_{x_2\in\Lambda_N} \langle \phi_{x_1}\phi_{x_2} \rangle_{V^N}^N = 0 \text{ for all } N,$$

(B) $$|\langle \phi_{x_1}\phi_{x_2} \rangle_V| \leqq c[1+|x_1-x_2|]^{-d},$$

(C) $$\sum_{x_2\in\mathbb{Z}^d} |\langle \phi_{x_1}\phi_{x_2} \rangle| = \infty.$$

Remark. (C) shows that (B) is the best polynomial bound. Recall that ϕ is the analogue of $\nabla\phi$, ϕ a scalar field. (A)–(B) are the properties of $\langle \nabla\phi_{x_1} \nabla\phi_{x_2} \rangle$ in massless free theory.

Finally, we show how the RG can be used to study general truncated expectations. We derive convergent inductive expansions for them, which could be used to study their long distance behaviour. Since this behaviour is not particularly illuminating even in the free case we will not tackle that problem here.

2. The Free Model

Let us first show why our results are true in the free case. For this we need the following result proven in [1] using a high temperature cluster expansion. For later use, we state it in the interacting case. Let $|\tilde{V}^k|_A < \eta$ and define the expectation

$$\langle - \rangle_k = \mathcal{N}^{-1} \int - e^{-V^k(z^k)} dv_k(Z^k). \tag{1}$$

Then

Proposition 2. *Let D, A be large enough and κ, η small enough. Then uniformly in k*

$$|\langle Z_{\bar{u}_1}; \ldots; Z_{\bar{u}_p}\rangle_k^T| \leqq \prod M_r! \prod_{j=1}^{p} e^{D(|\bar{u}_j| + L(\underline{u}_j))} \cdot \exp[-\tfrac{1}{2} A L(\underline{u}_1; \ldots; \underline{u}_p)], \tag{2}$$

where M_r are the numbers of the sequences $\bar{u}_1, \ldots, \bar{u}_p$ equal up to permutations and $L(\underline{u}_1; \ldots; \underline{u}_1)$ the length of the shortest graph on the points of $\bigcup_j \underline{u}_j$ and possibly other points connected with respect to the groups $\underline{u}_j$.

To establish the thermodynamic limit of the free model we use (2) to write

$$\left\langle \prod_{i=1}^{m} \phi_{x_i} \right\rangle_0^N = \sum_{n_1, \ldots, n_m = 0}^{[N/2]} L^{-\frac{d}{2}\Sigma n_i} \left\langle \prod_{i=1}^{m} z_{x_i}^{n_i} \right\rangle_0 + R_N$$
$$= \sum_{n_1, \ldots, n_m} L^{-\frac{d}{2}\Sigma n_i} \prod_j \int z_{\bar{x}_j}^{m_j} dv_{m_j}(Z^{m_j}) + R_N. \tag{3}$$

The thermodynamic limit for the $\int z_{\bar{x}_j}^{m_j} dv_{m_j}$'s is standard, given the cluster expansion of [1]. Moreover

$$\left| \left\langle \prod_{i=1}^{m} z_{u_i}^{m_i} \right\rangle_0^N \right| \leqq c^m \text{ for all } u_i$$

uniformly in N. Hence the existence of the $N \to \infty$ limit follows by the dominated convergence theorem, since $R_N \to 0$.

For Theorem 2 note that since

$$\langle \phi_{x_1} \phi_{x_2} \rangle_0^N = \sum_{k=0}^{N-1} L^{-kd} \mathscr{A}([L^{-k}x_1] - L[L^{-k-1}x_1]) \cdot \mathscr{A}([L^{-k}x_2] - L[L^{-k-1}x_2]) \langle Z^k_{[L^{-k-1}x_1]} Z^k_{[L^{-k-1}x_2]} \rangle_k, \tag{4}$$

part A follows because $\mathscr{A}$ has zero mean. For part B we use Proposition 2 to get

$$|\langle Z^k_{[L^{-k-1}x_1]} Z^k_{[L^{-k-1}x_2]} \rangle_k| \leqq c e^{-\varepsilon L^{-k-1}|x_1 - x_2|}, \tag{5}$$

so that

$$|\langle \phi_{x_1} \phi_{x_2} \rangle| \leqq c \sum_{k=0}^{\infty} L^{-kd} \exp[-\varepsilon L^{-k-1}|x_1 - x_2|]$$
$$\leqq c[1 + |x_1 - x_2|]^{-d}. \tag{6}$$

To prove part C for the free model, consider first the local case, i.e. $U = 0$. Then $\langle Z_x^k Z_y^k \rangle = 0$ if $\left[\frac{x}{L}\right] \neq \left[\frac{y}{L}\right]$. Thus, letting N be such that $x_1 \in b_0^{N_1}$ and x_2 such that

$x_2 \notin b_0^{N_1}$ we get from (4)

$$\langle \phi_{x_1} \phi_{x_2} \rangle_0^N = \left[L^{-d(j-1)} \frac{\mathscr{A}(0)}{L^d - 1} ((L^d-1)\mathscr{A}(x) + \mathscr{A}(0)) - L^{-d(N-1)} \frac{\mathscr{A}(0)^2}{L^d - 1} \right] \int Z^2 dx(Z), \tag{7}$$

where j is the smallest integer such that $x_2 \varepsilon b_0^j$ and $x = [L^{-(j-1)} x_2]$. Since $\mathscr{A}$ by definition is nonconstant in b_0^1 with $\mathscr{A}(0) \neq 0$ and zero mean

$$\sum_{x \in b_0^1} |(L^d - 1)\mathscr{A}(x) + \mathscr{A}(0)| \geqq \varepsilon |\mathscr{A}(0)|. \tag{8}$$

Take now $L^{-N/2} < \frac{\varepsilon}{2}$, whence

$$\sum_{x_2} |\langle \phi_{x_1} \phi_{x_2} \rangle_0^N| \geqq \frac{\varepsilon}{2} \frac{\mathscr{A}(0)|^2}{L^d - 1} \sum_{j=N_1+2}^{N/2} \int Z^2 dx(Z). \tag{9}$$

Upon taking N to infinity we get C.

For the nonlocal case, $U \neq 0$, we get instead of (7)

$$\begin{aligned} \langle \phi_{x_1} \phi_{x_2} \rangle_0^N = & \sum_{k=0}^{j-2} L^{-kd} \mathscr{A}([L^{-k} x_1] - L[L^{-k-1} x_1]) \mathscr{A}([L^{-k} x_2] - L[L^{-k-1} x_2]) \\ & \cdot \langle Z^k_{[L^{-k-1}x_1]} Z^k_{[L^{-k-1}x_2]} \rangle_0^N + L^{-(j-1)d} \mathscr{A}(0) \mathscr{A}(x) \langle (Z^k_{[L^{-j}x_1]})^2 \rangle_0^N \\ & + \sum_{k=j}^{N-1} L^{-kd} \mathscr{A}(0)^2 \langle (Z^k_{[L^{-k-1}x_1]})^2 \rangle_0^N . \end{aligned} \tag{10}$$

Now, since $y_1 \equiv [L^{-k-1} x_1] \neq [L^{-k-1} x_2] \equiv y_2$ for $0 \leqq k \leqq j-2$, we get

$$\langle Z^k_{y_1} Z^k_{y_2} \rangle_0^N = -\int_0^1 d\lambda \langle Z^k_{y_1} Z^k_{y_2} ; U^k(Z^k) \rangle_{0,\lambda}^N, \tag{11}$$

where on the right hand side we replace U^k in (1.10) by λU^k. Using Proposition 2 and (1.5) we easily bound (11) by

$$|\langle Z^k_{y_1} Z^k_{y_2} \rangle_0^N| \leqq c\kappa \exp[-\varepsilon |y_1 - y_2|], \tag{12}$$

and thus the first term A on the right hand side of (10) is bounded by

$$A \leqq C\kappa [1 + |x_1 - x_2|]^{-d} \leqq C\kappa L^{-(j-1)d}, \tag{13}$$

since $|x_1 - x_2| \geqq \frac{1}{2}(L^{j-1} - L^{N_1}) \geqq \frac{1}{4} L^{j-1}$ if $j > N_1 + 1$.

Similarly one shows that uniformly in k

$$|\langle (Z^k_y)^2 \rangle_0^N - \int Z^2 dx(z)| \leqq C\kappa. \tag{14}$$

Thus combining (10), (13), and (14) we get for κ small enough (9) with ε replaced by $\frac{\varepsilon}{2}$ (say) and Theorem 2 is proved in the free case.

We will now proceed with establishing the renormalization group transformation properties of the correlation functions.

3. RG Transformations for Correlation Functions

Let F be a function of ϕ. We define the first RG-transformation of F, S_1F by computing

$$\begin{aligned}\langle F\rangle_{V^N}^N &= \langle Fe^{-V^N}\rangle_0^N / \langle e^{-V^N}\rangle_0^N = \langle\langle Fe^{-V^N}\rangle_{v_N}\rangle_0^{N-1} / \langle\langle e^{-V^N}\rangle_{v_N}\rangle_0^{N-1} \\ &= \left\langle \frac{\langle Fe^{-V^N}\rangle_{v_N}}{\langle e^{-V^N}\rangle_{v_N}} \langle e^{-V^N}\rangle_{v_N} \right\rangle_0^{N-1} \Big/ \langle\langle e^{-V^N}\rangle_{v_N}\rangle_0^{N-1} \\ &= \langle S_1 F\rangle_{T,V^N}^{N-1},\end{aligned} \tag{1}$$

where

$$(S_1F)(\phi') = \frac{\left\langle F\left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}\cdot]} + z^0_{\cdot}\right) e^{-V^N\left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}\cdot]} + z^0_{\cdot}\right)}\right\rangle_{v_N}}{\left\langle e^{-V^N\left(L^{-\frac{d}{2}}\phi^1_{[L^{-1}\cdot]} + z^0_{\cdot}\right)}\right\rangle_{v_N}}, \tag{2}$$

and we denoted

$$\langle - \rangle_{v_N} = \int - \, dv_N\,. \tag{3}$$

The successive transformations S_k are defined similarly by replacing N in (2) by $N-k+1$, ϕ^1 by ϕ^k and z^0 by z^{k-1}. So

$$\langle F\rangle_{V^N}^N = \langle S_k \dots S_1 F\rangle_{T_k \dots T_1 V}^{N-k}. \tag{4}$$

4. The Local Model

As a motivation to the general case we will establish Theorem 2 in a local model defined by $U=0$ and

$$V^N(\phi) = \sum_{x\in\Lambda_N} v(\phi_x). \tag{1}$$

From (1.1), (1.10), and (3.2) we see that T_1 preserves locality

$$(TV^N)(\phi') = \sum_{x\in\Lambda_{N-1}} tv(\phi'_x), \tag{2}$$

with

$$\begin{aligned} tv(\phi) = &-\log\int \exp\left[-\sum_{x\in b_0^1} v\left(\mathrm{L}^{-\frac{d}{2}}\phi + \mathscr{A}(x)Z\right)\right] dx(Z) \\ &+ \log\int \exp\left[-\sum_x v(\mathscr{A}(x)Z)\right] dx(Z).\end{aligned} \tag{3}$$

Since ϕ is bounded we will consider t as a transformation on $C_e^4(-\beta,\beta)$, where e stands for even and β is large enough. Let us write

$$v(\phi) = \tfrac{1}{2}c\phi^2 + \tilde{v}(\phi), \tag{4}$$

$$tv(\phi) = \tfrac{1}{2}c'\phi^2 + \tilde{v}'(\phi), \tag{5}$$

with

$$\frac{d^2\tilde{v}}{d\phi^2}(0) = \frac{d^2\tilde{v}'}{d\phi^2}(0) = 0. \tag{6}$$

An analogue of Proposition 1, proven in a simple manner in [1] is

Proposition 3. *There are* $0<\delta<1$ *and* $\alpha>0$ *such that for* $0<\eta$ *small enough*

$$\left|\frac{d^4\tilde{v}}{d\phi^4}\right| \leqq \eta \tag{7}$$

implies that

$$\left|\frac{d^4\tilde{v}'}{d\phi^4}\right| \leqq \delta\eta \tag{8}$$

and

$$|c'-c| \leqq \alpha\eta . \tag{9}$$

Let us consider S_1 first on a localized F:

$$F(\phi)=f(\phi_y).$$

Let $y_n=[L^{-n}y]$. Then, from (2) $S_n \dots S_1 F$ is localized at y_n:

$$S_n \dots S_1 F = f_n(\phi^n_{y_n}) \tag{10}$$

and

$$\begin{aligned} f_{n+1}(\phi) = S^{y_{n+1}}_{v_n} f_n(\phi) \equiv \int f_n\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(y_{n+1})Z\right) e^{-\sum_x v_n\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(x)Z\right)} dx(Z) \\ \cdot \left[\int \exp\left[-\sum_x v_n\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(x)Z\right)\right] dx(Z)\right]^{-1}, \end{aligned} \tag{11}$$

where $v_n=t^n v$. We often suppress the y-dependence of s^y_v below.

We will consider s_v, given by (11), as a transformation on $C^4(-\beta,\beta)$. We deal separately with the even and odd functions, C^4_e, C^4_0. Let $f\in C^4_0$. We write

$$f(\phi)=\mu\phi+\bar{f}(\phi), \quad \frac{d^k\bar{f}}{d\phi^k}(0)=0 \quad k=0,1,2. \tag{12}$$

For even g,

$$\begin{aligned} g(\phi)=\alpha+\beta\phi^2+\bar{g}(\phi), \\ \frac{d^k\bar{g}}{d\phi^k}(0)=0, \quad k=0,1,2,3. \end{aligned} \tag{13}$$

We need

Lemma 1. (a) *Let* $f\in C^4_0$, $v\in C^4_e$ *be such that for* $k=3,4$

$$\left|\frac{d^k\bar{f}}{d\phi^k}\right| \leqq \eta, \quad \left|\frac{d^4\tilde{v}}{d\phi^4}\right| \leqq \varepsilon\eta . \tag{14}$$

There exist $\varepsilon>0$, $\eta_0>0$ *such that for* $0<\eta<\eta_0$

$$(S_v f)(\phi)=L^{-\frac{d}{2}}(\mu'\phi+\bar{f}'(\phi)), \tag{15}$$

where

$$|\mu-\mu'|\leqq c\eta\,,\qquad \left|\frac{d^k\bar{f}'}{d\phi^k}\right|\leqq\delta\eta\,,\qquad k=3,4 \tag{16}$$

for some $0<\delta<1$.

(b) *Let* $g,\, v\in C_e^4$ *with*

$$\left|\frac{d^4\bar{g}}{d\phi^4}\right|\leqq\eta\,,\qquad \left|\frac{d^4\tilde{v}}{d\phi^4}\right|\leqq\varepsilon\eta\,. \tag{17}$$

There are $\varepsilon>0$, $\eta_0>0$ *such that for* $\eta<\eta_0$

$$(s_v^y g)(\phi)=\alpha'+L^{-d}[\beta'\phi^2+\bar{g}'(\phi)]\,, \tag{18}$$

where

$$|\alpha'-\alpha-\mathscr{A}(y)^2\beta\int z^2dx_c|\leqq c\eta$$

with

$$dx_c(z)=\mathscr{N}^{-1}\exp\left[-\frac{c}{2}\sum_x\mathscr{A}(x)^2Z^2\right]dx(Z) \tag{19}$$

and

$$\left|\frac{d^4\bar{g}'}{d\phi^4}\right|\leqq\delta\eta\,,\qquad 0<\delta<1\,. \tag{20}$$

Proof. (a) From (11) we get

$$(s_v^y f)(\phi)=L^{-\frac{d}{2}}\mu\phi+\left\langle\mu\mathscr{A}(y)Z+\bar{f}\left(L^{-\frac{d}{2}}\phi+\mathscr{A}(y)Z\right)\right\rangle_\phi\,, \tag{21}$$

where $\langle-\rangle_\phi$ is in $\mathscr{N}^{-1}\exp\left[-\sum_x\tilde{v}\left(L^{-\frac{d}{2}}\phi+\mathscr{A}(x)Z\right)\right]dx(z)$.

Hence

$$\mu'=\mu+\left\langle\frac{d\bar{f}}{d\phi}(\mathscr{A}(y)Z)\right\rangle_{\phi=0}-\sum_{x\in b_0}\left\langle\bar{f}(\mathscr{A}(y)Z);\frac{d\tilde{v}}{d\phi}(\mathscr{A}(x)Z)\right\rangle^T_{\phi=0}\,, \tag{22}$$

and

$$\frac{d^k\bar{f}^1}{d\phi^k}=L^{-\frac{d(k-1)}{2}}\sum_{p=1}^{k}(-1)^{p-1}\sum_{\{I_j\}_{j=1}^p}\sum_{\substack{\{x_j\}_{j=2}^p\\ x_j\in b_0}}\left\langle\frac{d^{|I_1|}f}{d\phi^{|I_1|}};\frac{d^{|I_2|}\tilde{v}}{d\phi^{|I_2|}}(\phi');\ldots;\frac{d^{|I_p|}\tilde{v}}{d\phi^{|I_p|}}(\phi')\right\rangle^T_\phi \tag{23}$$

for $k=3,4$, where $\phi'_j=L^{-d/2}\phi+\mathscr{A}(x_j)Z$ and $\{I_j\}$ run through the partitions of $\{1,\ldots,k\}$, with I_1 possibly empty. Since $\bar{f}(\phi)=\frac{1}{2}\int_0^\phi\frac{d^3\bar{f}}{d\phi^3}(\psi)(\psi-\phi)^2d\psi$, the $p=1$ term gives η by (14), the $p=2$, $I_1=\emptyset$ gives $0(\eta\varepsilon)$, and the rest of the terms are $0(\eta^2)$. Thus

$$\left|\frac{d^k f^1}{d\phi^k}\right| = L^{-\frac{d(k-1)}{2}}(\eta + c\eta\varepsilon + c\eta^2) \leqq \delta\eta \tag{24}$$

for ε, η small, $k=3,4$. The claim for μ follows similarly.

(b) We get from (11)

$$(s_v^y g)(\phi) = \alpha + L^{-d}\beta\phi^2 + \left\langle 2L^{-\frac{d}{2}}\beta\mathscr{A}(y)\phi Z + \mathscr{A}(y)^2\beta Z^2 + \bar{g}\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(y)Z\right)\right\rangle_\phi . \tag{25}$$

Noting that $\langle Z^2\rangle_{\phi=0} = \int Z^2 d\chi + 0(\varepsilon\eta)$, we can proceed as in (a) □

Consider now the two point function $\langle\phi_{x_1}\phi_{x_2}\rangle$. If $F=f^1f^2$ with f^i localized at x_i, $\left[\frac{x_1}{L}\right] \neq \left[\frac{x_2}{L}\right]$ then

$$S_1 F = s_v^{x_1} f^1 s_v^{x_2} f^2 . \tag{26}$$

In fact, we may iterate (26). Let b_0^j be the smallest block containing x_1 and x_2. Then

$$\langle\phi_{x_1}\phi_{x_2}\rangle = \langle f_{j-1}^1 f_{j-1}^2\rangle , \tag{27}$$

where

$$f_{j-1}^i = s_{v_{j-2}} \dots s_v f^i \tag{28}$$

and we suppressed the $\left[\frac{x_i}{L^m}\right]$'s.

Let v now satisfy the assumptions of Lemma 1. Taking $\delta<1$ suitably we can apply Proposition 3 and Lemma 1 to obtain

$$f_{j-1}^i = L^{-\frac{d}{2}(j-1)}[\mu^i\phi_{[L^{-j+1}x_i]} + \bar{f}^i(\phi_{[L^{-j+1}x_i]})] , \tag{29}$$

where

$$|\mu^i - 1| \leqq c\eta , \tag{30}$$

$$\left|\frac{d^k\bar{f}^i}{d\phi^k}\right| \leqq \delta^{j-1}\eta . \tag{31}$$

Next we compute, denoting $[L^{-j+1}x_i]$ by y_i:

$$\begin{aligned} S_j(f_{j-1}^1 f_{j-1}^2)(\phi) = L^{-d(j-1)}\Big[&L^{-d}\mu^1\mu^2\phi^2 + \Big\langle \mu^1\mu^2\mathscr{A}(y_1)\mathscr{A}(y_2)Z^2 \\ &+ \mu^1\mu^2 L^{-\frac{d}{2}}\mathscr{A}(y_2)\phi Z + \mu_1\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(y_1)Z\right)\bar{f}^2\left(L^{-\frac{d}{2}}\phi + \mathscr{A}(y_2)Z\right) \\ &+ (1\leftrightarrow 2) + \bar{f}^1\bar{f}^2\Big\rangle_{\phi, v_{j-1}}\Big] \equiv L^{-d(j-1)}[\alpha + L^{-d}\beta\phi^2 + \bar{g}(\phi)] . \end{aligned} \tag{32}$$

Similarly as in Lemma 1 we get

$$\alpha = [\mathscr{A}(y_1)\mathscr{A}(y_2) + 0(\eta)] \int Z^2 dx_c, \tag{33}$$

$$\beta = 1 + 0(\eta), \tag{34}$$

$$\left|\frac{d^4\bar{g}}{d\phi^4}\right| \leqq c\delta^j\eta. \tag{35}$$

Now, since $\frac{d^4 v_i}{d\phi^4} \leqq \varepsilon\delta^j\eta$, we can apply Lemma 1 and Proposition 3 to iterate (32). Let $x_1 \in b_0^{j-1}$ so that $y_1 = 0$. Then we get

$$\langle\phi_{x_1}\phi_{x_2}\rangle = L^{-d(j-1)}\mathscr{A}(0)\mathscr{A}(y_2)[1 + 0(\eta)] \int Z^2 dx + \sum_{k=j}^{N-1} L^{-dk}\mathscr{A}(0)^2 \beta_k \int Z^2 dx + L^{-d(N-1)} 0(\delta^{N-1}\eta), \tag{36}$$

where

$$|\beta_n - 1| \leqq c\eta \quad \text{for all } n. \tag{37}$$

Now (36) yields $|\langle\phi_{x_1}\phi_{x_2}\rangle| \leqq CL^{-dj} \leqq C'[1 + (|x_1 - x_2|)]^{-d}$ which is B of Theorem 2. (36) is up to $0(\eta)$ terms (7) and we can proceed as there to establish the claim C of Theorem 2. □

5. The Nonlocal Model – Localized Expectations

As in the local case we will start with localized F:s, when studying $S_1 F$ of (3.2). For this purpose let F be given by its Taylor series at $\phi = 0$:

$$F(\phi) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\bar{x} \in \Lambda^m} F_m(\bar{x})\phi_{\bar{x}}, \tag{1}$$

where

$$F_m(x_1, \ldots, x_m) = F_m(x_{\pi(1)}, \ldots, x_{\pi(m)}). \tag{2}$$

We say that F is localized at point s, with constants α, D if for all m.

$$\sum_{\bar{x}} \exp\left[\frac{1}{L} AL(s \cup \underline{x})\right] |F_m(\bar{x})| \leqq D\alpha^m m! \tag{3}$$

Let now $S^V F$ be the RG transform of F:

$$S^V F(\phi) = \frac{\left\langle F\left(L^{-\frac{d}{2}}\phi_{[L^{-1}\cdot]} + z\right) \exp\left[-V\left(L^{-\frac{d}{2}}\phi_{[L^{-1}\cdot]} + z\right)\right]\right\rangle_v}{\left\langle \exp\left[-V\left(L^{-\frac{d}{2}}\phi_{[L^{-1}\cdot]} + z\right)\right]\right\rangle_v}, \tag{4}$$

where for generality we drop all the indices, i.e. N is general as is k (in S_k), and v is a measure involving U satifying (1.5)–(1.7). As an analogue of Lemma 1 in the local case we prove

Lemma 2. *Let F satisfy* (3) *and let $|V|_A<\eta$. There exist A_0, $\alpha_0(A)$ such that for $A\geqq A_0$, $\alpha\leqq\alpha_0(A)$ and $\eta\leqq\alpha$*

$$\sum_{\bar{x}}|S^V F_m(\bar{x})|\exp\left[-\frac{A}{L}L([L^{-1}s]\cup\underline{x})\right] \leqq De^{C\alpha^{1/2}}(\delta\alpha)^m m!\cdot\begin{cases}L^{-\frac{d}{2}}\delta^{-1}, & m=1\\ L^{-d}\delta^{-2}, & m\geqq 2\end{cases} \tag{5}$$

for some $0<\delta<1$. Moreover,

$$S^V(C\phi_s+F)=CL^{-\frac{d}{2}}\phi_{[L^{-1}s]}+F' \tag{6}$$

and if $\eta<\alpha^2$, then

$$S^V(C\phi_s^2+F)=CL^{-d}(\phi_{[L^{-1}s]})^2+F'', \tag{7}$$

where F', F'' satisfy (5).

Proof. Differentiating (4) with respect to ϕ we obtain

$$(S^V F)_m(\bar{x})=L^{-\frac{dm}{2}}\sum_{I_0\subset\{1,\dots,m\}}\sum_{k=0}^{m-|I_0|}(-1)^k\sum_{\{I_j\}_{j=1}^k}{}' \cdot\sum_{\bar{y}:[L^{-1}y_i]=x_i}\left\langle\frac{\delta^{|I_0|}F}{\delta\phi_{\bar{y}_{I_0}}};\frac{\delta^{|I_1|}V}{\delta\phi_{\bar{y}_{I_1}}};\dots;\frac{\delta^{|I_k|}V}{\delta\phi_{\bar{y}_{I_k}}}\right\rangle^T. \tag{8}$$

where $\sum'_{\{I_j\}}$ is the sum over partitions of $\{1,\dots,m\}\backslash I_0$ into sets I_j. The set I_0 may be empty. $\delta\phi_{\bar{y}_I}=\prod_{i\in I}\delta\phi_{y_i}$ and $\langle-\rangle^T$ is the truncated expectation in

$$\langle-\rangle=\mathcal{N}^{-1}\int -e^{-V(z)}dv(z). \tag{9}$$

Now

$$\frac{\delta^{|I_0|}F}{\delta\phi_{\bar{y}_{I_0}}}(z)=\sum_{m_0\geqq|I_0|}((m_0-|I_0|)!)^{-1}\sum_{(v_1,\dots,v_{m_0-|I_0|})}F_{m_0}(\bar{y}_{I_0},\bar{v})z_{\bar{v}}, \tag{10}$$

and similarly for V. Using (1.1) we write

$$z_{\bar{v}}=\prod_{i=1}^{m_0-|I_0|}\mathcal{A}(v_i-Lu_i)\prod_{i=1}^{m_0-|I_0|}Z_{\bar{u}_i}\equiv\mathcal{A}(\bar{v},\bar{u})Z_{\bar{u}}, \tag{11}$$

where

$$u_i=[L^{-1}v_i]. \tag{12}$$

Inserting to (8) we obtain

$$(S^V F)_m(\bar{x}) = L^{-\frac{md}{2}} \sum_{I_0, k \geqq 1} (-1)^k \sum_{\{I_j\}} \sum_{\substack{\{m_j\} \\ m_j \geqq |I_j|+1}} \sum_{\bar{y}:[L^{-1}\bar{y}]=\bar{x}}$$

$$\cdot \sum_{\{\bar{v}_j\}} \prod_j \left[\frac{1}{(m_j - |I_j|)!} \mathscr{A}(\bar{v}_j, \bar{u}_j)\right] F_{m_0}(\bar{y}_{I_0}, \bar{v}_i) \prod_{j=1}^{k} \tilde{V}_{m_i}(\bar{y}_{I_j}, \bar{v}_j)$$

$$\cdot \left\langle \prod_{i=0}^{k} Z_{\bar{u}_i} \right\rangle^T + L^{-\frac{md}{2}} \sum_{m_0 \geqq m} \sum_{\bar{y}:[L^{-1}\bar{y}]=\bar{x}} \sum_{\bar{v}} ((m_0 - m)!)^{-1}$$

$$\cdot F_{m_0}(\bar{y}, \bar{v}) \langle z_{\bar{v}} \rangle \equiv F_m^{(1)}(\bar{x}) + F_m^{(2)}(\bar{x}). \tag{13}$$

The reader is invited to compare (13) with (5.18) of [1]. The estimation proceeds now analogously as there. Consider first the second term $F_m^{(2)}$. Let $H_m^{(1)}(\bar{x}) = \exp\left[\frac{1}{L} AL\left(\left[\frac{s}{L}\right] \cup \bar{x}\right)\right] F_m^{(1)}(\bar{x})$ (and $H_m^{(2)}$ respectively) and $H_m(\bar{x}) = \exp\left[\frac{1}{L} AL([L^{-1}s] \cup \bar{x})\right] F_m(\bar{x})$.

Then using Proposition 2

$$\sum_{\bar{x}} |H_m^{(2)}(\bar{x})| \leqq L^{-\frac{md}{2}} \sum_{m_0 \geqq m} \frac{C^{m_0 - m}}{(m_0 - m)!} \sum_{\bar{y}, \bar{v}} |H_{m_0}(\bar{y}, \bar{v})|$$

$$\cdot \exp\left[-\frac{A}{L}(L(s \cup \underline{y} \cup \underline{v}) - L([L^{-1}s] \cup [L^{-1}\underline{y}]))\right]. \tag{14}$$

Some straightforward tree estimates (see Lemma 2 in [1]) bound

$$L(s \cup \underline{y} \cup \underline{v}) - L\left(\left[\frac{s}{L}\right] \cup [L^{-1}\underline{y}]\right) \geqq 0. \tag{15}$$

Thus by assumption (3)

$$\sum_{\bar{x}} |H_m^{(2)}(\bar{x})| \leqq L^{-\frac{md}{2}} \sum_{m_0 = m}^{\infty} \frac{C^{m_0 - m}}{(m_0 - m)!} D\alpha^{m_0} m_0! = D\left(L^{-\frac{d}{2}}\alpha\right)^m m! \sum_{\ell=0}^{\infty} (C\alpha)^{\ell} \binom{m+\ell}{m}$$

$$= D\left(L^{-\frac{d}{2}}\alpha\right)^m m!(1 - C\alpha)^{-m-1}. \tag{16}$$

For the first term in (13) we get by Proposition 2

$$\sum_{\bar{x}} |H_m^{(1)}(\bar{x})| \leqq L^{-\frac{md}{2}} \sum_{I_0, k} \sum_{\{I_j\}} \sum_{\{m_j\}} \sum_{\bar{y}, \{\bar{v}_j\}} |H_{m_0}(\bar{y}_{I_0}, \bar{v}_0)| \prod_{j=1}^{k} |\tilde{W}(\bar{x}_{I_j}, \bar{v}_j)| \prod_{j=0}^{k} \frac{C^{m_j - |I_j|}}{(m_j - |I_j|)!} \prod_r M_r!$$

$$\cdot \exp\left[\frac{A}{L} L([L^{-1}s] \cup [L^{-1}\underline{y}]) + D\sum_j L(\underline{u}_j) - \frac{A}{L}\sum_j L(\underline{y}_{I_j} \cup \underline{v}_j) - \frac{A}{2} L(u_1; \ldots; u_k)\right], \tag{17}$$

where we denoted by $\tilde{W}(\bar{x}) \equiv e^{\frac{1}{L}AL(x)} \tilde{V}(\bar{x})$. Now, it was shown in [1] (Lemmas 2–4) that one can obtain from the last two terms in the exponent in (17) enough decay to control the $\bar{y}$, $\bar{v}$ sums in (17) in terms of the estimates (2) and $|V|_A < \eta$:

$$\sum_{\bar{y}} \sum_{\{\bar{v}_j\}} |H_{m_0}(\bar{y}_{I_0}, \bar{v}_0)| \prod_{j=1}^{k} \tilde{W}(x_{I_j}, \bar{v}_j) \prod M_r! \exp[\ldots] \leqq \prod_{j=0}^{k} C^{m_j - |I_j|} D\alpha^{m_0} m_0! \prod_{j=1}^{k} \eta^{m_j} m_j! (k+1)! \, e^{CAk}, \tag{18}$$

and thus, since $\eta \leqq \alpha$:

$$\sum_{\bar{x}} |H_m^{(1)}(\bar{x})| \leqq DL^{-\frac{md}{2}} \sum_{I_0} \sum_{k=1}^{m-|I_0|} e^{CAk}(k+1)! \sum_{\{I_j\},\{m_j\}} \cdot \prod_{j=0}^{k} \left(\frac{C^{m_j - |I_j|}}{(m_j - |I_j|)!} m_j! \alpha^{m_j} \right) \leqq L^{-\frac{md}{2}} D \sum_{n_0=0}^{m} \binom{m}{n_0} \sum_{k=1}^{m-n_0} e^{CAk}(k+1)!$$

$$\sum_{\substack{(n_1,\ldots,n_k): \sum n_j = m - n_0 \\ n_j \geqq 1}} (k!)^{-1} \frac{(m-n_0)!}{\prod_1^k n_j!} \sum_{\{m_j \geqq n_j + 1\}_0^k} \prod_{j=0}^{k} \frac{C^{m_j - n_j}}{(m_j - n_j)!} m_j! \alpha^{m_j}$$

$$\leqq Dm! \left(L^{-\frac{d}{2}} \alpha\right)^m \sum_{n_0, k} e^{CAk} \sum_{\{n_j\}} \prod_j \left(\sum_{m_j > n_j} (\alpha C)^{m_j - n_j} \binom{m_j}{n_j} \right). \tag{19}$$

Using

$$\sum_{m_j > n_j} (C\alpha)^{m_j - n_j} \binom{m_j}{n_j} = (1 - C\alpha)^{-n_j - 1} - 1 \leqq C\alpha^{1/2} e^{C\alpha^{1/2} n_j}, \tag{20}$$

Eq. (19) gives

$$\sum_{\bar{x}} |H_m^{(1)}(\bar{x})| \leqq Dm! \left(L^{-\frac{d}{2}} \alpha\right)^m \sum_{n_0, k} e^{CAk} \alpha^{\frac{k+1}{2}} \binom{m - n_0 - 1}{k-1} e^{C\alpha^{1/2} m}$$

$$\leqq Dm! \left(L^{-\frac{d}{2}} \alpha\right)^m m\alpha e^{C\alpha^{1/2} m} \leqq Dm! \left(L^{-\frac{d}{2}} \alpha\right)^m \alpha^{1/2} e^{C\alpha^{1/2} m} \tag{21}$$

for $\alpha_0(A)$ sufficiently small. Equations (16) and (21) yield (5). For (6) and (7) note that $F'(F'')$ involve $k \geqq 1$ terms in (8) and the extra η's can be used to compensate for the lack of α in $C\phi(C\phi\phi)$. □

6. The Nonlocal Model, Two-Point Function

Consider now the two-point function $\langle \phi_{x_1} \phi_{x_2} \rangle_{V^N}^N$. Use the notation

$$\langle - \rangle_{\phi^k} \equiv \langle - \rangle_{T^k V^N}^{N-k}, \tag{1}$$

$$\langle F \rangle_{Z^k} \equiv (S_{k+1} F)(\phi^{k+1}) = \frac{\left\langle (F e^{-V^{N-k}}) \left(L^{-\frac{d}{2}} \phi_{[L^{-1}\cdot]}^{k+1} + z^k \right) \right\rangle_{v^k}}{\left\langle \exp\left[- V^{N-k} \left(L^{-\frac{d}{2}} \phi_{[L^{-1}\cdot]}^{k+1} + z^k \right) \right] \right\rangle_{v^k}}, \tag{2}$$

$$V^{N-k} = T^k V^N, \tag{3}$$

and

$$F_i^0(\phi^0) = \phi_{x_i}. \tag{4}$$

Then

$$\begin{aligned}\langle \phi_{x_1}\phi_{x_2}\rangle &= \langle F_1^0 ; F_2^0\rangle_\phi \\ &= \langle \langle F_1^0\rangle_{Z^0} ; \langle F_2^0\rangle_{Z^0}\rangle_{\phi^1} + \langle\langle F_1^0 ; F_2^0\rangle_{Z^0}\rangle_{\phi^1} \\ &= \langle F_1^n ; F_2^n\rangle_{\phi^n} + \sum_{\ell=0}^{n-1} \langle\langle F_1^\ell ; F_2^\ell\rangle_{Z^\ell}\rangle_{\phi^{\ell+1}},\end{aligned} \tag{5}$$

where

$$F_i^n = \langle F_i^{n-1}\rangle_{Z^n} = S_n F^{n-1} = S^{V^{N-n}} F^{n-1}. \tag{6}$$

By Lemma 2

$$F_i^1 = L^{-\frac{d}{2}} \phi^1_{[L^{-1}x_i]} + \tilde{F}_i^1(\phi^1), \tag{7}$$

with

$$\sum_{\bar{x}} |\tilde{F}_m^1(\bar{x})| e^{\frac{A}{L} L(\underline{x})} \leqq D\eta^m m! \tag{8}$$

Since $|V^{N-1}|_A \leqq \delta\eta$, we can iterate

$$F_i^n = L^{-n\frac{d}{2}} \phi^n_{[L^{-n}x_i]} + \tilde{F}_i^n(\phi^n), \tag{9}$$

with

$$\begin{aligned}\sum_{\bar{x}} |\tilde{F}_m^n(\bar{x})| e^{\frac{A}{L} L([L^{-n}x_i]\cup \underline{x})} &\leqq D L^{-(n-1)\frac{d}{2}} m!\eta^m \\ &\cdot \delta^{(n-1)m-1} \exp \sum_{\ell=1}^{n-1} C(\eta\delta^\ell)^{1/4} \leqq C L^{-n\frac{d}{2}} (\eta\delta^{n-1})^m m! \delta^{1-n},\end{aligned} \tag{10}$$

where C is independent on n. Let $x_1 \in b_0^{n-1} \not\ni x_2 \in b_0^n$. Hence $[L^{-n}x_i] = 0$.

Then

$$F_1^n F_2^n = L^{-nd}\phi_s^2 + \tilde{\tilde{F}}, \tag{11}$$

where

$$\sum_{\bar{x}} |\tilde{\tilde{F}}_m(\bar{x})| \exp\left[\frac{A}{L} L([L^{-n}x_1]\cup x)\right] \leqq C L^{-nd} (2\eta\delta^{n-1})^m m! \tag{12}$$

Indeed, consider e.g.

$$(\tilde{F}_1^n \tilde{F}_2^n)_m(\bar{x}) \equiv C_m(\bar{x}) = \sum_{I_1\cup I_2 = \{1,\dots,m\}} \tilde{F}^n_{1|I_1|}(\bar{x}_{I_1}) \tilde{F}_{2|I_2|}(\bar{x}_{I_2}). \tag{13}$$

Then

$$\sum_{\bar{x}} |C_m(\bar{x})| e^{\frac{A}{L} L(s \cup \underline{x})} \leqq \sum_{I_1, I_2} \sum_{\bar{x}_{I_1}, \bar{x}_{I_2}} \prod_{i=1}^{2} |\tilde{H}^n_{i|I_i|}(\bar{x}_{I_i})|$$

$$\cdot \exp\left[\frac{A}{L}(L(s \cup \underline{x}) - L(s \cup \underline{x}_{I_1}) - L(s \cup \underline{x}_{I_2}))\right] \leqq C \sum_{\substack{n_1 + n_2 = m \\ n_1 > 0}} \frac{m!}{n_1! n_2!}$$

$$\cdot L^{-nd}(\eta \delta^{n-1})^m n_1! n_2! \leqq C L^{-nd}(2\eta \delta^{n-1})^m m!, \tag{14}$$

where $\eta^{1/2} \leqq \alpha \leqq \alpha_0(A)$. The other 2 terms have similar bounds.

Since $\langle F^n_1 ; F^n_2 \rangle_{\phi^n} = \langle F^n_1 F^n_2 \rangle_{\phi^n}$ we may iterate (5)

$$\langle \phi_{x_1} \phi_{x_2} \rangle = (S_N S_{N-1} \ldots (F^n_1 F^n_2))(0) + \sum_{\ell=0}^{n-1} \langle \langle F^\ell_1 ; F^\ell_2 \rangle_{Z^\ell} \rangle_{\phi^{\ell+1}}. \tag{15}$$

For the first term we apply (7) of Lemma 2. Denoting

$$S_M \ldots S_{n+1} F^n_1 F^n_2 \equiv G^M,$$

we get

$$G^M(\phi) = L^{-Md} \phi^2 + \tilde{G}^M(\phi), \tag{16}$$

with ($m \geqq 2$)

$$\sum_{\bar{x}} |\tilde{G}^M_m(\bar{x})| e^{\frac{A}{L}} L(\underline{x}) \leqq C L^{-Md}(2\eta \delta^{M-1})^m m! \delta^{1-M}, \tag{17}$$

where C is uniform in M. Thus

$$G^N(0) = C \sum_{M=n}^{N-1} L^{-Md} \mathscr{A}(0)^2 \langle (Z^M)^2 \rangle_M + \sum_{M=n}^{N-1} \sum_{m=1}^{\infty} \sum_{\bar{x}} ((2m)!)^{-1} \tilde{G}^M_{2m}(\bar{x}) \langle z^M_{\bar{x}} \rangle_M, \tag{18}$$

where

$$\langle - \rangle_M = \mathscr{N}^{-1} \int - \exp[-V^{N-M}(z^M)] \, dv_M(Z^M). \tag{19}$$

Using Proposition 2, (18) and proceeding as in Sect. 5 one gets

$$G^N(0) = \sum_{M=n}^{N-1} L^{-Md} \mathscr{A}(0)^2 (\int Z^2 dx + 0(\eta)). \tag{20}$$

For the second term in (15) we compute again, denoting

$$\langle F^\ell_1 ; F^\ell_2 \rangle_{Z^\ell} \quad \text{by} \quad K^\ell(\phi^{\ell+1}):$$

$$K^\ell_m(\bar{x}) = L^{-\frac{md}{2}} \sum_{\substack{I_1, I_2 \subset \{1, \ldots, m\} \\ I_1 \cap I_2 = \emptyset}} \sum_{k=2}^{m+2-|I_1|-|I_2|} \sum_{\{I_j\}_{j=3}^k} \sum_{\{m_j \geqq I_j + 1\}}$$

$$\cdot \sum_{[L^{-1}\bar{y}] = \bar{x}} \sum_{\{\bar{v}_j\}} \left(\prod_j \frac{1}{(m_j - |I_j|)!} \right) F^\ell_{1m_1}(\bar{y}_{I_1}, v_1) F^\ell_{2m_2}(\bar{y}_{I_2}, v_2)$$

$$\cdot \prod_{j=3}^{k} \tilde{V}^{N-\ell}_{m_j}(\bar{y}_{I_j}, \bar{v}_j) \left\langle \prod_{j=1}^{k} z^\ell_{\bar{v}_j} \right\rangle^T_\ell. \tag{21}$$

As in Lemma 1, we convert the F's to H's and $\tilde{V}$'s to W's and gain an exponential

$$\exp\left[-\frac{A}{L}\left(\sum_{i=1}^{2} L([L^{-\ell}x_i]\cup \underline{y}_{I_i}\cup \underline{v}_i\right)+\sum_{i=3}^{k} L(\underline{y}_{I_i},\underline{v}_i)\right.$$
$$\left.+LL([L^{-1}\underline{v}_1];\dots;[L^{-1}\underline{v}_k])-D\sum_{j=1}^{k} L([L^{-1}\underline{v}_j])\right]. \tag{22}$$

Let $m>0$. As explained in the proof of Lemma 2, we may extract from (22) an $\exp\{-\varepsilon A|[L^{-\ell}x_1]-[L^{-\ell}x_2]|\}e^{ACk}$ factor, together with enough tree structure to bound the $\bar{y}$ and $\bar{v}_j$ sums:

$$\sum_{\bar{x}}|K_m^\ell(\bar{x})|\leqq L^{-\frac{md}{2}}\sum_{I_1I_2k}\sum_{\{I_i\}\{m_j\}}\prod\frac{C^{m_j-|I_j|}}{(m_j-|I_j|)!}$$
$$\cdot e^{ACk}k!CL^{-\ell d}(\eta\delta^{\ell-1})^{m_1+m_2-2}m_1!m_2!\prod_{j=3}^{k}(\eta\delta^\ell)^{m_j}m_j!$$
$$\cdot\exp[-\varepsilon AL^{-\ell}|x_1-x_2|], \tag{23}$$

where we used (9) and (10). We may now proceed as in Lemma 2. Noticing that if $m_1=m_2=1$, k must be $\geqq 3$ since $m>0$, there is always an $\eta^{1/4}$ factor to kill e^{AC}. We get

$$\sum_{\bar{x}}|K_m^\ell(\bar{x})|\leqq C\eta^{1/2}L^{-\ell d}\exp[-\varepsilon AL^{-\ell}|x_1-x_2|]\frac{\eta^{(m-2)/2}}{m!}. \tag{24}$$

Let $m=0$. Then

$$K_0^\ell=\sum_{m_1m_2>0}\sum_{\bar{v},\bar{v}_2}F_{1m_1}^\ell(\bar{v}_1)F_{2m_2}^\ell(\bar{v}_2)\langle z_{\bar{v}_1}^\ell;z_{\bar{v}_2}^\ell\rangle_\ell^T$$
$$=L^{-\ell d}\langle z_{x_{\ell 1}}^\ell;\,z_{x_{\ell 2}}^\ell\rangle_\ell^T+\tilde{K}_0^\ell, \tag{25}$$

where we used (9) and denoted $x_{\ell i}=[L^{-\ell}x_i]$. $\tilde{K}_0^\ell$ involves terms of $0(\eta)$ and will easily be shown to satisfy $\tilde{K}_0^\ell\leqq C\eta^{1/4}L^{-\ell d}\exp[-\varepsilon AL^{-\ell}|x_1-x_2|]$. For the first term we have

$$\langle z_{x_{\ell 1}}^\ell;z_{x_{\ell 2}}^\ell\rangle_\ell^T=\mathscr{A}(x_{\ell 1}-Lx_{\ell+1\,1})\mathscr{A}(x_{\ell_2}-Lx_{\ell+1\,2})\langle Z_{x_{\ell+1\,1}}^\ell;Z_{x_{\ell+1\,2}}^\ell\rangle_\ell^T. \tag{26}$$

For $\ell<n-1$

$$|\langle Z_{x_{\ell+1\,1}}^\ell;Z_{x_{\ell+1\,2}}^\ell\rangle_\ell^T|\leqq 0(\eta)e^{-\varepsilon AL^{-\ell}|x_1-x_2|}, \tag{27}$$

and for $\ell=n-1$ $x_{\ell+1\,i}=s$ and

$$\langle Z_s^{n-1};Z_s^{n-1}\rangle_{n-1}^T=\int Z^2dx+0(\eta). \tag{28}$$

Combining (21)–(28) we obtain

$$\sum_{\ell=0}^{n-1}\langle\langle F_1^\ell;F_2^\ell\rangle\rangle=\sum_{\ell=0}^{n-1}\langle K^\ell(\phi^{\ell+1})\rangle_{\phi^{\ell+1}}=L^{-(n-1)d}\mathscr{A}([L^{-n+1}x_1])$$
$$\cdot\mathscr{A}([L^{-n+1}x_2])(\int Z^2dx+0(\eta))+0(\eta)L^{-(n-1)d}. \tag{29}$$

Combining (15), (20), and (29) we finally get

$$\langle\phi_{x_1}\phi_{x_2}\rangle = L^{-nd}0(\eta) + L^{-(n-1)d}\mathscr{A}(0)\mathscr{A}([L^{-n+1}x_1])(\int Z^2dx + 0(\eta)) + \sum_{j=n}^{N-1} L^{-jd}\mathscr{A}(0)^2[\int Z^2dx + 0(\eta)]. \tag{30}$$

We may now proceed as in Sect. 2 to establish the claim. □

7. The Thermodynamic Limit

We will now use Lemma 2 to establish Theorem 1. Let $\{F_i^0\}_{i=1}^s$ be a family of functionals of ϕ localized at points z_j respectively. Using the notation (6.1) and (6.2) it is easy to establish that

$$\left\langle \prod_{i=1}^s F_i^0 \right\rangle^T = \sum_{k=1}^s \sum_{\{I_j\}_{j=1}^k} \left\langle \prod_{j=1}^k \left\langle \prod_{i\in I_j} F_i^0 \right\rangle^T_{Z_0} \right\rangle^T_{\phi_1} \equiv \sum_k \sum_{\{I_j\}} \left\langle \prod_{j=1}^k F_{I_j}^1 \right\rangle^T_{\phi_1}, \tag{1}$$

where we defined

$$F_{I_j}^1 \equiv \left\langle \prod_{i\in I_j} F_i^0 \right\rangle^T_{Z_0}. \tag{2}$$

The basic idea will be to show that the $F_{I_j}^i$'s are localized whence we can use induction in s and the properties of S^V established in Lemma 2 to iterate (1). Thus we start with

Lemma 3. *Let $\{F^j\}_{j=1}^{\ell}$ satisfy*

$$\sum_{\bar{x}} |F_m^j(\bar{x})| \exp\left[\frac{A}{L} L(z_j \cup \underline{x})\right] \leqq D_j \alpha^m m! \tag{3}$$

and let $|V|_A \leqq \eta \leqq \alpha$. Then the function

$$F \equiv \left\langle \prod_{j=1}^{\ell} F^j \right\rangle^T_Z \tag{4}$$

satisfies

$$\sum_{\bar{x}} |F_m(\bar{x})| e^{\frac{A}{L}L([L^{-1}z_j]\cup \underline{x})} \leqq C(\ell, A) \prod_j D_j m! \alpha^m. \tag{5}$$

for all j. $C(\ell, A)$ is independent of $\{z_j\}_{j=1}^{\ell}$.

Proof. As before we compute

$$F_m(\bar{x}) = L^{-\frac{md}{2}} \sum_{(I_1,\dots,I_\ell)} \sum_{k=0}^{m-\sum|I_i|} \sum_{\{I_j\}_{\ell+1}^{\ell+k}} \sum_{\{m_j \geqq 1+|I_j|\}} \sum_{\bar{y},\{\bar{v}_j\}} \prod_j \frac{1}{(m_j - |I_i|)!} \cdot \prod_{j=1}^{\ell} F_{m_j}^j(y_{I_j}, \bar{v}_j) \prod_{j=\ell+1}^{\ell+k} \tilde{V}_{m_j}(y_{I_j}, \bar{v}_j) \langle \prod z_{\bar{v}_j} \rangle^T, \tag{6}$$

where I_j, $j=1\dots\ell$ may be empty. As in Lemma 2

$$
\begin{aligned}
&\sum_{\bar{x}} |F_m(\bar{x})| \exp\left[\frac{A}{L} L([L^{-1}z_j]\cup x)\right] \leqq L^{-\frac{md}{2}} \sum_{\{I_j\},k,\{m_j\}} \sum_{\bar{y},\{\bar{v}_j\}} \\
&\cdot \prod_j \frac{C^{m_j-|I_j|}}{(m_j-|I_j|)!} \prod H^j_{m_j}(\bar{y}_{I_j},\bar{v}_j) \prod W(\bar{y}_{I_j},\bar{v}_j) \prod M_r! \\
&\cdot \exp\left[-\frac{A}{L}\sum_{j=1}^{\ell} L(z_j\cup \underline{y}_{I_j}\cup \underline{v}_j) - \frac{A}{L}\sum_{j=\ell+1}^{\ell+k} L(\underline{y}_{I_j}\cup \underline{v}_j)\right. \\
&-\frac{A}{2} L([L^{-1}\underline{v}_1];\dots;[L^{-1}\underline{v}_{\ell+k}]) \\
&\left.+\frac{A}{L} L\left(\left[\frac{z_1}{L}\right]\cup[L^{-1}\underline{y}]\right) + D\sum L([L^{-1}\underline{v}_j])\right],
\end{aligned}
\tag{7}
$$

where

$$
H^j_{m_j}(\bar{x}) = |F^j_{m_j}(\bar{x})| \exp\left[\frac{A}{L} L([L^{-1}z_j]\cup \underline{x})\right]. \tag{8}
$$

Denoting $[\bar{y}L^{-1}]$ by $\bar{x}$, $[L^{-1}\bar{v}]$ by $\bar{u}$, $[L^{-1}\bar{z}]$ by $\bar{w}$ and using Lemma 2 of [1] to estimate the trees we get

$$
\begin{aligned}
&\exp[\dots] \leqq \exp\left[-\frac{A(1+\varepsilon')}{L}\left(\sum_{j=1}^{\ell} L(w_j\cup \underline{x}_{Ij}\cup \underline{u}_j)\right.\right. \\
&\left. + \sum_{j=\ell+1}^{\ell+k} L(\underline{x}_{Ij}\cup \underline{u}_j)\right) + \frac{A(\ell+k)}{2} + \frac{A}{L} L(w_j\cup \underline{x}) + D\sum L(\underline{u}_j) \\
&\left.-\frac{A}{2} L(\underline{u}_1;\dots;\underline{u}_{\ell+k})\right] \leqq \exp\left[\frac{A(\ell+k)}{2} - \frac{A\varepsilon'}{2L} L(\underline{w})\right. \\
&-\frac{A\varepsilon'}{2L}\left(\sum_{j=1}^{\ell} L(w_j\cup \underline{x}_{Ii}\cup \underline{u}_i) + \sum_{j=\ell+1}^{\ell+k} L(\underline{x}_{I_j}\cup \underline{u}_j)\right) + D\sum L(\underline{u}_j) \\
&\left.-\frac{\varepsilon'\mathrm{A}}{2\mathrm{L}} \mathrm{L}(\underline{u}_1;\dots;\underline{u}_{\ell+k})\right] \leqq \exp\left[\frac{A(\ell+k)}{2} - A\varepsilon'' L(\underline{w})\right. \\
&\left. - A\varepsilon'' L(\underline{u}_{\mathscr{J}}) - A\varepsilon'' \sum_{i=1}^{\ell} |w_i - u_{j_i}|\right],
\end{aligned}
\tag{9}
$$

where $\mathscr{J}$ denotes a sequence $j_1\dots j_\ell$, $1\leqq j_i \leqq m_i - |I_i|$. Now

$$
\prod M_r! \exp[\dots] \leqq \sum_{\mathscr{J}} \prod N_s! \exp[\text{right hand side of (9)}], \tag{10}
$$

where N_s are the multiplicities of the j_i's. Let $\mathscr{J}^1 = (j_1,\dots,j_\ell)$ and $\mathscr{J}^2 = (j_\ell,\dots,j_{\ell+k})$. Then

$$
L(u_{\mathscr{J}}) \geqq \tfrac{1}{2} L(u_{\mathscr{J}^1}) + \tfrac{1}{2} L(u_{\mathscr{J}^2}), \tag{11}
$$

and by Lemma 3 of [1]

$$e^{-\frac{A\varepsilon''}{2}L(\underline{u}_{\mathcal{J}^i})}\prod N_s^i! \leqq C^{k+\ell}\sum_{\tau} e^{-A\varepsilon L_\tau(\underline{u}_{\mathcal{J}^i})}, \tag{12}$$

τ running through trees on $\mathcal{J}^1$ and no other points. But

$$\prod N_s! = \prod (N_s' + N_s^2)! \leqq \prod N_s^1! N_s^2! 2^{N_s^1+N_s^2} \leqq C^{k+\ell}\prod N_s^1! N_s^2!. \tag{13}$$

Combining (9)–(13) we obtain

$$\begin{aligned}&\prod M_r! \exp[\ldots]\\ &\leqq C^{A(k+\ell)}\left[e^{-A\varepsilon L(\underline{w})}\sum_{\mathcal{J}'}\sum_{\tau}\exp\left[-A\varepsilon\left(L_\tau(\bar{u}_{\mathcal{J}'}) + \sum_{i=1}^{\ell}|u_{j_i} - w_i|\right)\right]\right]\\ &\cdot \sum_{\mathcal{J}^2}\sum_{\tau} e^{-A\varepsilon L_\tau(\bar{u}_{\mathcal{J}^2})}.\end{aligned} \tag{14}$$

Using

$$\sum_{j_i} 1 = m_j - |I_j| \tag{15}$$

and

$$e^{-A\varepsilon L(\underline{w})}\sup_{\bar{u}_{\mathcal{J}^1}}\sum_{\tau} e^{-A\varepsilon[L_\tau(\bar{u}_{\mathcal{J}^1}) + \sum |u_{j_i} - w_i|]} \leqq C(\ell), \tag{16}$$

we get finally

$$\prod M_r! \exp[\ldots] \leqq C^{A(k+\ell)}C(\ell)\sum_{\tau, \mathcal{J}^2} e^{-A\varepsilon L_\tau(\bar{u}_{\mathcal{J}^2})}. \tag{17}$$

Using the assumption (3) for F^j and $|V|_A \leqq \eta \leqq \alpha$, the $\bar{y}$, $\bar{v}_j$ sums in (7) are now easily estimated, see [1], giving

$$\begin{aligned}&\sum_{\bar{x}}|F_m(\bar{x})| \exp\left[\frac{A}{L}L([L^{-1}z_j]\cup \underline{x}\right] \leqq C^{A\ell}C(\ell)L^{-\frac{md}{2}}\\ &\cdot \sum_{\{I_j\}, k, \{m_i\}} C^{Ak}k!\prod_j \frac{C^{m_j - |I_j|}}{(m_j - |I_j|)!} m_j!\alpha^{m_j}.\end{aligned} \tag{18}$$

Now,

$$\begin{aligned}&\sum_{\{I_j\}, k, \{m_j\}} \cdots = \sum_{q=0}^{m}\sum_{p=1}^{\min(\ell, q)}\binom{m}{q}\binom{\ell}{p}\sum_{\substack{\sum_1^p n_i = q\\ n_i \geqq 1}}\frac{q!}{\prod n_i!}\sum_{k=0}^{m-q} C^{Ak}\\ &\cdot \sum_{\sum_{\ell+1}^{\ell+k} n_i = m-q}(k!)^{-1}\frac{(m-q)!}{\prod n_i!}\sum_{\{m_j\}}\prod_{j=1}^{\ell+k}\frac{C^{m_j - n_j}}{(m_j - n_j)!} m_j!\alpha^{m_j}k!,\end{aligned} \tag{19}$$

where by definition $n_i = 0$, $i = p+1, \ldots, \ell$. Since

$$\sum_{m_j > n_j}(C\alpha)^{m_j - n_j}\binom{m_j}{n_j} = (1 - C\alpha)^{-n_j - 1} - 1 \leqq \begin{cases} Cn_j\alpha e^{Cn_j\alpha} \\ C\alpha^{1/2}e^{Cn_j\alpha^{1/2}},\end{cases} \tag{20}$$

we get as in Lemma 2

$$(19) \leqq C^{\ell} m! \alpha^m e^{C\alpha^{1/2}m} L^{-\frac{md}{2}} \sum_{q,p} \alpha^{\frac{1}{2}(1-\delta_{mq})} \sum_{\sum n_i = q} \prod n_i \alpha^{\ell}$$

$$\leqq C^{\ell} m! \alpha^m e^{C\alpha^{1/2}m} L^{-\frac{md}{2}} \sum_{q,p} \left(\frac{q+p}{p}\right)^p \binom{q-1}{p-1} \alpha^{\frac{1}{2}(1-\delta_{mq})+\ell}. \tag{21}$$

If $m \leqq \ell$, $\left(\frac{q+p}{p}\right)^p \binom{q-1}{p-1} \leqq C^{\ell}$ whence

$$(21) \leqq C^{\ell} m! \alpha^{m+\ell} e^{C\alpha^{1/2}m} L^{-\frac{md}{2}}, \tag{22}$$

and the claim follows, for α small.

Let $m > \ell$. Since for $p \leqq q$

$$\left(\frac{q+p}{p}\right)^p < \binom{q+p}{p} 2^p,$$

We get

$$\sum_{q=0}^{\ell} \sum_{p=1}^{q} \left[\frac{p+q}{p}\right]^p \binom{q-1}{p-1} \alpha^{\ell} \leqq C^{\ell} \alpha^{\ell},$$

and

$$\sum_{q=\ell+1}^{m} \alpha^{\frac{1}{2}(1-\delta_{qm})+\ell} \sum_{p=1}^{\ell} \left[\frac{p+q}{p}\right]^p \binom{q-1}{p-1} \leqq C\alpha^{\frac{1}{2}} m C^{\ell} e^{C\alpha m} \leqq C^{\ell} e^{C\alpha^{\frac{1}{2}} m}, \tag{23}$$

thus giving the claim for $m > \ell$. □

Remark. From (16) it is not hard to see that we may replace $C(\ell, A)$ by

$$\prod R_s! \exp[-\varepsilon A L([L^{-1} \underline{z}])] C(A)^{\ell}.$$

This would be useful if we studied the infrared behaviour of n-point functions in more detail.

Now we turn to the study of the thermodynamical limit of our model and the RG-transformation in this limit, that is, Theorem 1. We start from Part (A), i.e. the thermodynamic limit of correlation functions.

We consider first one-point functions. The idea is then to proceed inductively in s using (1). Since we will compute the correlations by iterating the RG, let us start with general localized "one-point functions." Let for each N $F^N(x, \phi)$ be localized at x with constants D, α, given by translation invariant kernels $F_m^N(x, \bar{x}) \bar{x} \in \Lambda_N^m$ and such that $F_m^N(x, \bar{x}) \xrightarrow[N \to \infty]{} F_m(x, \bar{x})$ for each $x, \bar{x}, m$.

An example of such $F^N(x, \phi)$ is ϕ_x^n. We now want to show that the limit of $\langle F^N(x, \phi) \rangle_{V^N}^N$ as $N \to \infty$ exists.

Denoting $S_n \ldots S_1 F^N$ by $F^{N,n}$, we get

$$\langle F^N(x, \phi) \rangle_{V^N}^N = F_0^N(x) + \sum_{\ell=0}^{N-1} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\bar{x}} F_m^{N,\ell}(x, \bar{x}) \langle z_{\bar{x}}^{\ell} \rangle_{\ell} \equiv F_0^N(x) + \sum_{l=0}^{N-1} \mathscr{F}_N^{\ell}(x), \tag{24}$$

where $\langle Z_{\bar{x}}^{\ell}\rangle_{\ell}$ is taken in $\frac{1}{\mathcal{N}}e^{-V^{N-\ell}(z^{\ell})}dv(Z^{\ell})$ and

$$\mathscr{F}_N^{\ell}(x)=\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{x}F_m^{N,\ell}(x,\bar{x})\langle z_{\bar{x}}^{\ell}\rangle_{\ell}. \tag{25}$$

By Lemma 2

$$|\mathscr{F}_N^{\ell}(x)|\leqq CD\delta^{\ell}, \qquad \delta<1. \tag{26}$$

Hence, to establish the limit $N\to\infty$ for $\langle F\rangle_{V^N}^N$ it suffices to show that for each ℓ $\mathscr{F}_N^{\ell}(x)$ converges as $N\to\infty$. This we prove inductively in ℓ. Recall the recursion formulae (5.13) and ([1], (5.12)):

$$\begin{aligned}F_m^{N,\ell+1}([L^{-1}x],\bar{x})=L^{-\frac{md}{2}}&\sum_{k,\{I_j\},\{m_j\}}\prod_{\bar{y},\{\bar{v}_j\}}\prod_{j}\frac{\mathscr{A}(\bar{v}_j,[L^{-1}\bar{v}_j])}{(m_j-|I_j|)!}\\&\cdot F_{m_1}^{N,\ell}(x,\bar{y}_{I_1},\bar{v}_1)\prod_{j=2}^{k}\tilde{V}_{m_j}^{N-\ell}(\bar{y}_{I_j},\bar{v}_j)\left\langle\prod_{j=0}^{k}Z_{[L^{-1}\bar{v}_j]}^{\ell}\right\rangle_{\ell}^{T},\end{aligned} \tag{27}$$

$$\begin{aligned}\tilde{V}_m^{N-\ell-1}(\bar{x})=L^{-\frac{md}{2}}&\sum_{k,\{I_j\},\{m_j\}}\sum_{\bar{y},\{\bar{v}_j\}}\prod_{j}\frac{\mathscr{A}(\bar{v}_j,[L^{-1}\bar{v}_j]}{(m_j-|I_j|)!}\\&\cdot\prod_{j}\tilde{V}_{m_j}^{N-\ell}(\bar{y}_{I_j},\bar{v}_j)\left\langle\prod_{j=1}^{k}Z_{[L^{-1}\bar{v}_j]}^{\ell}\right\rangle_{\ell}^{T}.\end{aligned} \tag{28}$$

Assume inductively that for each $m,\bar{x},y,F_m^{N,\ell}(y,\bar{x})$ and $\tilde{V}_m^{N-\ell}(\bar{x})$ converge to some limits $F_m^{\ell}(y,\bar{x})$, $\tilde{V}_m^{\ell}(\bar{x})$. Since the sums in (27), (28) converge absolutely and are uniformly bounded in N (this was the content of Lemma 2 and an analogous one in [1]!), the induction step follows, provided the correlations $\langle Z_x^{\ell}\rangle_{\ell}$ have $N\to\infty$ limit for each $\bar{x}$. But this is standard, given the cluster expansions for $\langle-\rangle_{\ell}$, established in [1].

To start the induction, let $\ell=0$. From (25) we see that $\mathscr{F}_N^0(x)$ converges since $F_m^{N,0}$ does by definition. $\tilde{V}_m^N(\bar{x})$ converges too, since it is the periodization of $\tilde{V}_m$ which satisfies $|\tilde{V}|_A\leqq\eta$. Hence the thermodynamic limit of $\langle F^N(x,\phi)\rangle_{V^N}^N$ exists, in particular that of $\langle(\phi_x)^n\rangle_{V^N}^N$ does.

Now consider a general expectation

$$G^N(\bar{y})=\langle F_1^N(\phi);\ldots;F_s^N(\phi)\rangle_{V^N}^{T,N}, \tag{29}$$

where the F_i^N are as above. We proceed by induction in s, iterating the expansion (1) (we drop the subscripts from s_i):

$$\begin{aligned}G^N(\bar{y})&=\left\langle\sum_{i=1}^{S}SF_i^N\right\rangle_{V^{N-1}}^{T,N-1}+\sum_{k=1}^{s-1}\sum_{\{I_j\}}\left\langle\prod_{j=1}^{k}F_{I_j}\right\rangle_{V^{N-1}}^{T,N-1}\\&=\sum_{\ell=1}^{[N/2]}\sum_{k=1}^{s-1}\sum_{\{I_j\}}\left\langle\prod_{j=1}^{k}(S^{\ell-1}F)_{I_j}\right\rangle_{V^{N-\ell}}^{T,N-\ell}\\&\quad+\left\langle\prod_{i=1}^{S}S^{[N/2]}F_i^N\right\rangle_{V^{N-\left[\frac{N}{2}\right]}}^{T,N-\left[\frac{N}{2}\right]}.\end{aligned} \tag{30}$$

By the recursion formula (6) the various kernels of $(S^{\ell-1}F)_{I_j}$ have $N\to\infty$ limit and thus by induction the sum on the right hand side has, since it is uniformly bounded in N (as follows from Lemma 3). The second term satisfies

$$\left|\left\langle \prod_{i=1}^{S} S^{[N/2]}F_i^N \right\rangle_{N-\left[\frac{N}{2}\right]}^{T,N-\left[\frac{N}{2}\right]}\right| \leqq C\delta^{sN/2} \xrightarrow[N\to\infty]{} 0 \tag{31}$$

as follows from Lemma 2. Thus $\lim\limits_{N\to\infty} G^N(\bar{y})$ exists. Finally, a general correlation is a linear combination of products of truncated ones and Theorem 1 (A) follows.

Next, we turn to Part B of Theorem 1. We have already shown that

$$\lim_{N\to\infty} (TV^N)_m(\bar{x}) \equiv \lim_{N\to\infty} V_m^{N-1}(\bar{x})$$

exist for all $m, \bar{x}$. Call this limit $(TV)_m(\bar{x})$. We have $|TV|_A \leqq \delta\eta$. Indeed, TV is given by the $N\to\infty$ limit of (28) with $\ell=0$, where $\tilde{V}_{m_j}^N$ will then be replaced by $\tilde{V}_{m_j}$. But we assumed that $|\tilde{V}|_A \leqq \eta$. Thus the same proof which showed that

$$\sum_{\bar{x}} |\tilde{V}_m^{N-1}(\bar{x})| e^{\frac{A}{L}L_p(\bar{x})} \leqq m!(\delta\eta)^m \tag{32}$$

(where we denoted explicitly by p that the distance is taken on the torus Λ_{N-1}) can now be repeated to show

$$\sum_{\bar{x}} |\widetilde{TV}_m(\bar{x})| e^{\frac{A}{L}L(\bar{x})} \leqq m!(\delta\eta)^m \tag{33}$$

(we also need Proposition 2 in the $N\to\infty$ limit, but this is straightforward given the cluster expansion of [1]). Thus there is a state $\langle - \rangle_{TV}$ and $\langle \phi_{\bar{x}} \rangle_{TV}$ is given by the $N\to\infty$ limit of the expansions (27), (28), and (30) (as applied to $\langle \prod F_i \rangle_{TV}$ to start with), i.e. it is uniquely determined by the kernels $(TV)_m$. But so is $\lim\limits_{N\to\infty} \langle \phi_{\bar{x}} \rangle_{VN}^{N-1}$ whose existence follows as that of $\lim\limits_{N\to\infty} \langle \phi_{\bar{x}} \rangle_{VN}^{N}$ proved above. Part B is thus proved.

To prove Part C we proceed similarly. First we write the infinite volume versions of (24), (27), (28), and (30):

$$\langle F(x,\phi)\rangle_V = F_0(x) + \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\bar{x}} (S^\ell F)_m([L^{-1}x], \bar{x}) \langle z_{\bar{x}}^\ell \rangle_\ell, \tag{34}$$

$$\begin{aligned}(S^{\ell+1}F)_m([L^{-\ell-1}x],\bar{x}) &= L^{-\frac{md}{2}} \sum_{k,\{I_j\},\{m_j\}} \sum_{\bar{y},\{\bar{v}_j\}} \\ &\cdot \left(\prod_j \frac{1}{(m_j-|I_j|)!}\right) (S^\ell F)_{m_1}([L^{-\ell}x], \bar{y}_{I_1}, \bar{v}_1) \\ &\cdot \prod_{j=2}^{k} (\widetilde{T^\ell V})_{m_j}(\bar{y}_{I_j}, \bar{v}_j) \langle \prod z_{\bar{v}_j}^\ell \rangle_\ell^T,\end{aligned} \tag{35}$$

$$\begin{aligned}(\widetilde{T^{\ell+1}V})_m(\bar{x}) &= L^{-\frac{md}{2}} \sum_{k,\{I_j\},\{m_j\}} \sum_{\bar{y},\{v_j\}} \prod_j \frac{1}{(m_j-|I_j|)!} \\ &\cdot \prod_j (\widetilde{T^\ell V})_{m_j}(\bar{y}_{I_j}, \bar{v}_j) \left\langle \prod_j z_{\bar{v}_j}^\ell \right\rangle_\ell^T.\end{aligned} \tag{36}$$

Of course $\langle - \rangle_\ell$ is taken in the thermodynamic limit given by the cluster expansion of [1]. Since $|\tilde{V}_n|_A \leqq 2\eta$ for $n \geqq n_0$ and C_n are uniformly bounded, the sums converge absolutely, uniformly in n, when written for $V = V_n$. Since the individual terms converge, we may proceed by induction to show that

$$\langle F(x, \phi) \rangle_{V_n} \xrightarrow[n \to \infty]{} \langle F(x, \phi) \rangle_V .$$

For generalized n-point functions $\langle F_1(x_1, \phi); \ldots; F_s(x_s, \phi) \rangle_V^T$ we write the infinite volume limits of (30) and (6):

$$\left\langle \prod_{i=1}^{s} F_i \right\rangle_V^T = \sum_{\ell=1}^{\infty} \sum_{k=1}^{s-1} \sum_{\{I_j\}} \left\langle \prod_{j=1}^{k} (S^{\ell-1} F)_{I_j} \right\rangle_{T^\ell V}^T , \tag{37}$$

$$\begin{aligned} ((S^{\ell-1} F)_{I_j})_m(\bar{x}) = & \sum_{(\ldots)} \left(\prod_j \frac{1}{(m_j - |I_j|)!} \right) \prod_j (S^{\ell-1} F_j)_{m_j}(\bar{y}_{I_j}, \bar{v}_j) \\ & \cdot \prod_j \widetilde{(T^{\ell-1} V)}_{m_j}(\bar{y}_{I_j}, \bar{v}_j) \left\langle \prod_j z_{\bar{v}_j} \right\rangle_\ell^T . \end{aligned} \tag{38}$$

Again, by Lemmas 2 and 3, the sums are absolutely convergent uniformly in n. Hence the convergence when $n \to \infty$ follows by induction in s. Specializing to $F_i(x_i, \phi) = \phi_{x_i}$ we get the claim of Theorem 1, Part C, the convergence being uniform in $\{x_i\}$ since our bounds are. □

Notice that (34)–(38) together with the cluster expansion for $\langle - \rangle_\ell^T$ provide a sort of inductive infinite volume expansion for our model.

References

1. Gawedzki, K., Kupiainen, A.: Renormalization group study of a critical lattice model I. Convergence to the line of fixed points. Commun. Math. Phys. **82**, 407–433 (1981)

Communicated by A. Jaffe

Received August 5, 1981